

Phase structure of the Kerr-AdS black hole

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We study the critical phenomena of Kerr-AdS black hole. Phase structures are observed at different temperatures, T_L , T_{c1} and T_{c2} with various features. We discuss the thermal stability considering the isothermal compressibility and how phase transitions related to each other. The asymptotic value of the angular momentum also has an implication on separating stable and unstable part. Near critical temperature T_{c1} , the order parameter is determined to calculate the critical exponents. All the critical exponents $(\alpha, \beta, \gamma, \delta) = (0, \frac{1}{2}, 1, 3)$ are identical to that of mean field systems. We plot the phase diagram near this critical point, and discuss the scaling symmetry of the free energy.

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I. INTRODUCTION

During the last few decades, black hole thermodynamics has been playing the role of a “thinking experiment” to understand quantum gravity. The discovery of Hawking radiation shows that the analogy between black hole mechanical laws and the laws of thermodynamics is physically meaningful. Based on this analogy, Davis pioneered to consider the phase transition of RN black holes [1]. Hawking and Page later investigated the phase transition of Schwarzschild-AdS black holes in [2]. Following their path-breaking research, many works have been done along this direction and rich phase structures have been discovered [3–9]. The later-established AdS/CFT duality [10,11] further inspired people to focus on the asymptotically anti-de Sitter (AdS) black holes. Critical phenomena were discovered in asymptotically AdS black holes (see [12–19]).

In [16], Chamblin *et al.* studied the phase structures of the RN-AdS black hole. They identified a critical point in the RN-AdS black hole by considering the divergence of heat capacity. Near this critical point, the behavior of isotherms are similar to that of the van der Waals liquid/gas system. However, the critical exponents of RN-AdS black hole are different from that of the van der Waals case as it has been shown in [18]. A detailed investigation of the phase structure of Kerr-AdS black hole is needed to be compared to the previous results. We hope the report of critical phenomena can help us to achieve the ultimate goal

of finding a microscopic description of the black hole phase structure.

Another motivation of this work came from the ideas of holographic superconductors (see [20,21]) and their rotating extension (see [22]). In [22], Sonner studied the superconducting phase transition on the boundary of Kerr-Newman-AdS black hole. The phase structure of the background field may affect some properties of the rotating holographic superconductor. And the knowledge of phase transitions of Kerr-AdS and RN-AdS black holes could be essential to fully understand the holographic superconductors.

In this paper, we study the phase structure of Kerr-AdS black hole. Rich phase structures were discovered at three diverse critical temperatures, and this multicritical phenomenon in the Kerr-AdS black hole has not been carefully discussed in the previous literatures.

To begin with, we briefly discuss the phase structure by plotting the isotherm and describe the three critical temperatures. We then detailed discuss the critical behavior of each isotherm near the three critical temperature T_L , T_{c1} and T_{c2} , respectively. We determined the asymptotic value of the angular momentum, which is important to understand the thermal stability of the Kerr-AdS black hole.

At a certain temperature T_{c1} , we discovered van der Waals-like phase transition. Unlike the previous case in [18,19], the critical exponents of the Kerr-AdS black hole are found identical to the van der Waals liquid/gas system and the Weiss ferromagnet. It provides a strong evidence that Kerr-AdS black hole system belongs to the universality class which contains these two systems. We also discuss the scaling symmetry of the free energy near this critical point.

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The paper is organized in the following manner. In Sec. II, we introduce the necessary basics of Kerr-AdS black hole and define its thermodynamics quantities. The asymptotic behavior of angular velocity is briefly discussed. The critical isotherms are plotted in Sec. III, in which we briefly discuss the phase structure of Kerr-AdS black hole at T_L , T_{c1} and T_{c2} , respectively. The asymptotic angular velocity has an implication on the thermal stability of the Kerr-AdS black hole through the phase transition happens at T_{c2} . In Sec. IV, we calculate the critical exponents of the newly discovered critical point at T_{c1} can compare it to the well-known phase transition systems, such as Weiss ferromagnet system and van der Waals liquid/gas system. We make the analogy of the free energy and discuss its scaling symmetry. The paper is concluded in Sec. V with discussions and future outlooks.

II. THERMODYNAMICS OF THE KERR-ADS BLACK HOLE

The Kerr-AdS black hole is a rotating black hole in AdS space-time. The exact Kerr-AdS black hole solution of the Einstein equations is given by [23] in the Boyer-Lindquist coordinates as

$$ds^2 = -\frac{\Delta_r}{\Sigma} \left(dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right)^2 + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta \sin^2 \theta}{\Sigma} \left(a dt - \frac{r^2 + a^2}{\Xi} d\phi \right)^2, \quad (1)$$

where

$$\Delta_r = (r^2 + a^2) \left(1 + \frac{r^2}{l^2} \right) - 2Mr, \quad \Xi = 1 - \frac{a^2}{l^2},$$

$$\Delta_\theta = 1 - \frac{a^2}{l^2} \cos^2 \theta, \quad \Sigma = r^2 + a^2 \cos^2 \theta. \quad (2)$$

Here, M is the mass of the Kerr-AdS black hole and a is the rotational parameter related to the angular momentum of the black hole. If we take $a = 0$, the above metric reduces to the Schwarzschild metric.

