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Class. Quantum Grav. **29** (2012) 045004 (28pp) doi:10.1088/0264-9381/29/4/045004

No more CKY two-forms in the NHEK

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Received 1 November 2011, in final form 8 December 2011
Published 30 January 2012
Online at stacks.iop.org/CQG/29/045004

Abstract
We show that in the near-horizon limit of a Kerr–NUT–AdS black hole, the space of conformal Killing–Yano two-forms does not enhance and remains of dimension 2. The same holds for an analogous polar limit in the case of extremal NUT charge. We also derive the conformal Killing–Yano p-form equation for any background in an arbitrary dimension in the form of parallel transport.

PACS numbers: 04.20.−q, 04.70.Bw, 02.40.−k

1. Introduction and conclusion

Killing–Yano (KY) forms suitably generalize the notion of Killing vectors to higher degree differential forms [1]. They are related to constants of geodesic motion [2], symmetries of the Dirac equation [3], exotic supersymmetries of the superparticle [4], ADM-like charges [5], and the integrability of the Hamilton–Jacobi equations and the Klein–Gordon equation [6]. Two recent, complementary, reviews are given in [4] and [6]. Conformal KY (CKY) forms are the conformal generalization of KY forms and can describe the symmetries of massless or conformally invariant equations. In four dimensions, the only degree of CKY forms to study other than 1 is, by the Hodge duality, the degree of two-forms.

Rasmussen shows in [7] that the principal KY two-form of the $d = 4$ Kerr–NUT–(anti)-de Sitter black hole has a smooth limit under the near-horizon spacetime limit. At the same time, the CKY equation can be written in the form of parallel transport under some connection D on a bundle of p-forms [8]. Then, Geroch's result on spacetime limits and the holonomy of D [9] implies that the vector space of CKY two-forms cannot reduce its dimension under the near-horizon limit, in accordance with [7]. The space of CKY two-forms could enhance, though. It was this question we wanted to answer definitely. We find that there are only two independent CKY two-forms in the near-horizon geometry: the KY two-form and its Hodge dual in [7].

For the near-horizon geometry, there is an enhancement of isometries from rank 2, the time translations and rotations, to rank 4 that form $\mathfrak{sl}(2,\mathbb{R})\oplus\mathbb{R}$. This is expected by construction of the limit that can be written as a limit of a finite diffeomorphism:

$$g_0 = \lim_{\zeta\to 0^+} \mathrm{e}^{\frac{1}{\zeta}(\tau\partial_\tau - x\partial_x)} g.$$

In this expression, an infinitesimal diffeomorphism of the form $\tau\partial_\tau - x\partial_x$, which generates the finite diffeomorphism, becomes in the limit $\zeta \to 0^+$ a Killing vector of g_0. Given that the Killing vectors enhance, we asked whether the same holds for the KY two-forms, and more generally whether the CKY two-forms enhance. There are for instance relations between KY forms of various degrees in the definition of differential structure, e.g., see [8]. Note though that the near-horizon limit is not the same as the BPS limits of black holes in [10], which are known to have extra structure, e.g., see [11].

A recent result is the classification of spacetimes under the existence of a closed CKY two-form in [12–14]. However, our motivation was more in line with [15]. In particular, we were interested in the construction and consequences of a non-trivial (graded) algebra of Killing vectors and CKY two-forms. This could be achieved by studying the (graded) commutator of the symmetry operators on the Dirac equation [3] as in [15] or by lifting them, in the case of special KY, to parallel objects on the cone [8] as in [16]. In the context of recent interest in the near-horizon geometry [17], but undoubtedly beyond that, extra structure or symmetries in the near-horizon geometry would be of considerable interest.

We also derive explicitly the CKY transport equation of p-forms, which was described in [8]. Here, we were tempted to solve the equation explicitly. However, the calculation would have been quite involved and not illuminating. Since the derivation of the transport equation itself is straightforward but intense, we give the result in appendix A. Instead, we solve the problem at hand by making use of the symmetry $\mathfrak{sl}(2,\mathbb{R})\oplus\mathbb{R}$ that renders the background of cohomogeneity 1. This allows us to write an ansatz on both sides of the CKY two-form defining equation, which is shown to not pass the test.

In section 2, we introduce CKY two-forms. We show how for $d = 4$ Einstein solutions each CKY two-form is mapped to a pair of Killing vectors. Furthermore, the map transforms equivariantly under the isometry algebra. In section 3, we introduce the Kerr–NUT–AdS black hole and the coordinate ranges that we use. In section 4, we introduce the near-horizon limit of these black holes. In addition to [7], we discuss when the near-horizon limit has well-defined coordinate ranges, as inherited from the black hole. This happens only when the NUT charge is zero. We can thus differentiate the limit from a solution-generating technique to a limit that can describe a physical process when there is no NUT charge. We also discuss in parallel another spacetime limit, which we dub the polar limit. It is similar to the near-horizon limit but with the role of radial and polar coordinates exchanged. We find that the polar limit has well-defined coordinate ranges for any nonzero NUT charge. In these two sections, sections 3 and 4, we briefly comment on the positive cosmological constant case.

In section 5, we show that there are no parallel two-forms in these two limits because the Lévi-Cività holonomy is not special. In section 6, we show that the CKY two-form equation in the near-horizon extremal Kerr (NHEK) background can have either two or eight independent solutions. This follows because the CKY two-forms should transform under $\mathrm{SL}(2,\mathbb{R})$ and that there are no parallel two-forms. If there are eight independent solutions, we can write an ansatz for the 'new' CKY two-forms by using the symmetries of the metric. However, the ansatz does not pass the test of the CKY equation as we show in the concluding section section 7.

The appendices contain supplementary material for our work. Appendix A derives the connection D under which CKY p-forms are parallel. Appendix B describes a diffeomorphism

of AdS_4, which motivates the coordinate range of anti-de Sitter black holes with the NUT charge. Appendix C repeats the analysis of section 3 for a positive cosmological constant. Appendix D is a brief classification of AdS_3 coordinates and appendix E solves the Einstein equations for a specific form of the metric.

2. Conformal Killling–Yano two-forms

A two-form K on a d-dimensional manifold M with the metric g is a conformal KY two-form (CKY) if it satisfies

$$\nabla_\mu K_{\nu\rho} = A_{\mu\nu\rho} + \tfrac{1}{2} g_{\mu\nu} B_\rho - \tfrac{1}{2} g_{\mu\rho} B_\nu, \tag{1}$$

where ∇ is the Lévi-Cività derivative, A is a three-form, and B is a one-form. A CKY two-form with $B = 0$ is called a KY two-form. CKY two-forms are in one-to-one correspondence with D-parallel sections of $\mathcal{E}$:

$$K \overset{=}{\mapsto} E = K + A + B + C \in \Lambda^2 \oplus \Lambda^3 \oplus \Lambda^1 \oplus \Lambda^2 = \mathcal{E}, \tag{2}$$

where Λ^p is the space of p-forms and D is a connection on $\mathcal{E}$. In appendix A, we derive the connection D for the general case of a CKY p-form[1].

The transport equation was described in [8], the calculation tool preferred instead being Bär's cone construction. However, the latter construction is possible only for the so-called special CKY p-forms. The first-order equations $D_\mu E = 0$ can be, in principle, solved up to the obstructions given by the holonomy of D. The transport equations have the rank $d(d+1)(d+2)/6$. For instance, in $d = 4$, there are 20 first-order equations and solving them for a background of interest is computationally involved.

In the case of an Einstein space of dimension $d = 4$, we will use the simplification that both B and the Hodge dual of the three-form A in (1) are Killing one-forms, as noted already in [18]. The connection D is then given by (1) and

$$\nabla_\mu A_{\nu_1\nu_2\nu_3} = -\tfrac{3}{2} R_{[\nu_1\nu_2|\mu}{}^{\sigma} K_{\sigma|\nu_3]} - \tfrac{3}{4} g_{\mu[\nu_1} C_{\nu_2\nu_3]}, \tag{3}$$

$$\nabla_\mu B_\nu = \tfrac{1}{2} C_{\mu\nu}, \tag{4}$$

$$\nabla_\mu C_{\nu_1\nu_2} = -2 R_{\nu_1\nu_2\mu}{}^{\sigma} B_\sigma. \tag{5}$$

The Lévi-Cività derivative on the one-form B, i.e. $\nabla_\mu B_\nu$, is antisymmetric in its indices, and thus, by definition, B_μ is a Killing one-form. This result was already shown by Tachibana in [19]. The content of (5) is no more than Killing's identity for a Killing vector. Furthermore, if K satisfies

$$\nabla_X K = i_X A + \tfrac{1}{2} X^\flat \wedge B \tag{6}$$

as in (1), then its Hodge dual[2] is again a CKY two-form and satisfies

$$\nabla_X * K = -\tfrac{1}{2} i_X * B + X^\flat \wedge *A. \tag{7}$$

By the same token as we used for B, $\nabla_\mu (*A)_\nu$ is antisymmetric in its indices and is also a Killing one-form. It is this simplification that we will use in this paper.

[1] For $p = 1$, these are the (metric dual of) conformal Killing vectors. Nevertheless, for $p > 1$, one cannot in general associate with the p-form a derivation generalizing the Lie derivative.

[2] The square of the Hodge dual in $d = 4$ Lorentzian signature is $*^2|_{\Lambda^2} = -1$ and $*^2|_{\Lambda^1 \oplus \Lambda^3} = +1$. With indices these relations are $1/(2!)^2\, \epsilon_{\mu\nu}{}^{\rho\sigma} \epsilon_{\rho\sigma}{}^{\tilde{\mu}\tilde{\nu}} = -\delta_\mu^{[\tilde{\mu}} \delta_\nu^{\tilde{\nu}]}$, $1/3!\, \epsilon_\mu{}^{\nu_1\cdots\nu_3} \epsilon_{\nu_1\cdots\nu_3}{}^{\tilde{\mu}} = \delta_\mu^{\tilde{\mu}}$ and $1/3!\, \epsilon_{\mu\nu\rho}{}^{\sigma} \epsilon_\sigma{}^{\tilde{\mu}\tilde{\nu}\tilde{\rho}} = \delta_\mu^{[\tilde{\mu}} \delta_\nu^{\tilde{\nu}} \delta_\rho^{\tilde{\rho}]}$, where $\epsilon_{\mu\nu\rho\sigma}$ is the Lévi-Cività tensor.

From the above, the CKY two-form K on a four-dimensional Einstein space (M, g) is mapped to a pair of Killing vectors. We will write this as $K \overset{\pi}{\mapsto} (\tilde{\xi}, \xi)$ with $A = *g(\tilde{\xi}, -)$ and $B = \frac{1}{2}g(\xi, -)$. Note that the kernel of this map is given by those CKY two-forms that are parallel. It is easy to show, using (7), that if $K \overset{\pi}{\mapsto} (\tilde{\xi}, \xi)$, then its Hodge dual is a CKY two-form with $*K \overset{\pi}{\mapsto} (-\xi, \tilde{\xi})$. That is, the complex structure of CKY two-forms $iK := *K$ is compatible with the complex structure of the Killing vector doublets $i(\tilde{\xi}, \xi) := (-\xi, \tilde{\xi})$.

Knowledge of the isometries of the metric simplifies the task of finding the CKY two-forms, since the unknowns on the right-hand side of (6) are now in terms of a finite number of constants, a linear combination of the known Killing vectors. Killing vectors are easier to find in general, whereas their maximal rank is $d(d+1)/2 = 12$. However, the problem can be reduced further. The Lie derivative along a Killing vector k commutes with the Hodge operator, and its commutator with the Lévi-Cività derivative satisfies $[\mathcal{L}_k, \nabla_X] = \nabla_{[k,X]}$ for all vectors X. By using (6), we can show that if K is a CKY two-form with $K \overset{\pi}{\mapsto} (\tilde{\xi}, \xi)$, then its Lie derivative along a Killing vector k is also a CKY two-form with $\mathcal{L}_k K \overset{\pi}{\mapsto} ([k, \tilde{\xi}], [k, \xi])$:

$$\begin{aligned} 0 &= \mathcal{L}_k \left(\nabla_X K - i_X A - \tfrac{1}{2} X^\flat \wedge B\right) \\ &= \nabla_{[k,X]} K - i_{[k,X]} A - \tfrac{1}{2}[k, X]^\flat \wedge B + \nabla_X \mathcal{L}_k K - i_X \mathcal{L}_k A - \tfrac{1}{2} X^\flat \wedge \mathcal{L}_k B \\ &= \nabla_X \mathcal{L}_k K - i_X \mathcal{L}_k A - \tfrac{1}{2} X^\flat \wedge \mathcal{L}_k B. \end{aligned} \tag{8}$$

Therefore, CKY two-forms form a representation under the isometry algebra of the metric and the map $K \overset{\pi}{\mapsto} (\tilde{\xi}, \xi)$ is equivariant under the action of $\mathcal{L}_k$. We will use the power of this result in what follows.

