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Applying snapback repellers in resource budget models

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In Satake's generalized resource budget model of ecology, which was modified from Isagi's resource budget model, Satake and Iwasa illustrated, by computing the positive Lyapunov exponent, that if the depletion coefficient is greater than one, then the system is chaotic. However, a positive Lyapunov exponent implies only sensitivity in Devaney's chaos. Therefore, this work presents mathematical viewpoints and numerical analysis on Satake's generalized resource budget model to rigorously prove that the generalized resource budget model is chaotic in Devaney's sense by using the snapback repeller theory and the topological entropy theory. Moreover, this work also investigates that there is a significant difference between the behaviors of positive odd depletion coefficients and positive even depletion coefficients under numerical computations.
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In this paper, we present mathematical viewpoints and numerical analysis on Satake's generalized resource budget model, which describes the growth of plants in ecology, to rigorously prove that the model is chaotic in Devaney's sense by using the snapback repeller theory and the topological entropy theory. Moreover, this work also investigates that there is a significant difference between the behaviors of positive odd depletion coefficients and positive even depletion coefficients under numerical computations.

I. INTRODUCTION

Several explanations of the masting phenomenon have been proposed.[1–22] They involve environmental fluctuations, weather conditions, swamping predators, the weight of young deer, bird populations, the reproductive success of bears, increased efficiency of wind pollination, attraction to seed distributions, cue masting, and the dispersing of animals. However, most of these hypotheses explain neither the mechanism of masting nor the mechanism by which the timing of reproduction varies among individuals.[23]

A. Isagi's resource budget model

Isagi, Sugimura, Sumida, and Ito proposed a simple model of the mechanism of masting that was based on the resource budget of an individual tree.[24] They assumed that a constant amount of photosynthate is produced by each tree annually, given that the environmental conditions are constant from year to year. Photosynthate (P_S) is consumed for the growth and the maintenance of the tree; any that is not used by the plant is stored in a pool within the tree. The amount of P_S was constant from year to year. In one year when the accumulated P_S exceeded a threshold (L_T), the amount of accumulated P_S minus L_T was used for flowering and is regarded as the cost of flowering C_f. Hence, whenever the amount of photosynthate accumulated in preceding years was large, the tree was inclined to flower more, and the amount of flowering in a year also depended on the amount of photosynthetic products that had accumulated in the previous years. The amount of accumulated P_S was decreased to L_T after the flowering. The flowers were pollinated and bore fruits at a cost of C_a. The ratio C_a/C_f was assumed to be constant R_C. After the fruiting had been completed, the amount accumulated was $L_T - C_a = L_T - R_C C_f$. In the model, P_S accumulates annually, until the tree flowers again when the amount exceeds L_T.

B. Satake's generalized resource budget model

Let $S^{(t)}$ be the amount of energy reserved at the beginning of year t. If the sum $S^{(t)} + P_S$ is below the threshold L_T, then the tree does not reproduce and saves all of its reserved energy for the following year. If the sum exceeds L_T, then the tree uses energy for flowering. Isagi *et al.*[24] assumed that the energy expenditure for flowering exactly equals the excess, $S^{(t)} + P_S - L_T$. Satake and Iwasa[23] generalized Isagi's model by introducing a non-dimensionalized variable $Y^{(t)} = (S^{(t)} + P_S - L_T)/P_S$, and the resource budget model was rewritten as

$$Y^{(t+1)} = \begin{cases} Y^{(t)} + 1 & \text{if } Y^{(t)} \le 0, \\ -\kappa Y^{(t)} + 1 & \text{if } Y^{(t)} > 0, \end{cases} \quad t = 0, 1, \ldots, \qquad (1)$$

where $Y^{(0)} \in \mathbb{R}$ and κ denotes the degree of resource depletion after a reproductive year divided by the excess amount of energy in reserve before that year and is called the depletion coefficient. Notably, the quantity $Y^{(t)}$ is positive if and only if the tree exhibits some reproductive activity in year t.

The generalized resource budget model (1) includes only one parameter κ. It is clear that $Y^{(t+1)}$ goes to infinity eventually at $\kappa < 0$. On the other hand, $Y^{(t+1)}$ belongs in $[-\kappa + 1, 1]$ as t large enough at $\kappa \ge 0$. Satake and Iwasa[23]

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illustrated trajectories for three different values of κ. When $\kappa \in [0,1)$, $Y^{(t+1)}$ quickly converges to the stable equilibrium $1/(\kappa+1)$. There are a number of two-point cycles corresponding to different initial conditions when κ is exactly equal to 1. When $\kappa > 1$, $Y^{(t+1)}$ keeps fluctuating with a chaotic time series. Further, the authors studied the model of the coupling of trees and found perfectly synchronized periodic reproduction, synchronized reproduction with a chaotic time series, clustering phenomena, and chaotic reproduction of trees without synchronization over individuals.

Satake and Iwasa[23] identified chaos by computing a positive Lyapunov exponent as the depletion coefficient $\kappa > 1$. It is true[25–28] that some investigations regard the positive Lyapunov exponent as the definition of chaos because *sensitivity* is the most important property of chaotic systems and is easily observed. However, a positive Lyapunov exponent just implies that the map has sensitive dependence on initial conditions.[26,28] The goal here is to prove chaos by identifying *density* and *transitivity* rather than *sensitivity* as in the chaos of Devaney (defined in Sec. II A).

In this paper, we would like to point out that the generalized resource budget model (1) is chaotic in the sense of Devaney. This paper is organized as follows. In Sec. II, we first list essential preliminaries. In Sec. III, we prove the existence of the snapback repeller of the generalized resource budget model, whenever the depletion coefficient κ becomes greater than one. Numerical analysis of numerical simulations of the generalized resource budget model are presented in Sec. IV. Finally, a conclusion is given in Sec. V.