The curvature radius l is related to the negative cosmological constant Λ by $\Lambda = -3l^{-2}$. And the radius of horizon r_+ is defined by taking $\Delta_r = 0$, i.e.

$$(r_+^2 + a^2) \left(1 + \frac{r_+^2}{l^2} \right) - 2Mr_+ = 0. \quad (3)$$

So the area of event horizon A is

$$A = \frac{4\pi(r_+^2 + a^2)}{\Xi}. \quad (4)$$

A. Thermodynamic quantities of the Kerr-AdS black hole

The thermodynamic quantities of the Kerr-AdS black hole [24] can be expressed in terms of the radius of horizon

r_+ , the rotational parameter a and the cosmological constant Λ ($\Lambda = -3/l^2$). The Hawking temperature of the Kerr-AdS black hole is now

$$T = \frac{3r_+^4 + (a^2 + l^2)r_+^2 - l^2 a^2}{4\pi l^2 r_+ (r_+^2 + a^2)}, \quad (5)$$

while the Bekenstein-Hawking entropy is $S = A/4$.

Considering the first law of the Kerr-AdS black hole

$$dE = TdS + \Omega dJ, \quad (6)$$

one will have to choose the angular velocity Ω measured relative to a frame which is nonrotating at infinity. This quantity is defined as

$$\Omega = \frac{a(1 + r_+^2 l^{-2})}{r_+^2 + a^2}. \quad (7)$$

Now for Eq. (6), one has the ‘‘physical’’ mass (or energy) E and angular momentum J defined as

$$E = \frac{M}{\Xi^2}, \quad J = \frac{Ma}{\Xi^2}. \quad (8)$$

By solving Eq. (3) to determine M , we have

$$J = \frac{a(1 + \frac{r_+^2}{l^2})(a^2 + r_+^2)}{2r_+(1 - \frac{a^2}{l^2})^2}. \quad (9)$$

B. Equation of state and asymptotic behavior of Ω in large J

Because Λ is a constant, we can rescale each quantity to simplify the expressions and calculations,

$$lT \rightarrow T, \quad l\Omega \rightarrow \Omega, \quad \frac{J}{l} \rightarrow J,$$

$$\frac{M}{l} \rightarrow M, \quad \frac{a}{l} \rightarrow a, \quad \frac{r_+}{l} \rightarrow r_+. \quad (10)$$

Thus J , Ω , and T become

$$J = \frac{a(1 + r_+^2)(a^2 + r_+^2)}{2r_+(1 - a^2)^2}, \quad (11)$$

$$\Omega = \frac{a(1 + r_+^2)}{r_+^2 + a^2}, \quad (12)$$

$$T = \frac{3r_+^4 + a^2 r_+^2 + r_+^2 - a^2}{4\pi r_+(r_+^2 + a^2)}. \quad (13)$$

Here we have three equations which govern the phase structure of the Kerr-AdS black hole, which can be regarded as ‘‘equations of state.’’ The functions Ω , J , and T are complicatedly dependent on r_+ and a , so that it is difficult to solve $r_+(J, \Omega)$ and $a(J, \Omega)$ analytically to get the equation of state $T = T(\Omega, J)$ directly.

To begin with, we solve a as a function of T and r_+ to get

$$a = \sqrt{\frac{r_+^2 + 3r_+^4 - 4\pi r_+^3 T}{1 - r_+^2 + 4\pi r_+ T}}. \quad (14)$$

By putting a back into (11) and (12), we obtain the following parametrized expressions for J and Ω :

$$J = \frac{r_+^2 \sqrt{(1 - r_+^2 + 4\pi r_+ T)(1 + 3r_+^2 - 4\pi r_+ T)}}{(1 - 3r_+^2 + 4\pi r_+ T)^2}, \quad (15)$$

$$\Omega = \frac{\sqrt{(1 - r_+^2 + 4\pi r_+ T)(1 + 3r_+^2 - 4\pi r_+ T)}}{2r_+}. \quad (16)$$

These two parametrized equations allow us to discuss the asymptotic behavior of Ω on an isotherm when J goes to infinity. For a fixed T , angular momentum J goes to infinity as $r_+ \rightarrow \frac{1}{3}(2\pi T + \sqrt{3 + 4\pi^2 T^2})$. Taking r_+ to this value, we get

$$\Omega \rightarrow 1. \quad (17)$$

Thus we have the asymptotic value of Ω as one when J goes to infinity.

