J Comb Optim (2012) 23:61–78
DOI 10.1007/s10878-010-9341-7

A new approach to solve open-partition problems

Huilan Chang · Frank K. Hwang · Uriel G. Rothblum

Published online: 3 August 2010

Abstract A partition problem in one-dimensional space is to seek a partition of a set of numbers that maximizes a given objective function. In some partition problems, the partition size, i.e., the number of nonempty parts in a partition, is fixed; while in others, the size can vary arbitrarily. We call the former the size-partition problem and the latter the open-partition problem. In general, it is much harder to solve open problems since the objective functions depend on size. In this paper, we propose a new approach by allowing empty parts and transform the open problem into a size problem allowing empty parts, called a relaxed-size problem. While the sortability theory has been established in the literature as a powerful tool to attack size partition problems, we develop the sortability theory for relaxed-size problems as a medium to solve open problems.

Keywords Partition · Objective function · Partition property · Sortability

1 Introduction

Consider a finite set Θ of n distinct numbers ($\theta^1 \leq \theta^2 \leq \cdots \leq \theta^n$). A *partition* of Θ is a finite collection of sets $\pi = (\pi_1, \ldots, \pi_p)$ where $\pi_1, \ldots, \pi_p$ are pairwise disjoint

H. Chang (✉)
Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30010, Taiwan, ROC
e-mail: huilan0102@gmail.com

H. Chang
DIMACS, Rutgers University, 96 Frelinghuysen Road, Piscataway, NJ 08854, USA

F.K. Hwang
6306 Westover Way, Somerset, NJ 08873, USA

U.G. Rothblum
Faculty of Industrial Engineering and Management, Technion, Haifa 32000, Israel

nonempty sets whose union is $\mathcal{N} \equiv \{1, \ldots, n\}$ (namely, it is not Θ but $\mathcal{N}$ that is partitioned). In this case we refer to $p \equiv |\pi|$ as the *size* of π and to the sets $\pi_1, \ldots, \pi_p$ as the *parts* of π. Further, if $n_1, \ldots, n_p$ are the sizes of $\pi_1, \ldots, \pi_p$, respectively, the *shape* of π is the vector $(n_1, \ldots, n_p)$, in particular, $\sum_{j=1}^{p} n_j = n$.

In many applications, one seeks an optimal partition in a set of partitions that is restricted by constraints on the shape or on the size, (e.g., Chakravarty et al. 1982; Hwang et al. 1985; Boros and Hammer 1989; Anily and Federgruen 1991; Gal and Klots 1995). We refer to a *shape* (*size*) *partition problem*, or briefly, to a *shape* (*size*) *problem*, as an optimization problem over the set of partitions that have a prescribed shape (size) and call the set of partitions a *shape-family* (*size-family*). We refer to an *open partition problem*, or briefly, to an *open problem*, as one where there are no restrictions at all on the allowed partitions, and call the set of partitions an *open-family*. At times we restrict attention to partitions where empty parts are prohibited and at other times we wish to allow for empty parts. Our default position will be that empty parts are prohibited and we refer to *relaxed problems* over a *relaxed-family* when we allow for empty parts. In particular, we refer to *relaxed-size problems* over a *relaxed-size-family* when we allow empty parts in a size problem. Of course, for shape problems the question whether or not the problem is "relaxed" is implicitly stated in the prescribed shape—it is relaxed if and only if the prescribed shape has 0-entries. We use the term *t-family* to refer to an unspecified family where $t \in \{$shape, size, open, relaxed-size$\}$.

We shall use the notation $F(\cdot)$ for the objective function in a partition problem. In open problems, where the sizes of the partitions can vary, any result about optimal partitions requires some structure that facilitates comparison of values of objective functions for partitions of different sizes. Thus the objective function has the partition's size as a parameter. A major difficulty in addressing open families is the fact that they are not finite. The following example demonstrates that a partition problem over an open family need not have an optimal solution.

Example Let $n = 1$ and $F(\pi) = \sum_j j\theta_{\pi_j}$ for each partition π, whose parts π_j are allowed to be empty. (With $\theta^1 = 1$, $\theta_{\pi_j} = \sum_{i \in \pi_j} \theta^i$ for each j.) Then no optimal partition exists for this problem.

A most natural dependence of $F(\cdot)$ on the size is the "*reduction assumption*" that asserts that $F(\cdot)$ is invariant under the permutation/elimination of empty parts. For example, in many applications the value of the objective function at a partition is expressed by $F(\pi) = f_p(g(\pi_1), \ldots, g(\pi_p))$; the reduction assumption is then satisfied when f_p is the sum/max/min function, where $g(\pi_j) = 0/-\infty/\infty$ if π_j is empty, and g only depends on the elements of a part. The reduction assumption implicitly implies that the underlying problem is relaxed – empty parts are allowed and they can be eliminated. A natural assumption that accompanies the reduction assumption is symmetry of F under part-permutations.

When the reduction assumption is satisfied, a partition with p nonempty parts can effectively be viewed as a partition (many choices) of size $p' > p$ that has $p' - p$ empty parts. In particular, an open problem can be embedded in a relaxed-size problem with n as the prescribed size. This approach is useful as it converts the variable-size-family of partitions into a fixed-size (relaxed-size) -family.

In most realistic partition problems, n is large but p is small. As it is usually difficult to obtain an optimal partition analytically, a useful approach is to identify a property of partitions such that the set of partitions that satisfy it is guaranteed to contain an optimal partition and where the number of partitions satisfying this property is polynomial in n. An optimal partition can then be found in polynomial time by examining all partitions satisfying the property. For example, a partition is *consecutive* if each part consists of consecutive integers (all from $\mathcal{N}$). Some partition problems for which there exist consecutive optimal partitions were identified in Hwang (1981), Chakravarty et al. (1982) and it is easy to verify that the number of consecutive partitions of size p is $p!\binom{n-1}{p-1}$, an expression which is polynomial in n.

Consider a partition property Q, $t \in \{\text{shape, size, open}\}$ and a t-family Π. A strategy to prove the existence of a partition that is optimal over Π and satisfies Q is to show that for any partition π that is optimal over Π and does not satisfy Q, there is a finite sequence of transformations on partitions in Π starting at π such that the optimality is preserved and the transformations guarantee ending at a partition satisfying Q. In particular, if Π^* is the set of optimal partitions in Π, then the goal of preserving optimality is achieved by keeping the transformations inside Π^*. In Hwang et al. (1996), the goal of Π satisfying Q, i.e., having transformations ending at a partition satisfying Q, and the goal of having transformations staying inside the family are treated separately by two notions–sortability and invariance in the following.

A *k-subpartition* of a partition π is a set of k nonempty parts of π. The index set of such a subpartition K will be denoted by $J(K)$. The transformation usually used in the literature is the (*local*) *k-sorting*, first introduced in Hwang et al. (1996) and then generalized in Chang et al. (1999), which is to sort a k-subpartition K of π not satisfying Q into a partition K' such that K' satisfies Q; a *size-sorting* is a sorting with $J(K) = J(K')$; *shape-sorting* preserves not only the index set but also the shape; *relaxed-size-sorting* differs from size-sorting only in the admission of empty parts; also, *open-sorting* refers to a sorting where K' can be any partition of $\mathcal{N}(K) \equiv \bigcup_{\pi_i \in K} \pi_i$ that satisfies Q where K' can drop some indices of K (meaning the corresponding parts become empty) and add some indices of $\mathcal{N}$ which represent empty parts in π.