Throughout this paper, the composition of two functions is defined as $f \circ g(x) = f(g(x))$. The n-fold composition of f with itself recurs repeatedly in the sequel, f^n, and it is defined as $f^n(x) = f \circ \cdots \circ f(x)$, where n is the number of iterations.

II. PRELIMINARIES

A. Devaney's chaos

The chaos of a map has been defined in several ways.[29] Although the comment "so many authors, so many definitions," is true, a basic component of all definitions is the unpredictability of the behavior of the trajectory which is determined with some certain error. (The associated phenomenon is usually described in terms of sensitive dependence on initial conditions.) The definition of the chaos of Devaney is considered herein because it is fundamental and widely accepted.

Definition 1: (Devaney's chaos[30]) *Let X be a metric space. A continuous map $f: X \to X$ is said to be chaotic on X if*

(Sensitivity): *f has sensitive dependence on initial conditions, meaning that there exists $\delta > 0$ such that, for any $x \in X$ and any neighborhood N_x of x, there exists $y \in N_x$ and $n \in \mathbb{N}$ such that $|f^n(x) - f^n(y)| > \delta$;*

(Density): *periodic points are dense in X;*

(Transitivity): *f is topologically transitive. That is, for any pair of nonempty open sets $U, V \subset X$, there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.*

A chaotic map possesses three ingredients, which are unpredictability, an element of regularity, and indecomposability. The system is unpredictable because of the sensitive dependence on initial conditions.[30] In the midst of this random behavior, however, is an element of regularity, which is exhibited by the periodic points that are dense. A chaotic system cannot be broken down or decomposed into two subsystems (two invariant open subsets) that do not interact under f because of topological transitivity.

B. Snapback repellers

Generally, proving that a dynamical system has chaotic behavior is difficult. Most techniques for making such a determination involve computer simulations, which apply the arithmetic of the Lyapunov exponent, find a period doubling bifurcation, and perform other tasks that are associated with numerical dynamical systems. However, obtaining such results by rigorous mathematical proofs is difficult.

A dynamical system with diffeomorphism has chaotic behavior that can be proved by using known methods, such as the existence of Smale horseshoe, transversal homoclinic orbits, or heteroclinic orbits. Noninvertible maps have chaotic behavior that can be identified by the existence of snapback repellers. However, for general focus problems, applying the above methods without computer assistance is difficult. In most cases, the verification must be carried out with the aid of a computer.[31]

In 1978, Marotto defined the snapback repeller.[32] The existence of snapback repellers implies that a system is chaotic.

Definition 2: *Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be differentiable in $B_r(x^*)$ and x^* be a fixed point of f with all eigenvalues of $Df(x^*)$ exceeding 1 in norm, and there exists a constant $s > 1$ such that $\| f(x) - f(y) \| > s \| x - y \|$ for all $x, y \in B_r(x^*)$. Suppose there exists a point $x_0 \in B_r(x^*)$ with $x_0 \neq x^*$ and some positive integer m such that $f^m(x_0) = x^*$ and* $\det(Df^m(x_0)) \neq 0$. *Then x^* is called a* ***snapback repeller*** *of f* (Ref. 33).

Remark:

(1) In one-dimensional space $\mathbb{R}$, the Jacobi matrix $Df(x^*) = f'(x^*)$ and

$$\begin{aligned}\det(Df^m(x_0)) &= (f^m)'(x_0) = f'(f^{m-1}(x_0)) \cdot f'(f^{m-2}(x_0)) \\ &\quad \cdots f'(f(x_0)) \cdot f'(x_0) \\ &= f'(x_{m-1}) \cdot f'(x_{m-2}) \cdots f'(x_1) \cdot f'(x_0),\end{aligned}$$

where $x_j = f^j(x_0)$, $1 \le j \le m-1$.

(2) Let snapback repeller x^*, f, m, and x_0 be the same as Definition 2. x^* is said to be a **nondegenerate snapback repeller** of f if there exist positive constants μ and δ_0 such that $B_{\delta 0}(x_0) \subset B_{r0}(x^*)$ and $\| f^m(x) - f^m(y) \| \ge \mu \| x - y \|$ for all $x, y \in B_{\delta 0}(x_0)$; x^* is called a **regular snapback repeller** of f if $f(B_{r0}(x^*))$ is open and there exists a positive constant δ_0^* such that $B_{\delta_0^*}(x_0) \subset B_{r_0}(x^*)$ and x^* is an interior point of $f^m(B_\delta(x_0))$ for any positive constant $\delta \le \delta_0^*$.[34,35]

The snapback repeller in Marotto's theorem is nondegenerate and regular.

Theorem 3: *Let snapback repeller x^*, f, m, and x_0 be the same as Definition 2. If f is C^1 in some neighborhood of x_j ($x_j = f^j(x_0)$), $\det(Df(x_j)) \neq 0$, $0 \le j \le m-1$, then f is chaotic in the sense of Devaney* (Refs. 34–38).

C. Topological entropy

Topological entropy was defined by Adler, Konheim, and McAndrew for topologically conjugate invariance in 1965.[39] If the space is compact metric, then the following definition is equivalent to the definition of Adler, Konheim, and McAndrew,[40] and it is more useful.[41]

Definition 4: *Let $f: X \to X$ be a continuous map on the space X with metric d. A set $S \subset X$ is called (n,ϵ)-separated for f for n a positive integer and $\epsilon > 0$ provided that for every pair of distinct points $x, y \in S$, $x \neq y$, there is at least one k with $0 \le k < n$ such that $d(f^k(x), f^k(y)) > \epsilon$* (Refs. 26, 40, and 42). *The number of different orbits of length n (as measured by ϵ) is defined by*

$$r(n,\epsilon,f) = \max\{\#(S) : S \subset X \text{is} (n,\epsilon) - \text{separated set for } f\},$$

where #(S) is the cardinality of elements in S. Let

$$h_{\text{top}}(\epsilon,f) = \limsup_{n\to\infty} \frac{\log(r(n,\epsilon,f))}{n}$$

and define the ***topological entropy*** *of f as*

$$h_{\text{top}}(f) = \lim_{\epsilon\to 0,\epsilon>0} h_{\text{top}}(\epsilon,f).$$

Consider the continuous map on the compact interval, the relationship between positive topological entropy ($h_{\text{top}}(f) > 0$) and Devaney's chaos is equivalent.