3. Kerr (NUT) anti-de Sitter black holes

Kerr's black holes in anti-de Sitter space with the cosmological constant $-3/\ell^2$ are described by their mass M, a rotational parameter a and the NUT charge L. The metric is given by

$$\mathrm{d}s^2 = -\frac{\Delta_r}{r^2+y^2}(\mathrm{d}\hat{t} + y^2\,\mathrm{d}\psi)^2 + \frac{\Delta_y}{r^2+y^2}(\mathrm{d}\hat{t} - r^2\,\mathrm{d}\psi)^2 + \frac{r^2+y^2}{\Delta_r}\,\mathrm{d}r^2 + \frac{r^2+y^2}{\Delta_y}\,\mathrm{d}y^2, \tag{9}$$

where the metric functions are

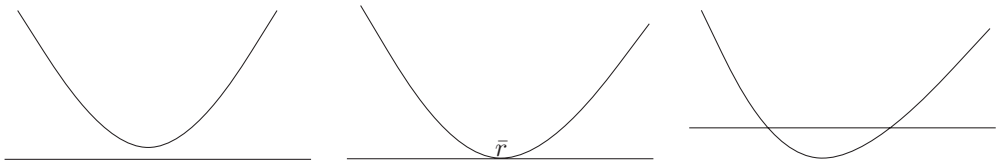
$$\Delta_r = \left(1 + \frac{r^2}{\ell^2}\right)(r^2 + a^2) - 2Mr, \tag{10a}$$

$$\Delta_y = (a^2 - y^2)\left(1 - \frac{y^2}{\ell^2}\right) + 2Ly. \tag{10b}$$

These metrics can be generalized to the Plebański–Demiański [20] family of type-D Einstein–Maxwell solutions[3]. By using the symmetry of Chen *et al* [10], which inverts $a/\ell \mapsto \ell/a$, we will always take $0 \leqslant a \leqslant \ell$.

When $M = L = 0$, the space is isometric to anti-de Sitter in two different ways; see appendix B. Each diffeomorphism corresponds to either the range $|y| < a$ or $|y| > \ell$, the two regions where the function Δ_y is positive. In particular, the range $|y| < a$ covers the whole of AdS_4. This implies that we can focus on $|y| \leqslant a$ since the two ranges are isometric. As we add the nonzero mass and NUT charge, we will continue to take y in the finite region between the two middle roots of Δ_y. With the zero mass, Δ_r is positive for all r. Above some critical mass M_* there are roots to Δ_r so that the singularities at $r = y = 0$ are hidden from large distances by a horizon. The coordinate range of (r, y) is thus determined from the roots of Δ_r and Δ_y.

[3] See [21], which discusses their CKY two-forms.



(*a*) $M < M_*$, no roots (*b*) $M = M_*$, one extremal root (*c*) $M > M_*$, two roots

Figure 1. The graph of $\Delta_r(r)$ in Kerr–AdS for fixed a.

One of our tasks is to give the different profiles of the graphs of these two functions as we vary M and L for fixed a and ℓ.

The value of NUT charge also affects the periodicity of the coordinates. When $L = 0$, we shift $\hat{t} = t - a^2\psi$ so that

$$\mathrm{d}s^2 = -\frac{\Delta_r}{r^2+y^2}(\mathrm{d}t + (y^2-a^2)\,\mathrm{d}\psi)^2 + \frac{(a^2-y^2)\big(1-\frac{y^2}{\ell^2}\big)}{r^2+y^2}(\mathrm{d}t - (r^2+a^2)\,\mathrm{d}\psi)^2 + \frac{r^2+y^2}{\Delta_r}\,\mathrm{d}r^2 + \frac{r^2+y^2}{(a^2-y^2)\big(1-\frac{y^2}{\ell^2}\big)}\,\mathrm{d}y^2. \tag{11}$$

Smoothness close to $y = \pm a$ at constant (t, r) requires ψ to be periodic with

$$\psi = \psi + \frac{2\pi}{a\big(1-\frac{a^2}{\ell^2}\big)}. \tag{12}$$

We are also interested in how the periodicities change with $L \neq 0$. As usual, a spacetime with the NUT charge will require closed timelike curves. In this section, we present the allowed coordinate ranges of (r, y, t, ψ) for the various allowed choices of parameters (a, M, L).

Profiles

For $M = 0$, Δ_r is always positive and has one extremum at $r = 0$ with $\Delta_r(0) = a^2$. The number of local extrema is preserved for all M, because otherwise there would be a value of M such that $\Delta_r' = \Delta_r'' = 0$ has a solution. However, Δ_r'' is positive for all M. As we increase M, the graph of Δ_r will deform and cross the horizontal axis for the value of an extremal mass $M = M_*$. Solving $\Delta_r = \Delta_r' = 0$, we find that this happens only once, with

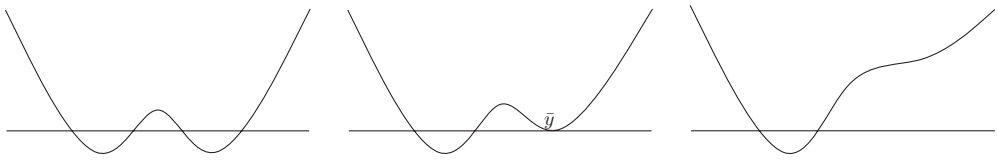
$$M_* = \frac{\ell}{\sqrt{6}}\left(-\left(1+\frac{a^2}{\ell^2}\right) + \sqrt{\left(1+\frac{a^2}{\ell^2}\right)^2 + 12\frac{a^2}{\ell^2}}\right)^{1/2} \times \left(\frac{2}{3}\left(1+\frac{a^2}{\ell^2}\right) + \frac{1}{3}\sqrt{\left(1+\frac{a^2}{\ell^2}\right)^2 + 12\frac{a^2}{\ell^2}}\right). \tag{13}$$

The profile of the graph of Δ_r is shown in figure 1.

We parametrize the values of (M, a) at extremality in terms of the double root $\bar{r}$:

$$M_* = \bar{r}\frac{\big(1+\frac{\bar{r}^2}{\ell^2}\big)^2}{1-\frac{\bar{r}^2}{\ell^2}}, \qquad a_*^2 = \bar{r}^2\frac{1+3\frac{\bar{r}^2}{\ell^2}}{1-\frac{\bar{r}^2}{\ell^2}}. \tag{14}$$

Note that given a there is always one extremal value for the mass given by (13). Since the relation $M_*(\bar{r})$ for $0 < \bar{r} < \ell$ is one-to-one and onto $\mathbb{R}^+$, there is also a unique critical



(a) $|L| < L_+$, four roots (b) $|L| = L_+$, three roots (c) $|L| > L_+$, two roots

Figure 2. Graph of $\Delta_y(y)$ in Kerr–AdS for fixed a.

value of acceleration for any choice of mass. A black hole with shielded singularities requires $M > M_*(a)$ and r is taken larger than the biggest root of Δ_r.

For $L = 0$, there are three extrema and four roots to $\Delta_y = 0$. We deform the graph by turning on L. For some value of L, the three extrema will degenerate to one. However, we are interested in the transition from four roots to two roots, which occurs at a lower value of L. Since $\Delta_y(y = 0) = a^2$ for all L and this is the local maximum for $L = 0$, the transition from four to two roots involves one of the two local minima, rather than the local maximum, crossing the horizontal line $\Delta_y = 0$.

Solving $\Delta_y = \Delta_y' = 0$, we find a unique solution up to sign, $|L| = L_*$ with

$$L_* = \frac{\ell}{\sqrt{6}}\left(1 + \frac{a^2}{\ell^2} + \sqrt{\left(1 + \frac{a^2}{\ell^2}\right)^2 + 12\frac{a^2}{\ell^2}}\right)^{\frac{1}{2}} \times \left(\frac{2}{3}\left(1 + \frac{a^2}{\ell^2}\right) - \frac{1}{3}\sqrt{\left(1 + \frac{a^2}{\ell^2}\right)^2 + 12\frac{a^2}{\ell^2}}\right). \tag{15}$$

We parametrize the extremal values of (L, a) with respect to the double root $\bar{y}$:

$$L_* = \bar{y}\frac{\left(\frac{\bar{y}^2}{\ell^2} - 1\right)^2}{1 + \frac{\bar{y}^2}{\ell^2}}, \qquad a_*^2 = \bar{y}^2\frac{3\frac{\bar{y}^2}{\ell^2} - 1}{1 + \frac{\bar{y}^2}{\ell^2}}. \tag{16}$$

The profile of Δ_y is shown in figure 2.

Periodicities

Let us take $0 < |L| < L_*$, in which case there are four roots to Δ_y, $y_1 < y_- < y_+ < y_2$. We define $t = t_\pm - y_\pm^2\psi_\pm$ and $\psi = \psi_\pm$ and expand the metric with $y = y_\pm \mp \rho^2$ close to $\rho^2 = 0$. At constant $(t_\pm, r)$, the metric becomes

$$\mathrm{d}s^2\big|_{r,t_\pm} \approx \frac{4(r^2 + y_\pm^2)}{\mp\Delta_y'(y_\pm)} \times \left(\mathrm{d}\rho^2 + \rho^2\left(\frac{\Delta_y'(y_\pm)}{2}\right)^2 \mathrm{d}\psi_\pm^2\right). \tag{17}$$

Smoothness at $y = y_\pm$ requires the periodicity $\psi_\pm = \psi_\pm + 2\pi T_\pm$, with

$$T_\pm = \frac{2}{|\Delta_y'(y_\pm)|}. \tag{18}$$

The two coordinate systems $(t_\pm, \psi_\pm)$ are patched together, away from the roots $y = y_\pm$, by

$$t_+ = t_- + (y_+^2 - y_-^2)\psi_-, \tag{19a}$$

$$\psi_+ = \psi_-. \tag{19b}$$

It follows from (18) and (19) that at fixed (y, r) the two patches describe torus fibers with periodicities

$$(t_\pm, \psi_\pm) = (t_\pm, \psi_\pm + 2\pi\, T_\pm) = \left(t_\pm \pm 2\pi \left(y_+^2 - y_-^2\right) T_\mp, \psi_\pm + 2\pi\, T_\mp\right). \quad (20)$$

We see that a nonzero NUT charge generically necessitates both the existence of closed timelike curves, for instance the curve at fixed $(\psi_\pm, r, y)$, and the non-existence of a global coordinate system to describe the $t - \psi$ part of the metric; see [22, 23]. When $L = 0$, (19) becomes $t_+ = t_-$ and $\psi_+ = \psi_-$, and the periodicity is simply

$$(t_+, \psi_+) = \left(t_+, \psi_+ + \frac{2\pi}{a\left(1 - \frac{a^2}{\ell^2}\right)}\right). \quad (21)$$

There is however one more case we want to consider, i.e. when two roots degenerate at $|L| = L_*$.

If there is a double root, say $y_+ = y_2 =: \bar{y}$, the torus fibers essentially 'uncompactify' in one direction. This is because expanding close to the double root, the metric is approximately

$$\mathrm{d}s^2\big|_{r,t_\pm} \approx \left[-\frac{\Delta_r}{r^2 + \bar{y}^2} 4\bar{y}^2 + \frac{\Delta_y''(y_+)}{2}(r^2 + \bar{y}^2)\right](y - \bar{y})^2\, \mathrm{d}\psi_+^2 + 2\frac{r^2 + \bar{y}^2}{\Delta_y''(y_+)} \frac{\mathrm{d}y^2}{(y - \bar{y})^2}. \quad (22)$$

The bracket in $g_{\psi_+\psi_+}$ can become negative, but this is inconsequential[4]. What is important is that there is an infinite throat that does not impose any periodicity on ψ_+. We still have

$$(t_-, \psi_-) = (t_-, \psi_- + 2\pi\, T_-) \quad (23)$$

from expanding close to the other root. This enforces the periodicity on the (t_+, ψ_+) patch

$$(t_+, \psi_+) = \left(t_+ + \left(\bar{y}^2 - y_-^2\right) 2\pi\, T_-, \psi_+ + 2\pi\, T_-\right). \quad (24)$$

This periodicity is inherited by the polar limit that we define in section 4.

The case of de Sitter

The Kerr family of black holes in de Sitter space, with the cosmological constant $3g^2$, can be obtained from the Kerr–anti-de Sitter metric by simply substituting $\ell^2 = -1/g^2$. We will consider this case as a side note, with a few more details given in appendix C. Here, we present the qualitative differences to what was done previously.

We find that unless[5] $a^2g^2 < 7 - 2\sqrt{12}$, the function Δ_r always has two roots and it is positive in a bounded region between these roots. Furthermore, the 'origin' $r = 0$ is in the bounded region. When $a^2g^2 < 7 - 2\sqrt{12}$, there are two critical masses $M_\pm$ such that Δ_r has four roots (three of which are positive) when $M_-(a) < M < M_+(a)$. When $M < M_-$ or $M > M_+$, there are again only two roots and the origin is in between them. The situation for Δ_y is, in a sense, opposite to that of Δ_r. It always has two roots unless $a^2g^2 > 7 + 2\sqrt{12}$. For $a^2g^2 > 7 + 2\sqrt{12}$, there are four roots only when $L_- < |L| < L_+$ for two critical values $L_\pm$.

