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A Finiteness Theorem for Maximal Independent Sets

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Abstract. Denote by $mi(G)$ the number of maximal independent sets of G. This paper studies the set $S(k)$ of all graphs G with $mi(G) = k$ and without isolated vertices (except $G \cong K_1$) or duplicated vertices. We determine $S(1)$, $S(2)$, and $S(3)$ and prove that $|V(G)| \leq 2^{k-1} + k - 2$ for any G in $S(k)$ and $k \geq 2$; consequently, $S(k)$ is finite for any k.

1. Introduction

All graphs in this paper are simple, i.e., finite, undirected, loopless, and without multiple edges. In graph G, an *independent set* is a subset of $V(G)$ in which every two distinct vertices are nonadjacent. A *maximal independent set* is an independent set which is not a proper subset of any other independent set. A *clique* is a subset of vertices in which every two distinct vertices are adjacent. A *maximal clique* is a clique which is not a proper subset of any other clique. Let $MI(G)$ denote the set of all maximal independent sets of G and $mi(G)$ the size of $MI(G)$.

Erdös and Moser raised the problem of determining the maximum number $f(p)$ of maximal independent sets possible in a graph with p vertices and that of determining which graphs have this many maximal independent sets. Later, Moon and Moser [7] gave a complete answer to this problem, which is that for $p \geq 2$,

$$f(p) = \begin{cases} 3^t, & \text{if } p = 3t \text{ for } t \geq 1, \\ 4 \cdot 3^{t-1}, & \text{if } p = 3t + 1 \text{ for } t \geq 1, \\ 2 \cdot 3^t, & \text{if } p = 3t + 2 \text{ for } t \geq 0, \end{cases}$$

and $mi(G) = f(p)$ if and only if

$$G \cong \begin{cases} tK_3, & \text{if } p = 3t, \\ (t-1)K_3 \cup K_4 \text{ or } (t-1)K_3 \cup 2K_2, & \text{if } p = 3t + 1, \\ tK_3 \cup K_2, & \text{if } p = 3t + 2. \end{cases}$$

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Erdös and Moser actually raised their problem in terms of maximal cliques, which are maximal independent sets in complement graphs. About two decades later, a number of authors studied the same problem for trees [4, 6, 8–10], connected graphs [1, 2], triangle-free graphs [3], and bipartite graphs [5].

Instead of determining an upper bound for $mi(G)$, this paper studies $mi(G)$ from another point of view. For a fixed positive integer k, our problem is to determine all graphs G satisfying $mi(G) = k$. In a graph G, the *neighborhood* of a vertex x is

$$N_G(x) = \{y \in V(G) : x \text{ is adjacent to } y \text{ in } G\}.$$

A vertex x is *isolated* if $N_G(x) = \varnothing$. Two vertices x and y are *duplicated* if $N_G(z) = N_G(y)$. The following lemmas are trivial.

Lemma 1.1. *If x is an isolated vertex in G, then $mi(G - x) = mi(G)$.*

Lemma 1.2. *If x and y are duplicated vertices in G, then $mi(G - x) = mi(G)$.*

Proof. The lemma follows from the fact that for any $A \in MI(G)$, $x \in A$ if and only if $y \in A$. □

By Lemmas 1.1 and 1.2, deleting an isolated vertex or a duplicated vertex from graph G does not change $mi(G)$, so we shall consider only those graphs without isolated or duplicated vertices. Denote by $S(k)$ the set of all graphs G with $mi(G) = k$ and without isolated vertices (except $G \cong K_1$) or duplicated vertices. In this paper, we determine $S(1)$, $S(2)$, and $S(3)$. We also prove that $|V(G)| \leq 2^{k-1} + k - 2$ for any G in $S(k)$ and $k \geq 2$; consequently, $S(k)$ is finite for any k.

2. Graphs G with $mi(G) = k$

In this section we first determine $S(1)$, $S(2)$, and $S(3)$. The following idea is useful in this paper: For an independent set B of G there exists at least one $A \in MI(G)$ such that $B \subseteq A$.