We name this asymptotic value $\Omega = 1$ as Ω_{c2} because it relates to the critical temperature T_{c2} . This critical behavior at T_{c2} will be further discussed in Sec. III B. In the next section, we will plot the isotherms to study the phase structure of the Kerr-AdS black hole.

III. CARVING THE PHASE STRUCTURE

We present the isotherms at various temperatures in Fig. 1. For each fixed temperature T , we vary r_+ to get an isotherm of (J, Ω) by using Eq. (15) and (16).

The first apparent feature we observed of these isotherms is that above a certain temperature $T_L = 0.2757$, the isotherms become L -shaped with their cusp located on $(J, \Omega) = (0, 0)$, as one can see in Fig. 1. The Kerr-AdS black hole reduces to the Schwarzschild-AdS black hole at

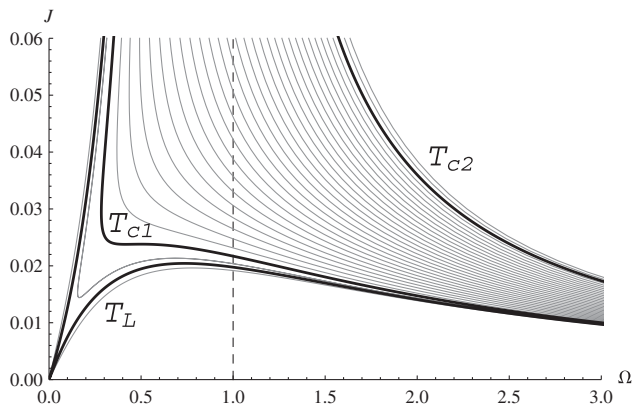


FIG. 1. The isotherms on the (J, Ω) space. Three isotherms have been labeled by their temperatures, T_L , T_{c1} , and T_{c2} . $T_L > T_{c1} > T_{c2}$.

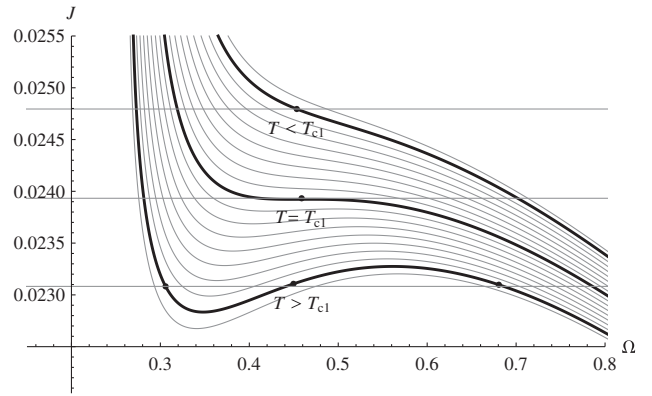


FIG. 2. The critical isotherms near the critical temperature T_{c1} .

this point. The isotherm has positive slope near the $(0, 0)$ point, which makes this part thermally unstable as far as the isothermal compressibility κ_T being considered. This L -shaped critical phenomena also destroy the phase structure of T_{c1} at T_{c1}' ($T_{c1} < T_{c1}' < T_L$), as we will discuss in section IV.

There are two other isotherms being specifically labeled in Fig. 1 with their temperatures T_{c1} and T_{c2} . Two different critical phenomena occur at lower temperatures. To see the phase structure at T_{c1} and T_{c2} clearly, we plot the isotherms around the critical temperatures in different scales in Figs. 2 and 3.

A. Phase structure near critical point at T_{c1}

First, we discuss the phase transition taking place at $T_{c1} = 0.270$. This temperature is determined by solving the standard critical point equations (see [25]),

$$\left(\frac{\partial J}{\partial \Omega}\right)_{T_{c1}} = 0, \quad \left(\frac{\partial^2 J}{\partial \Omega^2}\right)_{T_{c1}} = 0. \quad (18)$$

By using the above equations of state Eq. (15) and (16), we determine $(\Omega_{c1}, J_{c1}) = (0.459, 0.024)$. The isotherms

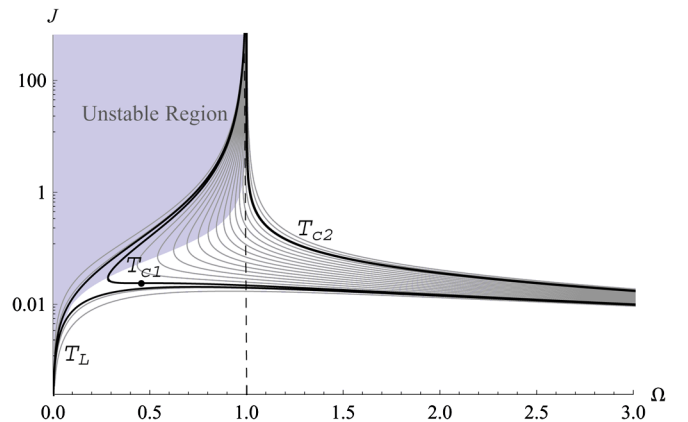


FIG. 3 (color online). The diagram contains all three critical phenomena with their features. Here, J is plot on the log coordinate.

around this critical point are plotted in Fig. 2. At this critical point, the isothermal compressibility $\kappa_T = (\frac{\partial \Omega}{\partial J})_T$ and the specific heat C_J both diverge. And above T_{c1} , a single J corresponds to multiple Ω s for each fixed temperature.