A black hole requires that the singularities at $r = 0$ are hidden behind a horizon. The parameter space of physical interest is thus given by $a^2g^2 < 7 - 4\sqrt{3}$ and $M_-(a) < M < M_+(a)$, as in figure 3. The region where Δ_r is positive covers the exterior of the black hole but within the cosmological horizon [24]. The periodicity of (ψ, t) is analyzed identically to the negative cosmological constant case without any surprises.

[4] We can equivalently Kaluza–Klein reduce under the isometry generated by ∂_{t_+}.
[5] Note that one cannot choose $a^2\, g^2 \leqslant 1$ as we did for ℓ^2, $a^2 \leqslant \ell^2$.

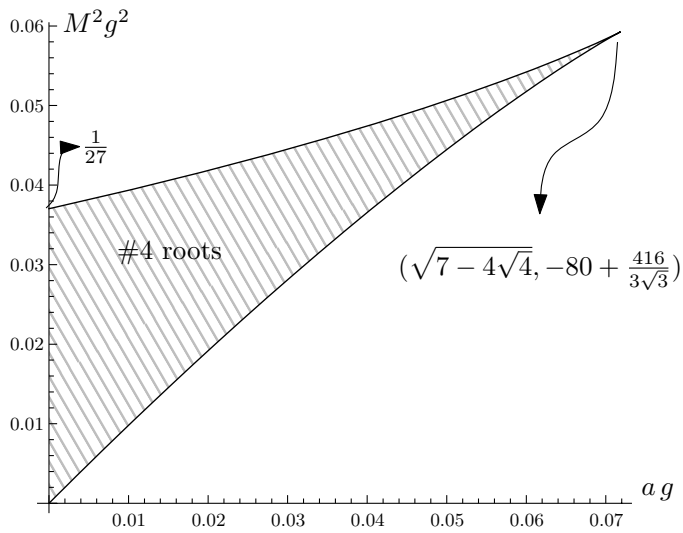


Figure 3. Physical regions in the (M^2, a) plane.

4. Extremal limits

Now that we have introduced the black hole parameters (a, M, L) and how they affect the range of coordinates, we proceed to define the extremal limits. The first limit, which is known as the near-horizon extremal limit, corresponds to blowing up the region close to the extremal horizon, i.e. when $M = M_*$. A systematic treatment of the near-horizon limit for supersymmetric backgrounds was presented in [25], but the notion has been known since at least [9]; see also [26]. We shall also consider a second spacetime limit, which corresponds to blowing up the region of the throat $y = \bar{y}$, when $L = L_*$. For the lack of a better name, we will call this an (extremal) polar limit.

Near-horizon limit

When the mass attains its lower bound, $M = M_*$, the horizon is extremal in the sense that past and future event horizons do not intersect; equivalently, the Hawking temperature is zero. In this case, we expand Δ_r close to its horizon $r = \bar{r}$ as

$$\Delta_r = \frac{1}{\beta^2}(r-\bar{r})^2 + \mathcal{O}(r-\bar{r})^3, \tag{25}$$

where

$$\beta^2 = \frac{1-\frac{\bar{r}^2}{\ell^2}}{1+6\frac{\bar{r}^2}{\ell^2}-3\frac{\bar{r}^4}{\ell^2}}. \tag{26}$$

The metric in the $(t_\pm, \psi_\pm)$ patch is

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{\Delta_r}{r^2+y^2}\left(\mathrm{d}t_\pm + \left(y^2-y_\pm^2\right)\mathrm{d}\psi_\pm\right)^2 \\ &+\frac{\Delta_y}{r^2+y^2}\left(\mathrm{d}t_\pm - \left(r^2+y_\pm^2\right)\mathrm{d}\psi_\pm\right)^2 + \frac{r^2+y^2}{\Delta_r}\mathrm{d}r^2 + \frac{r^2+y^2}{\Delta_y}\mathrm{d}y^2. \end{aligned} \tag{27}$$

We define the diffeomorphism $(t_\pm, r, \psi_\pm, y) \mapsto (\tau, x, \bar{\phi}, y)$ for any $\epsilon > 0$ by

$$r = \epsilon x + \bar{r}, \tag{28a}$$

$$t_\pm = \frac{\beta^2(\bar{r}^2 + y_\pm^2)\tau}{\epsilon}, \tag{28b}$$

$$\psi_\pm = T_\pm \bar{\phi} + \frac{t_\pm}{(\bar{r}^2 + y_\pm^2)}. \tag{28c}$$

Taking the limit $\epsilon \to 0^+$ gives the NHEK metric

$$\mathrm{d}\bar{s}^2 = \Omega^2 \left(-x^2\,\mathrm{d}\tau^2 + \frac{\mathrm{d}x^2}{x^2} + \Lambda^2 (T_u\,\mathrm{d}\bar{\phi} + x\,\mathrm{d}\tau)^2 \right) + \frac{\bar{r}^2 + y^2}{\Delta_y}\,\mathrm{d}y^2, \tag{29}$$

where

$$\Omega^2 = \beta^2(\bar{r}^2 + y^2), \qquad \Omega^2\Lambda^2 = \frac{\Delta_y}{\bar{r}^2 + y^2} 4\bar{r}^2\beta^4, \qquad T_u = \frac{\bar{r}^2 + y_\pm^2}{2\bar{r}\beta^2} T_\pm. \tag{30}$$

We make the observation that if $L \neq 0$, then the limit does not inherit a well-defined coordinate range. That is, the lattice in (20) becomes degenerate in the coordinates $(\tau, x, \bar{\phi})$ of (28) when $\epsilon \to 0^+$. This is reminiscent of the notion of the pinching manifold that was defined in another context in [27]. If however $L = 0$, then $\bar{\phi}$ is simply periodic with $\bar{\phi} = \bar{\phi} + 2\pi$ and the coordinate ranges are well defined.

The NHEK metric (29) on a constant slice of y is

$$\mathrm{d}\bar{s}^2\big|_y = \Omega^2 \left(-x^2\,\mathrm{d}\tau^2 + \frac{\mathrm{d}x^2}{x^2} + \Lambda^2 (\mathrm{d}u + x\,\mathrm{d}\tau)^2 \right). \tag{31}$$

For $u \in \mathbb{R}$, this is the so-called spacelike-warped AdS_3 metric, a metric deformation of AdS_3, an exposition of which can be found in [28]. When $\Lambda^2 = 1$, the space is precisely AdS_3. The coordinates used here are closely related to the self-dual coordinates of [29]; see appendix D. As a group manifold, $\mathrm{AdS}_3 = SL(2, \mathbb{R})$ is preserved by its right action and left action, generated, respectively, by the Killing vectors

$$r_0 = -\frac{1}{x}\partial_u - \tau x \partial_x + \frac{1}{2}\left(1 + \frac{1}{x^2} + \tau^2\right)\partial_\tau, \tag{32}$$

$$r_1 = x\partial_x - \tau\partial_\tau, \tag{33}$$

$$r_2 = \frac{1}{x}\partial_u + \tau x \partial_x + \frac{1}{2}\left(1 - \frac{1}{x^2} - \tau^2\right)\partial_\tau \tag{34}$$

and

$$l_0 = -\frac{\cosh u}{x}\partial_\tau - x \sinh u \partial_x + \cosh u \partial_u, \tag{35}$$

$$l_1 = \frac{\sinh u}{x}\partial_\tau + x \cosh u \partial_x - \sinh u \partial_u, \tag{36}$$

$$l_2 = \partial_u. \tag{37}$$

However, spacelike-warped AdS_3, i.e. when $\Lambda^2 \neq 1$, is preserved only by r_a, $a = 0, 1, 2$, and l_2. The NHEK metric (29) is thus also preserved by the Killing vectors r_a and l_2. We will henceforth use small latin indices $a, b, c, \ldots$ that take values 0, 1, 2 and are raised or lowered with a flat Lorentzian metric η_{ab}.

Before we move on to the extremal polar limit, let us make a few more remarks. A metric of the form

$$\mathrm{d}\bar{s}^2 = \Omega^2(y)\left(-x^2\,\mathrm{d}\tau^2 + \frac{\mathrm{d}x^2}{x^2} + \Lambda^2(y)(\mathrm{d}u + x\,\mathrm{d}\tau)^2\right) + F^2(y)\,\mathrm{d}y^2 \tag{38}$$

is precisely AdS_4 only if $\Lambda^2 = 1$. This follows easily from inspection of the curvature. The other two functions, $\Omega^2(y)$ and $F^2(y)$, are uniquely determined up to a diffeomorphism $y \mapsto y'(y)$, e.g., with $F = \ell$ and $\Omega^2 = \ell^2\cosh^2\left(y/\ell^2\right)$. However, the NHEK solution (30) can have $\Lambda^2 = 1$ only if $\bar{r}^2 = -\ell^2$ and $L = 0$. In these coordinates, the AdS_4 metric becomes

$$\mathrm{d}\bar{s}^2 = \frac{y^2 - \ell^2}{4}\left(-x^2\,\mathrm{d}\tau^2 + \frac{\mathrm{d}x^2}{x^2} + (\mathrm{d}u + x\,\mathrm{d}\tau)^2\right) + \frac{1}{\ell^2(y^2 - \ell^2)}\,\mathrm{d}y^2. \tag{39}$$

Since the parameter $\bar{r}$ is a positive real number, the NHEK geometry seems to be 'disconnected' from AdS_4. On the other hand, we can ask when does a metric of the form (38) satisfy the Einstein equations of motion. We find that up to diffeomorphisms of $y \mapsto y'(y)$, the most general solution is determined uniquely by two integration constants; see appendix E. Therefore, by replacing the parameters L and $\bar{r}^2$ with any real value, the NHEK geometry (29) with metric functions (30) is locally the most general Einstein solution of the form (38).

Polar limit

Next, we consider the extremal limit where we blow up the double root $y = \bar{y}$ of Δ_y at extremality $L = L_*$. With

$$\Delta_y = \frac{1}{\beta^2}(y - \bar{y})^2 + \mathcal{O}(r - \bar{r})^3, \tag{40}$$

where

$$\beta^2 = \frac{1 + \frac{\bar{y}^2}{\ell^2}}{-1 + 6\frac{\bar{y}^2}{\ell^2} + 3\frac{\bar{y}^4}{\ell^2}}, \tag{41}$$

the metric is

$$\begin{aligned}\mathrm{d}s^2 = &-\frac{\Delta_r}{r^2 + y^2}(\mathrm{d}t_+ + (y^2 - \bar{y}^2)\,\mathrm{d}\psi_+)^2 + \frac{\Delta_y}{r^2 + y^2}(\mathrm{d}t_+ - (r^2 + \bar{y}^2)\,\mathrm{d}\psi_+)^2 \\ &+ \frac{r^2 + y^2}{\Delta_r}\,\mathrm{d}r^2 + \frac{r^2 + y^2}{\Delta_y}\,\mathrm{d}y^2.\end{aligned} \tag{42}$$

We define the diffeomorphism $(t_+, r, \psi_+, y) \mapsto (t_+, r, \bar{\psi}, x)$ for any $\epsilon > 0$ by

$$\epsilon x = y - \bar{y}, \tag{43}$$

$$\epsilon\psi_+ = \beta^2\bar{\psi}. \tag{44}$$

After taking the limit $\epsilon \to 0^+$, we arrive at the metric

$$\mathrm{d}\bar{s}'^2 = \Omega^2(r)\left(+x^2\,\mathrm{d}\bar{\psi}^2 + \frac{\mathrm{d}x^2}{x^2} - \Lambda^2(r)\left(\frac{1}{2\bar{y}\beta^2}\,\mathrm{d}t_+ + x\,\mathrm{d}\bar{\psi}\right)^2\right) + \frac{\bar{y}^2 + r^2}{\Delta_r}\,\mathrm{d}r^2, \tag{45}$$

with

$$\Omega^2 = \beta^2(\bar{y}^2 + r^2), \tag{46}$$

$$\Omega^2\Lambda^2 = \frac{\Delta_r}{\bar{y}^2 + r^2}4\bar{y}^2\beta^4. \tag{47}$$

By combining (43) and (24), we find that the extremal limit inherits the periodicity

$$(t_+, \bar{\psi}) = (t_+ + (\bar{y}^2 - y_-^2)2\pi\, T_-, \bar{\psi}). \tag{48}$$

In this case, the polar extremal limit has a well-defined coordinate range. Recall that this was not the case for the NHEK unless $L = 0$.