Lemma 2.1. *If G is an induced subgraph of H, then $mi(G) \leq mi(H)$.*

Proof. For any $B \in MI(G)$, B is an independent set of H and so there exists at least one $A \in MI(H)$ such that $B \subseteq A$. Therefore, there exists a function f from $MI(G)$ to $MI(H)$ such that $f(B) \in MI(H)$ and $B \subseteq f(B)$ for any $B \in MI(G)$. Since B is a maximal independent set of G and $B \subseteq f(B)$,

$$B = f(B) \cap V(G). \tag{2.1}$$

Consequently, f is a one-to-one function and so $mi(G) \leq mi(H)$. □

Lemma 2.2. *For any two disjoint graphs G and H, $mi(G \cup H) = mi(G)mi(H)$.*

It is straightforward to check that $mi(K_n) = n$ for any $n \geq 1$, $mi(P_2) = mi(P_3) = 2$, $mi(P_4) = 3$, $mi(P_5) = 4$, $mi(C_3) = 3$, $mi(C_4) = 2$, and $mi(C_5) = 5$. For the values of $mi(P_n)$ and $mi(C_n)$ for general n, see [1].

Lemma 2.3. *Suppose G is a graph without duplicated vertices. If G has a cycle of length ≥ 4, then $mi(G) \geq 4$. If G has a cycle of length ≥ 3,* then $mi(G) \geq 3$.

Proof. We first consider the case where $G = (V, E)$ has a cycle of length ≥ 4. Choose such a cycle of minimum length n. For the case of $n \geq 5$, by the minimality of n, the cycle has no chord, i.e., C_n is an induced subgraph of G. If $n = 5$, then $mi(G) \geq mi(C_5) = 5 > 4$. If $n \geq 6$, then $mi(G) \geq mi(C_n) \geq mi(P_5) = 4$. Thus we may assume that G has a 4-cycle C: v_1, v_2, v_3, v_4, v_1. Now consider the following three cases.

Case 1. C has two chords v_1v_3 and v_2v_4. In this case, $\{v_1, v_2, v_3, v_4\}$ is a clique and so $mi(G) \geq mi(K_4) \geq 4$.

Case 2. C has exactly one chord, say $v_1v_3 \notin E$ and $v_2v_4 \in E$. Since G has no duplicated vertices, there exists a vertex y not in C that is adjacent to exactly one vertex of $\{v_1, v_3\}$, say $v_1y \in E$ and $v_3y \notin E$. Choose four maximal independent sets A_1, A_2, A_3, and A_4 of G that include $\{v_2\}$, $\{v_4\}$, $\{v_3, v_1\}$, and $\{v_3, y\}$, respectively. Since $\{v_2, v_3, v_4\}$ is a clique and v_1 is adjacent to y, these four maximal independent sets are distinct. Thus $mi(G) \geq 4$.

Case 3. C has no chord, i.e., $v_1v_3 \notin E$ and $v_2v_4 \notin E$. Since G has no duplicated vertices, there exist vertices y and z not in C that are adjacent to exactly one vertex of $\{v_1, v_3\}$ and $\{v_2, v_4\}$, respectively, say, $v_1y \in E$, $v_3y \notin E$, $v_2z \in E$, and $v_4z \notin E$. Choose four maximal independent sets A_1, A_2, A_3, and A_4 of G that include $\{v_3, v_1\}$, $\{v_3, y\}$, $\{v_4, v_2\}$, and $\{v_4, z\}$, respectively. Since v_3 is adjacent to v_4, v_1 is adjacent to y, and v_2 is adjacent to z, these four maximal independent sets are distinct. Thus $mi(G) \geq 4$.

Finally, for the case where G has a cycle of length 3, $mi(G) \geq mi(C_3) = 3$. □

Since any graph G with at least one edge has $mi(G) \geq 2$, $S(1) = \{K_1\}$.

Theorem 2.4. $S(2) = \{P_2\}$.

Proof. It is clear that $mi(P_2) = 2$ and P_2 has no isolated or duplicated vertices. On the other hand, suppose G is in $S(2)$. By Lemma 2.2 and the assumption that G has no isolated vertices, G is connected. If G has a cycle, then $mi(G) \geq 3$ by Lemma 2.3, which is impossible. Since $mi(P_4) = 3$, the maximum distance between two vertices of G is at most two. Therefore G is a star and so in fact is P_2, as G has no duplicated vertices. □

Besides P_4 and K_3, the two graphs G_1 and G_2 in Fig. 2.1 are such that $mi(G) = 3$.