The notion of invariance was developed to attain the goal of staying inside the family. A t-family Π is *Q-k-invariant* if for every partition in Π not satisfying Q and a k-subpartition K not satisfying Q, there exists a Q-k-t-sorting of K which yields a new partition also in Π. There are four levels $l \in \{\text{strong, part-specific, sort-specific, weak}\}$ of Q-k-invariance depending on the strength of requirement: Π is *$(strong, k, t)$-invariant* if for every subpartition K not satisfying Q and every Q-k-sorting of K, π' is in Π; Π is *$(part\text{-}specific, k, t)$-invariant* if K is specific but the sorting is arbitrary; Π is *$(sort\text{-}specific, k, t)$-invariant* if K is arbitrary but the sorting is specific; Π is *$(weak, k, t)$-invariant* if both K and the sorting are specific. Of course, Π with each partition satisfying Q is Q-(l, k, t)-invariant which we refer to as a *trivial* invariant family.

The notion of sortability, complementing the notion of invariance, was developed to attain the goal of Π satisfying Q. Again, sortability is classified into four levels depending on the strength of requirement. Match {strong, weak} as a pair and {part-specific, sort-specific} as the other pair. If l is a member of a pair, then l^{-1} denote the other member of that pair. Then Q is (l,k,t)-*sortable* if and only if there exists a non-trivial Q-(l^{-1},k,t)-invariant family and every such family satisfies Q.

In this paper we propose a new approach for addressing problems over the open family for objective functions that satisfy the reduction assumption. Such problems can be reduced to relaxed-size problems with $p=n$. This approach induces one to study "relaxed-size-sortability" of properties Q, which has not been discussed before. We develop relaxed-size sortability results for useful properties that have been studied in the literature. Note that these properties were defined for partitions not allowing empty parts. Thus we need to expand these definitions to partitions allowing empty parts, which we refer to as *relaxed partitions* (and a part allowed to be empty as a *relaxed part*). Given a relaxed partition, the subpartition consisting of its nonempty parts is called its *induced partition*. Then a relaxed partition satisfies Q if and only if its induced partition satisfies Q.

2 The main results

Numerous partition properties which have been discussed in the literature (and are also the subjects of this paper) depend on the notion "penetration": a part π_i is said to *penetrate* another part π_j, written $\pi_i \to \pi_j$, if there exist $a<b<c$ such that $b\in\pi_i$ and $a,c\in\pi_j$. We now list some properties of interest.

Noncrossing (NC):	$\pi_i \to \pi_j$ implies $\pi_j \nrightarrow \pi_i$.
Nested (N):	For all i and j, either $\pi_i \to \pi_j, \pi_j \nrightarrow \pi_i$ or vice versa.
Nearly nested (NN):	Nested except that singleton parts do not have to penetrate other parts.
Consecutive (C):	For all i and j, $\pi_i \nrightarrow \pi_j$.
Size-consecutive (S):	Consecutive and larger elements go to parts of larger sizes.
Extremal (E):	Size-consecutive with at most one part of size larger than one.
Order-consecutive (O):	Parts can be indexed such that $\pi_i \nrightarrow \bigcup_{j=1}^{i-1}\pi_j$ for all i.

We note that NC was first introduced in Kreweras (1972), N in Boros and Hammer (1989), NN in Gal and Klots (1995), C and O in Chakravarty et al. (1982), S in Hwang et al. (1985), and E in Anily and Federgruen (1991).

Related to k-sorting is the notion of k-consistency. A property of partitions Q is k-*consistent* if a partition must satisfy Q whenever all of its k-subpartitions satisfy Q. This condition implies that if π does not satisfy Q, then there must exist some k-subpartition that does not satisfy Q on which we can do the k-sorting. It has been proved in Hwang et al. (1996) that if Q is k-consistent, then it is k'-consistent for all $k'>k$. Thus we need only to know the minimum k-consistency of Q, which is completely solved in Chang et al. (1999) and Hwang et al. (1996).

Theorem 2.1 *The minimum consistency index is* 2 *for* $Q\in\{E,S,C,NC,N,NN\}$ *and* 4 *for* O.

The following lemma from Hwang et al. (1996) connects sortability and consistency.

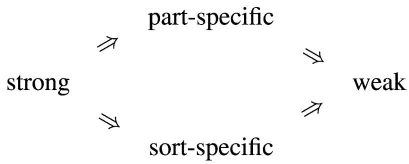

Fig. 1 Implications among levels of sortability

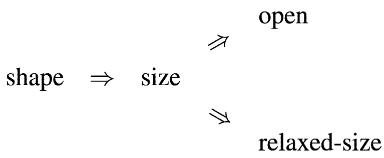

Fig. 2 Implications among types of sortability

Lemma 2.2 *If Q is (l,k,t)-sortable for some l and t, then Q is k-consistent.*

Hwang and Rothblum (2010) considered the t-regularity of a partition property. Q is *t-regular* if a t-family always contains a partition satisfying Q. Note that if there is a t-family not satisfying Q, then there is no point doing any sorting on partitions in that family since we will never get a partition satisfying Q. Furthermore, for $t \in \{\text{shape, size, relaxed-size, open}\}$, if Q is t-regular for all t, then there exists a non-trivial Q-(l,k,t)-invariant family for any l. We summarize the findings into two simple results.

Lemma 2.3 *All above properties are t-regular for $t \in \{$shape, size, relaxed-size, open$\}$ except*

(i) *N is not shape-regular for any shape containing more than one 1-entry and not p-size-regular if $n < 2p - 1$.*
(ii) *E is not shape-regular for any shape containing more than two entries larger than one.*

Subsequently, we simply assume Q is t-regular in general discussion.

When k and $t \in \{\text{open, relaxed-size, size, shape}\}$ are fixed, the implications among the four levels of sortability are given by Fig. 1 (where $x \Rightarrow y$ means that level-x sortable implies level-y sortable).

For a t-family and a t'-family where t and t' belong to {shape, size, relaxed-size, open}, $t' \Rightarrow t$ means that a member of the t'-family is also a member of the t-family. The partial order among the above four families is given by Fig. 2.

For a given Q, let π be a partition not satisfying Q and K a k-subpartition of π also not satisfying Q. Then a k-open-sorting of K can yield a 1-subpartition (which always satisfies Q discussed in this paper) and thus reduce the number of parts by $k-1$. For $k \geq 3$, then a sequence of k-open-sortings can yield a partition π' with size k' less than k even though $|\pi|$ is much larger than k. It is possible that this π' still does not satisfy Q, yet we don't know how to continue the k-sorting since π' does not have k parts. (Note that this situation cannot occur for $k = 2$.) Five positive results about open-sortability have been proved in the literature (Hwang et al. 1996; Chang et al. 1999):

(i) E is (strong, k, open)-sortable for all $k \geq 2$,
(ii) C is (strong, 2, open)-sortable,
(iii) C is (part-specific, k, open)-sortable for all $k \geq 2$,
(iv) NC is (part-specific, k, open)-sortable for all $k \geq 2$,
(v) S is (sort-specific, k, open)-sortable for all $k \geq 2$.