At constant r, the metric in (45) becomes

$$\mathrm{d}\bar{s}^2\big|_r = \Omega^2 \left(+x^2\, \mathrm{d}\psi^2 + \frac{\mathrm{d}x^2}{x^2} - \Lambda^2 (\mathrm{d}u + x\, \mathrm{d}\bar{\psi})^2 \right). \tag{49}$$

For $u \in \mathbb{R}$, this describes the so-called timelike-warped AdS_3. For $\Lambda^2 = 1$, this is again precisely AdS_3. However, the coordinates of AdS_3 in (49) are different from those used for the AdS_3 in (31) for $\Lambda^2 = 1$. The coordinates of AdS_3 used to describe timelike or spacelike 'warping' should not intimidate the uninitiated reader. We give a brief classification of the various AdS_3 coordinates in appendix D. Timelike-warped AdS_3 and by extension the polar limit (45) are preserved by only four Killing vectors of AdS_3: r_0, r_1, r_2 and l_0. Again, the extremal polar limit is locally AdS_4 only if $\Lambda^2 = 1$, which can be obtained from the polar limit with $\bar{r}^2 = a^2 = \ell^2$ and $M = 0$.

Limits for a positive cosmological constant

The extremal limits can also be performed for a positive cosmological constant. When $a^2 g^2 < 7 - 4\sqrt{3}$, the near-horizon limit gives

$$\mathrm{d}\bar{s}^2 = \Omega^2 \left(-x^2\, \mathrm{d}\tau^2 + \frac{\mathrm{d}x^2}{x^2} + \Lambda^2 (T_u\, \mathrm{d}\bar{\phi} + x\, \mathrm{d}\tau)^2 \right) + \frac{\bar{r}^2 + y^2}{\Delta_y}\, \mathrm{d}y^2, \tag{50}$$

where

$$\Omega^2 = \beta^2 (\bar{r}^2 + y^2), \qquad \Omega^2 \Lambda^2 = \frac{\Delta_y}{\bar{r}^2 + y^2} 4\bar{r}^2 \beta^4, \tag{51}$$

$$T_u = \frac{\bar{r}^2 + y_\pm^2}{2\bar{r}\beta^2} T_\pm, \quad \beta^2 = \frac{1 + g^2 \bar{r}^2}{1 - 6g^2\bar{r}^2 - 3g^4\bar{r}^4}. \tag{52}$$

Here, β^2 and Ω^2 can now be of either sign, corresponding to the two extremal masses $M = M_\pm$, where the sign of $\Delta_r''(\bar{r})$ is different. Similar to the case of a negative cosmological constant, unless $L = 0$, the coordinates are not well defined by the limit. The extremal polar limit can also be performed in the unphysical region $a^2 g^2 > 7 - 4\sqrt{3}$, when $|L| = L_\pm$ but we will not pursue this here. We simply note that in all cases, the (positive cosmological constant) extremal limits can be obtained from the negative cosmological constant extremal limits with the substitution $g^2 = -1/\ell^2$.

We will focus on the negative cosmological constant case. The aim of the subsequent sections is to show that in the near-horizon limit

$$\mathrm{d}\bar{s}^2 = \Omega^2(y) \left(-x^2\, \mathrm{d}\tau^2 + \frac{\mathrm{d}x^2}{x^2} + \Lambda^2(y)\, (\mathrm{d}u + x\, \mathrm{d}\tau)^2 \right) + F^2(y)\, \mathrm{d}y^2 \tag{53}$$

the space of CKY two-forms is not enhanced from dimension 2. Our result can be easily repeated for the polar limit

$$\mathrm{d}\bar{s}^2 = \Omega^2(r) \left(+x^2\, \mathrm{d}\psi^2 + \frac{\mathrm{d}x^2}{x^2} - \Lambda^2(r)\, (\mathrm{d}u + x\, \mathrm{d}\psi)^2 \right) + F^2(r)\, \mathrm{d}r^2. \tag{54}$$

5. Holonomy of extremal limits

A parallel two-form, $\nabla_\mu K_{\nu\rho} = 0$, solves trivially the conformal KY equation. Although the black hole metrics do not allow parallel p-forms, the situation could potentially change for the—highly symmetric—extremal limits. In this section, we cast away with such doubt and show that this is not the case. We will work primarily with the near-horizon geometry, but the result also applies to the polar extremal limit.

If there is a parallel two-form K, then one has the integrability condition

$$[\nabla_\mu, \nabla_\nu]K_{\rho\sigma} = R_{\mu\nu\rho}{}^{\tau}K_{\sigma\tau} + R_{\mu\nu\sigma}{}^{\tau}K_{\tau\rho} = 0. \tag{55}$$

That is, parallel two-forms are stabilized by the curvature at any point and, more generally, by the holonomy algebra of the Lévi-Cività connection. Our main task is to show that the holonomy algebra, which to first order is generated by the curvature, is not special but spans $\mathfrak{so}(1,3)$. Since the curvature acts on two-forms in the adjoint representation and $\mathfrak{so}(1,3)$ has no central elements, there can be no parallel two-forms.

Let us first introduce the set of one-forms:

$$\theta^0 = -x\cosh u\,\mathrm{d}\tau + \frac{\sinh u}{x}\,\mathrm{d}x, \tag{56}$$

$$\theta^1 = \frac{\cosh u}{x}\,\mathrm{d}x - x\sinh u\,\mathrm{d}\tau, \tag{57}$$

$$\theta^2 = \mathrm{d}u + x\,\mathrm{d}\tau. \tag{58}$$

A relation we shall use soon is the Maurer–Cartan structure equation

$$\mathrm{d}\theta^a - \tfrac{1}{2}\epsilon^a{}_{bc}\theta^b \wedge \theta^c = 0. \tag{59}$$

Here, θ^a forms a basis for the right-invariant Maurer–Cartan one-forms of $SL(2,\mathbb{R})$. In particular, the metric of spacelike-warped AdS_3 is equal to

$$\mathrm{d}s^2_{\mathrm{wAdS}_3} = \Omega^2(-\theta^0\otimes\theta^0 + \theta^1\otimes\theta^1 + \Lambda^2\theta^2\otimes\theta^2). \tag{60}$$

Here, θ^a is dual to the right-invariant Killing vectors of AdS_3 by $\theta^a(l_b) = \delta^a_b$. Note how the right action of r_a leaves the basis θ^a invariant, whereas l_2 preserves separately the first two and the last combination in the summand of the spacelike-warped AdS_3 metric. When $\Lambda^2 = 1$, the metric of AdS_3 can also be written similarly to (60) in terms of the left-invariant Maurer–Cartan one-forms $\tilde{\theta}^a$, for which $\tilde{\theta}^a(r_b) = \delta^a_b$. This implies that $\tilde{\theta}^a = \tilde{M}^a{}_b\theta^b$ for a matrix $\tilde{M}_a{}^b$ that is an element of $O(1,2)$. Since $\tilde{M}_{ac}\tilde{M}^c{}_b = \eta_{ab}$, we have also $\tilde{M}_{ab} = \theta_b(r_a)$.

It is convenient to define an orthonormal basis $\hat{\theta}^A$ for a metric of the form

$$\mathrm{d}\bar{s}^2 = \mathrm{e}^{2\omega(y)}\left(-x^2\,\mathrm{d}\tau^2 + \frac{\mathrm{d}x^2}{x^2} + \mathrm{e}^{2\lambda(y)}(\mathrm{d}u + x\,\mathrm{d}\tau)^2\right) + \mathrm{e}^{2f(y)}\,\mathrm{d}y^2 \tag{61}$$

by using the one-forms of AdS_3:

$$\hat{\theta}^0 = \mathrm{e}^{\omega}\theta^0, \qquad \hat{\theta}^1 = \mathrm{e}^{\omega}\theta^1, \qquad \theta^2 = \mathrm{e}^{\omega+\lambda}\theta^2, \qquad \hat{\theta}^y = \mathrm{e}^{f}\mathrm{d}y. \tag{62}$$

For future use we collect the first three relations in $\hat{\theta}^a = \hat{M}^a{}_b\theta^b$ by defining the diagonal matrix $\hat{M}^a{}_b$. We first calculate the spin coefficients from

$$\mathrm{d}\hat{\theta}^A + \hat{\omega}^A{}_B \wedge \hat{\theta}^B = 0 \tag{63}$$

and then calculate the curvature.

By using (59), the solution to (63) is found to be

$$\begin{aligned}
&\hat{\omega}^0{}_1 = \left(-\mathrm{e}^{-\omega-\lambda} + \tfrac{1}{2}\mathrm{e}^{-\omega+\lambda}\right)\hat{\theta}^2, && \hat{\omega}^2{}_0 = \tfrac{1}{2}\mathrm{e}^{-\omega+\lambda}\hat{\theta}^1,\\
&\hat{\omega}^1{}_2 = \tfrac{1}{2}\mathrm{e}^{-\omega+\lambda}\hat{\theta}^0, && \hat{\omega}^0{}_z = \dot{\omega}\,\mathrm{e}^{-f}\hat{\theta}^0,\\
&\hat{\omega}^1{}_z = \dot{\omega}\,\mathrm{e}^{-f}\hat{\theta}^1, && \hat{\omega}^2{}_z = (\dot{\omega} + \dot{\lambda})\,\mathrm{e}^{-f}\hat{\theta}^2.
\end{aligned} \tag{64}$$

The curvature two-form $R^{\nabla}{}_{AB}$ is calculated from the right-hand side of

$$R^{\nabla}{}_{AB} = \tfrac{1}{2} R_{ABMN}\, \mathrm{d}x^M \wedge \mathrm{d}x^N = \mathrm{d}\hat{\omega}_{AB} + \hat{\omega}_{AC} \wedge \hat{\omega}^C{}_B.$$

We find the following components:

$$R^{\nabla 1}{}_0 = \dot{\lambda}\, \mathrm{e}^{-\omega+\lambda-f} \hat{\theta}^y \wedge \hat{\theta}_2 - \left(\mathrm{e}^{-2\omega} - \frac{3}{4}\, \mathrm{e}^{-2\omega+2\lambda} + \dot{\omega}^2\, \mathrm{e}^{-2f} \right) \hat{\theta}^1 \wedge \hat{\theta}_0, \tag{65a}$$

$$R^{\nabla 2}{}_y = \dot{\lambda}\, \mathrm{e}^{-\omega+\lambda-f} \hat{\theta}^0 \wedge \hat{\theta}^1 + (\ddot{\omega} + \ddot{\lambda} + (\dot{\omega} + \dot{\lambda})(\dot{\omega} + \dot{\lambda} - \dot{f}))\, \mathrm{e}^{-2f} \hat{\theta}^y \wedge \hat{\theta}^2, \tag{65b}$$

$$R^{\nabla 0}{}_y = \frac{\dot{\lambda}}{2}\, \mathrm{e}^{-\omega+\lambda-f} \hat{\theta}^1 \wedge \hat{\theta}^2 + (\ddot{\omega} + \dot{\omega}^2 - \dot{\omega}\dot{f})\, \mathrm{e}^{-2f} \hat{\theta}^y \wedge \hat{\theta}^0, \tag{65c}$$

$$R^{\nabla 2}{}_1 = -\frac{\dot{\lambda}}{2}\, \mathrm{e}^{-\omega+\lambda-f} \hat{\theta}^y \wedge \hat{\theta}^0 + \left(\dot{\omega}(\dot{\omega} + \dot{\lambda})\, \mathrm{e}^{-2f} + \frac{1}{4}\, \mathrm{e}^{-2\omega+2\lambda} \right) \hat{\theta}^1 \wedge \hat{\theta}_2, \tag{65d}$$

$$R^{\nabla 2}{}_0 = \frac{\dot{\lambda}}{2}\, \mathrm{e}^{-\omega+\lambda-f} \hat{\theta}^y \wedge \hat{\theta}^1 - \left(\frac{1}{4}\, \mathrm{e}^{-2\omega+2\lambda} + \dot{\omega}(\dot{\omega} + \dot{\lambda})\, \mathrm{e}^{-2f} \right) \hat{\theta}^0 \wedge \hat{\theta}_2, \tag{65e}$$

$$R^{\nabla 1}{}_y = \frac{\dot{\lambda}}{2}\, \mathrm{e}^{-\omega+\lambda-f} \hat{\theta}^0 \wedge \hat{\theta}_2 + (\ddot{\omega} + \dot{\omega}(\dot{\omega} - \dot{f}))\, \mathrm{e}^{-2f} \hat{\theta}^y \wedge \hat{\theta}^1. \tag{65f}$$

Note that the metric is locally AdS_4, $R^{\nabla}{}_{AB} = -\frac{1}{4\ell^2} \hat{\theta}^A \wedge \hat{\theta}^B$, only if $\dot{\lambda} = 0$.