Figure 2 is similar to the liquid/gas PVT diagram [25]. This van der Waals-like phase transition can be clearly visualized as we choose the $J \rightarrow P$, $\Omega \rightarrow V$ correspondence. We will make a detailed study of this critical point in Sec. IV.

B. Phase structure near critical point at T_{c2}

The other critical temperature, $T_{c2} = 0.159$, is determined when an isotherm has its slope go to infinity ($\kappa_T \rightarrow 0$) when $J \rightarrow \infty$. Below the temperature T_{c2} , the isothermal compressibility κ_T is always negative along each isotherm, which means that the black hole is stable under this consideration. But as one raises the temperature to above T_{c2} , positive isothermal compressibility emerge on the upper part of the curves, making the black hole thermally unstable. We highlight the unstable region as the shadowed region in Fig. 3 as well as the critical isotherm at T_{c2} . The J value is plotted on the log coordinate, while Ω is on the original coordinate, to better visualize the critical behavior of isotherms near T_L , T_{c1} , and T_{c2} .

One can understand the critical behavior at T_{c2} in another manner. The asymptotic angular velocity, Ω_{c2} , also marks the critical angular momentum where the points on the isotherms could become thermally unstable in the upper part, as we can see in Fig. 3. On the right side of the $\Omega = \Omega_{c2}$ line, all the isotherms are stable on the upper part. But on the left side of $\Omega = \Omega_{c2}$, the unstable parts start to emerge on the upper part of the isotherms. It is an important feature: the Kerr-AdS black hole not only has an asymptotic angular velocity, but this asymptotic angular velocity also determine the thermal stability as far as isothermal compressibility is considered.

At the end of this section, we summarize the phase structure of the Kerr-AdS black hole with the aid of Fig. 3. The isotherm at T_L has its cusp located on $(0, 0)$. The critical isotherm at T_{c1} has zero slope at its critical point (the black point in the figure) and then slowly bending downward as Ω increases. The isotherm at T_{c2} has infinite slope when J is very large. One can also see the asymptotic/critical value $\Omega_{c2} = 1$ which separates the stable and the unstable regions in Fig. 3.

IV. THE PHASE STRUCTURE NEAR T_{c1}

In this section, we consider the phase transition which happens above critical temperature T_{c1} . In Fig. 2, this van der Waals-like phase structure can be clearly visualized. This critical point happens when we choose a grand canonical (fixed horizon angular velocity Ω) ensemble. The critical phenomena has not been discussed carefully in the

previous literatures. As pointed out by [26], the phase structure of black holes strongly depends on the choice of the ensembles. We study the quantitative properties of this newly found second-order phase transition, and compare it to well-known systems.

A. Determination of the critical point

The first critical point can be determined by of the following conditions:

$$\left(\frac{\partial J}{\partial \Omega}\right)_T = 0, \quad \left(\frac{\partial^2 J}{\partial \Omega^2}\right)_T = 0, \quad (19)$$

which can be expressed as functions of r_+ and T by Eq. (15) and (16),

$$\left(\frac{\partial J}{\partial \Omega}\right)_T = \frac{(\frac{\partial J}{\partial r_+})_T}{(\frac{\partial \Omega}{\partial r_+})_T} = J_\Omega(r_+, T), \quad (20)$$

$$\left(\frac{\partial^2 J}{\partial \Omega^2}\right)_T = \frac{(\frac{\partial^2 J}{\partial r_+^2})_T}{(\frac{\partial \Omega}{\partial r_+})_T} = J_{\Omega^2}(r_+, T). \quad (21)$$

Solving the above two equations, we find that at the critical point $T = T_{c1} \equiv 0.270$ and $r_+ = r_{c1} \equiv 0.459$. Then we obtain $(\Omega_{c1}, J_{c1}) = (0.459, 0.024)$.

