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The initial value problem for some hyperbolic-dispersive system

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We consider the initial value problem for some nonlinear hyperbolic-dispersive systems in one space dimension. Combining the classical energy method and the smoothing estimates for the Airy equation, we guarantee the time local well-posedness for this system. We also discuss the extension of our results to more general hyperbolic-dispersive system.

Keywords: hyperbolic and dispersive system; smoothing effect

1. Introduction

We consider local well-posedness for the initial value problem of the hyperbolic-dispersive system

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x v + v\partial_x u = 0, & t, x \in \mathbf{R}, \\ \partial_t v + \partial_x w + u\partial_x u + v\partial_x v = 0, & t, x \in \mathbf{R}, \\ \partial_t w + \partial_x^3 w + u\partial_x u + w\partial_x w = 0, & t, x \in \mathbf{R}, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad w(0,x) = w_0(x), & x \in \mathbf{R}, \end{cases} \tag{1}$$

where u, v, and w are real valued unknown functions. The notion of well-posedness used here includes the existence and uniqueness of a solution, and its continuous dependence upon the initial data.

System (1) is the modified version of the following system proposed by Lin-Wong [1]:

$$\begin{cases} \partial_t u + i\partial_x^2 u + u\partial_x v + v\partial_x u = 0, & t, x \in \mathbf{R}, \\ \partial_t v + \partial_x w + \frac{1}{2}\partial_x |u|^2 + v\partial_x v = 0, & t, x \in \mathbf{R}, \\ \partial_t w + \partial_x^3 w + \frac{1}{2}\partial_x |u|^2 + w\partial_x w = 0, & t, x \in \mathbf{R}, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad w(0,x) = w_0(x), & x \in \mathbf{R}, \end{cases} \tag{2}$$

where u is the complex valued function, and v and w are the real valued functions. They derived (2) to study the zero-dispersion limit of the water wave equations that arise in modeling surface waves in the presence of both gravity and capillary modes. We replaced the second derivative $i\partial_x^2 u$ in the first equation of (2) by the third derivative $\partial_x^3 u$. We briefly explain why we make the above modification. Letting $u_1 = \Re u$, $u_2 = \Im u$ be the real and imaginary parts of u and $u_3 = v$ in (2), we obtain the following system:

$$\begin{pmatrix} \partial_t u_1 \\ \partial_t u_2 \\ \partial_t u_3 \end{pmatrix} + \begin{pmatrix} 0 & -\partial_x^2 & 0 \\ \partial_x^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} u_3 & 0 & u_1 \\ 0 & u_3 & u_2 \\ u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} \partial_x u_1 \\ \partial_x u_2 \\ \partial_x u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\partial_x w \end{pmatrix},$$

$$\partial_t w + \partial_x^3 w + u_1\partial_x u_1 + u_2\partial_x u_2 + w\partial_x w = 0. \tag{3}$$

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There are several results on the solvability for the system of hyperbolic and dispersive equations. In [2], they study the time local well-posedness of the interaction equation between short and long waves by applying Bourgain's Fourier restriction norm method. In [3–5], they study the local and global solvability for Benney type system by successfully using the energy method. We notice that the equations treated in those papers do not contain the linear coupled derivative term. As will be discussed in detail below, the coupled term $-\partial_x w$ in (2) prevents us from using the classical energy method of the quasi-linear hyperbolic system or the contraction principle via the integral equation. Therefore, it seems that it is extremely difficult to prove the well-posedness of (2). In this note, modifying the system (2) into (1) and employing the theory of dispersive equation, we prove the well-posedness for (1). This is an intermediate step of the study of the zero-dispersion limit of the water-wave equation. In addition to proving the well-posedness of (1), we consider the well-posedness of more general hyperbolic-dispersive systems including the linear coupled derivative term.

The notable difference between the hyperbolic equations and the dispersive equations is the gain of the regularity of their solutions. More precisely, Kenig–Ponce–Vega [6] derived the following smoothing property of a solution to the linear dispersive equations: Let $\{V_m(t)\}_{t\in\mathbf{R}}$ be a unitary group generated by $i(-i\partial_x)^m$. Then we have

$$\|D_x^{(m-1)/2}V_m(t)\phi\|_{L_x^\infty L_T^2} \le C\|\phi\|_{L_x^2}, \tag{4}$$

$$\|D_x^{m-1}\int_0^t V_m(t-\tau)F(\tau)\mathrm{d}\tau\|_{L_x^\infty L_T^2} \le C\|F\|_{L_x^1 L_T^2}. \tag{5}$$

Those estimates tell us that for large m, the operator $\{V_m(t)\}_{t\in\mathbf{R}}$ induces the strong smoothing effect. Thanks to the smoothing property of solutions to the Airy equation $\partial_t u + \partial_x^3 u = 0$ (Lemma 2.2 below), we can control the worse term $-\partial_x w$ and guarantee the local well-posedness of (1).

Before stating main theorem, we introduce several notations and function spaces. We denote the Fourier and its inverse transforms by $\mathcal{F}$ and $\mathcal{F}^{-1}$:

$$\mathcal{F}[f](\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{-ix\cdot\xi}f(x)\mathrm{d}x, \quad \mathcal{F}^{-1}[f](x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{+ix\cdot\xi}f(\xi)\mathrm{d}\xi.$$

Let $\langle x\rangle^n = (1+|x|^2)^{n/2}$. The operators D_x^m and $\langle D_x\rangle^m$ are given by $D_x^m = \mathcal{F}^{-1}|\xi|^m\mathcal{F}$ and $\langle D_x\rangle^m = \mathcal{F}^{-1}\langle\xi\rangle^m\mathcal{F}$. Let $H_x^{m,n}$ be the weighted Sobolev space: $H_x^{m,n} = \{u \| u\|_{H_x^{m,n}} = \|\langle x\rangle^n\langle D_x\rangle^m u\|_{L_x^2} < \infty\}$. For $1 \le p,q \le \infty$, let $L_T^pL_x^q = L_T^p(0,T;L_x^q(\mathbf{R}))$ and $L_x^pL_T^q = L_x^p(\mathbf{R};L_t^q(0,T))$. The main result in this note is the following:

Theorem 1.1
There exists $\epsilon > 0$ such that for any $(u_0,v_0,w_0) \in (H_x^{6,0}\cap H_x^{3,1})\times(H_x^{6,0}\cap H_x^{1,1})\times(H_x^{7,0}\cap H_x^{2,1})$ and $\|u_0\|_{H_x^{6,0}} + \|u_0\|_{H_x^{3,1}} + \|v_0\|_{H_x^{6,0}} + \|v_0\|_{H_x^{1,1}} + \|w_0\|_{H_x^{7,0}} + \|w_0\|_{H_x^{2,1}} < \epsilon$ the initial value problem (1) has a unique solution $(u(\cdot),v(\cdot),w(\cdot))$ defined in the interval $[0,T]$, $T = T(\|u_0\|_{H^{6,0}},\|v_0\|_{H_x^{6,0}},\|w_0\|_{H_x^{7,0}})$ satisfying

$$(u,v,w) \in C([0,T];H_x^{6,0}\cap H_x^{3,1})\times C([0,T];H_x^{6,0}\cap H_x^{1,1})\times C([0,T];H_x^{7,0}\cap H_x^{2,1}) \equiv X_T.$$

Moreover, for any $T' \in (0,T)$ there exists $\delta > 0$ such that the map $(\tilde{u}_0,\tilde{v}_0,\tilde{w}_0) \to (\tilde{u}(t),\tilde{v}(t),\tilde{w}(t))$ from $\{(\tilde{u}_0,\tilde{v}_0,\tilde{w}_0)\| \|\tilde{u}_0-u_0\|_{H_x^{6,0}} + \|\tilde{v}_0 - v_0\|_{H_x^{6,0}} + \|\tilde{w}_0 - w_0\|_{H_x^{7,0}} < \delta\}$ into $X_{T'}$ is Lipschitz continuous.

Remark 1.2
In Theorem 1.1, we assumed the smallness condition on the initial data. The reason is as follows: We establish Theorem 1.1 by applying the Banach fixed point theorem to the corresponding integral equation. In this proof, to close the estimate, we need to regard the coupling term $-\partial_x w$ as a perturbation and evaluate it by making use of the smoothing effect of the dispersive term. In this step, we have to impose the smallness condition not only on the time interval T but also on the initial data. The large data well-posedness of (1) is an issue in the future.

Remark 1.3
It is natural to arise the following question: Does (1) have a unique local solution for smooth large data? If the system has some symmetry and the coupling term has special structure such as the Benney type system, then the answer is affirmative. However, we do not know whether the general hyperbolic-dispersive system including (1) has a large smooth solution or not.

We give the outline of the proof of Theorem 1.1. As we explained in the introduction, the difficulty comes from the coupling term $-\partial_x w$ in the second equation on (1).

The second equation includes the linear term $\partial_x w$ and quadratic term $u\partial_x u$. Therefore, the reader might think that we need to assume $(u,w) \in L_T^\infty H_x^{s+1,0}\times L_T^\infty H_x^{s+1,0}$ to control the $L_T^\infty H^s$ norm of v. Indeed, we need to assume $w \in L_T^\infty H_x^{s+1,0}$ to guarantee that $L_T^\infty H^s$ norm of v is bounded because of the linear term $\partial_x w$. However, thanks to the quadratic structure, the smoothing effect of the Airy group, and the inclusion $L_x^2L_T^\infty\cdot L_x^\infty L_T^2 \subset L_x^2L_T^2$, it suffices to assume $u \in L_T^\infty H_x^{s,0}$ to guarantee $v \in L_T^\infty H_x^{s,0}$.

On the other hand, the third equation includes the derivative term $u\partial_x u$. Therefore, we meet the loss of the second order derivatives to evaluate $w \in L_T^\infty H_x^{s+1,0}$. Thanks to the smoothing effect for the Airy group of the inhomogeneous type that gains the *"two"* derivatives (see (5) with $m = 3$), we can estimate this term using only the fact $u_0 \in H_x^{s,0}$. We note that the weak smoothing property of the free

Schrödinger group does not overcome the loss of two derivatives. Therefore, we replaced the second derivatives in the first equation of (2) by the third derivatives. In the nonlinear estimate, the inclusion $L^1_x L^\infty_T \cdot L^\infty_x L^2_T \subset L^1_x L^2_T$ appears. Therefore, we need to estimate the maximal function ($L^1_x L^\infty_T$–type estimate, see Lemma 2.3 below). Because the quantity $\|\cdot\|_{L^1_x L^\infty_T}$ is not expected to be small even when $T \downarrow 0$, we need the smallness assumption on the initial datum. We note that the estimate of the maximal function gives the regularity and weight conditions on the initial datum.

We rewrite the first and third equations in (1) into the integral equations to apply the smoothing property of the Airy group, and we employ the standard energy method to the second equation in (1). Therefore, we introduce the following linearization that is not the standard linearization. As we explained in the introduction, the coupling term $\partial_x w$ prevents us from using the standard linearization.

For (u_0, v_0, w_0), we denote by $(u, v, w) = \Phi(\tilde{u}, \tilde{v}, \tilde{w})$ the solution to the linearized problem

$$\begin{cases} \partial_t u + \partial_x^3 u + \tilde{u}\partial_x \tilde{v} + \tilde{v}\partial_x \tilde{u} = 0, & t, x \in \mathbf{R}, \\ \partial_t v + \partial_x \tilde{w} + \tilde{u}\partial_x \tilde{u} + \tilde{v}\partial_x v = 0, & t, x \in \mathbf{R}, \\ \partial_t w + \partial_x^3 w + \tilde{u}\partial_x \tilde{u} + \tilde{w}\partial_x \tilde{w} = 0, & t, x \in \mathbf{R}, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad w(0,x) = w_0(x), & x \in \mathbf{R}. \end{cases} \tag{6}$$

Define

$$Z^a_T \equiv \{(u, v, w) |\ \|(u, v, w)\|_{Z_T} \le a\},$$

where

$$\begin{aligned} \|(u, v, w)\|_{Z_T} \equiv\ & \|u\|_{L^\infty_T H^{6,0}_x} + \|u\|_{L^\infty_T H^{3,1}_x} + \|\partial_x^7 u\|_{L^\infty_x L^2_T} + \|u\|_{L^1_x L^\infty_T} + \sum_{j=0}^{1} \|\partial_x^j u\|_{L^2_x L^\infty_T} \\ & + \|v\|_{L^\infty_T H^{6,0}_x} + \|v\|_{L^\infty_T H^{1,1}_x} + \|v\|_{L^2_x L^\infty_T} \\ & + \|w\|_{L^\infty_T H^{7,0}_x} + \|w\|_{L^\infty_T H^{2,1}_x} + \|\partial_x^8 w\|_{L^\infty_x L^2_T} + \sum_{j=0}^{1} \|\partial_x^j w\|_{L^2_x L^\infty_T}. \end{aligned}$$

It will be established that for appropriate a and T if $(\tilde{u}, \tilde{v}, \tilde{w}) \in Z^a_T$, then the solution $(u, v, w) = \Phi(\tilde{u}, \tilde{v}, \tilde{w})$ to (6) belongs to Z^a_T and Φ is a contraction on Z^a_T.