The curvature two-form acts on two-forms linearly through the adjoint action,

$$\hat{\theta}^A \wedge \hat{\theta}^B : \hat{\theta}^C \wedge \hat{\theta}^D \longmapsto \eta^{BC} \hat{\theta}^A \wedge \hat{\theta}^D - \eta^{BD} \hat{\theta}^A \wedge \hat{\theta}^C - \eta^{AC} \hat{\theta}^B \wedge \hat{\theta}^D + \eta^{AD} \hat{\theta}^B \wedge \hat{\theta}^C,$$

as is consistent with the integrability condition (55). From the form (65), the span of R^{∇}_{cy} and R^{∇}_{ab}, where a, b and c are all different, is either the direct sum of $\langle \hat{\theta}^a \wedge \hat{\theta}^b \rangle$ and $\langle \hat{\theta}^a \wedge \hat{\theta}^y \rangle$ or a one-dimensional subspace thereof, depending on the determinant of the matrix transformation in

$$\begin{pmatrix} R^{\nabla}_{cy} \\ R^{\nabla}_{ab} \end{pmatrix} = \begin{pmatrix} R_{cyab} & R_{cycy} \\ R_{abab} & R_{abcy} \end{pmatrix} \begin{pmatrix} \hat{\theta}^a \wedge \hat{\theta}^b \\ \hat{\theta}^a \wedge \hat{\theta}^y \end{pmatrix}. \tag{66}$$

All together, R^{∇}_{AB} span $\Lambda^2 = \mathfrak{so}(1,3)$ if and only if

$$R_{abab} R_{cycy} - (R_{abcy})^2 \neq 0 \tag{67}$$

for all permutations of $(a, b, c) = (0, 1, 2)$. So far we have not used the specific functions (λ, ω, f) of the NHEK geometry. We can check (67) at any point using the NHEK solution (30) and a computer calculation confirms its validity. Since $\mathfrak{so}(1,3)$ is centerless, there can be no two-form that is stabilized by the curvature two-form. Whence there are no parallel two-forms in the NHEK geometry. This result applies only to the NHEK solution, i.e. the most general Einstein solution of this form.

The derivation can be repeated for the polar limit, which is of the form

$$\mathrm{d}\bar{s}^2 = \mathrm{e}^{2\omega(y)} \left(x^2\, \mathrm{d}\psi^2 + \frac{\mathrm{d}x^2}{x^2} - \mathrm{e}^{2\lambda(y)} (\mathrm{d}u + x\, \mathrm{d}\psi)^2 \right) + \mathrm{e}^{2f(y)}\, \mathrm{d}y^2. \tag{68}$$

As we commented earlier, the geometry at constant y is the so-called timelike-warped AdS_3, so we can similarly use the right-invariant one-forms θ^a adapted to the timelike warping. The equivalent orthonormal frame to (62) is now

$$\hat{\theta}^0 = \mathrm{e}^{\omega+\lambda}\, (\mathrm{d}u + x\, \mathrm{d}\psi)\,, \qquad \hat{\theta}^1 = \mathrm{e}^{\omega} \left(\cos ux\, \mathrm{d}\psi + \sin u \frac{\mathrm{d}x}{x} \right), \tag{69}$$

$$\hat{\theta}^2 = \mathrm{e}^{\omega}\left(-\sin ux\,\mathrm{d}\psi + \cos u\frac{\mathrm{d}x}{x}\right), \qquad \hat{\theta}^y = \mathrm{e}^f\,\mathrm{d}y. \tag{70}$$

We simply remark here that the curvature two-form components are the same as in (65) with the interchange of flat indices $2 \leftrightarrow 0$. This explains why we chose, perhaps mysteriously, to present the curvature two-form in (65) with some flat indices up and others down. The algebraic relation (67) can then be used in place, giving the same result that there are no parallel two-forms in the extremal limit.

6. Larger than 2 is 8

In [7], Rasmussen showed that the KY two-form of the Kerr–(A)dS black hole, including the NUT charge, survives the near-horizon limit. Its Hodge dual is the so-called principal CKY two-form of the geometry K_p. That is, it is given by $K_p = \mathrm{d}b$, where

$$b = -\frac{y^2+\bar{r}^2}{2}x\,\mathrm{d}t - \frac{y^2}{2}\,\mathrm{d}u. \tag{71}$$

A similar result holds for the extreme polar limit: there are two CKY two-forms, $K_p = \mathrm{d}b$ and $*K_p$, where

$$b = -\frac{r^2+\bar{y}^2}{2}x\,\mathrm{d}\bar{\psi} - \frac{r^2}{2}\,\mathrm{d}u. \tag{72}$$

For definiteness, we will work with the near-horizon limit, but the section can be read for the polar limit instead.

In this section, we shall prove that there are either only two linearly independent CKY two-forms, K_p and $*K_p$, or else the space of CKY two-forms $\mathcal{K}$ is eight dimensional. In the latter case, the rank of $\mathcal{K}$ is the maximal allowed, i.e. twice the number of independent Killing vectors, r_a and ∂_u. Our ultimate aim is to show that the maximal case, $\dim\mathcal{K} = 8$, is not realized.

Recall that all CKY two-forms are mapped to a pair of Killing vectors. An explicit calculation shows that

$$*\,\mathrm{d}*\mathrm{d}b = 3\,T_u^2\,g(\partial_u, -), \tag{73}$$

and so we find that the known CKY two-forms are such that $*K_p \overset{\pi}{\mapsto} (T_u^2\partial_u, 0)$ and $K_p \overset{\pi}{\mapsto} (0, -T_u^2\partial_u)$. Since the space of CKY two-forms $\mathcal{K}$ is a vector space, either these two CKY two-forms span the entire space or the space is bigger. If $\dim\mathcal{K} > 2$, then there is at least one CKY two-form K such that $K \overset{\pi}{\mapsto} (\tilde{r}, r)$ with

$$\tilde{r} = \tilde{A}r_0 + \tilde{B}r_1 + \tilde{C}r_2, \tag{74}$$

$$r = Ar_0 + Br_1 + Cr_2. \tag{75}$$

r_a transforms in the adjoint (vector) representation V of $\mathfrak{so}(1,2)$. It will then follow that the image $\pi\left(\mathcal{K}/\langle K_p, *K_p\rangle\right)$ is the entire $V \oplus V$.

Assume a CKY two-form K as before, $K \overset{\pi}{\mapsto} (\tilde{r}, r)$. If r and $\tilde{r}$ are linearly dependent, $r = c\,\tilde{r}$ with $c \neq 0$, then we also have the CKY two-form

$$K' = \frac{1}{c^2+1}(K - c*K) \overset{\pi}{\mapsto} (\tilde{r}, 0).$$

If on the other hand r and $\tilde{r}$ are nonzero and linearly independent, the CKY two-form $K' = \mathcal{L}_r K$ is such that $K' \overset{\pi}{\mapsto} (r', 0)$ with $r' = [r, \tilde{r}]$ nonzero[6]. Finally, if either r or $\tilde{r}$ is zero, the action of

[6] It follows from the properties of $\mathfrak{so}(1,2)$ that r' cannot be zero, $\mathrm{stab}(r) = \langle r\rangle$.

the Hodge duality allows us to consider the case, where $K \overset{\pi}{\mapsto} (\tilde{r}, 0)$ in any case. The vector representation V is irreducible, though. Whence through the action of $\mathfrak{so}(1,2)$ and linearity, all $\tilde{r} \in V$ can be obtained.

We can be more explicit by using the properties of V. The action of $\mathfrak{so}(1,2)$ can be integrated on both sides of $\mathcal{L}_\xi K \overset{\pi}{\mapsto} (\mathcal{L}_\xi \tilde{r}, 0)$. Then, any $\tilde{r}$ as in (74) can be transformed into a Killing vector $\tilde{r}'$ proportional to r_0, r_2 or $r_0 \pm r_2$, depending on whether the length $\tilde{A}^2 - \tilde{B}^2 - \tilde{C}^2$ is, respectively[7], positive, negative, or zero. We can then act on $K' \overset{\pi}{\mapsto} (\tilde{r}', 0)$ with any of r_a so that we obtain all of the characteristic elements of V under the action of $\mathfrak{so}(1,2)$. By the action of the Hodge duality, the same is true for the closed conformal KY two-forms.

To summarize, either there are two CKY two-forms as found in [7], or the space of CKY two-forms $\mathcal{K}$ is augmented so that its image under the map π spans the entire double copy of the space of Killing vectors $V \oplus \mathbb{R}$. In the latter case, we showed that the KY two-forms transform in the same representation as $V \oplus \mathbb{R}$, as do the the closed conformal KY two-forms. Explicitly, if there are more than two independent CKY two-forms, then there are three independent KY two-forms K_a with $K_a \overset{\pi}{\mapsto} (r_a, 0)$ for each $a = 0, 1, 2$. This will allow us to write down an ansatz for the most general $K \in \mathcal{K}/\langle K_p, *K_p\rangle$ that fails to satisfy the CKY equation.

Part of these results can be immediately generalized. Given a four-dimensional Einstein manifold (M^4, g) and a reductive isometry algebra $\mathfrak{g} = \oplus_i^N \mathfrak{g}_i$, i.e. each $\mathfrak{g}_i$ is a simple Lie algebra, the space of conformal KY two-forms modulo the space of parallel two-forms can be decomposed under $\mathfrak{g}$ into a direct sum of a subset of the prime ideals $\oplus_{i\in S}\mathfrak{g}_i^{\mathcal{K}}$, where each $\mathfrak{g}_i^{\mathcal{K}}$ is either $\mathfrak{g}_i \oplus \mathfrak{g}_i$ or $\mathfrak{g}_i$. The first case, $\mathfrak{g}_i^{\mathcal{K}} = \mathfrak{g}_i \oplus \mathfrak{g}_i$, is when the CKY two-forms can be decomposed into KY and closed conformal KY two-forms. The second case, $\mathfrak{g}_i^{\mathcal{K}} = \mathfrak{g}_i$, is when the two-forms cannot be decomposed like that, while Goursat's lemma associates with each such case an automorphism $\mathfrak{g}_i \to \mathfrak{g}_i$. However, we cannot always decompose the CKY two-forms into KY forms and closed conformal KY forms as we did for the extremal limits.

7. No more CKY two-forms

In this section, we work with the assumption that there is a closed conformal KY two-form K_a such that

$$K_a \overset{\pi}{\mapsto} (0, r_a). \tag{76}$$

The CKY two-form equation (1) becomes

$$\nabla_X K_a = X^\flat \wedge (r_a)^\flat . \tag{77}$$

We first work with (76) in order to derive an ansatz for K_a that fails to pass the test of (77). This is an ansatz in the sense that if K_a satisfies (76), then it has to be of this form. The failure of the ansatz to pass the test proves that there are only two CKY two-forms: K_p and its Hodge dual.

Birth of an ansatz

The nth action of the Lie derivative on K_a along a Killing vector r, where r is a linear combination of the right action as in (75), is also a CKY two-form with $\mathcal{L}_r^n K_a \overset{\pi}{\mapsto} (0, \mathcal{L}_r^n r_a)$. If we integrate this, we arrive at

$$\mathrm{e}^{\epsilon \mathcal{L}_r} K \overset{\pi}{\mapsto} (0, \mathrm{e}^{\epsilon \mathcal{L}_r} r_a). \tag{78}$$

[7] It is useful to think of the familiar Lorentz transformations acting on the three-dimensional Minkowski space $\langle r_a\rangle = \mathbb{R}^{1,2}$, with $\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{so}(1,2)$.

We can exponentiate the adjoint action $[r_a, r_b] = -\epsilon_{ab}{}^c r_c$ and define the matrix $S_{\epsilon a}{}^b \in SO(1,2)$ by $\mathrm{e}^{\epsilon \mathcal{L}_r} r_a = S_{\epsilon a}{}^b r_b$. Subtracting (78) from $S_{\epsilon a}{}^b$ times (76), we obtain

$$\mathrm{e}^{\epsilon \mathcal{L}_r} K_a - S_{\epsilon a}{}^b K_b \overset{\pi}{\mapsto} (0,0). \tag{79}$$

We have shown that there are no parallel two-forms and so

$$\mathrm{e}^{\epsilon \mathcal{L}_r} K_a = S_{\epsilon a}{}^b K_b. \tag{80}$$

In order to proceed, we make the observation that r_a act transitively at constant y on the near-horizon geometry. The orbit is the space known as spacelike-warped AdS_3, which is diffeomorphic (as a manifold) to $SL(2,\mathbb{R})$. The infinitesimal action on a group can be integrated for all $\epsilon \in \mathbb{R}$, which defines the flow $\phi_\epsilon : \mathrm{wAdS}_3 \to \mathrm{wAdS}_3$ according to $r(f)|_p = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \phi_\epsilon)\big|_{t=0}$ for any function f and at any point p; see, e.g., [30]. On a vector field X and a one-form a, the exponential of the Lie derivative is related to the pushforward and pullback of ϕ_ϵ, respectively, by

$$\mathrm{e}^{\epsilon \mathcal{L}_r}\, X|_p = \phi_{-\epsilon *}\, X|_{\phi_\epsilon(p)}, \tag{81}$$

$$\mathrm{e}^{\epsilon \mathcal{L}_r}\, a|_p = \phi_\epsilon{}^*\, a|_{\phi_\epsilon(p)}. \tag{82}$$

We henceforth fix a slice $y = y_0$ and a point $p \in \mathrm{wAdS}_3$. At any other point $\phi_\epsilon(p) \in \mathrm{wAdS}_3$ of the same slice of y, (80) implies that K_a is given by

$$K_a|_{\phi_\epsilon(p)} = S_{\epsilon a}{}^b \phi_{-\epsilon}{}^*\, K_b|_p\,. \tag{83}$$

This equation is what allows us to write an ansatz for K_a.