B. Order parameter and law of equal area

We will define the order parameter to describe the critical behavior near the critical point. Above the critical temperature T_{c1} , there are three points on an isotherm having the same J but with the different Ω s. Like the van der Waals system, one can define an order parameter by Maxwell's equal-area law. As in Fig. 4, we choose an isotherm with $T > T_{c1}$ and draw a horizontal line which intersect the isotherm at three points a, m and b . When the area A is equal to the area B, the value $\eta = (\Omega_b - \Omega_a)/2$ is defined as the order parameter. This method also allows

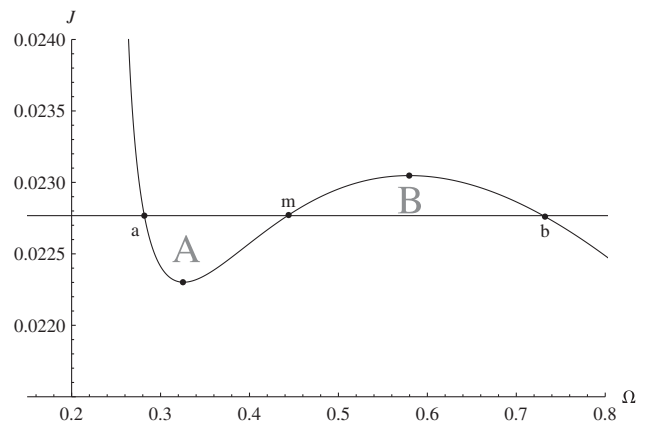


FIG. 4. Applying the equal-area law on an isotherm at a temperature $T > T_{c1}$.

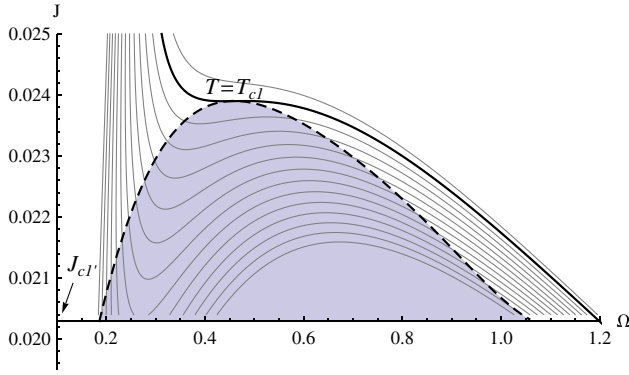


FIG. 5 (color online). The dashed curve is the coexistence curve of phase transition above T_{c1} . The coexistence curve ends at $J_{c1'}$, because the stable area is too small to apply the equal-area law at the corresponding temperature $T_{c1'}$.

us to plot the coexistence curve as presented in Fig. 5. At $T_{c1'} = 0.2735$ (determined numerically), the area A of the left part becomes too small to achieve Maxwell's equal-area law, therefore the phase transition ends at the temperature $T_{c1'}$.

For the mean field systems, spontaneous magnetism is the order parameter in Weiss ferromagnetic system, while the half width of the coexistence curve is the order parameter in the van der Waals system [25].

C. Critical exponents

With the order parameter $\eta = (\Omega_a - \Omega_b)/2$ which we defined in the previous section, we will calculate the following well-known critical exponents in this section.

Degree of critical isotherm:

$$J - J_{c1} = A_\delta |\Omega - \Omega_{c1}|^\delta \text{sign}(\Omega - \Omega_{c1}), \quad T = T_{c1}. \quad (22)$$

Degree of coexistence curve:

$$\eta = -A_\beta (T - T_{c1})^\beta, \quad T > T_{c1}. \quad (23)$$

Degree of heat capacity ($\Omega = \Omega_{c1}$):

$$C_\Omega = \begin{cases} c]llA_{\alpha'} \{-(T - T_{c1})\}^{-\alpha'}, & T < T_{c1} \\ A_{\alpha} \{+(T - T_{c1})\}^{-\alpha}, & T > T_{c1} \end{cases}. \quad (24)$$

Degree of isothermal compressibility:

$$\kappa_T = \begin{cases} A_{\gamma'} \{-(T - T_{c1})\}^{-\gamma'}, & T < T_{c1} \\ A_{\gamma} \{+(T - T_{c1})\}^{-\gamma}, & T > T_{c1} \end{cases}. \quad (25)$$

Following the discussion in [25], the definitions of the degree of isothermal compressibility κ_T for $T < T_{c1}$ and $T > T_{c1}$ are different. For $T < T_{c1}$, γ' is defined along the isoangular momentum line, i.e., $\Omega = \Omega_{c1}$; when $T > T_{c1}$, γ is defined along the coexistence curve, which is illustrated in Fig. 5.

Next, we will calculate these critical exponents of the phase transitions in the Kerr-AdS black hole system one by one.

1. Degree of critical isotherm

At this critical point $T = T_{c1}$, the first and second derivatives of J over Ω satisfy

$$\left(\frac{\partial J}{\partial \Omega} \right)_T \Big|_{c1} = \left(\frac{\partial^2 J}{\partial \Omega^2} \right)_T \Big|_{c1} = 0. \quad (26)$$

The third order derivative can be calculated as

$$\left(\frac{\partial^3 J}{\partial \Omega^3} \right)_T \Big|_{c1} = -0.539 \neq 0, \quad (27)$$

which is not vanishing, therefore we get $\delta = 3$.