However, among the above five results, (i), (iii) and (iv) all overlooked the fact that the sorting sequence can come across a partition of size less than k. We propose a simple-minded remedy by prescribing: when we do open-k-sorting to a partition with less than k parts, then the new partition can be any member from the set of all partitions satisfying Q. Thus the three positive results (i), (iii) and (iv) remain correct since the overlooked case still leads to a partition satisfying Q. We also extend this rule to a relaxed-size partition which has fewer than k nonempty parts. We refer to this rule as the *k-deficiency rule*. By adopting this rule, we would not discuss in the future the case $|\pi| < k$ for open partition. Note that in (ii) where $k = 2$, the "overlooked" situation cannot occur. In (v), the sortability is proved by using the (strong, k, shape)-sortability of S (Chang et al. 1999) and the implication given in Fig. 1 to derive its (sort-specific, k, shape)-sortability and then using the following theorem (with $Q = Q'$).

For two properties Q and Q', $Q' \Rightarrow Q$ means that if a partition satisfies Q', it satisfies Q. For convenience, we refer to a part which can be empty as a *relaxed part* and a partition consisting of (k) relaxed parts as a *relaxed* (k-)*partition*.

Theorem 2.4 *Suppose Q and Q' are two k-consistent properties with $Q' \Rightarrow Q$ while t and t' either both refer to relaxed or both refer to non-relaxed with $t' \Rightarrow t$. Then if Q' is (sort-specific, k, t')-sortable for all t', Q is (sort-specific, k, t)-sortable.*

Proof It suffices to prove that every Q-(part-specific, k, t)-invariant family satisfies Q. Let Π be such a family and $\pi \in \Pi$. If π satisfies Q, we are done. So assume π doesn't satisfy Q, hence doesn't satisfy Q'. By the Q-(part-specific, k, t)-invariance of Π, there exists a k-subpartition K not satisfying Q such that any Q-t-k-sorting of K would yield a partition in Π. Since the set of all Q-t-k-sortings of K contains the set of all its Q'-t'-k-sortings, Π contains all partitions yielded from Q'-t'-sortings of K. Hence Π contains a subfamily Π' which is Q'-(part-specific, k, t')-invariant. By the (sort-specific, k, t')-sortability of Q', Π', hence Π, satisfies Q'. Consequently Π satisfies Q. □

Note that size $\Rightarrow$ relaxed-size is excluded from Theorem 2.4 as t' and t. The reason is that the specific relaxed K may contain empty parts where a size-sorting does not apply, but for $k = 2$, this case does not occur since a relaxed-size 2-subpartition not satisfying Q cannot have an empty part. Thus we have

Corollary 2.5 *Suppose Q and Q' are two k-consistent properties with $Q' \Rightarrow Q$. Then if Q' is (sort-specific, 2, size)-sortable, Q is (sort-specific, 2, relaxed-size)-sortable.*

Corollary 2.6 *For $Q \in \{E, S, C, O, NC, NN\}$ and $l \in \{$sort-specific, weak$\}$, Q is (l, k, open)-sortable for all $k \geq$the minimum consistency index of Q.*

Proof It was proved in Chang et al. (1999) that for $Q \in \{E, S, C, O, NC, NN\}$, Q is (sort-specific, k, size)-sortable for the specified set of k's. By Theorem 2.4, Q is (sort-specific, k, open)-sortable. Further, by the implication scheme among levels shown before, Q is also (weak, k, open)-sortable. □

Note that Corollary 2.6 does not apply to N due to non-universality of its size-sortability. We will give a direct proof of its (sort-specific, k, open)-sortability in Sect. 3.

Chang et al. (1999) first proved Theorem 2.4, with $t \in$ {shape, size, open} for the two special cases $t = t'$ and $Q = Q'$. Moreover, they stated the two results not just for $l =$ sort-specific, but also for $l =$ weak. We now show that in the case $t = t'$ the proof does not cover for the weak case. This is because then we have to show the existence of a subfamily Π^* in Π which is Q'-(strong, k, t)-invariant, i.e., for any k-subpartition K^* not satisfying Q', any Q'-sorting will stay inside Π^*. But if K^* satisfies Q, then K^* is not a candidate as a k-subpartition to be Q-sorted in Π. Hence we can borrow no sorting from Π for this Q'-sorting; consequently, there is no argument that a Q'-sorting of K^* will stay in Π^*. However, for $Q' = Q$, this concern, i.e., K^* satisfies Q but not Q', of course doesn't exist and the proof of Theorem 2.4 goes through for the weak case as proved in Chang et al. (1999).

Theorem 2.7 *For $t' \Rightarrow t$, if Q is (weak, k, t')-sortable for all t', then Q is (weak, k, t)-sortable.*

The argument for proving Theorem 2.4 also does not work for $l \in$ {strong, part-specific} for a different reason even for $Q' = Q$. Let π be a partition in the t-family not satisfying Q, and K be a k-subpartition not satisfying Q. Then a Q-t-sorting of K may not be its Q-t'-sorting. So an (l^{-1}, k, t)-invariant family dealing with a specific t-sorting at each step may not contain any (l^{-1}, k, t')-invariant subfamily.

Theorem 2.8 *If Q is (strong, 2, size)-sortable, then Q is (strong, 2, relaxed-size)-sortable.*

Proof Suppose to the contrary that there exists a Q-(weak, 2, relaxed-size)-invariant family not satisfying Q. Let $\Pi = \{\pi^1 \to \pi^2 \to \cdots \to \pi^r \to \pi^1\}$ be such a family which minimizes (p, r) lexicographically. We also use $|\pi|$ to denote the number of nonempty parts in π for π in a relaxed family. Since a relaxed partition containing at most one nonempty part satisfies Q, the two parts selected for sorting must be nonempty. Hence $|\pi^i| \geq |\pi^{i+1}|$, where index $i + 1$ is assumed to be modulo r. Indeed, $|\pi^i|$ must be a constant for otherwise $|\pi^{i'}| > |\pi^{i'+1}| \geq \cdots \geq |\pi^{i'}|$ for some i', an absurdity and thus empty parts are not involved in the sortings. Hence by the definition of Π, no partition of Π contains empty parts. We conclude that Π is also a Q-(weak, 2, size)-invariant family not satisfying Q, contradicting the (strong, 2, size)-sortability of Q. □

Theorem 2.9 *For $l \in$ {strong, part-specific} and $t' \Rightarrow t$, if Q is (l, k, t)-sortable, then Q is (l, k, t')-sortable.*