Next, we need an expression for the transformation matrix $S_{\epsilon a}{}^b$. r_a transform the same as their dual one-forms, $\tilde{\theta}^a(r_b) = \delta^a_b$, whereas according to the discussion above (60), we have the relation $\tilde{\theta}^a = \tilde{M}^a{}_b \theta^b$, where $\tilde{M}_{ab}$ is an $O(1,2)$ matrix and θ^a is invariant under the action of r_a. By contracting the relation with r_c, we arrive at $\tilde{M}_{ab} = \theta_b(r_a)$. Now consider the relation $\tilde{\theta}^a|_p = \tilde{M}_p{}^a{}_b \theta^b|_p$ at a fixed point p. It transforms as $S_{\epsilon a}{}^b \tilde{\theta}_b|_p = \tilde{M}_{\phi_\epsilon(p)}{}^a{}_b \theta^b|_p$. We hence deduce that

$$S_{\epsilon a}{}^b = \tilde{M}_{\phi_\epsilon(p)a}{}^c \tilde{M}_p{}^b{}_c = \theta^c(r_a)|_{\phi_\epsilon(p)}\, \tilde{M}_p{}^b{}_c, \tag{84}$$

which we can insert into (83).

At fixed y and fixed $p \in \mathrm{wAdS}_3$, we write K_a as

$$K_a|_p = H(y)_{ab} \tfrac{1}{2} \epsilon^b{}_{cd} \hat{\theta}^c|_p \wedge \hat{\theta}^d|_p + G(y)_{ab} \hat{\theta}^y|_p \wedge \hat{\theta}^b|_p. \tag{85}$$

At any other point $\phi_\epsilon(p)$ of the same slice of y, K_a is given by (83):

$$K_a|_{\phi_\epsilon(p)} = S_{\epsilon a}{}^b \big(H(y)_{bc} \tfrac{1}{2} \epsilon^c{}_{de} \phi_{-\epsilon}{}^* \hat{\theta}^d|_p \wedge \phi_{-\epsilon}{}^* \hat{\theta}^e|_p + G(y)_{bc} \phi_{-\epsilon}{}^* \hat{\theta}^y|_p \wedge \phi_{-\epsilon}{}^* \hat{\theta}^c|_p \big). \tag{86}$$

However, the orthonormal basis $\hat{\theta}^A$ is given by (62), and in particular, it is right invariant. If we also use the expression for $S_{\epsilon a}{}^b$ in (84) and absorb the matrix $\tilde{M}_p{}^a{}_b$ multiplying the left of $H(y)_{ab}$ and $G(y)_{ab}$ in their definition, then (86) becomes

$$K_a = \theta^b(r_a) \left(H(y)_{bc} \tfrac{1}{2} \epsilon^c{}_{de} \hat{\theta}^d \wedge \hat{\theta}^e + G(y)_{bc} \hat{\theta}^y \wedge \hat{\theta}^c \right), \tag{87}$$

which is valid at any point of the fixed slice y. Furthermore, the equality should vary smoothly over y. Our ansatz is thus that K_a is given by (87), with the unknown functions of y: the matrices $H(y)_{ab}$ and $G(y)_{ab}$. It should hold for any CKY two-form that satisfies our initial assumption (76).

Death of the ansatz

We have come a long way since we wrote down the CKY defining equation. So far we have shown that there are either two independent CKY two-forms in the NHEK background, or else there are three more linearly independent KY two-forms of the form (87). It is a straightforward calculation to check whether (87) satisfies the CKY equation. We will explicitly demonstrate here that, unless a background of the form (61) is precisely (locally) AdS_4, there is no solution to the matrices $H(y)_{bc}$ and $G(y)_{bc}$.

We begin with taking the derivative of K_a with respect to $X = \partial_y$, in which case the right-hand side of the CKY equation (77) becomes

$$\nabla_y K_a = \hat{\theta}_b(r_a)\,\mathrm{e}^f \hat{\theta}^y \wedge \hat{\theta}^b. \tag{88}$$

Let us use the diagonal matrix $\hat{M}$ by $\hat{\theta}^a = \hat{M}^a{}_b \theta^b$, i.e. from (62):

$$\hat{M}^a{}_b = \mathrm{e}^{\omega} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{\lambda} \end{pmatrix}. \tag{89}$$

We observe that the spin connection has no y-component, $\omega_{AB}(\partial_y) = 0$, so ∇_y acts in the orthonormal basis as ∂_y. By using the ansatz (87) and comparing to (88), we find

$$\partial_y H_{bc} = 0, \tag{90a}$$

$$\partial_y G_{bc} = \mathrm{e}^f \hat{M}_{cb}. \tag{90b}$$

At this point, H_{bc} is a constant matrix and the y-dependence of G_{bc} is fixed.

Next, we take the covariant derivative of K_a with respect to l_b. We can use the spin connection as found in (64) and $\theta^a(l_b) = \delta^a_b$. The right-hand side of the CKY equation is

$$\nabla_{l_b} K_a = (l_b)^\flat \wedge (r_a)^\flat = \theta^e(r_a)\hat{M}_{cb}\hat{M}_{de}\hat{\theta}^c \wedge \hat{\theta}^d. \tag{91}$$

In order to calculate the derivative $\nabla_{l_b} K_a$, we make use[8] of $\mathrm{d}\tilde{M}_{ab} = \mathrm{d}\,(\theta_b(r_a)) = \tilde{M}_{ad}\epsilon^d{}_{bc}\theta^c$. We thus calculate the left-hand side of (91) as

$$\begin{aligned}\nabla_{l_b} K_a = {} & \tilde{M}_a{}^c \left(\epsilon_{cdb}\tfrac{1}{2}H^d{}_e\epsilon^e{}_{fg} - H_{cd}\epsilon^d{}_{eg}\hat{\omega}^e{}_f(l_b) - G_{cg}\hat{\omega}^y{}_f(l_b)\right)\hat{\theta}^f \wedge \hat{\theta}^g \\ & + \tilde{M}_a{}^c\left(\epsilon_{cdb}G^d{}_e - H_{cd}\epsilon^d{}_{fe}\hat{\omega}^f{}_y(l_b) - G_{cd}\hat{\omega}^d{}_e(l_b)\right)\hat{\theta}^y \wedge \hat{\theta}^e.\end{aligned} \tag{92}$$

Equating (91) and (92) gives two sets of equations[9]

$$\epsilon_{abd}H^d{}_c - H_{ad}\hat{\omega}_c{}^d(l_b) - G_{ad}\hat{\omega}^y{}_e(l_b)\epsilon^{ed}{}_c = \hat{M}_{ad}\hat{M}_{eb}\epsilon^{ed}{}_c, \tag{93}$$

$$\epsilon_{adc}G^d{}_b - H_{ad}\epsilon^d{}_{eb}\hat{\omega}^e{}_y(l_c) - G_{ad}\hat{\omega}^d{}_b(l_c) = 0. \tag{94}$$

Setting $b = c$ in these two equations, we find that the off-diagonal components of G_{ab} and H_{ab} are zero. It is easy to see this. The elements $\hat{\omega}^e{}_y(l_b)$ are nonzero only when $e = b$ and $\hat{\omega}^e{}_d(l_b)$ are nonzero only when e, d and b are all different. So with $b = c$ only the first terms, $\epsilon_{adc}G^d{}_b$ and $\epsilon_{abd}H^d{}_c$, survive and they give that the off-diagonal G_{ab} and H_{ab} are zero. The same result is obtained whenever one or more of a, b and c are the same. The rest of the equation components are

$$H_{22} + H_{00}\tfrac{1}{2}\,\mathrm{e}^{\lambda} + G_{00}\dot{\omega}\,\mathrm{e}^{\omega - f} = -\mathrm{e}^{2\omega}, \tag{95a}$$

$$-H_{11} - H_{00}\left(1 - \tfrac{1}{2}\,\mathrm{e}^{2\lambda}\right) - G_{00}(\dot{\omega} + \dot{\lambda})\,\mathrm{e}^{-f+\omega+\lambda} = \mathrm{e}^{2\omega+\lambda}, \tag{95b}$$

[8] This follows from $\mathrm{d}f(Y) = \mathcal{L}_Y f$ with $Y = l_a$ and $f = \tilde{M}_{ab} = \theta_b(r_a)$.

[9] We use $\epsilon^{abc}\epsilon_{abd} = -2\delta^c_d$ and $\epsilon^{abc}\epsilon_{a'b'c} = 2\delta^{[a}_{a'}\delta^{b]}_{b'}$.

$$-H_{22}+H_{11}\tfrac{1}{2}\,\mathrm{e}^{\lambda}+G_{11}\dot{\omega}\,\mathrm{e}^{-f+\omega}=\mathrm{e}^{2\omega}, \tag{95c}$$

$$-H_{00}-H_{11}\left(1-\tfrac{1}{2}\,\mathrm{e}^{2\lambda}\right)-G_{11}(\dot{\omega}+\dot{\lambda})\,\mathrm{e}^{-f+\omega+\lambda}=-\mathrm{e}^{2\omega+\lambda}, \tag{95d}$$

$$H_{11}-H_{22}\tfrac{1}{2}\,\mathrm{e}^{\lambda}-G_{22}\dot{\omega}\,\mathrm{e}^{-f+\omega}=-\mathrm{e}^{2\omega+\lambda}, \tag{95e}$$

$$H_{00}+H_{22}\tfrac{1}{2}\,\mathrm{e}^{\lambda}+G_{22}\dot{\omega}\,\mathrm{e}^{-f+\omega}=\mathrm{e}^{2\omega+\lambda}, \tag{95f}$$

$$G_{00}-H_{11}(\dot{\omega}+\dot{\lambda})\,\mathrm{e}^{-f+\omega+\lambda}+G_{11}\left(1-\tfrac{1}{2}\,\mathrm{e}^{2\lambda}\right)=0, \tag{95g}$$

$$G_{22}+H_{11}\dot{\omega}\,\mathrm{e}^{-f+\omega}-G_{11}\tfrac{1}{2}\,\mathrm{e}^{\lambda}=0, \tag{95h}$$

$$-G_{00}+H_{22}\dot{\omega}\,\mathrm{e}^{-f+\omega}-G_{22}\tfrac{1}{2}\,\mathrm{e}^{\lambda}=0, \tag{95i}$$

$$-G_{11}-H_{22}\dot{\omega}\,\mathrm{e}^{-f+\omega}+\tfrac{1}{2}G_{22}\,\mathrm{e}^{\lambda}=0, \tag{95j}$$

$$G_{11}-H_{00}(\dot{\omega}+\dot{\lambda})\,\mathrm{e}^{-f+\omega+\lambda}+G_{00}\left(1-\tfrac{1}{2}\,\mathrm{e}^{2\lambda}\right)=0 \tag{95k}$$

and

$$-G_{22}+H_{00}\dot{\omega}\,\mathrm{e}^{\omega-f}-\tfrac{1}{2}G_{00}\,\mathrm{e}^{\lambda}=0. \tag{95l}$$

Rather than attempt to solve these, we make the following observation. First, it is easy to show that $H_{00}=-H_{11}$ and $G_{00}=-G_{11}$. Combining (95*d*) and (95*g*), we obtain

$$\tfrac{1}{2}H_{11}\,\mathrm{e}^{2\lambda}+2H_{11}(\dot{\omega}+\dot{\lambda})^2\,\mathrm{e}^{-2f+2\omega}=-\mathrm{e}^{2\omega+\lambda}. \tag{96}$$

However, H_{11} is a constant and $\mathrm{e}^{2\omega}$, $\mathrm{e}^{2\lambda}$ and e^{2f} are rational polynomials of y. The left-hand side of (96) is then a rational polynomial and so should be its right-hand side. However, this is true only when e^{λ} is a rational polynomial. This is not true for the NHEK geometry, but it is true for AdS_4, in which case $\mathrm{e}^{\lambda}=1$. This concludes what we sought to confirm.

Since the NHEK geometry is the most general solution to the Einstein equations for a metric of such a form (61), it is difficult to generalize this result. For instance, we can take a metric of the NHEK form so that it does not satisfy the Einstein equations of motion. We must still impose the condition $R_{\sigma\mu}K^{\sigma}{}_{\nu}+R_{\sigma\nu}K^{\sigma}{}_{\mu}=0$ for a CKY two-form so that K still maps to a doublet of Killing vectors. We then ask if the image of the map covers $\mathfrak{sl}(2,\mathbb{R})$ or not. Modulo parallel two-forms we arrive at the same set of equations given in (95). One can then carry on and show that these equations are consistent only if $\lambda=0$ and the space is conformal to AdS_4.