2. Degree of coexistence curve

In Fig. 5, we plot the curve of the coexisting states by Maxwell's equal-area law. This curve indicates the temperature dependence of the order parameter right after the order parameter emerges.

We expand J in terms of Ω and T to the third order as

$$\begin{aligned} J - J_{c1} &\approx (\partial_T J)_\Omega|_c (T - T_{c1}) + (\partial_T (\partial_\Omega J)_T)_\Omega|_c \\ &\times (\Omega - \Omega_{c1})(T - T_{c1}) + \frac{1}{2} (\partial_T (\partial_\Omega^2 J)_T)_\Omega|_c \\ &\times (\Omega - \Omega_{c1})^2 (T - T_{c1}) + \frac{1}{2} (\partial_T^2 (\partial_\Omega J)_T)_\Omega|_c \\ &\times (\Omega - \Omega_{c1})(T - T_{c1})^2 + \frac{1}{3!} (\partial_\Omega^3 J)_T|_c \\ &\times (\Omega - \Omega_{c1})^3. \end{aligned} \quad (28)$$

To simplify the calculation, we define

$$j = J - J_{c1}, \quad t = T - T_{c1}, \quad \omega = \Omega - \Omega_{c1}, \quad (29)$$

so that

$$j = c_{10}t + c_{11}\omega t + c_{21}\omega t^2 + c_{12}\omega^2 t + c_{03}\omega^3. \quad (30)$$

All coefficients c_{ij} in Eq. (30) can be calculated by the standard partial differentiation. The results are

$$\begin{aligned} c_{10} &= -1.027, & c_{11} &= 3.684, & c_{12} &= -15.792, \\ c_{21} &= -458.833, & c_{03} &= -0.090. \end{aligned} \quad (31)$$

As we will see in the following discussions, the critical exponents are not sensitive to the exact values of these constants.

By applying the conditions of equal angular momentum and equal area,

$$j(\omega_a, t) = j(\omega_b, t), \quad \int_{\omega_b}^{\omega_a} \omega dj = 0, \quad (32)$$

we found that

$$-(\omega_a - \omega_b)[c_{11}t + c_{21}t^2 + c_{12}t(\omega_a + \omega_b) + c_{03}(\omega_a^2 + \omega_a\omega_b + \omega_b^2)] = 0, \quad (33)$$

$$-(\omega_a - \omega_b)[6c_{11}t(\omega_a + \omega_b) + 6c_{21}t^2(\omega_a + \omega_b) + 8c_{12}t(\omega_a^2 + \omega_a\omega_b + \omega_b^2) + 9c_{03}(\omega_a^3 + \omega_a^2\omega_b + \omega_a\omega_b^2 + \omega_b^3)] = 0. \quad (34)$$

Changing the variables as

$$\omega_- \equiv \omega_b - \omega_a = \Omega_b - \Omega_a, \quad \omega_+ \equiv \omega_b + \omega_a, \quad (35)$$

one can solve ω_+ and ω_- as

$$\omega_+ = -\frac{2c_{12}t}{3c_{03}}, \quad (36)$$

$$\omega_- = \sqrt{\frac{-4c_{11}t + \left(\frac{4c_{12}^2}{3c_{03}} - 4c_{21}\right)t^2}{c_{03}}}. \quad (37)$$

Now we can expand t in term of ω_- as

$$t = -\frac{c_{11}c_{03}}{4c_{11}^2}\omega_-^2 + O(\omega_-^4), \quad (38)$$

to get the order parameter

$$\frac{\Omega_b - \Omega_a}{2} \approx A_\beta(T - T_{c1})^{(1/2)}. \quad (39)$$

Thus we read $\beta = \frac{1}{2}$.

3. Critical exponent of heat capacity

We now consider the critical exponent of the heat capacity along the constant angular velocity line $\Omega = \Omega_c$. By the black hole thermal dynamic laws, the role of the internal energy Q is played by the black hole mass E of the Kerr-AdS black hole. E is given by Eq. (8) and is rescaled by Eq. (10) as

$$E = \frac{r_+^3 - r_+^5 + 4\pi r_+^4 T}{(1 - 3r_+^2 + 4\pi r_+ T)^2}. \quad (40)$$

The heat capacity C_Ω can be calculated as

$$C_\Omega = \left(\frac{\partial E}{\partial T}\right)_\Omega \Big|_{c_1} = -2.880 \neq 0. \quad (41)$$

Therefore, α and α' are both zero because the heat capacity neither diverges nor vanishes, i.e. $\alpha = \alpha' = 0$.