In the next section, we list some linear estimates including the estimate for the smoothing effect and the maximal function for the Airy group. In the third section, we show the crucial nonlinear estimate and prove Theorem 1.1.

We can extend Theorem 1.1 to more general hyperbolic-dispersive system. We shall discuss the generalization of our result in Section 4.

2. Linear estimates

In this section, we consider the properties of a solution to the Airy equation. Let

$$V(t)\phi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ix\xi + it\xi^3} \hat{\phi}(\xi) \mathrm{d}\xi. \tag{7}$$

Lemma 2.1 (Strichartz estimates)

$$\|V(t)\phi\|_{L^\infty_T L^2_x} \le C\|\phi\|_{L^2_x}, \tag{8}$$

$$\|\partial_x \int_0^t V(t-\tau)F(\tau)\mathrm{d}\tau\|_{L^\infty_T L^2_x} \le C\|F\|_{L^1_x L^2_T}. \tag{9}$$

Proof of Lemma 2.1
The inequality (8) follows from the standard energy method. For the proof of (9) see [6, Theorem 3.5 (ii)]. □

Lemma 2.2 (Local smoothing effects)

$$\|\partial_x V(t)\phi\|_{L^\infty_x L^2_T} \le C\|\phi\|_{L^2_x}, \tag{10}$$

$$\|\partial_x^2 \int_0^t V(t-\tau)F(\tau)\mathrm{d}\tau\|_{L^\infty_x L^2_T} \le C\|F\|_{L^1_x L^2_T}. \tag{11}$$

Proof of Lemma 2.2
For the proof of (10) and (11), see [6, Theorem 3.5 (i)] and [6, Theorem 3.5 (iii)], respectively. □

Lemma 2.3 (Estimates for the maximal functions)

$$\|V(t)\phi\|_{L_x^2L_T^\infty} \leq C(1+T)\|\phi\|_{H_x^{1,0}}, \tag{12}$$

$$\|V(t)\phi\|_{L_x^1L_T^\infty} \leq C(1+T)^2(\|\phi\|_{H_x^{5,0}} + \|\phi\|_{H_x^{3,1}}). \tag{13}$$

Proof of Lemma 2.3

For the proof of (12), see [7, Corollary 2.9]. To prove (13), we employ the method due to [8, Proposition 3.7]. By Sobolev's inequality with respect to t variable, we see

$$\begin{aligned}\|V(t)\phi\|_{L_x^1L_T^\infty} &\leq CT^{-1}\|V(t)\phi\|_{L_x^1L_T^1} + C\|\partial_t V(t)\phi\|_{L_x^1L_T^1}\\ &= CT^{-1}\|V(t)\phi\|_{L_T^1L_x^1} + C\|V(t)\partial_x^3\phi\|_{L_T^1L_x^1}\\ &\leq CT^{-1}(\|V(t)\phi\|_{L_T^1L_x^2} + \|xV(t)\phi\|_{L_T^1L_x^2}) + C(\|V(t)\partial_x^3\phi\|_{L_T^1L_x^2} + \|xV(t)\partial_x^3\phi\|_{L_T^1L_x^2}).\end{aligned}$$

Because $xV(t) = V(t)J(-t)$ where $J(t) = x - 3it\partial_x^2$ and $V(t)$ is unitary in L^2, we obtain (13). □

3. Nonlinear estimates

Proof of Theorem 1.1

Hereafter, we assume $0 < T < 1$. Firstly, we estimate u. The first equation in (6) is rewritten as

$$u(t) = V(t)u_0 - \int_0^t V(t-\tau)(\tilde{u}\partial_x\tilde{v} + \tilde{v}\partial_x\tilde{u})(\tau)\mathrm{d}\tau,$$

where $\{V(t)\}_{t\in\mathbf{R}}$ is the free Airy group. By a simple calculation, we see that

$$\partial_x^5(\tilde{u}\partial_x\tilde{v} + \tilde{v}\partial_x\tilde{u}) = \tilde{u}\partial_x^6\tilde{v} + \tilde{v}\partial_x^6\tilde{u} + R_1,$$

where R_1 is some quadratic function with respect to $(\partial_x^j\tilde{u})_{j=0}^5$ and $(\partial_x^j\tilde{v})_{j=1}^5$. By Hölder's and Sobolev's inequalities, we easily see that

$$\|\partial_x R_1\|_{L_T^1L_x^2} \leq CT\|\tilde{u}\|_{L_T^\infty H_x^{6,0}}\|\tilde{v}\|_{L_T^\infty H_x^{6,0}}.$$

Applying (8) and (10) for u_0 and R_1, and (9) and (11) for $u\partial_x^6\tilde{v} + \tilde{v}\partial_x^6 u$, we obtain

$$\begin{aligned}\|\partial_x^6 u\|_{L_T^\infty L_x^2} + \|\partial_x^7 u\|_{L_x^\infty L_T^2} &\leq C\|u_0\|_{H_x^6} + C\|\tilde{u}\partial_x^6\tilde{v}\|_{L_x^1L_T^2} + C\|\tilde{v}\partial_x^6\tilde{u}\|_{L_x^1L_T^2} + C\|\partial_x R_1\|_{L_T^1L_x^2}\\ &\leq C\|u_0\|_{H_x^6} + C\|\tilde{u}\|_{L_x^2L_T^\infty}\|\partial_x^6\tilde{v}\|_{L_x^2L_T^2} + C\|\tilde{v}\|_{L_x^2L_T^\infty}\|\partial_x^6\tilde{u}\|_{L_x^2L_T^2}\\ &\quad + C\|\partial_x R_1\|_{L_T^1L_x^2}\\ &\leq C\|u_0\|_{H_x^6} + CT^{1/2}\|\tilde{u}\|_{L_x^2L_T^\infty}\|\tilde{v}\|_{L_T^\infty H_x^{6,0}} + CT^{1/2}\|\tilde{v}\|_{L_x^2L_T^\infty}\|\tilde{u}\|_{L_T^\infty H_x^{6,0}}\\ &\quad + CT\|\tilde{u}\|_{L_T^\infty H_x^{6,0}}\|\tilde{v}\|_{L_T^\infty H_x^{6,0}}\\ &\leq C\|u_0\|_{H_x^{6,0}} + CT^{1/2}\|\!|(\tilde{u},\tilde{v},\tilde{w})|\!\|_{Z_T}^2.\end{aligned} \tag{14}$$