Acknowledgments

We would like to thank Shoichi Kawamoto for pointing out some relevant literature.

Appendix A. Conformal KY transport

In this section, we rewrite the definition of a conformal KY tensor into a form of parallel transport equation. By the definition of a conformal KY p-form K, there exist a $(p+1)$-form A and a $(p-1)$-form B, which satisfy

$$\nabla_\mu K_{\nu_1\cdots\nu_p} = A_{\mu\,\nu_1\cdots\nu_p} + g_{\mu[\nu_1}B_{\nu_2\cdots\nu_p]}.$$

We thus have that $A_{\mu\,\nu_1\cdots\nu_p} = \nabla_{[\mu}K_{\nu_1\cdots\nu_p]}$ and

$$B_{\nu_2\cdots\nu_p} = \frac{p}{d-(p-1)}\nabla^\mu K_{\mu\,\nu_2\cdots\nu_p}, \tag{A.1}$$

where d is the spacetime dimension. Our aim is to add a p-form C and write the definition of K using a covariant derivative $D_\mu O$, where O will be a section $K+A+B+C \in \Lambda^p \oplus \Lambda^{p+1} \oplus \Lambda^{p-1} \oplus \Lambda^p$.

From the B components of the transport equation, we will see that in the case of $p=2$, B_μ is a Killing vector for a large class of manifolds, which is an ingredient in the main part. A possible application of the full transport equation is to put it on a computer and search for CKY tensors.

A.1. A identity

We begin by writing

$$\nabla_\mu A_{\nu\,\nu_1\cdots\nu_p} - \nabla_\nu A_{\mu\,\nu_1\cdots\nu_p} = [\nabla_\mu, \nabla_\nu]K_{\nu_1\cdots\nu_p} - 2\nabla_{[\mu}g_{\nu][\nu_1}B_{\nu_2\cdots\nu_p]}.$$

We do the same with indices exchanged

$$\nabla_\nu A_{\mu\,\nu_1\,\nu_2\cdots\nu_p} - \nabla_{\nu_1}A_{\mu\,\nu\,\nu_2\cdots\nu_p} = -[\nabla_\nu, \nabla_{\nu_1}]K_{\mu\,\nu_2\cdots\nu_p} + 2\nabla_{[\nu}g_{\nu_1][\mu}B_{\nu_2\cdots\nu_p]}, \tag{A.2}$$

$$\nabla_{\nu_1}A_{\mu\,\nu\,\nu_2\cdots\nu_p} + \nabla_\mu A_{\nu\,\nu_1\cdots\nu_p} = [\nabla_{\nu_1}, \nabla_\mu]K_{\nu\,\nu_2\cdots\nu_p} - 2\nabla_{[\nu_1}g_{\mu][\nu}B_{\nu_2\cdots\nu_p]}, \tag{A.3}$$

and add the three equations together to obtain

$$\begin{aligned}2\nabla_\mu A_{\nu\,\nu_1\cdots\nu_p} &= [\nabla_\mu, \nabla_\nu]K_{\nu_1\cdots\nu_p} - [\nabla_\nu, \nabla_{\nu_1}]K_{\mu\,\nu_2\cdots\nu_p} + [\nabla_{\nu_1}, \nabla_\mu]K_{\nu\,\nu_2\cdots\nu_p}\\ &\quad - 2\nabla_{[\mu}g_{\nu][\nu_1}B_{\nu_2\cdots\nu_p]} + 2\nabla_{[\nu}g_{\nu_1][\mu}B_{\nu_2\cdots\nu_p]} - 2\nabla_{[\nu_1}g_{\mu][\nu}B_{\nu_2\cdots\nu_p]}.\end{aligned} \tag{A.4}$$

In the following, we often use the identity

$$X_{[a_1\cdots a_k]} = \frac{1}{k}\left(X_{|a_1|[a_2\cdots a_k]} - X_{[a_2|a_1|a_3\cdots a_k]} + \cdots + (-1)^{k+1}X_{[a_k a_2\cdots a_{k-1}]a_1}\right)$$

and any other symmetries of the expression for X. Using them, we write (A.4) as

$$\begin{aligned}2\nabla_\mu A_{\nu\,\nu_1\cdots\nu_p}\frac{\mathrm{d}x^{\nu\nu_1\cdots\nu_p}}{(p+1)!} &= \Big\{pR_{\mu\nu\nu_1}{}^\sigma K_{\sigma\,\nu_2\cdots\nu_p}\\ &\quad - R_{\nu\nu_1\mu}{}^\sigma K_{\sigma\,\nu_2\cdots\nu_p} - (p-1)R_{\nu\nu_1\nu_2}{}^\sigma K_{\mu\sigma\,\nu_3\cdots\nu_p} + pR_{\nu_1\mu\nu}{}^\sigma K_{\sigma\,\nu_2\cdots\nu_p}\\ &\quad + g_{\mu\nu_1}\nabla_\nu B_{\nu_2\cdots\nu_p} + \frac{2}{p}g_{\nu_1\mu}\nabla_\nu B_{\nu_2\cdots\nu_p} - g_{\mu\nu}\nabla_{\nu_1}B_{\nu_2\cdots\nu_p}\Big\}\frac{\mathrm{d}x^{\nu\nu_1\cdots\nu_p}}{(p+1)!}.\end{aligned} \tag{A.5}$$

Using the algebraic Bianchi identity, we collect our first main identity

$$\nabla_\mu A_{\nu\,\nu_1\cdots\nu_p}\frac{\mathrm{d}x^{\nu\nu_1\cdots\nu_p}}{(p+1)!} = \left(-\frac{p+1}{2}R_{\nu\nu_1\mu}{}^\sigma K_{\sigma\,\nu_2\cdots\nu_p} - \frac{p+1}{p}g_{\mu\nu}\nabla_{\nu_1}B_{\nu_2\cdots\nu_p}\right)\frac{\mathrm{d}x^{\nu\nu_1\cdots\nu_p}}{(p+1)!}.$$

We can also write this as

$$\nabla_\mu A = -\frac{p+1}{2} R_{\nu\nu_1\mu}{}^{\sigma} K_{\sigma\nu_2\cdots\nu_p} \frac{\mathrm{d}x^{\nu\nu_1\cdots\nu_p}}{(p+1)!} - \frac{1}{p^2} g_{\mu\nu} \mathrm{d}x^\nu \wedge \mathrm{d}B.$$

We also want one more expression for this. It is

$$\begin{aligned}\nabla_\mu A_{\nu\nu_1\cdots\nu_p} = &-R_{\mu\sigma\nu[\nu_1} K^\sigma{}_{\nu_2\cdots\nu_p]} - \frac{p-1}{2} R_{\mu\sigma[\nu_1\nu_2} K^\sigma{}_{|\nu|\nu_3\cdots\nu_p]} \\ &- \frac{1}{p^2} g_{\mu\nu}\, \mathrm{d}B_{\nu_1\cdots\nu_p} + \frac{1}{p} g_{\mu[\nu_1}\, \mathrm{d}B_{|\nu|\nu_2\cdots\nu_p]}.\end{aligned}$$

Using this last expression, we derive

$$\nabla_\mu A^\mu{}_{\nu_1\cdots\nu_p} = -R_{\sigma[\nu_1} K^\sigma{}_{\nu_2\cdots\nu_p]} + \frac{p-1}{2} R_{\mu\sigma[\nu_1\nu_2} K^{\mu\sigma}{}_{\nu_3\cdots\nu_p]} - \frac{d-p}{p^2}\, \mathrm{d}B_{\nu_1\cdots\nu_p}. \tag{A.6}$$

A.2. B identity

We also need to find an expression for the derivative on B. We begin from (A.1) to obtain

$$\begin{aligned}\nabla_{\nu_1} B_{\nu_2\cdots\nu_p} &= \frac{p}{d-(p-1)} \nabla_{\nu_1} \nabla_\mu K^\mu{}_{\nu_2\cdots\nu_p} \\ &= \frac{p}{d-(p-1)} [\nabla_{\nu_1}, \nabla_\mu] K^\mu{}_{\nu_2\cdots\nu_p} + \frac{p}{d-(p-1)} \nabla^\mu \nabla_{\nu_1} K_{\mu\nu_2\cdots\nu_p} \\ &= \frac{p}{d-(p-1)} [\nabla_{\nu_1}, \nabla_\mu] K^\mu{}_{\nu_2\cdots\nu_p} + \frac{p}{d-(p-1)} \nabla^\mu (-A_{\mu\nu_1\cdots\nu_p} + g_{\nu_1[\mu} B_{\nu_2\cdots\nu_p]}).\end{aligned} \tag{A.7}$$

However, we also have

$$[\nabla_{\nu_1}, \nabla_\mu] K^\mu{}_{\nu_2\cdots\nu_p} = R_{\nu_1\mu}{}^{\mu\sigma} K_{\sigma\nu_2\cdots\nu_p} + (p-1) R_{\nu_1\mu[\nu_2|\sigma|} K^{\mu\sigma}{}_{\nu_3\cdots\nu_p]} \tag{A.8}$$

$$= -R_{\sigma\nu_1} K^\sigma{}_{\nu_2\cdots\nu_p} + \frac{p-1}{2} R_{\mu\sigma\nu_1[\nu_2} K^{\mu\sigma}{}_{\nu_3\cdots\nu_p]} \tag{A.9}$$

where we used the algebraic Bianchi identity. Using identity (A.6), and putting it all together, we arrive at

$$\begin{aligned}\nabla_{\nu_1} B_{\nu_2\cdots\nu_p} = &\frac{p}{d-(p-1)} \left(-R_{\sigma\nu_1} K^\sigma{}_{\nu_2\cdots\nu_p} + \frac{p-1}{2} R_{\mu\sigma\nu_1[\nu_2} K^{\mu\sigma}{}_{\nu_3\cdots\nu_p]} \right) \\ &- \frac{p}{d-(p-1)} \left(-R_{\sigma[\nu_1} K^\sigma{}_{\nu_2\cdots\nu_p]} + \frac{p-1}{2} R_{\mu\sigma[\nu_1\nu_2} K^{\mu\sigma}{}_{\nu_3\cdots\nu_p]} \right) \\ &+ \frac{1}{d-(p-1)} \left((d-p) \nabla_{[\nu_1} B_{\nu_2\cdots\nu_p]} + \nabla_{\nu_1} B_{\nu_2\cdots\nu_p} \right) \\ &- \frac{p-1}{d-(p-1)} g_{\nu_1[\nu_2} \nabla_{|\mu|} B^\mu{}_{\nu_3\cdots\nu_p]}.\end{aligned} \tag{A.10}$$

The trace part with respect to ν_1 and ν_2 gives $\nabla_\mu B^\mu{}_{\nu_3\cdots\nu_p} = 0$, and collecting terms, we obtain the B identity

$$\begin{aligned}\nabla_\mu B_{\nu_2\cdots\nu_p} = &+\nabla_{[\mu} B_{\nu_2\cdots\nu_p]} + \frac{p}{d-p} \left(R_{\sigma[\mu} K^\sigma{}_{\nu_2\cdots\nu_p]} - R_{\sigma\mu} K^\sigma{}_{\nu_2\cdots\nu_p} \right) \\ &+ \frac{p(p-1)}{2(d-p)} \left(R_{\sigma\tau\mu[\nu_2} K^{\sigma\tau}{}_{\nu_3\cdots\nu_p]} - R_{\sigma\tau[\mu\nu_2} K^{\sigma\tau}{}_{\nu_3\cdots\nu_p]} \right).\end{aligned} \tag{A.11}$$

A.3. dB identity

Define now $C = \mathrm{d}B$. We have

$$C_{\mu\nu_2\cdots\nu_p} = p\nabla_\mu B_{\nu_2\cdots\nu_p} - \frac{p^2}{d-p}\left(R_{\sigma[\mu}K^\sigma{}_{\nu_2\cdots\nu_p]} - R_{\sigma\mu}K^\sigma{}_{\nu_2\cdots\nu_p}\right) - \frac{p^2(p-1)}{2(d-p)}\left(R_{\sigma\tau\mu[\nu_2}K^{\sigma\tau}{}_{\nu_3\cdots\nu_p]} - R_{\sigma\tau[\mu\nu_2}K^{\sigma\tau}{}_{\nu_3\cdots\nu_p]}\right). \tag{A.12}$$

Next define

$$\Omega_{\mu\nu\rho\nu_3\cdots\nu_p} = \nabla_\mu C_{\nu\rho\nu_3\cdots\nu_p} - \nabla_\nu C_{\mu\rho\nu_3\cdots\nu_p}$$

so that

$$2\nabla_\mu C_{\nu\rho\nu_3\cdots\nu_p} = \Omega_{\mu\nu\rho\nu_3\cdots\nu_p} + \Omega_{\rho\nu\mu\nu_3\cdots\nu_p} + \Omega_{\rho\mu\nu\nu_3\cdots\nu_p}.$$