Theorem 5: *Let f be a continuous map of a compact interval I to itself. f has positive topological entropy if and only if f is chaotic in the sense of Devaney* (Refs. 43–46).

The basic result following that is used to help calculate the entropy, and relates the entropy of a map f to a n-fold composition of f, f^n.

Theorem 6: *Assume $f: X \to X$ is uniformly continuous or X is compact, and n is an integer with $n \ge 1$. Then $h_{\text{top}}(f^n) = n \cdot h_{\text{top}}(f)$* (Ref. 26).

III. MATHEMATICAL ANALYSIS

In this section, we will prove that the generalized resource budget model is chaotic in the sense of Devaney (defined in Definition 1) by using the preliminaries, the snapback repeller theory and the topological entropy theory (mentioned in Definition 2 and Definition 4).

Theorem 7: *The generalized resource budget model (1) is chaotic in the sense of Devaney when the depletion coefficient κ is greater than 1.00026.*

Proof: The generalized resource budget model (1) can be represented as a map g,

$$g(x) = \begin{cases} x+1 & \text{if } x \le 0 \\ -\kappa x + 1 & \text{if } x > 0 \end{cases}, \tag{2}$$

where κ is the depletion coefficient. Then, we would like to prove that the map g is chaotic in the sense of Devaney when $\kappa > \kappa_{11} \approx 1.0002538$. In this proof, there are three stages. First, try to find a snapback repeller of g. There exists the snapback repeller of g when $\kappa > \kappa_0$ with $\kappa_0 = \frac{1+\sqrt{5}}{2} \approx 1.6180$. Therefore, a result will be revealed that the map g is chaotic in the sense of Devaney as $\kappa > \kappa_0$ by Theorem 3. Second, improve the result in the first stage to calculate snapback repellers of g^2. There exists a snapback repeller of g^2 when $\kappa > \kappa_1$ with $\kappa_1 = \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} \approx 1.3247$. It implies that g^2 is chaotic in the sense of Devaney as $\kappa > \kappa_1$ by Theorem 3. Then, according to Theorems 5 and 6, the map g^2 has positive topological entropy, $h_{\text{top}}(g^2) > 0$, and $h_{\text{top}}(g^2) = 2 \cdot h_{\text{top}}(g)$, meaning that, $h_{\text{top}}(g) > 0$. Therefore, the map g is chaotic in the sense of Devaney as $\kappa > \kappa_1$ by Theorem 5 again. Finally, apply the technique in the second stage to the map g^{2p} with $p \in \mathbb{N}$. Here, it is not easy to find the snapback repellers of g^{2p}. We make a recurrent formula (3) for representing the map g^{2p} partially in a specific interval,

$$g^{2^p}(x) = \begin{cases} L_{2^p}(x), & x \in \left[\alpha_{p-3}\left(\frac{1}{\kappa}\right), \alpha_{p-2}\left(\frac{1}{\kappa}\right)\right] \\ R_{2^p}(x), & x \in \left[\alpha_{p-2}\left(\frac{1}{\kappa}\right), 1\right] \end{cases}, \tag{3}$$

where

$$L_{2^p}(x) = \begin{cases} -\kappa R_{2^p}(x) + \kappa + 1, & p \text{ is odd} \\ \dfrac{-R_{2^p}(x) + \kappa + 1}{k}, & p \text{ is even} \end{cases},$$
$$R_{2^p}(x) = L_{2^{p-1}} \circ R_{2^{p-1}}(x),$$
$$R_1(x) = -\kappa x + 1,$$
$$L_1(x) = x + 1,$$

and $j \in \mathbb{N}$,

$$\alpha_j(z) = \begin{cases} \alpha_{j-1} \circ \gamma \circ \gamma \circ \alpha_{j-1}(z), & j \text{ is odd} \\ \alpha_{j-1} \circ \beta \circ \alpha_{j-1}(z), & j \text{ is even} \end{cases},$$

with $\alpha_0(z) = \alpha_{-1} \circ \beta \circ \alpha_{-1}(z)$, where $\alpha_{-1}(z) = z$, $\alpha_{-2}(z) = 0$, $\beta(z) = \frac{1}{\kappa}(2 - z)$, and $\gamma(z) = \frac{1}{\kappa}(1 - z)$. Then, for different p, the snapback repeller of g^{2p} can be found from the formula (3) when the depletion coefficient $\kappa > \kappa_p$, where κ_p is computed by determining the roots of a polynomial with degree 2^{p+1} and listed in Table I. Hence, the result shows that the map g can possess Devaney's chaos for the depletion coefficient $\kappa > 1.00026$. The details of the proof are in Appendix A. □

In the proof of Theorem 7, we consider that the iterative number of the map g is only two to the power of any natural number to obtain the lower κ_p. As $1 < \kappa \le \kappa_0$ for any positive odd iterative number n the map g^n has only one fixed point, $\frac{1}{1+\kappa}$, but it is not a snapback repeller of the map g^n. At the same time, as $1 < \kappa \le \kappa_1$, the map g^m has only two fixed points, $\frac{1}{1+\kappa}$ and $\frac{2}{1+\kappa}$, for any positive even iterative number m but two to the power. However, these two fixed points both