Using (12), we have

$$\begin{aligned}\sum_{j=0}^{1}\|\partial_x^j u\|_{L_x^2L_T^\infty} &\leq C\|u_0\|_{H_x^{2,0}} + C\|\tilde{u}\partial_x\tilde{v} + \tilde{v}\partial_x\tilde{u}\|_{L_T^1H_x^{2,0}}\\ &\leq C\|u_0\|_{H_x^{2,0}} + CT\|\tilde{u}\|_{L_T^\infty H_x^{2,0}}\|\tilde{v}\|_{L_T^\infty H_x^{3,0}} + CT\|\tilde{v}\|_{L_T^\infty H_x^{2,0}}\|\tilde{u}\|_{L_T^\infty H_x^{3,0}}\\ &\leq C\|u_0\|_{H_x^{2,0}} + CT\|\!|(\tilde{u},\tilde{v},\tilde{w})|\!\|_{Z_T}^2.\end{aligned} \tag{15}$$

Because $xV(t) = V(t)J(-t)$, we see that

$$x\partial_x^3 u(t) = V(t)J(-t)(\partial_x^3 u_0 - \int_0^t V(-\tau)\partial_x^3(\tilde{u}\partial_x\tilde{v} + \tilde{v}\partial_x\tilde{u})(\tau)\mathrm{d}\tau).$$

Hence,

$$\begin{aligned}\|x\partial_x^3 u\|_{L_T^\infty L_x^2} &\le C\|x\partial_x^3 u_0\|_{L_x^2} + CT\|\partial_x^5 u_0\|_{L_x^2}\\ &\quad + C\|x\partial_x^3(\tilde{u}\partial_x\tilde{v} + \tilde{v}\partial_x\tilde{u})\|_{L_T^1L_x^2} + CT\|\partial_x^5(\tilde{u}\partial_x\tilde{v} + \tilde{v}\partial_x\tilde{u})\|_{L_T^1L_x^2}\\ &\le C(\|u_0\|_{H_x^{5,0}} + \|u_0\|_{H_x^{3,1}})\\ &\quad + CT(\|\tilde{u}\|_{L_T^\infty H_x^{4,0}}\|\tilde{v}\|_{L_T^\infty H_x^{1,1}} + \|\tilde{v}\|_{L_T^\infty H_x^{4,0}}\|\tilde{u}\|_{L_T^\infty H_x^{2,1}})\\ &\quad + CT^2(\|\tilde{u}\|_{L_T^\infty H_x^{5,0}}\|\tilde{v}\|_{L_T^\infty H_x^{6,0}} + \|\tilde{v}\|_{L_T^\infty H_x^{5,0}}\|\tilde{u}\|_{L_T^\infty H_x^{6,0}})\\ &\le C(\|u_0\|_{H_x^{5,0}} + \|u_0\|_{H_x^{3,1}}) + CT\|\!|(\tilde{u},\tilde{v},\tilde{w})|\!\|_{Z_T}^2.\end{aligned} \tag{16}$$

By Lemma 2.3 (13),

$$\begin{aligned}\|u\|_{L_x^1L_T^\infty} &\le C(\|u_0\|_{H_x^{5,0}} + \|u_0\|_{H_x^{3,1}})\\ &\quad + C(\|\tilde{u}\partial_x\tilde{v} + \tilde{v}\partial_x\tilde{u}\|_{L_T^1H_x^{5,0}} + \|\tilde{u}\partial_x\tilde{v} + \tilde{v}\partial_x\tilde{u}\|_{L_T^1H_x^{3,1}})\\ &\le C(\|u_0\|_{H_x^{5,0}} + \|u_0\|_{H_x^{3,1}}) + CT\|\!|(\tilde{u},\tilde{v},\tilde{w})|\!\|_{Z_T}^2.\end{aligned} \tag{17}$$

Next, we estimate v. By a standard energy estimate, we obtain

$$\begin{aligned}\|v\|_{L_T^\infty H_x^{6,0}} &\le \|v_0\|_{H_x^{6,0}} + T\|\tilde{w}\|_{L_T^\infty H_x^{7,0}} + \|\tilde{u}\partial_x\tilde{u}\|_{L_T^1H_x^{6,0}}\\ &\quad + T\|v\|_{L_T^\infty H_x^{6,0}}\|\tilde{v}\|_{L_T^\infty H_x^{6,0}}.\end{aligned}$$

We notice that

$$\partial_x^6(\tilde{u}\partial_x\tilde{u}) = \tilde{u}\partial_x^7\tilde{u} + R_2,$$

where R_2 is some quadratic function of $(\partial_x^j\tilde{u})_{j=0}^6$ and

$$\begin{aligned}\|\tilde{u}\partial_x^7\tilde{u}\|_{L_T^1L_x^2} &\le T^{1/2}\|\tilde{u}\partial_x^7\tilde{u}\|_{L_T^2L_x^2} = T^{1/2}\|\tilde{u}\partial_x^7\tilde{u}\|_{L_x^2L_T^2}\\ &\le T^{1/2}\|\tilde{u}\|_{L_x^2L_T^\infty}\|\partial_x^7\tilde{u}\|_{L_x^\infty L_T^2},\\ \|R_2\|_{L_T^1L_x^2} &\le CT\|\tilde{u}\|_{L_T^\infty H_x^{6,0}}^2.\end{aligned}$$