Proof Suppose Q is not (l,k,t')-sortable for $l \in \{\text{strong, part-specific}\}$. Then there exists a t'-family Π which is Q-(l^{-1},k,t')-invariant but not satisfying Q. We show that Π is also a Q-(l^{-1},k,t)-invariant family, contradicting the assumption that Q is (l,k,t)-sortable. Let $\pi \in \Pi$ be a partition not satisfying Q and K be a (relaxed) k-subpartition of π not satisfying Q (if t is relaxed). In particular, when t' is non-relaxed and t is relaxed, every partition in Π has no empty part so every relaxed k-subpartition is just a k-subpartition and thus is considered. Since Π is Q-(l^{-1},k,t')-invariant, there exists a Q-t'-sorting of K yielding a partition also in Π. Since a Q-t'-k-sorting is also a Q-t-k-sorting; hence Π is also Q-(l^{-1},k,t)-invariant. □

The special cases ($t'=$ shape, $t=$ size) and ($t'=$ size, $t=$ open) were first proved in Chang et al. (1999). The reason that the argument for Theorem 2.9 does not work for $l \in \{\text{sort-specific, weak}\}$ is that a Q-(l^{-1},k,t)-sorting of a k-subpartition K may not be a Q-(l^{-1},k,t')-sorting of K so a partition obtained from a Q-(l^{-1},k,t)-sorting may not be in Π, implying that Π is not necessarily Q-(l^{-1},k,t)-invariant.

We now study the relations between open families and relaxed-size families. Let Π be a relaxed-size family. We call Π' its *induced family* if a partition π' is in Π' if and only if it is an induced partition of a partition in Π. An open family Π is said to be *p-bounded* if for every partition $\pi \in \Pi$, its index-set (of parts) is a subset of $\{1,\dots,p\}$. Given a p-bounded open family Π, we can construct a relaxed-size(size p) family Π' such that a partition π' is in Π' if and only if it is obtained from a partition π in Π by setting all parts whose indices are not in $J(\pi)$ to be empty parts. Π' is called the relaxed p-family of Π. In particular, relaxed n-family is often used when the reduction assumption holds and p is bounded by the number n of elements in the partition set.

Lemma 2.10 *Given a k-subpartition K of a partition π, there exists a one-to-one mapping between the set of relaxed-size Q-k-sortings of K and the set of open Q-k-sortings of K into K' such that $J(K)$ contains $J(K')$.*

Proof Obvious by setting $J(K) \setminus J(K')$ to empty parts. □

For any two families belonging to {shape, size, relaxed-size, open}, we have seen that one contains the other except open-family and relaxed-size-family. We now study sortability implications between open-family and relaxed-size-family.

Theorem 2.11

(i) *If Q is (part-specific, k, open)-sortable, then Q is (part-specific, k, relaxed-size)-sortable.*
(ii) *If Q is (sort-specific, k, relaxed-size)-sortable, then Q is (sort-specific, k, open)-sortable.*

Proof Case (i). Let Π be a (sort-specific, k, relaxed-size)-invariant family and Π' be the induced family of Π. We prove that Π', hence Π, satisfies Q. Let $\pi' \in \Pi'$. By the k-deficient rule, we may assume $|\pi'| \geq k$. Let K be a k-subpartition of π' not satisfying Q and let π be the partition in Π corresponding to π'. Since Π is (sort-specific, k, relaxed-size)-invariant, there exists a relaxed-size sorting of K into K^*, resulting in a new partition π^* also in Π. But this sorting is also an open k-sorting since K contains no empty part, while the new partition, the induced partiton of π^*, is in Π'. This proves that Π' is also (sort-specific, k, open)-invariant. Since Q is (part-specific, k, open)-sortable, Π' satisfies Q.

Case (ii). Let Π be a (part-specific, k, open)-invariant family. Let $\pi \in \Pi$ not satisfy Q. By the k-deficient rule, we may assume that π has at least k (nonempty) parts. Then there exists a k-subpartition K of π not satisfying Q such that the partition obtained from π by an arbitrary open-sorting of K stays inside Π. Let Π' be the relaxed n-family of Π and π' the corresponding partition of π. By Lemma 2.10, every relaxed-size sorting stays inside Π'. Thus Π' is (part-specific, k, relaxed-size)-invariant. Since Q is (sort-specific, k, relaxed-size)-sortable, Π', hence Π, satisfies Q. □

We can not extend the argument for proving Theorem 2.11(i) to $l =$ strong since although there exists a relaxed k-subpartition K of π, K may contain less than k nonempty parts and hence cannot be a candidate of a k-subpartition to be sorted in Π'. However, for $k = 2$, both families consider only 2-subpartitions not satisfying Q (thus no empty part). Hence we have

Theorem 2.12 *If Q is 2-consistent and (strong, 2, open)-sortable, then Q is (strong, 2, relaxed-size)-sortable.*

Nor can we extend the argument for proving Theorem 2.11(ii) to $l =$weak since then we have to consider the (strong, k, open)-invariant family Π and its relaxed n-family Π'. To prove that Π' is (strong, k, relaxed-size)-invariant, we have to show that for $\pi' \in \Pi'$ not satisfying Q, every relaxed k-subpartition K can be relaxed-size sorted into another partition in Π'. This includes K containing some empty parts for which we cannot borrow any sorting in Π. Note that for $k = 2$, this issue does not occur since a relaxed 2-subpartition not satisfying Q cannot have empty parts. Thus we have

Theorem 2.13 *If Q is (weak, 2, relaxed-size)-sortable, then Q is (weak, 2, open)-sortable.*

A property Q is called *hereditary* if π satisfies Q implies all its subpartitions satisfy Q. All properties discussed in this paper are hereditary. The following result was first proved in Chang et al. (1999) for $t =$ shape.

Lemma 2.14 *Let $Q \in \{NC, N, NN, C, S, E, O\}$ and $k' > k \geq$ the minimum consistency of Q. Then if Q is (strong, k', t)-sortable, it is (strong, k, t)-sortable.*

Proof It is sufficient to consider the case where $k' = k + 1$. First assume that $Q \in \{NC, NN, C, O\}$. Suppose that Q is not (strong, k, t)-sortable. Let Π be a Q-(weak, k, t)-invariant family not satisfying Q. Let $\mathcal{N}'$ be the set obtained from $\mathcal{N}$ by

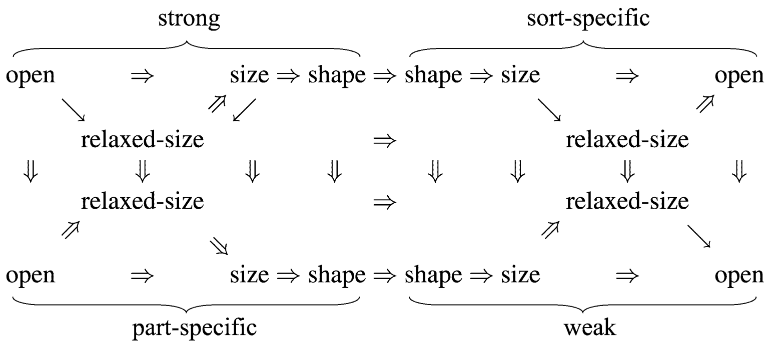

Fig. 3 Implications among sortabilities

augmenting it with a single element $n+1$. For each partition $\pi \in \Pi$, let π' be the partition $\pi \cup \{n+1\}$ (of $\mathcal{N}'$); in particular, the map from π to π' is a one-to-one map of Π onto Π'. We will show that Π' is Q-(weak, k', t)-invariant and does not satisfy Q.