TABLE I. κ_p is computed by determining the roots of a polynomial with degree 2^{p+1} in MAPLE 12 with the representation extended to 100 digits. κ_0 and κ_1 are solved exactly by the formulas of solving roots in polynomials with the degrees 2 and 4, respectively. However, there is no formula to solve exactly a polynomial with the degree 2^{p+1} for $p \geq 2$.

p	κ_p
0	**1.61803**398874989484820458683436563811772030917980576
1	**1.32471**795724474602596090885447809734073440405690173
2	**1.13472**413840151949260544605450647284027966722638280
3	**1.06829**718892084127636942958832387828209363101692083
4	**1.03277**096644104290932949288833474485665205837114040
5	**1.01644**386405941707209228020194178727791066232145413
6	**1.00814**003202116634233667531140811820889364490896404
7	**1.00407**366638869274027495235413584575421112130983612
8	**1.00203**177633341699708893271971142972647918937489 17
9	**1.00101**611635023998785395963563019367524570627032394
10	**1.00050**774307450011494818934717772385917913582101851
11	**1.00025**388579930649764694803800094131925950701465139

are not snapback repellers of the map g^m as m is even but not two to the power. Hence, it is a unique way to obtain lower κ_p by finding the snapback repeller of the map g^{2p} with $p \in \mathbb{N}$.

It is fortunate for $p=0$ or 1 that κ_0 and κ_1 can be solved exactly by determining roots of the polynomial with degrees 2 and 4, respectively. However, there is no general formula to solve the roots of a polynomial with degree 2^{p+1} with $p \geq 2$. Therefore, we use numerical computations to obtain κ_p in Table I by the software MAPLE 12 with the representation extended to 100 digits. The computations have to be done at a higher order of precision by extending the number of the digits of the representation since the degree 2^{p+1} of the polynomial is very large, even when p is small (for example, $p=10$ and then the degree is $2^{11}=2048$). Further, it can be observed that the sequence $\{\kappa_p\}$ converges linearly to $\kappa_\infty = 1$ at a rate of convergence of $\lim_{p\to\infty} \frac{\kappa_{p+1}-\kappa_\infty}{\kappa_p-\kappa_\infty} = \frac{1}{2}$. Hence, from a numerical computation point of view, the generalized resource budget model (1) is chaotic in the sense of Devaney when the depletion coefficient κ is greater than 1.

This section mathematically interprets that the generalized resource budget model (1) is chaotic in the sense of Devaney in Theorem 7. Section IV will analyze the generalized resource budget model in numerical simulations under a computer.

IV. NUMERICAL SIMULATIONS

The bifurcation diagram (Fig. 1) of the generalized resource budget model (1) with iterations given by the same random initial condition for the different depletion coefficient κ from 1 to 5 that Theorem 7 yielded rigorous mathematical results to show that the model is chaotic in the sense of Devaney. However, it eventually converges to a period cycle in Fig. 1 when the depletion coefficient κ is a positive even number. This is a strange result. From the derivative of the map (2), we know that the period cycle is unstable. In fact, this instability is true, and we will prove it later in Theorem 9.

Theorem 8: *For any initial value $Y^{(0)} \in \mathbb{Q}$ and the depletion coefficient $\kappa \in \mathbb{N}$, then the behavior of the generalized resource budget model (1) is a period cycle eventually.*

Proof: Without loss of generality, the initial value $Y^{(0)} \in \mathbb{Q} \cap [-\kappa+1, 1]$ and let $Y^{(0)} = \frac{n}{m} \in \mathbb{Q}$ with $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. Let $S = \left\{\frac{j}{m} \in [-\kappa+1, 1] : j \in \mathbb{Z}\right\}$, then we have $Y^{(0)} \in S$ and

$$Y^{(1)} = \begin{cases} \dfrac{j}{m} + 1 = \dfrac{j+m}{m}, & \text{if } Y^{(0)} \in [-\kappa+1, 0] \\ \dfrac{(-\kappa)j}{m} + 1 = \dfrac{(-\kappa)j+m}{m}, & \text{if } Y^{(0)} \in (0, 1] \end{cases}.$$

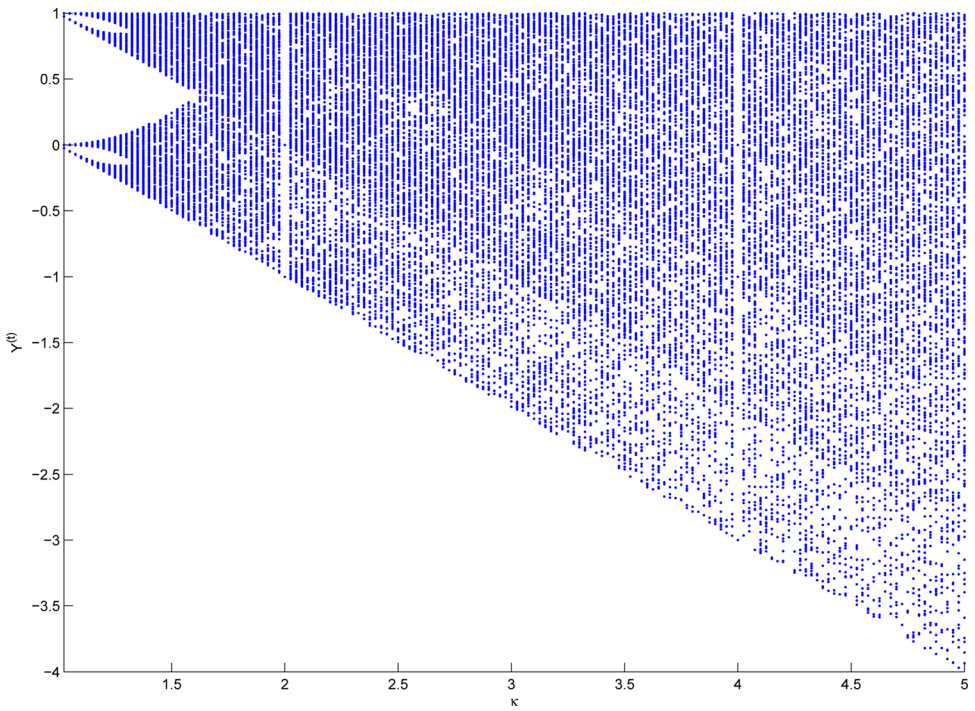

FIG. 1. (Color online) The bifurcation diagram of an individual tree. The horizontal axis represents the depletion coefficient κ, and the vertical axis represents $Y^{(t)}$ for $t > 1000$.