Therefore, we have

$$\|v\|_{L_T^\infty H_x^{6,0}} \le \|v_0\|_{H_x^{6,0}} + CT^{1/2}(1 + \|\!|(\tilde{u},\tilde{v},\tilde{w})|\!\|_{Z_T} + \|\!|(u,v,w)|\!\|_{Z_T})\|\!|(\tilde{u},\tilde{v},\tilde{w})|\!\|_{Z_T}.$$

Using the relation

$$v(t) = v_0 - \int_0^t (\partial_x\tilde{w} + \tilde{u}\partial_x\tilde{u} + \tilde{v}\partial_x v)(\tau)\,\mathrm{d}\tau,$$

we obtain

$$\begin{aligned}\|v\|_{L_T^\infty H_x^{1,1}} &\le \|v_0\|_{H_x^{1,1}} + T\|\tilde{w}\|_{L_T^\infty H_x^{2,1}} + CT\|\tilde{u}\|_{L_T^\infty H_x^{1,1}}\|\tilde{u}\|_{L_T^\infty H_x^{2,0}}\\ &\quad + CT\|\tilde{v}\|_{L_T^\infty H_x^{1,1}}\|v\|_{L_T^\infty H_x^{2,0}}\\ &\le \|v_0\|_{H_x^{1,1}} + CT(1 + \|\!|(\tilde{u},\tilde{v},\tilde{w})|\!\|_{Z_T} + \|\!|(u,v,w)|\!\|_{Z_T})\|\!|(\tilde{u},\tilde{v},\tilde{w})|\!\|_{Z_T},\end{aligned} \tag{18}$$

$$\begin{aligned}\|v\|_{L_x^2L_T^\infty} &\le \|v_0\|_{L_x^2} + T\|\partial_x\tilde{w}\|_{L_x^2L_T^\infty} + T\|\tilde{u}\partial_x\tilde{u}\|_{L_x^2L_T^\infty} + T\|\tilde{v}\partial_x v\|_{L_x^2L_T^\infty}\\ &\le \|v_0\|_{L_x^2} + T\|\partial_x\tilde{w}\|_{L_x^2L_T^\infty} + CT\|\tilde{u}\|_{L_x^2L_T^\infty}\|\partial_x\tilde{u}\|_{L_x^\infty L_T^\infty}\\ &\quad + CT\|\tilde{v}\|_{L_x^2L_T^\infty}\|\partial_x v\|_{L_x^\infty L_T^\infty}\\ &\le \|v_0\|_{L_x^2} + CT(1 + \|\!|(\tilde{u},\tilde{v},\tilde{w})|\!\|_{Z_T} + \|\!|(u,v,w)|\!\|_{Z_T})\|\!|(\tilde{u},\tilde{v},\tilde{w})|\!\|_{Z_T}.\end{aligned} \tag{19}$$

Finally, we estimate w. From the third equation in (6), we have

$$\partial_x^7 w(t) = V(t)\partial_x^7 w_0 - \int_0^t V(t-\tau)\partial_x^7(\tilde{u}\partial_x\tilde{u} + \tilde{w}\partial_x\tilde{w})(\tau)\mathrm{d}\tau.$$

Because

$$\begin{aligned}\partial_x^6(\tilde{u}\partial_x\tilde{u}) &= \tilde{u}\partial_x^7\tilde{u} + 7\partial_x\tilde{u}\partial_x^6\tilde{u} + R_3,\\ \partial_x^7(\tilde{w}\partial_x\tilde{w}) &= \tilde{w}\partial_x^8\tilde{w} + R_4,\end{aligned}$$

where R_3 and R_4 satisfy

$$\begin{aligned}\|\partial_x R_3\|_{L_T^1L_x^2} &\le CT\|\tilde{u}\|^2_{L_T^\infty H_x^{6,0}},\\ \|R_4\|_{L_T^1L_x^2} &\le CT\|\tilde{w}\|^2_{L_T^\infty H_x^{7,0}},\end{aligned}$$

applying (8) and (10) for $\partial_x^7 w_0$, $\partial_x R_3$, and $\partial_x^7(\tilde{w}\partial_x\tilde{w})$ and (9) and (11) for $\tilde{u}\partial_x^7\tilde{u}$ and $\partial_x\tilde{u}\partial_x^6\tilde{u}$, we obtain

$$\begin{aligned}\|\partial_x^7 w\|_{L_T^\infty L_x^2} + \|\partial_x^8 w\|_{L_x^\infty L_T^2} &\le C\|\partial_x^7 w_0\|_{L_x^2} + C\|\tilde{u}\partial_x^7\tilde{u}\|_{L_x^1L_T^2} + C\|\partial_x\tilde{u}\partial_x^6\tilde{u}\|_{L_x^1L_T^2}\\ &\quad + C\|\tilde{w}\partial_x^8\tilde{w}\|_{L_T^1L_x^2} + C\|\partial_x R_3\|_{L_T^1L_x^2} + C\|R_4\|_{L_T^1L_x^2}\\ &\le C\|w_0\|_{H_x^{7,0}} + C\|\tilde{u}\|_{L_x^1L_T^\infty}\|\partial_x^7\tilde{u}\|_{L_x^\infty L_T^2} + CT^{1/2}\|\partial_x\tilde{u}\|_{L_x^2L_T^\infty}\|\tilde{u}\|_{L_T^\infty H_x^{6,0}}\\ &\quad + CT^{1/2}\|\tilde{w}\|_{L_x^2L_T^\infty}\|\partial_x^8\tilde{w}\|_{L_x^\infty L_T^2} + CT\|\tilde{u}\|^2_{L_T^\infty H_x^{6,0}} + CT\|\tilde{w}\|^2_{L_T^\infty H_x^{7,0}}\\ &\le C\|w_0\|_{H_x^{7,0}} + C\|(\tilde{u},\tilde{v},\tilde{w})\|^2_{Z_T}.\end{aligned} \tag{20}$$

As in (15), we see

$$\begin{aligned}\sum_{j=0}^{1}\|\partial_x^j w\|_{L_x^2L_T^\infty} &\le C\|w_0\|_{H_x^{2,0}} + CT\|\tilde{u}\|_{L_T^\infty H_x^{2,0}}\|\tilde{u}\|_{L_T^\infty H_x^{3,0}}\\ &\quad + CT\|\tilde{w}\|_{L_T^\infty H_x^{2,0}}\|\tilde{w}\|_{L_T^\infty H_x^{3,0}}\\ &\le C\|w_0\|_{H_x^{2,0}} + CT\|(\tilde{u},\tilde{v},\tilde{w})\|^2_{Z_T}.\end{aligned} \tag{21}$$