For $Q \in \{NC, NN, C, O\}$, a partition π satisfies Q if and only if its augmented partition π' satisfies Q. Hence Π' does not satisfy Q. To see that Π' is Q-(weak, $k+1$, t)-invariant, consider a partition $\pi' \in \Pi'$ not satisfying Q where $\pi \in \Pi$. Then π does not satisfy Q and by the Q-(weak, k, t)-invariant of Π, there exist a k-subpartition K of π not satisfying Q and a Q-t-k-sorting of K into $\bar{K}$, resulting in a new partition $\bar{\pi}$ also in Π. Let K', $\bar{K}'$ and $\bar{\pi}'$ be the augmented partition of K, $\bar{K}$ and $\bar{\pi}$, respectively. Then $\bar{\pi}'$ can be viewed as obtained from a (specific) Q-t-$(k+1)$-sorting of K' into $\bar{K}'$. Since $\bar{\pi}'$ belongs to Π', Π' is Q-(weak, $k+1$, t)-invariant.

Next, consider $Q \in \{S, E\}$. We apply the arguments above for S and E with the following modifications. $\mathcal{N}'$ is set as $\mathcal{N} \cup \{0\}$ and partitions of $\mathcal{N}$ are augmented with a single part $\{0\}$ to obtain partitions of $\mathcal{N}'$. Then the above arguments are applicable because a partition of $\mathcal{N}$ satisfies Q if and only if the augmented partition satisfies Q.

Finally, let $Q = N$. Here, $\mathcal{N}'$ is set as $\mathcal{N} \cup \{0, n+1\}$ and partitions of $\mathcal{N}$ are augmented with a single part $\{0, n+1\}$ to obtain partitions of $\mathcal{N}'$. Again, the arguments are applicable because a partition of $\mathcal{N}$ satisfies Q if and only if the augmented partition satisfies Q. □

In Fig. 3, for a given Q, we have the implications ($\Rightarrow$) among (l, k, t)-sortabilities for fixed k and the implications ($\rightarrow$) for $k = 2$.

3 Relaxed-size-sortabilities of specific properties of partitions

For any $Q \in \{N, NN, E, S, C, NC, O\}$, Q is relaxed-size-regular, thus ensuring the existence of a non-trivial (l, k, relaxed-size)-invariant family. Hence, to prove their sortabilities, it suffices to prove that the associated invariant families satisfy Q.

A common technique used in the literature to prove that a Q-(l^{-1}, k, t)-invariant family Π satisfies Q, also used in this paper, is to establish the existence of a statistic

$s(\pi)$ such that for every partition in Π, a Q-sorting, meeting the (l,k,t) specification, of π into π' satisfies $s(\pi) > s(\pi')$. Since Π is finite, the strict decreasing of $s(\pi)$ forces the sorting sequence to end at a partition satisfying Q. We will call $s(\cdot)$ a *k-uniform statistic* if it applies to k'-sorting for all $k' \le k$ and > 1, and a *uniform statistic* if it applies to k'-sorting for all $k' > 1$.

We notice that Theorem 2.4 does not include the case that $t' =$ size and $t =$ relaxed-size. We observe a result with relaxed conditions for this case.

Theorem 3.1 *Suppose Q is 2-consistent. Then if Q is (sort-specific, k, size)-sortable for all $k \le k'$ with a k'-uniform statistic s, Q is (sort-specific, k, relaxed-size)-sortable for all $k \le k'$.*

Proof For any partition π not satisfying Q, any $k \le k'$ and any relaxed k-subpartition K not satisfying Q, the induced partition K^* of K is a $|K^*|$-subpartition not satisfying Q with $k' \ge k \ge |K^*| \ge 1$. Furthermore, Q being (sort-specific, $|K^*|$, size)-sortable guarantees a size-sorting of size $|K^*|$ which is also a relaxed-size sorting of K. Hence the theorem follows. □

Furthermore, Theorem 2.12 can be generalized as follows if the sortability is derived from a statistic.

Theorem 3.2 *Suppose Q is 2-consistent. Then if Q is (strong, k, open)-sortable for all $k \le k'$ with a k'-uniform statistic s, Q is (strong, k, relaxed-size)-sortable for all $k \le k'$.*

Proof Let π' be obtained from π by Q-relaxed-size-k-sorting some relaxed k-subpartition K not satisfying Q. Assume K is sorted into K'. Let K^* and K'^* be the induced partitions of K and K', respectively. Then this Q-relaxed-size-k-sorting of K can be viewed as Q-open-$|K^*|$-sorting K^* into K'^*. Since Q is (strong, $|K^*|$, open)-sortable with a k'-uniform statistic $s(\cdot)$, $s(\pi') < s(\pi)$. □

In this section, we study the relaxed-size sortabilities of the properties mentioned in Sect. 2. We summarize the sortability results in a table for each property which includes the sortability results for types other than relaxed-size (quoted from Chang et al. 1999) for easy reference and comparisons. The table has twelve cells each corresponding to a specific pair of (l,t). The convention used for the entries is that $k = x$ means (l,x,t)-sortable, $\bar{k} = x$ means not (l,x,t)-sortable and if a particular k does not appear in the entry, that means its (l,k,t)-sortability is not known one way or another. Note that entries in the relaxed-size column reflects not only results directly proved, but also results obtained by implications established in Sect. 2. Notice that all general results in this paper are applicable for any $Q \in \{N, NN, E, S, C, NC, O\}$ when considering relaxed-size sortability since they are relaxed-size-regular.

3.1 Nestedness and nearly nestedness

We first consider the strong and part-specific sortabilities.

Theorem 3.3 *For* $l =$ part-*specific or strong*, N *and* NN *are not* $(l, k,$ *relaxed-size*)-*sortable.*

Proof N and NN are proved not (part-specific, k, shape)-sortable in Chang et al. (1999). Thus by Theorem 2.9, the theorem follows. □

We next consider the sort-specific and weak sortabilities of N and NN. In Chang et al. (1999), N was proved (sort-specific, k, shape)-sortable for all k but the proof is incomplete. We remedy the result by adding a necessary and sufficient condition on the family along with a modified sorting and a modified statistic. For convenience, let $\pi(i)$ denote the part of π that contains $i \in \mathcal{N}$ and $j_i(\pi)$ denote its index and then π can be represented in the form $j_1(\pi) j_2(\pi) \cdots j_n(\pi)$.