It also implies that $Y^{(1)} \in S$. Therefore, for $t = 2,3,\ldots$ $Y^{(t)} \in S$, too. Next, Let S_1 be the set, $\{Y^{(0)}, Y^{(1)} Y^{(2)}, \ldots, Y^{(\kappa m+1)}\}$, then $S_1 \subseteq S$. The cardinality of S is denoted by $|S|$, and

$$|S| = \left| \left\{ \frac{j}{m} \in [-\kappa + 1, 1] : j \in \mathbb{Z} \right\} \right| = \kappa m + 1.$$

Since $S_1 \subseteq S$ and $|S| = \kappa m + 1$, $|S_1| \leq |S|$ and there exists $Y^{(i)} \in S$ for some i such that $Y^{(i)} = Y^{(\kappa m+1)}$ derived from the Pigeonhole Principle. It implies that $Y^{(t)}$ always is a period cycle of period at most $\kappa m + 1 - i$ for any rational initial value and the depletion coefficient $\kappa \in \mathbb{N}$. □

Further, there is no doubt that $Y^{(0)}$ can only be expressed using finite digits in binary representation in a computer. Therefore, for any simulation in the computer, the initial value is always a rational number such that the behavior of the generalized resource budget model (1) eventually goes a period cycle when the depletion coefficient $\kappa \in \mathbb{N}$. In fact, when the depletion coefficients κ are 2 and 4, these behaviors only converge to period cycles of period 3 and period 5 (see in Fig. 1), respectively. Satake and Iwasa explained these phenomena[23] as follows, *if κ is exactly the same as an integer, after a long transient the trajectory suddenly becomes a period cycle of period $\kappa + 1$; this pathological behavior would not be realized in real forest because there is always some noise*.

However, pathological behaviors are totally different in positive even depletion coefficients and positive odd depletion coefficients. In Fig. 1, $Y^{(t)}$ indeed converges to a period cycle of period $\kappa + 1$ and the period cycle is $\{-\kappa + 1, \ldots, 0, 1\}$ when κ is a positive even number (see Figs. 2(a) and 2(c)). But, the behavior of $Y^{(t)}$ is not like "lower" periodic when κ is a positive odd number (also see Figs. 2(b) and 2(d)). Next, we will propose good explanations in Theorem 9 and Theorem 10 for κ as a positive even number and a positive odd number, respectively.

Theorem 9: *Under a binary representation of finite digits, if the depletion coefficient κ is a positive even number, then the behavior of the generalized resource budget model converges to a period cycle $\{-\kappa + 1, -\kappa + 2, \ldots, 0, 1\}$ of period $\kappa + 1$.*

Proof: According to the result in Theorem 8, the behavior of the generalized resource budget model always converges to a period cycle of period at most $\kappa m + 1$ with $Y^{(t)} = \frac{n}{m} \in [0, 1]$ for some t and $n, m \in \mathbb{N}$. Here, $\frac{n}{m}$ is represented in the binary representation of ℓ finite digits. It implies that m has to be 2^i for $i \in \{0, 1, 2, \ldots, \ell\}$ and the period is at most $\kappa 2^i + 1$. Since κ is a positive even number,