As in (16), we have

$$\begin{aligned}\|w\|_{L_T^\infty H_x^{2,1}} &\le C(\|w_0\|_{H_x^{4,0}} + \|w_0\|_{H_x^{2,1}})\\ &\quad + CT(\|\tilde{u}\|_{L_T^\infty H_x^{1,1}}\|\tilde{u}\|_{L_T^\infty H_x^{3,0}} + \|\tilde{u}\|^2_{L_T^\infty H_x^{5,0}})\\ &\quad + CT(\|\tilde{w}\|_{L_T^\infty H_x^{1,1}}\|\tilde{w}\|_{L_T^\infty H_x^{3,0}} + \|\tilde{w}\|^2_{L_T^\infty H_x^{5,0}})\\ &\le C(\|w_0\|_{H_x^{5,0}} + \|w_0\|_{H_x^{2,1}}) + CT\|(\tilde{u},\tilde{v},\tilde{w})\|^2_{Z_T}.\end{aligned} \tag{22}$$

Collecting (14)–(22), we have that if $(u_0, v_0, w_0) \in Z_T^a$, then

$$\|(u,v,w)\|_{Z_T} \le C\epsilon_0 + Ca(1 + a + \|(u,v,w)\|_{Z_T}).$$

Therefore, if we choose ϵ_0, a, and T so small, then the map $\Phi : Z_T^a \to Z_T^a$ is well defined. Similarly, we can prove that the map Φ is a contraction on Z_T^a. This completes the proof of the main theorem. □

4. Generalization

In this section, we consider the hyperbolic-dispersive system

$$\begin{cases}\partial_t U + \partial_x^3 U + N_1(U, \partial_x U, V, \partial_x V, W, \partial_x W) = 0,\\ \partial_t V + \partial_x W + N_2(U, \partial_x U, W, \partial_x W) + M(U, V, W, \partial_x W)\partial_x V = 0,\\ \partial_t W + \partial_x^3 W + N_3(U, \partial_x U, W, \partial_x W) = 0,\\ U(0,x) = U_0(x), \quad V(0,x) = V_0(x), \quad W(0,x) = W_0(x),\end{cases} \tag{23}$$

where $U = {}^t(u_1, \cdots, u_l), V = {}^t(v_1, \cdots, v_m)$ and $W = {}^t(w_1, \cdots, w_m)$ are unknown vector-valued functions, $N_1 : \mathbf{R}^{2l+4m} \to \mathbf{R}^l$, $N_2 : \mathbf{R}^{2l+2m} \to \mathbf{R}^m$, $N_3 : \mathbf{R}^{2l+2m} \to \mathbf{R}^m$ are polynomials having no constant or linear terms, and M is a $m \times m$ symmetric matrix whose components are polynomials in $(U, V, W, \partial_x W)$ without constant. By the similar way to the argument in Section 3, we can prove the time local well-posedness for (23) in $(H_x^{7,0} \cap H_x^{4,1}) \times (H_x^{7,0} \cap H_x^{3,1}) \times (H_x^{8,0} \cap H_x^{4,1})$.

Theorem 4.1
There exists $\epsilon > 0$ such that for any $(U_0, V_0, W_0) \in (H_x^{7,0} \cap H_x^{4,1}) \times (H_x^{7,0} \cap H_x^{3,1}) \times (H_x^{8,0} \cap H_x^{4,1})$ and $\|U_0\|_{H_x^{7,0}} + \|U_0\|_{H_x^{4,1}} + \|V_0\|_{H_x^{7,0}} + \|V_0\|_{H_x^{3,1}} + \|W_0\|_{H_x^{8,0}} + \|W_0\|_{H_x^{4,1}} < \epsilon$ the initial value problem (23) has a unique solution $(U(\cdot), V(\cdot), W(\cdot))$ defined in the interval $[0, T]$, $T = T(\|U_0\|_{H^{7,0}}, \|V_0\|_{H_x^{7,0}}, \|W_0\|_{H_x^{8,0}})$ satisfying

$$(U, V, W) \quad \in \quad C([0,T]; H_x^{7,0} \cap H_x^{4,1}) \times C([0,T]; H_x^{7,0} \cap H_x^{3,1}) \times C([0,T]; H_x^{8,0} \cap H_x^{4,1}) \equiv Y_T.$$

Moreover, for any $T' \in (0, T)$ there exists $\delta > 0$ such that the map $(\tilde{U}_0, \tilde{V}_0, \tilde{W}_0) \to (\tilde{U}(t), \tilde{V}(t), \tilde{W}(t))$ from $\{(\tilde{U}_0, \tilde{V}_0, \tilde{W}_0) | \|\tilde{U}_0 - U_0\|_{H_x^{7,0}} + \|\tilde{V}_0 - V_0\|_{H_x^{7,0}} + \|\tilde{W}_0 - W_0\|_{H_x^{8,0}} < \delta\}$ into $Y_{T'}$ is Lipschitz continuous.

We only give the outline of proof for Theorem 4.1.
For (U_0, V_0, W_0) we denote by $(U, V, W) = \Phi(\tilde{U}, \tilde{V}, \tilde{W})$ the solution to the linearized problem

$$\begin{cases} \partial_t U + \partial_x^3 U + N_1(\tilde{U}, \partial_x \tilde{U}, \tilde{V}, \partial_x \tilde{V}, \tilde{W}, \partial_x \tilde{W}) = 0, \\ \partial_t V + \partial_x \tilde{W} + N_2(\tilde{U}, \partial_x \tilde{U}, \tilde{W}, \partial_x \tilde{W}) + M(\tilde{U}, \tilde{V}, \tilde{W}, \partial_x \tilde{W}) \partial_x V = 0, \\ \partial_t W + \partial_x^3 W + N_3(\tilde{U}, \partial_x \tilde{U}, \tilde{W}, \partial_x \tilde{W}) = 0, \\ U(0,x) = U_0(x), \quad V(0,x) = V_0(x), \quad W(0,x) = W_0(x), \end{cases}$$