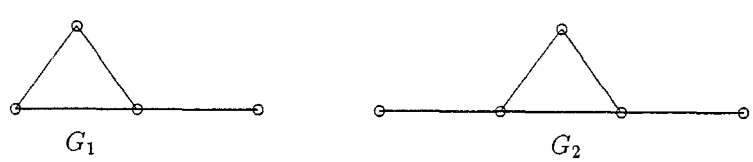

Fig. 2.1. $mi(G_1) = mi(G_2) = 3$.

Theorem 2.5. $S(3) = \{P_4, K_3, G_1, G_2\}$.

Proof. First of all, $mi(P_4) = mi(K_3) = mi(G_1) = mi(G_2) = 3$ and P_4, K_3, G_1, and G_2 have no isolated or duplicated vertices. On the other hand, suppose G is in $S(3)$. By Lemma 2.2 and the assumption that G has no isolated vertices, G is connected.

First, G has at most one block B which is not K_2 and all other blocks intersect B; otherwise, G contains $2K_2$ as an induced subgraph, which implies $mi(G) \geq mi(2K_2) = 4 > 3$, a contradiction. Second, B has 2 or 3 vertices, otherwise G has a cycle of length ≥ 4, which implies $mi(G) \geq 4 > 3$ by Lemma 2.3, again a contradiction. For the case where B is K_2, there are exactly two other blocks which are K_2 and intersect B at different vertices. This gives P_4. For the case where B is K_3, there are exactly 0, 1, or 2 blocks which are K_2 and intersect B at different vertices. This gives K_3, G_1, and G_2. □

To generalize the graphs in Theorems 2.4 and 2.5, we consider split graphs. A graph G is *split* if its vertex set can be partitioned into a clique $C \equiv \{v_1, v_2, \ldots, v_k\}$ and an independent set $I \equiv \{u_1, u_2, \ldots, u_m\}$. For the case of $\bigcup_{1 \leq i \leq m} N(u_i) \neq C$, $mi(G) = k$ and the maximal independent sets are $\{v_i\} \cup (I - N(v_i))$, $1 \leq i \leq k$. For the case of $\bigcup_{1 \leq i \leq m} N(u_i) = C$, $mi(G) = k + 1$ and besides the above k maximal independent sets, I is the $(k+1)$th maximal independent set. Note that the graphs P_2, P_4, K_3, G_1, and G_2 are all of this form.

For $k \geq 4$, it becomes hard to determine $S(k)$. However, we can prove that $|V(G)| \leq 2^{k-1} + k - 2$ for any G in $S(k)$; consequently, $S(k)$ is finite for any k.

Theorem 2.6. *If $k \geq 2$ and $G \in S(k)$, then $|V(G)| \leq 2^{k-1} + k - 2$.*

Proof. Without loss of generality, we may assume that G is in $S(k)$ and has as many vertices as possible. Let $MI(G) = \{A_1, A_2, \ldots, A_k\}$ and

$$B(v) = \{i : v \in A_i \in MI(G)\}$$

for all $v \in V(G)$. It is clear that each $B(v) \neq \varnothing$. Also, $B(v) \neq \{1, 2, \ldots, k\}$ since G has no isolated vertices. For any $u \neq v$, $N(u) \neq N(v)$ since G has no duplicated vertices. Assume that there exists some vertex $x \in N(u) - N(v)$. Then $\{x, v\} \subseteq A_j$ for some $A_j \in MI(G)$. Thus $j \in B(v) - B(u)$. This proves that $B(u) \neq B(v)$ whenever $u \neq v$. Denote by $\mathscr{P} = \{B(v): v \in V(G)\}$. Then $|V(G)| = |\mathscr{P}|$.

For any $i \in \{1, 2, \ldots, k\}$, we claim that $\{i\} \in \mathscr{P}$. Otherwise, suppose $\{i\} \notin \mathscr{P}$ and consider the graph G^* obtained from G by adding a new vertex v^*, which is adjacent to all vertices in $V(G) - A_i$. Note that $MI(G^*)$ is the same as $MI(G)$ except that A_i is replaced by $A_i \cup \{v^*\}$. Also, G^* is without isolated or duplicated vertices, a contradiction to the choice of G. Thus $\{i\} \in \mathscr{P}$ for all $1 \leq i \leq k$. We may assume that

$$V(G) = \{v_1, \ldots, v_k, \ldots, v_m\} \text{ and } B(v_i) = \{i\} \text{ for } 1 \leq i \leq k.$$

If $v_i v_j \in E(G)$, then v_i and v_j are not both in the same independent set; i.e., $B(v_i) \cap B(v_j) = \varnothing$. On the other hand, suppose $v_i v_j \notin E(G)$. Then $\{v_i, v_j\}$ is an independent set and so is a subset of some $A_r \in MI(G)$; i.e., $r \in B(v_i) \cap B(v_j)$. In conclusion, $v_i v_j \in E(G)$ if and only if $B(v_i) \cap B(v_j) = \varnothing$.