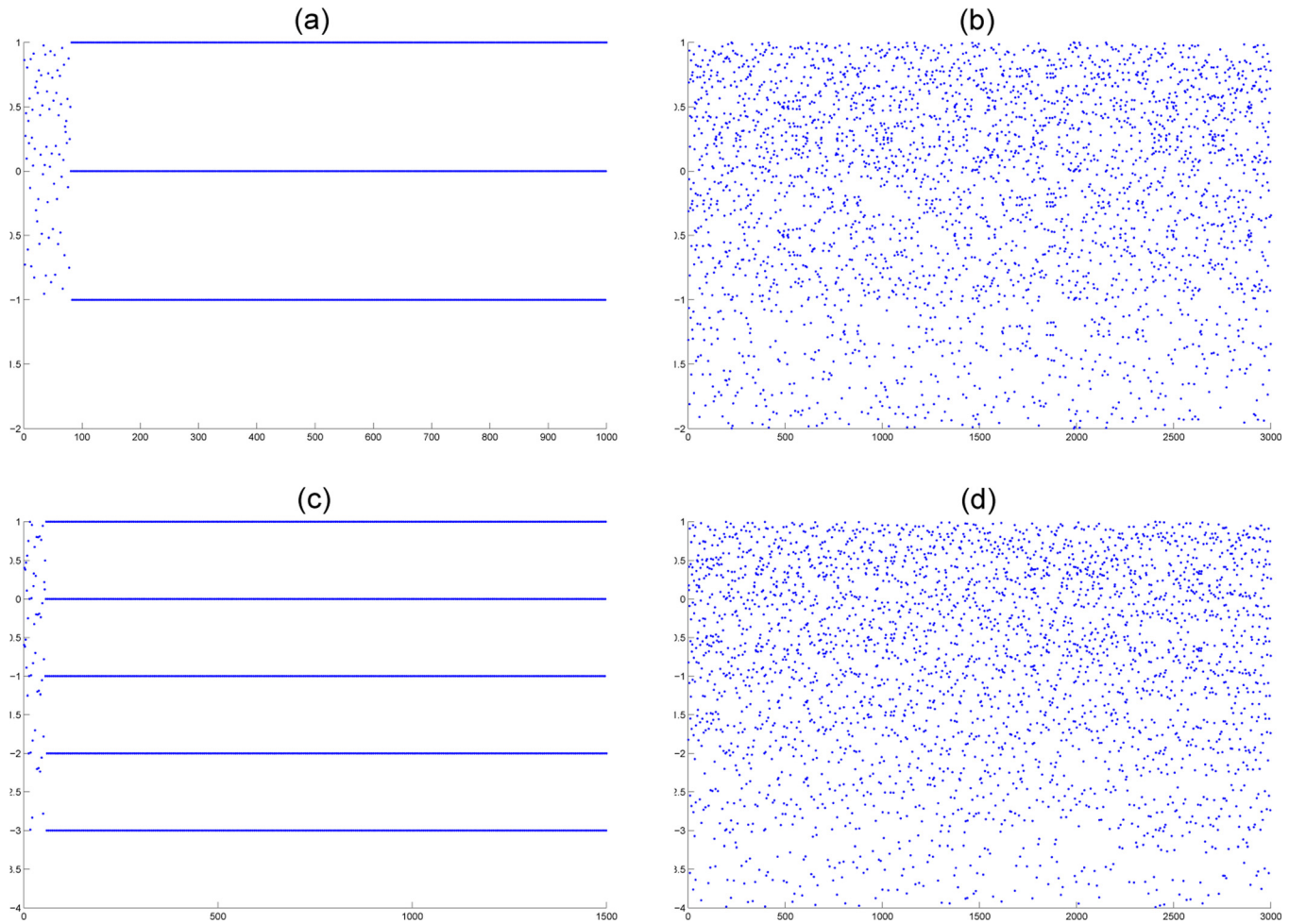

FIG. 2. (Color online) For the depletion coefficient κ is a positive even number ((a) $\kappa = 2$ and (c) $\kappa = 4$) or not ((b) $\kappa = 3$ and (d) $\kappa = 5$), the generalized resource budget model $Y^{(t)}$ converges to a lower period cycle of period $\kappa + 1$ or not.

$Y^{(t+1)} = -\kappa Y^{(t)} + 1$ should be $\frac{n_1}{m_1}$ with $n_1 \in \mathbb{Z}$ and $m_1 = 2^{i-1}$ such that the behavior of $Y^{(t+1)}$ converges to a period cycle of period at most $\kappa 2^{i-1} + 1$. Again, the period $\kappa 2^{i-1} + 1$ will be reduced to $\kappa + 1$ in finite iterations. Hence, we completely understand that the behavior of the generalized resource budget model eventually converges to the period cycle $\{-\kappa+1, -\kappa+2, \ldots, 0, 1\}$ of period $\kappa + 1$ under a binary representation of finite digits when the depletion coefficient κ is a positive even number. The details of the proof is in Appendix B. □

It is a key point that under a binary representation, a number can be represented in finite digits or not. For example, under the binary representation $02 = 0\overline{0011}$ cannot be represented in finite digits. In fact, the behavior of $Y^{(t)}$ is a period cycle {0.2,0.6, −0.2,0.8, −0.6,0.4} of period 6 when $Y^{(0)} = 0.2$ and $\kappa = 2$, not $\{-1, 0, 1\}$.

However, when the depletion coefficient κ is a positive odd number, the following theorem explains that the behavior of $Y^{(t)}$ is totally different to the positive even depletion coefficient.

Theorem 10: *Under a binary representation of finite digits, if the depletion coefficient κ is a positive odd number, then the behavior of the generalized resource budget model cannot converge to the period cycle* $\{-\kappa+1, -\kappa+2, \ldots, 0, 1\}$ *for any initial value but integer.*

Proof: Although the behavior of the generalized resource budget model converges to a period cycle of period at most $\kappa\mu + 1$ with $Y^{(\tau)} = \frac{\nu}{\mu} \in [0, 1]$ for some τ and $\nu, \mu \in \mathbb{N}$ by the result in Theorem 8, under the binary representation of finite digits, the behaviors of $Y^{(t)}$ are very different in an even κ and an odd κ. There is no chance to reduce the period $\kappa\mu + 1$ as κ is a positive odd number for almost all the initial values. The details of the proof is in Appendix C. □

V. CONCLUSIONS

Satake and Iwasa proved that the generalized budget resource model is chaotic when $\kappa > 1$ by computing the Lyapunov exponent.[23] A map possesses a positive Lyapunov exponent that implies only sensitive dependence on initial conditions. Although this result is very important and useful (it enables a single quantity to be computed to determine whether the process is highly sensitive to initial conditions[26,28]), it is just one of the necessary conditions in the definition of Devaney's chaos. In this paper, we clearly point out that the generalized resource budget model is chaotic in the sense of Devaney as the depletion coefficient $\kappa > 1$ by the relationship between Devaney's chaos, the topological entropy, and the snapback repeller.

At the same time, it is completely understood that computational simulations cause a lower period-$(\kappa + 1)$ cycle when the depletion coefficient κ is a positive even number. Further, all the trajectories will converge to periodic cycles when the initial value is a rational number and the depletion coefficient is a natural number. Based on these results of the generalized resource budget model for describing the growth of an individual tree, we will continue studying the model of the coupling of trees in future.