4. Degree of isothermal compressibility

The isothermal compressibility κ_T is defined as

$$\kappa_T = \left(\frac{\partial \Omega}{\partial J}\right)_T, \quad (42)$$

which is divergent at the critical point. To do the Taylor expansion, we consider the inverse of the isothermal compressibility

$$\kappa_T^{-1} = \begin{cases} A\{-(T - T_{c1})\}^{\gamma'}, & (T < T_{c1}) \\ A\{+(T - T_{c1})\}^{\gamma}, & (T > T_{c1}) \end{cases}. \quad (43)$$

Using Eq. (30), we have

$$\kappa_T^{-1} \propto \left(\frac{\partial j}{\partial \omega}\right)_t \approx c_{11}t. \quad (44)$$

The first order dependence is correct for both $T > T_{c1}$ (along the isoangular velocity line, on which $\omega = \omega_c$) and $T < T_{c1}$ (along the coexistence curve, on which $\omega \propto t^{(1/2)}$). Therefore, we get $\gamma = \gamma' = 1$.

D. Free energy and comparison with mean field models

In Table. I, we compared the critical exponents for various systems. It is interesting to take the point of view of the mean field theory to look at the results we got in the previous sections.

In Eq. (30), we expanded the equation of state of Kerr-AdS black hole in the vicinity of the critical point. With the definition of $\epsilon = (T - T_{c1})/T_{c1}$ and the order parameter $\eta = (\Omega - \Omega_{c1})/\Omega_{c1}$, the equation of state becomes

$$j = \tilde{c}_{10}\epsilon + \tilde{c}_{11}\epsilon\eta + \tilde{c}_{21}\epsilon^2\eta + \tilde{c}_{12}\epsilon\eta^2 + \tilde{c}_{03}\eta^3, \quad (45)$$

where \tilde{c}_{10} , \tilde{c}_{11} , \tilde{c}_{21} , \tilde{c}_{12} , \tilde{c}_{03} are constant coefficients.

The equation of state can be derived from the free energy,

$$g(j, \epsilon, \eta) = g_0(j, \epsilon) - (j - c_{10}\epsilon)\eta + (c_{11}\epsilon + c_{21}\epsilon^2) \times \frac{1}{2}\eta^2 + c_{12}\epsilon\frac{1}{3}\eta^3 + c_{03}\frac{1}{4}\eta^4. \quad (46)$$

Now we compare our results with some mean field systems.

The first system is the Weiss ferromagnet system, see [27]. The equation of state is

TABLE I. The comparison of critical exponents.

Exponent	Weiss/van der Waals	Kerr-AdS T_{c1}	Ising($d = 2$)	Ising($d = 3$)
α	0	0	0	0.110(5)
β	1/2	1/2	1/8	0.325 + 0.0015
γ	1	1	7/4	1.2405 + 0.0015
δ	3	3	15	4.82(4)

$$\frac{H}{k_B T} = M(1 - \tau) + M^3(\tau - \tau^2 + \tau^3/3 + \dots), \quad (47)$$

where M is the average magnetic moment, H is the external field, and τ^{-1} is defined as T/T_c .

In terms of $\epsilon = (T - T_c)/T_c$ as well as order parameter $\eta = M$, this equation of state can be written as

$$\frac{H}{k_B T} = \eta\epsilon + \eta^3 + O(\epsilon\eta^3), \quad (48)$$

which can be derived from the free energy Γ as

$$\Gamma(\eta, T, H) = \Gamma_0(T, H) - \frac{\eta H}{k_B T} + \frac{\epsilon\eta^2}{2} + \frac{1}{4}\eta^4. \quad (49)$$

Another mean field system we will consider is the van der Waals gas/liquid system, see also [27]. The equation of state is

$$\pi = \frac{P - P_c}{P_c} = 4\epsilon + 6\epsilon\eta + \frac{3}{2}\eta^3 + O(\eta^4, \eta^2\epsilon), \quad (50)$$

where $\epsilon = (T - T_c)/T_c$ and the order parameter $\eta = -(V - V_c)/V_c$.

The above equations of state can be derived from the Gibbs free energy G ,

$$G(p, T, \eta) = G_0(p, T) + \frac{N}{\rho_0^2} \left[-(\pi - 4\epsilon)\eta + 3\epsilon\eta^2 + \frac{3}{8}\eta^4 \right]. \quad (51)$$

The similarity of Eq. (46), (49), and (51) explains the identical critical exponents in different systems. All systems we considered above can be incorporated into the classical Landau theory [27].

E. Widom scaling

Now we consider the scaling symmetry of the free energy around the critical point. We first separate the free energy into two parts,

$$g(\epsilon, j) = g_r(\epsilon, j) + g_s(\epsilon, j). \quad (52)$$

Function $g_r(\epsilon, j)$ is the regular part which does not change when approaching the critical point, while $g_s(\epsilon, j)$ is the singular part which possesses the singular behavior of the system in the vicinity of the critical point. Assuming that the singular part is a generalized homogeneous function of its parameters, through Eq. (38), (45), and (46), we can write down the free energy near the critical point as

$$g_s(\epsilon, j) = c_\epsilon \epsilon^2 + c_j j^{4/3}, \quad (53)$$

which has the scaling symmetry,

$$g_s(\Lambda^p \epsilon, \Lambda^q j) = \Lambda g_s(\epsilon, j), \quad p = \frac{1}{2}, \quad q = \frac{3}{4}. \quad (54)$$