and let

$$Z_T^a \equiv \{(U, V, W) | \, \|\!|(U, V, W)\|\!|_{Z_T} \leq a\},$$

where

$$\begin{aligned} \|\!|(U,V,W)\|\!|_{Z_T} \equiv \; & \|U\|_{L_T^\infty H_x^{7,0}} + \|U\|_{L_T^\infty H_x^{4,1}} + \|\partial_x^8 U\|_{L_x^\infty L_T^2} + \sum_{j=0}^{1} \|\partial_x^j U\|_{L_x^1 L_T^\infty} + \sum_{j=0}^{1} \|\partial_x^j U\|_{L_x^2 L_T^\infty} \\ & + \|V\|_{L_T^\infty H_x^{7,0}} + \|V\|_{L_T^\infty H_x^{3,1}} + \sum_{j=0}^{1} \|\partial_x^j V\|_{L_x^2 L_T^\infty} \\ & + \|W\|_{L_T^\infty H_x^{8,0}} + \|W\|_{L_T^\infty H_x^{4,1}} + \|\partial_x^9 W\|_{L_x^\infty L_T^2} \sum_{j=0}^{1} \|\partial_x^j W\|_{L_x^1 L_T^\infty} + \sum_{j=0}^{2} \|\partial_x^j W\|_{L_x^2 L_T^\infty}. \end{aligned}$$

As in the preceding sections, we shall prove that for some a and T, $\Phi(\tilde{U}, \tilde{V}, \tilde{W})$ is a contraction map from Z_T^a into itself.
The first component U satisfies the integral equation

$$U(t) = e^{-t\partial_x^3} U_0 - \int_0^t e^{-(t-\tau)\partial_x^3} N_1(\tau) \mathrm{d}\tau,$$

and

$$\partial_x^6 N_1 \quad = \textstyle\sum_{j=1}^{l} \frac{\partial N_1}{\partial(\partial_x \tilde{U}_j)} \partial_x^7 \tilde{U}_j + \sum_{j=1}^{m} \frac{\partial N_1}{\partial(\partial_x \tilde{V}_j)} \partial_x^7 \tilde{V}_j + R_5,$$

where R_5 depends on $(\partial_x^j \tilde{U})_{j=0}^6$, $(\partial_x^j \tilde{V})_{j=0}^6$ and $(\partial_x^j \tilde{W})_{j=1}^7$. Therefore, by Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned} \|\partial_x^7 U\|_{L_T^\infty L_x^2} + \|\partial_x^8 U\|_{L_x^\infty L_T^2} \leq & \, C\|U_0\|_{H_x^7} + C \sum_{j=1}^{l} \|\frac{\partial N_1}{\partial(\partial_x \tilde{U}_j)} \partial_x^7 \tilde{U}_j\|_{L_x^1 L_T^2} \\ & + C \sum_{j=1}^{m} \|\frac{\partial N_1}{\partial(\partial_x \tilde{V}_j)} \partial_x^7 \tilde{V}_j\|_{L_x^1 L_T^2} + \|\partial_x R_5\|_{L_T^1 L_x^2}. \end{aligned} \tag{24}$$

The second term in the right hand side of (24) is evaluated as

$$\begin{aligned} \sum_{j=1}^{l} \|\frac{\partial N_1}{\partial(\partial_x \tilde{U}_j)} \partial_x^7 \tilde{U}_j\|_{L_x^1 L_T^2} \leq & \sum_{j=1}^{l} \|\frac{\partial N_1}{\partial(\partial_x \tilde{U}_j)}\|_{L_x^2 L_T^\infty} \|\partial_x^7 \tilde{U}_j\|_{L_x^2 L_T^2} \\ \leq & \, T^{1/2} \sum_{j=1}^{l} \|\frac{\partial N_1}{\partial(\partial_x \tilde{U}_j)}\|_{L_x^2 L_T^\infty} \|\partial_x^7 \tilde{U}_j\|_{L_T^\infty L_x^2}. \end{aligned}$$

By Hölder's and Sobolev's inequalities, we see that $\|\frac{\partial N_1}{\partial(\partial_x \tilde{U}_j)}\|_{L^2_x L^\infty_T}$ is evaluated in terms of $\|\tilde{U}\|_{L^\infty_T H^2_x}$, $\|\tilde{V}\|_{L^\infty_T H^2_x}$, $\|\tilde{W}\|_{L^\infty_T H^2_x}$, $\|\partial_x^k \tilde{U}\|_{L^2_x L^\infty_T}$, $\|\partial_x^k \tilde{V}\|_{L^2_x L^\infty_T}$, and $\|\partial_x^k \tilde{W}\|_{L^2_x L^\infty_T}$ $(k = 0, 1)$. The third term in the right hand side of (24) is evaluated by the similar way. We can estimate R_5 by the Hölder and Sobolev inequalities. Because the estimates for the other norms of U in Z_T are obtained as (15)–(17) in the Section 3, we omit the detail.

Next, we estimate the second component V. The standard energy method yields

$$\|V\|_{L^\infty_T H^{7,0}_x} \le \|V_0\|_{H^{7,0}_x} + T\|\tilde{W}\|_{L^\infty_T H^{8,0}_x} + \|N_2\|_{L^1_T H^{7,0}_x} + T(1 + \|\tilde{U}\|_{L^\infty_T H^{7,0}_x} + \|\tilde{V}\|_{L^\infty_T H^{7,0}_x} + \|\tilde{W}\|_{L^\infty_T H^{8,0}_x})^N \|V\|_{L^\infty_T H^{7,0}_x}$$

for some non-negative integer N. By a simple calculation we have

$$\partial_x^7 N_2 = \sum_{j=1}^{l} \frac{\partial N_1}{\partial(\partial_x \tilde{U}_j)} \partial_x^8 \tilde{U}_j + R_6(\tilde{U}, \cdots, \partial_x^7 \tilde{U}, \tilde{W}, \cdots, \partial^8 \tilde{W}),$$

and

$$\begin{aligned}\|\sum_{j=1}^{l} \frac{\partial N_2}{\partial(\partial_x \tilde{U}_j)} \partial_x^8 \tilde{U}_j\|_{L^1_T L^2_x} &\le T^{1/2} \sum_{j=1}^{l} \|\frac{\partial N_2}{\partial(\partial_x \tilde{U}_j)} \partial_x^8 \tilde{U}_j\|_{L^2_T L^2_x} \\ &= T^{1/2} \sum_{j=1}^{l} \|\frac{\partial N_2}{\partial(\partial_x \tilde{U}_j)} \partial_x^8 \tilde{U}_j\|_{L^2_x L^2_T} \\ &\le T^{1/2} \sum_{j=1}^{l} \|\frac{\partial N_2}{\partial(\partial_x \tilde{U}_j)}\|_{L^2_x L^\infty_T} \|\partial_x^8 \tilde{U}_j\|_{L^\infty_x L^2_T}\end{aligned}$$