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APPENDIX A: THE PROOF OF THEOREM 7

Suppose $\kappa > 1$. First, $x^* = \frac{1}{1+\kappa}$ is a fixed point of the map g in Eq. (2) with $|g'(x^*)| = \kappa$ exceeding 1 ($|g'(x)| = \kappa$ as $x \in (0, 1)$). Try to find $x_0 \in (0, 1)$ such that $g^2(x_0) = x^*$. Then, $x_0 = \frac{2\kappa+1}{\kappa^2+\kappa}$ and $x_0 < 1$, thus, $\frac{2\kappa+1}{\kappa^2+\kappa} < 1$ is a necessary condition. It implies that as $\kappa > \frac{1+\sqrt{5}}{2}$, there exists a positive integer $m = 2$ such that $g^m(x_0) = x^*$ and $\det(Dg^2(x_0)) = g'(x_1) \cdot g'(x_0) \neq 0$, where $x_1 = g(x_0)$. Therefore, x^* is a snapback repeller of g as $\kappa > \kappa_0 = \frac{1+\sqrt{5}}{2}$. Hence, the map g is chaotic in the sense of Devaney as $\kappa > \kappa_0$ by Theorem 3.

Second, $x^{**} = \frac{2}{1+\kappa}$ is a fixed point of g^2 with $|Dg^2(x^{**})| = \kappa$ exceeding 1. Here, $|Dg^2(x)| = \kappa$ as $x \in \left(\frac{1}{\kappa}, 1\right)$. Let $h = g^2$ and be restricted in the domain [0,1]. It means that

$$h(x) = \begin{cases} \kappa^2 x - \kappa + 1, & x \in \left[0, \frac{1}{\kappa}\right] \\ -\kappa x + 2, & x \in \left[\frac{1}{\kappa}, 1\right] \end{cases},$$

Try to find $x_0 \in \left(\frac{1}{\kappa}, 1\right)$ such that $h^2(x_0) = x^{**}$. Then, $x_0 = \frac{2\kappa^3+\kappa^2-1}{\kappa^3(1+\kappa)}$ and $x_0 < 1$, thus, $\frac{2\kappa^3+\kappa^2-1}{\kappa^3(1+\kappa)} < 1$ is a necessary condition. It implies that as $\kappa > \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3}$, there exists a positive integer $m = 2$ such that $h^m(x_0) = x^{**}$ and $\det(Dh^2(x_0)) = h'(x_1) \cdot h'(x_0) \neq 0$, where $x_1 = h(x_0)$. Therefore, x^{**} is a snapback repeller of g^2 as $\kappa > \kappa_1 = \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3}$. It shows that g^2 is chaotic in the sense of Devaney as $\kappa > \kappa_1$ by Theorem 3. Then, according to Theorems 5 and 6, the map g^2 has positive topological entropy, $h_{\text{top}}(g^2) > 0$, and $h_{\text{top}}(g^2) = 2 \cdot h_{\text{top}}(g)$, meaning that, $h_{\text{top}}(g) > 0$. Hence, the map g is chaotic in the sense of Devaney as $\kappa > \kappa_1$ by Theorem 5 again.

Finally, we focus on the map g^{2p} restricted in the domain $I_p = [\delta(\kappa), 1]$ with $0 < \delta(\kappa) < 1$ for $p \in \mathbb{N}$. For different p, the map g^{2p} defined in I_p is represented in (3).

$$g^{2^p}(x) = \begin{cases} L_{2^p}(x), & x \in \left[\alpha_{p-3}\left(\frac{1}{\kappa}\right), \alpha_{p-2}\left(\frac{1}{\kappa}\right)\right] \\ R_{2^p}(x), & x \in \left[\alpha_{p-2}\left(\frac{1}{\kappa}\right), 1\right] \end{cases},$$

where

$$L_{2^p}(x) = \begin{cases} -\kappa R_{2^p}(x) + \kappa + 1, & p \text{ is odd} \\ \dfrac{-R_{2^p}(x) + \kappa + 1}{k}, & p \text{ is even} \end{cases},$$
$$R_{2^p}(x) = L_{2^{p-1}} \circ R_{2^{p-1}}(x),$$
$$R_1(x) = -\kappa x + 1,$$
$$L_1(x) = x + 1,$$

and $j \in \mathbb{N}$,

$$\alpha_j(z) = \begin{cases} \alpha_{j-1} \circ \gamma \circ \gamma \circ \alpha_{j-1}(z), & j \text{ is odd} \\ \alpha_{j-1} \circ \beta \circ \alpha_{j-1}(z), & j \text{ is even} \end{cases},$$