Now, choose a maximal chain $C\colon B(v_{r_1}) \supset B(v_{r_2}) \supset \cdots \supset B(v_{r_s})$ in the poset defined on $\mathscr{P}$ under set inclusion. Note that $B(v_{r_s}) = \{r_s\}$ and $s \le k - 1$. Partition $2^{\{1,2,\ldots,k\}} - \{B(v_{r_s}), \varnothing\}$ into $C_1, C_2, \ldots, C_s$, where C_i is the set of all subsets S such that $S - B(v_{r_i}) \neq \varnothing$ but $S \subseteq B(v_{r_{i-1}})$, where $B(v_{r_0}) = \{1, 2, \ldots, k\}$. For each $1 \le i \le s$, partition C_i into pairs $\{S_j, T_j\}$ such that $S_j - B(v_{r_i}) = T_j - B(v_{r_i}) \neq \varnothing$ and $B(v_{r_s})$ is the disjoint union of $B_j \cap B(v_{r_i})$ and $T_j \cap B(v_{r_i})$. Consider the following cases:

Case 1. $S_j \cap B(v_{r_i}) \neq \varnothing$ and $T_j \cap B(v_{r_i}) \neq \varnothing$.

Suppose $S_j \in \mathscr{P}$ and $T_j \in \mathscr{P}$, say, $S_j = B(x)$ and $T_j = B(y)$. Note that $B(x)$, $B(y)$, $B(v_{r_i})$ are pairwise non-disjoint. Then $\{x, y, v_{r_i}\} \subseteq A_q$ for some $1 \le q \le k$. By the definition, $q \in B(x) \cap B(y) \cap B(v_{r_i}) = \varnothing$, a contradiction. Thus either S_j or T_j is not in $\mathscr{P}$.

Case 2. $S_j \cap B(v_{r_i}) = \varnothing$ or $T_j \cap B(v_{r_i}) = \varnothing$, say, $S_j \cap B(v_{r_i}) = B(v_{r_i})$ and $T_j \cap B(v_{r_i}) = \varnothing$.

In this case, $S_j \subseteq B(v_{r_{i-1}})$. If S_j is a proper subset of $B(v_{r_{i-1}})$, by the choice of the chain C, S_j is not in $\mathscr{P}$. If $S_j = B(v_{r_{i-1}})$ and $i = 1$, then $S_j = \{1, 2, \ldots, k\}$ is not in $\mathscr{P}$. If $S_j = B(v_{r_{i-1}})$ and $2 \le i \le s$, then S_j and T_j may be both in $\mathscr{P}$.

From the discussions in Cases 1 and 2, $2^{\{1,2,\ldots,k\}} - \{B(v_{r_s}), \varnothing\}$ can be partitioned into $2^{k-1} - 1$ pairs $\{S_j, T_j\}$ such that at least one in $\{S_j, T_j\}$, except possibly $s - 1$ pairs, is not in $\mathscr{P}$. Thus

$$|V(G)| = |\mathscr{P}| \le 1 + (2^{k-1} - 1) + s - 1 \le 2^{k-1} + k - 2. \qquad \square$$

The upper bound in Theorem 2.6 is sharp as the following example shows. Consider the split graph G^* whose vertex set $V(G^*)$ is partitioned into a clique $C = \{v_1, v_2, \ldots, v_k\}$ and an independent set $I = \{u_S\colon \varnothing \neq S \subset \{1, 2, \ldots, k-1\}\}$ such that $v_i u_S \in E(G^*)$ if and only if $i \in S$. It is clear that $G^* \in S(k)$ and $|V(G^*)| = 2^{k-1} + k - 2$. Note that Theorem 2.4 is also a consequence of Theorem 2.6.

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