Theorem 3.4 N *is* (*sort-specific*, k, *shape*)-*sortable if and only if the shape contains at most one* 1-*entry.*

Proof The necessary condition follows by that N is not shape-regular if the shape contains more than one 1-entry. Consider the sufficient condition. Suppose the shape contains at most one 1-entry. Then it is clear that there exists a non-trivial (part-specific, k shape)-invariant family. Let I be a set of integers. Define $\sigma(I) = \min_2(I) - \min(I)$, where $\min_2(I)$ is the second smallest element in I. For a p-partition π, order π_j's, according to $\min(\pi_j)$ such that $\pi_{[1]}$ has the smallest $\min(\pi_j)$, $\pi_{[2]}$ has the second smallest $\min(\pi_j)$, and so on, in particular, $\pi_{[p]}$ has the p-th smallest $\min(\pi_j)$. Define $s(\pi) = (\sigma(\pi_{[1]}), \ldots, \sigma(\pi_{[p]}))$ and the order of $s(\pi)$ is lexicographical.

Let Π be a (part-specific, k, shape)-invariant family. Suppose $\pi \in \Pi$ does not satisfy N. Then there exists a k-subpartition K not satisfying N. Assume that K' is obtained by an N-shape-k-sorting of K which sorts π into π'. Notice that $\sigma(\pi_j) = \sigma(\pi'_j)$ for j not in $J(K)$. Since $s(\cdot)$ is in lexicographical order, maximizing $s(\pi')$ requires to maximize $\sigma(K'_{[1]})$ first. To accomplish the maximization, $\min K'_{[1]}$ should be as small as possible and $\min_2 K'_{[1]}$ as large as possible. This implies that $\min K'_{[1]}$ should head the sequence of K' and the rest of $K'_{[1]}$ should be at the end of K' (consecutively). Further, $K'_{[1]}$ should be the smallest part among all non-singleton parts in K'. Next we maximize $\sigma(K'_{[2]})$. A similar argument then forces $K'_{[2]}$ to be the second smallest part among all non-singleton parts in K', $\min K'_{[2]}$ heads the subsequence $K' \setminus K'_{[1]}$ (the second position in K'), and the rest of $K'_{[2]}$ at the end of $K' \setminus K'_{[1]}$. The same goes to the determinations and configurations of $K'_{[3]}, \ldots, K'_{[k]}$.

To summarize, the above analysis leads to the following configuration of K': $n_u < n_v$ implies $\pi'_v \rightarrow \pi'_u$ for any $u, v \in J(K)$ except that a singleton part penetrates every other non-singleton part, and $\pi'_j \setminus \{\min(\pi'_j)\}$ is consecutive in the sequence K' for any $j \in J(K)$. For example, if we sort parts 1, 2, 3, 4 with shape (3, 4, 2, 1), then the new partition would be 3124222113. Continue the previous discussion, we know such a sorting uniquely maximizes $s(\cdot)$ over N-shape-k-sortings of K so $s(\pi) < s(\pi')$. Since K is arbitrary, the result is obtained under the shape. □

Corollary 3.5 *N is (weak, k, shape)-sortable if and only if the shape contains at most one* 1*-entry.*

Proof The corollary follows from Theorem 3.4 directly. □

Because N is not shape-regular whenever the shape has more than one 1-entry, Theorem 2.4 does not automatically apply to extend shape-sortability to size- and open-sortability. Indeed, a partition in a size- or open-family may contain more than one singleton and thus, we cannot rely on a shape-sorting to move ahead. This point was overlooked in Chang et al. (1999). We now give an analysis of size-sortability and open-sortability of N independent of its result on shape-sortability. Let $n(K)$ denote the number of elements in the union of parts in K.

Lemma 3.6 *Every partition π with $n \geq 2p-1$ contains a k-subpartition K with $n(K) \geq 2k-1$. In particular, for $k \geq 3$, if π does not satisfy N, then there exists such a K not satisfying N.*

Proof This statement is trivial if $p = k$ or π contains at most one singleton. Suppose π contains more than one singleton. Let π_i and π_j denote two singleton parts and let K' denote the set of the $k-2$ largest parts. Then $n(K') \geq 2k-3$ for otherwise each of the other $p-k$ parts can have at most two elements and thus $n = n(\pi) \leq 1+1+2k-4+2(p-k) = 2p-2$, contradicting our assumption $n \geq 2k-1$. Now suppose π does not satisfy N. Then $K' \cup \{\pi_i, \pi_j\}$ is a k-subpartition not satisfying N. □

Theorem 3.7 *N is (sort-specific, k, size)-sortable if and only if $n \geq 2p-1$.*

Proof It suffices to consider the sufficient condition. First, we will show the existence of a non-trivial (part-specific, k, size)-invariant family. Consider $k = 2$. Let $\Pi = \{\pi :$ π consists of 2-parts and exactly one singleton and removing singleton from π yields a partition satisfying $N\}$. For any $\pi \in \Pi$ not satisfying N, assume $\pi_0 = \{i_0\}$ is the singleton and let $\pi_j = \{i_1, i_2\}$ be the part where $i_0 + 1 = i_1 < i_2$ or $i_1 < i_2 = i_0 - 1$. Then $K = \{\pi_0, \pi_j\}$ does not satisfy N and any N-sorting of K yields a partition in Π. Thus Π is a non-trivial (part-specific, 2, size)-invariant family. Consider $k \geq 3$. Then by Lemma 3.6, any partition π not satisfying N must have a k-subpartition K not satisfying N with $n(K) \geq 2k-1$ which can be N-sorted, ensuring the existence of a non-trivial N-(part-specific, k, size)-invariant family.

Next, let $\delta(\pi)$ denote the number of singletons in a partition π. Define $s'(\pi) = (-\delta(\pi),\ s(\pi))$ where $s(\pi)$ is same as in the proof of Theorem 3.4. Let Π be an N-(part-specific, k, size)-invariant family and $\pi \in \Pi$ be a partition not satisfying N. We will show that $s'(\pi)$ is increasing lexicographically in some N-size-k-sorting of K. Then by the (part-specific, k, size)-invariance of Π, π' yielded by the above sorting belongs to Π. Since Π is finite and $s'(\pi') > s'(\pi)$, Π contains a partition satisfying N.

Suppose $\delta(K) \leq 1$. Then the shape-k-sorting in Theorem 3.4 preserves $\delta(K)$ and hence $\delta(\pi)$. Then $s'(\pi)$ increases in this sorting since $s(\pi)$ does.

Suppose $\delta(K) \geq 2$. Then there exists an N-size-sorting that sorts K into a partition with at most one singleton part, and such a sorting makes $-\delta(\pi)$ increase. □

Theorem 3.8 *N is (weak, k, size)-sortable if and only if $n \geq 2p - 1$.*

Proof It suffices to consider the sufficiency. Assume $n \geq 2p - 1$. Then every (strong, k, size)-invariant family Π must contain a (part-specific, k, size)-invariant family. Thus Π satisfies N by Theorem 3.7. It remains to show the existence of a non-trivial (strong, k, size)-invariant family. Let Π' be the set of all partitions each satisfying N and consisting of 2-parts and exactly one singleton. Let $\Pi = \{\pi = 12\cdots(p-2)p(p-1)(p-1)(p-2)\cdots 1\} \cup \Pi'$, where π is the only partition not satisfying N. Let K be a k-subpartition of π not satisfying N. Then $\{p-1, p\} \subseteq J(K)$ and $n(K) = 2k - 1$. Thus any N-sorting of K must yield a partition in Π', implying Π is N-(strong, k, size)-invariant. □

Theorem 3.9 *N is (sort-specific, k, open)-sortable for all k.*

Proof We use the statistic $s'(\pi)$ in Theorem 3.7. Suppose K does not contain any singleton. Then the shape-sorting in Theorem 3.4 would make $s'(\pi)$ increase. Obviously any K can be N-open-sorted into a partition without a singleton ($p' = 1$ is an extreme case) and thus if K contains a singleton, $s'(\pi)$ increases in such a sorting. □

None of the general results is applicable to deduce the relaxed-size sortability of N since the shape and size sortabilities are not universal. We give an independent proof for it, but with a similar idea.