with $\alpha_0(z) = \alpha_{-1} \circ \beta \circ \alpha_{-1}(z)$, where $\alpha_{-1}(z) = z, \alpha_{-2}(z) = 0$, $\beta(z) = \frac{1}{\kappa}(2 - z)$, and $\gamma(z) = \frac{1}{\kappa}(1 - z)$. Then, $I_p = \left[\alpha_{p-3}\left(\frac{1}{\kappa}\right), 1\right]$, and we can obtain a fixed point x^p of g^{2^p} in $\left(\alpha_{p-2}\left(\frac{1}{\kappa}\right), 1\right) \subset I_p$ and check $|Dg^{2^p}(x)| > 1$ as $x \in \left(\alpha_{p-2}\left(\frac{1}{\kappa}\right), 1\right)$. Try to find $x_0 \in \left(\alpha_{p-2}\left(\frac{1}{\kappa}\right), 1\right)$ such that $g^{2^p} \circ g^{2^p}(x_0) = x^p$. Thus, there exists x_0 under a necessary condition $\kappa > \kappa_p$, where κ_p is determined by a root of a polynomial with degree 2^{p+1}. Let $x_1 = Dg^{2^p}(x_0)$, then $x_1 \in \left(\alpha_{p-3}\left(\frac{1}{\kappa}\right), \alpha_{p-2}\left(\frac{1}{\kappa}\right)\right)$. At the same time, the derivatives of $L_{2^p}(x)$ and $R_{2^p}(x)$ are not equal to zeros on the domain $\left(\alpha_{p-3}\left(\frac{1}{\kappa}\right), \alpha_{p-2}\left(\frac{1}{\kappa}\right)\right)$ and $\left(\alpha_{p-2}\left(\frac{1}{\kappa}\right), 1\right)$, respectively. Hence, $\det(D(g^{2^p} \circ g^{2^p}(x_0))) = Dg^{2^p}(x_1) \cdot Dg^{2^p}(x_0) \neq 0$. It implies that x^p is a snapback repeller of g^{2^p} as $\kappa > \kappa_p$ and g^{2^p} is chaotic in the sense of Devaney as $\kappa > \kappa_p$ by Theorem 3. According to Theorem 5 and 6, the map g^{2^p} has positive topological entropy, $h_{\text{top}}(g^{2^p}) > 0$, and $h_{\text{top}}(g^{2^p}) = 2^p \cdot h_{\text{top}}(g)$, meaning that, $h_{\text{top}}(g) > 0$. It shows that the map g is chaotic in the sense of Devaney as $\kappa > \kappa_p$ by Theorem 5 again. In Table I, we use numerical computations to obtain κ_p, and in this paper, we arrive at $\kappa_{11} \approx 1.0002538$. Hence, the map g can possess Devaney's chaos for the depletion coefficient $\kappa > 1.00026$.

APPENDIX B: THE PROOF OF THEOREM 9

Under a binary representation with ℓ valid digits ($\ell \in \mathbb{N}$), for any non-integer number $y > 0$, it can be represented in $0.e_1e_2 \cdots e_\alpha d_1 d_2 \cdots d_\beta$ or $e_1e_2 \cdots e_\alpha.d_1 d_2 \cdots d_\beta$ for some positive integers α, β, with $\alpha + \beta \geq \ell$, where $e_i \in \{0, 1\}, i = 1, \ldots, \alpha$ and $d_j \in \{0, 1\}, j = 1, \ldots, \beta$. Then, κy will be represented in $\tilde{e}_1\tilde{e}_2 \cdots \tilde{e}_{\tilde{\alpha}}.\tilde{d}_2\tilde{d}_3 \cdots \tilde{d}_\beta$ or $\tilde{e}_1\tilde{e}_2 \cdots \tilde{e}_{\tilde{\alpha}}.\hat{e}_2\hat{e}_3 \cdots \hat{e}_\alpha\tilde{d}_1\tilde{d}_2 \cdots \tilde{d}_\beta$ for some positive integer $\tilde{\alpha}$, with $\tilde{\alpha} + \alpha + \beta \geq \ell$, where $\tilde{e}_i \in \{0, 1\}$, $i = 1, \ldots, \tilde{\alpha}$, $\tilde{d}_j \in \{0, 1\}, j = 1, \ldots, \beta$, and $\hat{e}_k \in \{0, 1\}, k = 2, \ldots, \alpha$, under the binary representation of ℓ valid digits, since κ is a positive even number. It means that the number of nonzero digits at the right hand side of the point will reduce at less than one after multiplying κ as κ is a positive even number. The result is true even if $y < 0$. Further, the operation (plus one) does not affect the number of nonzero digits at the right hand side of the point. Therefore, in the generalized resource budget model (1) with the positive even depletion coefficient κ, without loss of generality, for any initial value $Y^{(0)} \in (0, 1)$, the number of nonzero digits at the right hand side of the point of $\kappa Y^{(0)}$ has to be less than one or more than $Y^{(0)}$. It shows that nonzero digits at the right hand side of the point of $Y^{(t)}$ will disappear when t is large enough (after to multiply $\kappa\ell$ times at most), meaning that the behavior of $Y^{(t)}$ goes to a period cycle $\{-\kappa + 1, -\kappa + 2, \ldots, 0, 1\}$ of period $\kappa + 1$ in finite iterations.

APPENDIX C: THE PROOF OF THEOREM 10

Under a binary representation with ℓ valid digits ($\ell \in \mathbb{N}$ and $\ell > 3$), for $y \in (0, 1)$, let $y = 0.d_1d_2 \cdots d_\beta$ with $1 \leq \beta \leq \ell/2$ and $d_i \in \{0, 1\}, i = 1, \ldots, \beta$ but not all zeros. Assume that κ is lower than or equal to $2^{\ell/2}$ and $d_\beta = 1$. Then, Under the binary representation with ℓ valid digits, κy will be represented in $e_1e_2 \cdots e_\alpha.\tilde{d}_1\tilde{d}_2 \cdots \tilde{d}_{\beta-1}d_\beta$ with $1 \leq \alpha \leq \ell/2$ for some positive integer α, where $e_i \in \{0, 1\}$, $i = 1, \ldots, \alpha$, and $\tilde{d}_j \in \{0, 1\}$, $j = 1, \ldots, \beta - 1$. It means that the number of nonzero digits at the farthest right of the point will not change after to multiply κ, i.e., the βth digit at the right hand side of the point, d_β, is still equal to 1. The result is true even if $y < 0$. Further, the operation (plus one) does not affect the number of nonzero digits at the farthest right of the point. Therefore, in the generalized resource budget model (1) with the positive odd depletion coefficient κ, the number of nonzero digits at the farthest right of the point of $\kappa Y^{(0)}$ will be the same with $Y^{(0)}$'s for any initial value $Y^{(0)} \in (0, 1)$. It shows that nonzero digits at the farthest right of the point of $Y^{(t)}$ will not disappear for all t, meaning that the behavior of $Y^{(t)}$ cannot go to a period cycle $\{-\kappa + 1, -\kappa + 2, \ldots, 0, 1\}$ for any initial value but integer.

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