Following the discussion of [25], we can write the critical exponents in terms of p and q as

$$\alpha = \frac{2p - 1}{p}, \quad (55)$$

$$\beta = \frac{1 - q}{p}, \quad (56)$$

$$\gamma = \frac{2q - 1}{p}, \quad (57)$$

$$\delta = \frac{q}{1 - q}. \quad (58)$$

From the above equations, we find that the critical exponents satisfy the following expected relations:

$$\alpha + 2\beta + \gamma = 2, \quad \alpha + \beta(\delta + 1) = 2, \\ \gamma(\delta + 1) = (2 - \alpha)(\delta - 1), \quad \gamma = \beta(\delta - 1). \quad (59)$$

The above scaling relations stand as the consistency check for the critical exponents we obtained in the previous section.

F. Phase diagram

In Fig. 6, we present the phase diagram near critical point (T_{c1}, J_{c1}) on the T - J plane. One can see the phase transition line and the critical point. The liquid/gaslike phase transition occurs when crossing the phase transition line. This phase structure occurs at the temperature above T_{c1} , and ends when the temperature reaches T_{c1}' , at which the Maxwell's equal-area law fails to apply on the isotherm due to the fact that the stable part is too small.

Although this phase diagram is similar to that of the liquid/gas phase transition, this critical phenomenon happens when raising the temperature above T_{c1} , rather than lowering the temperature below the critical temperature. The phase transition line thus points to a different direction

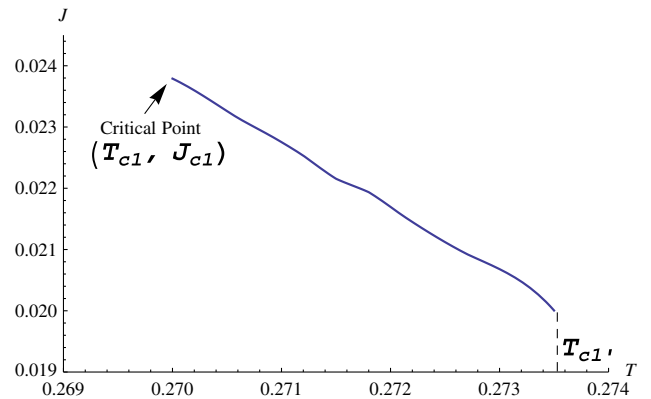


FIG. 6 (color online). Phase diagram near critical point T_{c1} . Similar to the phase diagram of liquid/gas phase transition [25,28].

as that of liquid/gas system [25,28]. This is an intriguing feature of the van der Waals-like critical point for both the RN-AdS black hole [18,19] and the Kerr-AdS black hole.

V. DISCUSSION

In this paper, we studied the phase structure of the Kerr-AdS black hole. Rich critical phenomena have been found at the different temperatures T_L , T_{c1} and T_{c2} . Based on the isotherms, we discussed the physical meaning of each critical point. The asymptotic angular momentum Ω_{c2} was also discussed and found to be associated to the critical behavior at T_{c2} . We have studied the critical behavior at the critical temperature $T = T_{c1}$ in a great detail and found the analogy between the critical behavior at T_{c1} and the van der Waals system. We also calculated the critical exponents. The critical exponents $(0, \frac{1}{2}, 1, 3)$ are the same as that of mean field systems, which means that they are in the same universality class. The corresponding scaling symmetry of free energy has also been discussed. We finally plot the phase diagram of the Kerr-AdS black hole.

The multicritical phenomena we described in this paper could be a guide which provides a restriction for a complete theory of underlying mechanism. We think that the similar van der Waals-like structure, but the different critical exponents of the RN-AdS and the Kerr-AdS black holes, deserves further study. The natural extension, the KN-AdS black hole, would be the next target to be understood.

Our results are interesting under the context of AdS/CFT duality. One can study a strong correlation system dual to the Kerr-AdS black hole. In [22], Sonner found that superconducting-like condensation also exists on the boundary of the KN-AdS black hole. He found that there is a critical value of rotation which could destroy the superconductivity in analogy to the critical magnetic field. It is natural to guess that the KN-AdS black hole has a similar phase structure as the Kerr-AdS and the RN-AdS black holes, which will in some way affect the thermal stability of the condensation. It would be interesting to conduct further study along this direction.

In recent years, people managed to rewrite the field equations of gravity into thermodynamics identities (for a review, see [29]). In [30], Verlinde suggested that the gravity could be understood as the entropy force induced by the inner freedom of the holographic screen. These observations hope to regard gravity theory as a result of more fundamental thermodynamic principles. It could be possible to use the ideas to explain the thermal stability and phase structure of the Kerr-AdS black hole.

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