Theorem 3.10 *N is (sort-specific, k, relaxed-size)-sortable for all k.*

Proof Again, we use the statistic $s'(\pi)$ in Theorem 3.7. Let K be any relaxed k-subpartition not satisfying N. Removing empty parts from K induces a k'-subpartition K' not satisfying N. Similar to Theorem 3.9, we can N-open-sort K' into a subpartition with at most k parts with $s'(\pi)$ increasing. Then insert enough empty parts to make it a relaxed k-subpartition. □

NN is t-regular for any $t \in$ {shape, size, relaxed-size, open} and thus to prove the sortability of NN, it remains to prove that the associated invariant families satisfy NN.

Theorem 3.11 *For $t \in$ {shape, size, open}, NN is (sort-specific, k, t)-sortable for all k.*

Proof Since the locations of the singleton parts do not matter and an N-sorting is also an NN-sorting, for any k-subpartition not satisfying NN we can use the N-shape-sorting in Theorem 3.4 on non-singleton parts to achieve a partition satisfying NN with s increasing. Furthermore, by Theorem 2.4, we can extend the shape-sortability to size- and open-sortabilities. □

Table 1 Nested and nearly nested. For nestedness, some sortabilities are satisfied with the following requirements: (†) any π in the size-family shall satisfy $n(\pi) \geq 2p-1$, and (‡) the shape shall have at most one 1-entry

	open	relaxed-size	size	shape
strong	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$
part-specific	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$
sort-specific	$k \geq 2$	$k \geq 2$	$k \geq 2^{\dagger}$	$k \geq 2^{\ddagger}$
weak	$k \geq 2$	$k \geq 2$	$k \geq 2^{\dagger}$	$k \geq 2^{\ddagger}$

Table 2 Extremal. (†) the shape shall have at most one entry larger than one

	open	relaxed-size	size	shape†
strong	$k \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$
part-specific	$k \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$
sort-specific	$k \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$
weak	$k \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$

Theorem 3.12 *NN is (sort-specific, k, relaxed-size)-sortable for all k.*

Proof Theorem 3.11 is proved with a uniform statistic. Then by Theorem 3.1, the theorem follows. □

We summarize the sortabilities of N and NN in Table 1.

3.2 Extremalness

Theorem 3.13 *E is (strong, k, relaxed-size)-sortable for all k.*

Proof E is proved (strong, k, open)-sortable for all k with a uniform statistic (Chang et al. 1999). Thus by Theorem 3.2, the theorem follows. □

We summarize the sortabilities of E in Table 2.

3.3 Consecutiveness and size-consecutiveness

Theorem 3.14 *C and S are (sort-specific, k, relaxed-size)-sortable for all k.*

Proof By Theorem 3.13, E is (sort-specific, k, relaxed-size)-sortable, and it is easily observed that $E \Rightarrow S \Rightarrow C$. Then by Theorem 2.4, the theorem follows. □

Theorem 3.15 *C is (strong, 2, relaxed-size)-sortable and not (strong, k, relaxed-size)-sortable for all $k \geq 3$.*

Proof C is not (strong, k, shape)-sortable for all $k \geq 3$ (Chang et al. 1999). Thus by Theorem 2.9, C is not (strong, k, relaxed-size)-sortable for $k \geq 3$. Next, C is proved (strong, 2, open)-sortable (Hwang et al. 1996). Then by Theorem 2.12, C is (strong, 2, relaxed-size)-sortable. □

Table 3 Consecutiveness

	open	relaxed-size	size	shape
strong	$k=2; \bar{k}\geq 3$	$k=2; \bar{k}\geq 3$	$k=2; \bar{k}\geq 3$	$k=2; \bar{k}\geq 3$
part-specific	$k\geq 2$	$k\geq 2$	$k\geq 2$	$k\geq 2$
sort-specific	$k\geq 2$	$k\geq 2$	$k\geq 2$	$k\geq 2$
weak	$k\geq 2$	$k\geq 2$	$k\geq 2$	$k\geq 2$

Theorem 3.16 *C is (part-specific, k, relaxed-size)-sortable for all k.*

Proof For any k, C is proved (part-specific, k, open)-sortable (Chang et al. 1999). Thus by Theorem 2.11, the theorem follows. □

We summarize the sortabilities of C in Table 3.

Theorem 3.17 *S is (strong, 2, relaxed-size)-sortable and not (strong, k, relaxed-size)-sortable for all $k\geq 3$.*

Proof In Chang et al. (1999), S is proved (strong, 2, size)-sortable and not (strong, k, size)-sortable for all $k\geq 3$. Thus by Theorems 2.8 and 2.9, the theorem follows. □

Theorem 3.18 *S is not (part-specific, k, relaxed-size)-sortable for all $k\geq 4$.*

Proof S is not (part-specific, k, size)-sortable for $k\geq 4$ (Chang et al. 1999); hence by Theorem 2.9, the theorem follows. □

Theorem 3.19 *S is (part-specific, 3, relaxed-size)-sortable.*

Proof Suppose to the contrary that there exists an S-(sort-specific, 3, relaxed-size)-invariant family Π not satisfying S. Since C is (part-specific, 3, relaxed-size)-sortable and an S-sorting is also a C-sorting, Π has a subfamily Π' of partitions satisfying C that is an S-(sort-specific, 3, relaxed-size)-invariant family. For $\pi\in\Pi'$, label the parts in increasing order as the elements are increasing and then ceaselessly label the empty parts. We shall prove by induction the claim that for any $\pi\in\Pi'$ and $\min(\pi_4)\leq t\leq n$ there exists $\pi'\in\Pi'$ such that $n'_1\leq n'_2\leq\cdots\leq n'_{j_t(\pi)-1}$ and $\pi(t)\subseteq\pi'(t)$ with $\max(\pi'(t))=\max(\pi(t))$, and $\pi'(i)=\pi(i)$ for $n\geq i>\max(\pi(t))$.

For $t\in\pi_4$, either $\pi'=\pi$ or π' obtained from an S-sorting on $\{\pi_1,\pi_2,\pi_3\}$ is as desired. Suppose the claim holds for $t\geq\max(\pi_4)$ for any $\pi\in\Pi'$ and consider $t+1$. By the induction hypothesis, there exists a $\pi'\in\Pi'$ such that $n'_1\leq n'_2\leq\cdots\leq n'_{j_t(\pi')-1}$ and $\pi(t)\subseteq\pi'(t)$ with $\max(\pi'(t))=\max(\pi(t))$, and $\pi'(i)=\pi(i)$ for $n\geq i>\max(\pi(t))$. If $t+1\in\pi(t)$, then π' is as desired. Thus it suffices to consider $t+1=\min(\pi(t+1))$.