$\|\frac{\partial N_2}{\partial(\partial_x \tilde{U}_j)}\|_{L^2_x L^\infty_T}$ is evaluated in terms of $\|\tilde{U}\|_{L^\infty_T H^2_x}$, $\|\tilde{W}\|_{L^\infty_T H^2_x}$, $\|\partial_x^k \tilde{U}\|_{L^2_x L^\infty_T}$, and $\|\partial_x^k \tilde{W}\|_{L^2_x L^\infty_T}$ $(k = 0, 1)$. Finally, we mention the estimation for the first component W. Because

$$\begin{aligned}\partial_x^8 W(t) &= e^{-t\partial_x^3} \partial_x^8 W_0 - \int_0^t e^{-(t-\tau)\partial_x^3} \partial_x^8 N_3(\tau) \mathrm{d}\tau, \\ \partial_x^8 N_3 &= \sum_{j=1}^{m} \frac{\partial N_3}{\partial(\partial_x \tilde{W}_j)} \partial_x^9 \tilde{W}_j \\ &\quad + \partial_x \left\{ \sum_{j=1}^{l} \frac{\partial N_3}{\partial(\partial_x \tilde{U}_j)} \partial_x^8 \tilde{U}_j + 7 \sum_{j=1}^{l} \sum_{k=1}^{l} \frac{\partial^2 N_3}{\partial \tilde{U}_j \partial(\partial_x \tilde{U}_k)} \partial_x \tilde{U}_j \partial_x^7 \tilde{U}_k \right. \\ &\quad + 7 \sum_{j=1}^{l} \sum_{k=1}^{l} \frac{\partial^2 N_3}{\partial(\partial_x \tilde{U}_j) \partial(\partial_x \tilde{U}_k)} \partial_x^2 \tilde{U}_k \partial_x^7 \tilde{U}_k + 7 \sum_{j=1}^{l} \sum_{k=1}^{m} \frac{\partial^2 N_3}{\partial(\partial_x \tilde{U}_j) \partial \tilde{W}_k} \partial_x^7 \tilde{U}_j \partial_x \tilde{W}_k \\ &\quad \left. + 7 \sum_{j=1}^{l} \sum_{k=1}^{m} \frac{\partial^2 N_3}{\partial(\partial_x \tilde{U}_j) \partial(\partial_x \tilde{W}_k)} \partial_x^7 \tilde{U}_j \partial_x^2 \tilde{W}_k \right\} \\ &\quad + \partial_x R_7(\tilde{U}, \cdots, \partial_x^6 \tilde{U}, \tilde{W}, \cdots, \partial^7 \tilde{W}) \\ &\equiv N_{3,1} + \partial_x N_{3,2} + \partial_x R_7.\end{aligned}$$

Applying (8) and (10) for W_0, $N_{3,1}$, and $\partial_x R_7$, and (9) and (11) for $N_{3,2}$, we obtain

$$\|\partial_x^8 W\|_{L^\infty_T L^2_x} + \|\partial_x^9 W\|_{L^\infty_x L^2_T} \le C\|W_0\|_{H^8_x} + \|N_{3,1}\|_{L^1_T L^2_x} + \|N_{3,2}\|_{L^1_x L^2_T} + \|\partial_x R_7\|_{L^1_T L^2_x}.$$

By the Hölder inequality,

$$\|N_{3,1}\|_{L^1_T L^2_x} \le T^{1/2} \sum_{j=1}^{m} \|\frac{\partial N_3}{\partial(\partial_x \tilde{W}_j)}\|_{L^2_x L^\infty_T} \|\partial_x^9 \tilde{W}_j\|_{L^\infty_x L^2_T}.$$

Because $\|\frac{\partial N_3}{\partial(\partial_x \tilde{W}_j)}\|_{L^2_x L^\infty_T}$ is evaluated in terms of $\|\tilde{U}\|_{L^\infty_T H^2_x}$, $\|\tilde{W}\|_{L^\infty_T H^2_x}$, $\|\partial_x^k \tilde{U}\|_{L^2_x L^\infty_T}$, and $\|\partial_x^k \tilde{W}\|_{L^2_x L^\infty_T}$ $(k = 0, 1)$, $\|N_{3,1}\|_{L^1_T L^2_x}$ is bounded by Z_T norm of $(\tilde{U}, \tilde{V}, \tilde{W})$.

The first term of $N_{3,2}$ is evaluated as

$$\sum_{j=1}^{l} \left\| \frac{\partial N_3}{\partial(\partial_x \tilde{U}_j)} \partial_x^8 \tilde{U}_j \right\|_{L_x^1 L_T^2} \leq \sum_{j=1}^{l} \left\| \frac{\partial N_3}{\partial(\partial_x \tilde{U}_j)} \right\|_{L_x^1 L_T^\infty} \|\partial_x^8 \tilde{U}_j\|_{L_x^\infty L_T^2}.$$

We easily see that $\| \frac{\partial N_3}{\partial(\partial_x \tilde{U}_j)} \|_{L_x^1 L_T^\infty}$ is evaluated in terms of $\|\tilde{U}\|_{L_T^\infty H_x^2}$, $\|\tilde{W}\|_{L_T^\infty H_x^2}$, $\|\partial_x^k \tilde{U}\|_{L_x^1 L_T^\infty}$, and $\|\partial_x^k \tilde{W}\|_{L_x^1 L_T^\infty}$ $(k = 0, 1)$. By similar way, we can estimate the other norms of $N_{3,2}$. Therefore, $\|N_{3,2}\|_{L_x^1 L_T^2}$ is bounded by Z_T norm of $(\tilde{U}, \tilde{V}, \tilde{W})$. The estimates for R_7 follows only using the Hölder and Sobolev inequalities.

The estimates for the other norms of W in Z_T follow from the similar argument as (21)–(22) in Section 3. Combination of above estimates and the contraction mapping principle guarantees the well-posedness of (23). This completes the proof of Theorem 4.1.

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