If $t\in\pi'_1$, then π' is as desired. Consider $t\in\pi'_2$. If π' is not as desired, then $\{\pi'_1,\pi'_2\}$ doesn't satisfy S and $\pi'_p=\emptyset$. Then π^* obtained from an S-sorting on $\{\pi'_1,\pi'_2,\pi'_p\}$ is as desired. Similar argument works well for $t\in\pi'_3$. Consider $j_t(\pi')=j\geq 4$. If $n'_{j-1}\leq n'_j$, then π' is as desired. Assume $n'_{j-1}>n'_j$. We may assume that

Table 4 Size-consecutiveness

	open	relaxed-size	size	shape
strong	$\bar{k} \geq 2$	$k = 2; \bar{k} \geq 3$	$k = 2; \bar{k} \geq 3$	$k \geq 2$
part-specific	$\bar{k} \geq 2$	$k = 2, 3; \bar{k} \geq 4$	$k = 2, 3; \bar{k} \geq 4$	$k \geq 2$
sort-specific	$k \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$
weak	$k \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$

π' is chosen such that $|\pi'(t)|$ is as large as possible. Let π^* be obtained from π' by S-sorting $\{\pi'_{j-2}, \pi'_{j-1}, \pi'_j\}$. Suppose $|\pi^*(t)| > |\pi'(t)|$. Then by induction hypothesis, there exists a $\pi^{*\prime} \in \Pi'$ with $n^{*\prime}_1 \leq n^{*\prime}_2 \leq \cdots \leq n^{*\prime}_{j_t(\pi^{*\prime})-1}$ and $\pi(t) \subseteq \pi'(t) \subsetneq \pi^*(t) \subseteq \pi^{*\prime}(t)$ with $\max(\pi^{*\prime}(t)) = \max(\pi^*(t)) = \max(\pi'(t)) = \max(\pi(t))$, and $\pi^{*\prime}(i) = \pi^*(i) = \pi'(i) = \pi(i)$ for $n \geq i > \max(\pi(t))$, contradicting the choice of π'; furthermore, if $\{\pi'_{j-2}, \pi'_{j-1}, \pi'_j\}$ is sorted into a partition with some empty part, then $|\pi^*(t)| > |\pi'(t)|$. Hence $|\pi^*(t)| \leq |\pi'(t)|$ where $\{\pi'_{j-2}, \pi'_{j-1}, \pi'_j\}$ is sorted into $\{\pi^*_{l-2}, \pi^*_{l-1}, \pi^*_l\}$ of three nonempty parts. Then $n^*_{l-2} = n'_{j-2} + n'_{j-1} + n'_j - n^*_{l-1} - n^*_l \geq n'_{j-2} \geq n'_{j-3} = n^*_{j-3}$ since $n'_{j-1} > n'_j \geq n^*_l \geq n^*_{l-1}$. So π^* is as desired.

Therefore, according to the claim, there exists $\pi \in \Pi'$ such that $n_1 \leq n_2 \leq \cdots \leq n_{j_n(\pi)-1}$. Now insert a pseudo part $\{n+1\}$ to π and consider $t = n+1$. By the above deduction, we obtain that there is a partition in Π' that satisfies S, a contradiction. □

We summarize the sortabilities of S in Table 4.

3.4 Noncrossing

Theorem 3.20 *NC is (part-specific, k, relaxed-size)-sortable and not (strong, k, relaxed-size)-sortable.*

Proof In Chang et al. (1999), NC is proved (part-specific, k, open)-sortable and not (strong, k, shape)-sortable. Thus by Theorems 2.11 and 2.9, the theorem follows. □

Theorem 3.21 *NC is (sort-specific, k, relaxed-size)-sortable for all k.*

Proof According to Theorem 3.14, C is (sort-specific, k, relaxed-size)-sortable and it is easily observed that $C \Rightarrow NC$. Thus by Theorem 2.4 with $Q' = C$ and $Q = NC$, the theorem follows. □

We summarize the sortabilities of NC in Table 5.

3.5 Order-consecutiveness

Theorem 3.22 *O is not (part-specific, k, relaxed-size)-sortable for all k.*

Proof O is proved not (part-specific, k, shape)-sortable for all k (Chang et al. 1999; Hwang et al. 1996); thus by Theorem 2.9, the theorem follows. □

Table 5 Noncrossing

	open	relaxed-size	size	shape
strong	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$
part-specific	$k \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$
sort-specific	$k \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$
weak	$k \geq 2$	$k \geq 2$	$k \geq 2$	$k \geq 2$

Table 6 Order-consecutive

	open	relaxed-size	size	shape
strong	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$
part-specific	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$	$\bar{k} \geq 2$
sort-specific	$k \geq 4; \bar{k} = 2, 3$	$k \geq 4; \bar{k} = 2, 3$	$k \geq 4; \bar{k} = 2, 3$	$k \geq 4; \bar{k} = 2, 3$
weak	$k \geq 4; \bar{k} = 2, 3$	$k \geq 4; \bar{k} = 2, 3$	$k \geq 4; \bar{k} = 2, 3$	$k \geq 4; \bar{k} = 2, 3$

Theorem 3.23 *O is (sort-specific, k, relaxed-size)-sortable for $k \geq 4$.*

Proof The minimum consistency index of O is 4 (Hwang et al. 1996) and it is easily observed that $C \Rightarrow O$. Hence by the (sort-specific, k, relaxed-size)-sortability of C (Theorem 3.14) and Theorem 2.4 with $Q' = C$ and $Q = O$, the theorem follows. □

The sortabilities of O are summarized in Table 6.

References

Anily S, Federgruen A (1991) Structured partitioning problems. Oper Res 39:130–149

Boros E, Hammer PL (1989) On clustering problems with connected optima in Euclidean spaces. Discrete Math 75:81–88

Chakravarty AK, Orlin JB, Rothblum UG (1982) A partitioning problem with additive objective with an application to optimal inventory grouping for joint replenishment. Oper Res 30:1018–1022

Chang GJ, Chen FL, Huang LL, Hwang FK, Nuan ST, Rothblum UG, Sun IF, Wang JW, Yeh HG (1999) Sortabilities of partition properties. J Comb Optim 2:413–427

Gal S, Klots B (1995) Optimal partitioning which maximizes the sum of the weighted averages. Oper Res 43:500

Hwang FK (1981) Optimal partitions. J Optim Theory Appl 34:1–10

Hwang FK, Rothblum UG (2010) Optimality and clustering. Series on appl math, vol 1. World Scientific, Singapore

Hwang FK, Rothblum UG, Yao YC (1996) Localizing combinatorial properties of partitions. Discrete Math 160:1–23

Hwang FK, Sun J, Yao EY (1985) Optimal set partitioning. SIAM J Algebra Discrete Math 6:163–170

Kreweras G (1972) Sur les partitions non croisées d'un cycle. Discrete Math 1:333–350