In calculating $\Omega_{\mu\nu_1\nu_2\nu_3\cdots\nu_p}$, we treat the ν_2 index separately from the $\nu_3, \ldots, \nu_p$ indices for later use. When ∇ acts on $K_{\nu_1\cdots\nu_p}$, we exchange it for A and B using the CKY equation. The end formula is

$$\begin{aligned}
\Omega_{\mu\nu_1\nu_2\nu_3\cdots\nu_p} \overset{[\nu_3\cdots\nu_p]}{=}\ & p[\nabla_\mu, \nabla_{\nu_1}]B_{\nu_2\cdots\nu_p} + \frac{p(p-1)}{d-p}((\nabla_\mu R_{\sigma\nu_1})K^\sigma{}_{\nu_2\cdots\nu_p} - (\mu \leftrightarrow \nu_1)) \\
& + \frac{p}{d-p}\left(\left(\nabla_\mu R_{\sigma\nu_2}\right)K^\sigma{}_{\nu_1\nu_3\cdots\nu_p} + (p-2)\left(\nabla_{\nu_1}R_{\sigma\nu_3}\right)K^\sigma{}_{\mu\nu_2\nu_4\cdots\nu_p} - (\mu \leftrightarrow \nu_1)\right) \\
& + \frac{p(p-1)}{d-p}R^\sigma_{\nu_1}A_{\mu\sigma\nu_2\cdots\nu_p} - (\mu \leftrightarrow \nu_1) \\
& - \frac{p-1}{d-p}\left(R^\sigma_{\nu_1}g_{\mu\nu_2}B_{\sigma\nu_3\cdots\nu_p} + (p-2)R^\sigma_\mu g_{\nu_1\nu_3}B_{\sigma\nu_2\nu_4\cdots\nu_p} - (\mu \leftrightarrow \nu_1)\right) \\
& + \frac{p}{d-p}\left(2R_{\sigma\nu_2}A_\mu{}^\sigma{}_{\nu_1\nu_3\cdots\nu_p} - 2(p-2)R_{\sigma\nu_3}A_\mu{}^\sigma{}_{\nu_1\nu_2\nu_4\cdots\nu_p}\right) \\
& + \frac{1}{d-p}\left(R_{\mu\nu_2}B_{\nu_1\nu_3\cdots\nu_p} + (p-2)R_{\nu_1\nu_3}B_{\mu\nu_2\nu_4\cdots\nu_p} + (p-2)R^\sigma_{\nu_2}g_{\mu\nu_3}B_{\sigma\nu_1\nu_4\cdots\nu_p}\right. \\
& \left. + (p-2)R^\sigma_{\nu_3}g_{\nu_1\nu_2}B_{\sigma\mu\nu_4\cdots\nu_p} + (p-2)(p-3)R^\sigma_{\nu_3}g_{\mu\nu_4}B_{\sigma\nu_1\nu_2\nu_5\cdots\nu_p} - (\mu \leftrightarrow \nu_1)\right) \\
& - \frac{p(p-2)}{2(d-p)}\left(\left(\nabla_\mu R_{\sigma\tau\nu_1\nu_2}\right)K^{\sigma\tau}{}_{\nu_3\cdots\nu_p} + (p-2)\left(\nabla_{\nu_1}R_{\sigma\tau\mu\nu_3}\right)K^{\sigma\tau}{}_{\nu_2\nu_4\cdots\nu_p}\right. \\
& \left. - 2\left(\nabla_\mu R_{\sigma\tau\nu_2\nu_3}\right)K^{\sigma\tau}{}_{\nu_1\nu_4\cdots\nu_p} - (p-3)\left(\nabla_\mu R_{\sigma\tau\nu_3\nu_4}\right)K^{\sigma\tau}{}_{\nu_1\nu_2\nu_5\cdots\nu_p} - (\mu \leftrightarrow \nu_1)\right) \\
& - \frac{p(p-2)}{2(d-p)}\left(R_{\sigma\tau\nu_1\nu_2}A_\mu{}^{\sigma\tau}{}_{\nu_3\cdots\nu_p} + (p-2)R_{\sigma\tau\mu\nu_3}A_{\nu_1}{}^{\sigma\tau}{}_{\nu_2\nu_4\cdots\nu_p} - (\mu \leftrightarrow \nu_1)\right) \\
& + \frac{p(p-2)}{2(d-p)}\left(4R_{\sigma\tau\nu_2\nu_3}A_\mu{}^{\sigma\tau}{}_{\nu_1\nu_4\cdots\nu_p} + 2(p-3)R_{\sigma_\tau\nu_3\nu_4}A_\mu{}^{\sigma\tau}{}_{\nu_1\nu_2\nu_5\cdots\nu_p}\right) \\
& - \frac{p-2}{2(d-p)}\left(2R^{\sigma\tau}{}_{\nu_1\nu_2}g_{\mu\sigma}B_{\tau\nu_3\cdots\nu_p} - 2(p-2)R^{\sigma\tau}{}_{\nu_1\nu_3}g_{\sigma\mu}B_{\tau\nu_2\nu_4\cdots\nu_p}\right. \\
& + (p-2)R^{\sigma\tau}{}_{\nu_1\nu_2}g_{\mu\nu_3}B_{\sigma\tau\nu_4\cdots\nu_p} - (p-2)R^{\sigma\tau}{}_{\nu_1\nu_3}g_{\mu\nu_2}B_{\sigma\tau\nu_4\cdots\nu_p} \\
& - (p-2)(p-3)R^{\sigma\tau}{}_{\nu_1\nu_4}g_{\mu\nu_3}B_{\sigma\tau\nu_2\nu_5\cdots\nu_p} - 4R^{\sigma\tau}{}_{\nu_2\nu_3}g_{\mu\sigma}B_{\tau\nu_1\nu_4\cdots\nu_p} \\
& + 2(p-3)R^{\sigma\tau}{}_{\nu_4\nu_3}g_{\mu\sigma}B_{\tau\nu_1\nu_2\nu_5\cdots\nu_p} - 2(p-3)R^{\sigma\tau}{}_{\nu_2\nu_3}g_{\mu\nu_4}B_{\sigma\nu_1\tau\nu_5\cdots\nu_p} \\
& + (p-3)R^{\sigma\tau}{}_{\nu_4\nu_3}g_{\mu\nu_2}B_{\sigma\nu_1\tau\nu_5\cdots\nu_p} + (p-3)(p-4)R^{\sigma\tau}{}_{\nu_5\nu_3}g_{\mu\nu_4}B_{\sigma\nu_1\tau\nu_2\nu_6\cdots\nu_p} \\
& \left. - (\mu \leftrightarrow \nu_1)\right),
\end{aligned} \tag{A.13}$$

where $\overset{[\nu_3\cdots\nu_p]}{=}$ indicates that we antisymmetrize the right-hand side with respect to $\nu_3, \ldots, \nu_p$. By alternating indices, and calculation of a few pages, we derive a differential condition for

$C_{\nu_1\cdots\nu_p}$,

$$\begin{aligned}\nabla_\mu C_{\nu_1\cdots\nu_p} \overset{[\nu_1\cdots\nu_p]}{=} & -\frac{p^2}{2}R_{\nu_1\nu_2\mu}{}^{\sigma}B_{\sigma\nu_3\cdots\nu_p} - \frac{p^2}{d-p}\nabla_{\nu_1}R^{\sigma}_{\mu}K_{\sigma\nu_2\cdots\nu_p} - \frac{p^2}{d-p}\nabla_{\nu_1}R^{\sigma}_{\nu_2}K_{\sigma\mu\nu_3\cdots\nu_p}\\ & + \frac{p^2}{d-p}R^{\sigma}_{\mu}A_{\sigma\nu_1\cdots\nu_p} - \frac{p^2}{d-p}R^{\sigma}_{\nu_1}A_{\mu\sigma\nu_2\cdots\nu_p}\\ & + \frac{p}{d-p}g_{\mu\nu_1}R^{\sigma}_{\nu_2}B_{\sigma\nu_3\cdots\nu_p} - \frac{p}{d-p}R_{\mu\nu_1}B_{\nu_2\cdots\nu_p}\\ & + \frac{p^2(p-2)}{4(d-p)}\nabla_\mu R_{\sigma\tau\nu_1\nu_2}K^{\sigma\tau}{}_{\nu_3\cdots\nu_p}\\ & + \frac{p^2(p-2)}{2(d-p)}R_{\sigma\tau\nu_1\nu_2}A_{\mu}{}^{\sigma\tau}{}_{\nu_3\cdots\nu_p} - \frac{p^2(p-2)}{2(d-p)}R_{\sigma\tau\mu\nu_1}A^{\sigma\tau}{}_{\nu_2\cdots\nu_p}\\ & + \frac{p(p-2)}{2(d-p)}R_{\mu}{}^{\tau}{}_{\nu_1\nu_2}B_{\tau\nu_3\cdots\nu_p} - \frac{p(p-2)}{2(d-p)}g_{\mu\nu_1}R_{\sigma\tau\nu_2\nu_3}B^{\sigma\tau}{}_{\nu_4\cdots\nu_p}. \end{aligned} \tag{A.14}$$

A.4. CKY two-forms

In the case of $p = 2$, a CKY two-form by definition satisfies

$$\nabla_\mu K_{\nu\rho} = A_{\mu\nu\rho} + \tfrac{1}{2}g_{\mu\nu}B_\rho - \tfrac{1}{2}g_{\mu\rho}B_\nu \tag{A.15}$$

and is in one-to-one correspondence with a section $K \oplus A \oplus B \oplus C$ that is parallel with respect to a connection D. This connection is given by the equation above, plus

$$\nabla_\mu A_{\nu_1\nu_2\nu_3} \overset{[\nu_1\nu_2\nu_3]}{=} -\tfrac{3}{2}R_{\nu_1\nu_2\mu}{}^{\sigma}K_{\sigma\nu_3} - \tfrac{3}{4}g_{\mu\nu_1}C_{\nu_2\nu_3}, \tag{A.16}$$

$$\nabla_\mu B_\nu = \tfrac{1}{2}C_{\mu\nu} - \frac{1}{d-2}(R_{\sigma\mu}K^{\sigma}{}_{\nu} + R_{\sigma\nu}K^{\sigma}{}_{\mu}), \tag{A.17}$$

$$\begin{aligned}\nabla_\mu C_{\nu_1\nu_2} = & -2R_{\nu_1\nu_2\mu}{}^{\sigma}B_\sigma\\ & + \frac{2}{d-2}\big(\nabla_{\nu_2}R_{\sigma\nu_1}K^{\sigma}{}_{\mu} - \nabla_{\nu_1}R_{\sigma\nu_2}K^{\sigma}{}_{\mu} + \nabla_{\nu_2}R_{\sigma\mu}K^{\sigma}{}_{\nu_1} - \nabla_{\nu_1}R_{\sigma\mu}K^{\sigma}{}_{\nu_2}\big)\\ & + \frac{1}{d-2}\big(4R_{\sigma\mu}A^{\sigma}{}_{\nu_1\nu_2} + 2R_{\sigma\nu_1}A^{\sigma}{}_{\mu\nu_2} - 2R_{\sigma\nu_2}A^{\sigma}{}_{\mu\nu_1}\big)\\ & + \frac{1}{d-2}\big(R_{\sigma\nu_2}B^{\sigma}g_{\mu\nu_1} - R_{\sigma\nu_1}B^{\sigma}g_{\mu\nu_2} + R_{\mu\nu_2}B_{\nu_1} - R_{\mu\nu_1}B_{\nu_2}\big). \end{aligned} \tag{A.18}$$

Equation (A.17) implies that B_μ is a Killing vector if $R_{\sigma\mu}K^{\sigma}_{\nu}$ is antisymmetric with respect to μ and ν. This is realized, for instance, if

$$R_{\mu\nu} = \sigma\,(x)\,g_{\mu\nu},$$

where $\sigma(x)$ is an arbitrary function.

Appendix B. Massless Kerr is anti-de Sitter

We give here, for reference, the isometry of the massless, NUT-less, Kerr black hole into AdS_4. For simplicity, we rescale here $\hat{t}$ and ψ by ℓ so that the metric becomes

$$\begin{aligned}\mathrm{d}s^2 = \ell^2\Bigg\{ & \frac{(r^2+a^2)(r^2+\tilde{\ell}^2)}{r^2+y^2}(\mathrm{d}\tilde{t} + y^2\,\mathrm{d}\tilde{\psi})^2 + \frac{r^2+y^2}{(r^2+a^2)(r^2+\tilde{\ell}^2)}\,\mathrm{d}r^2\\ & + \frac{(y^2-a^2)(y^2-\tilde{\ell}^2)}{r^2+y^2}(\mathrm{d}\tilde{t} - r^2\,\mathrm{d}\tilde{\psi})^2 + \frac{r^2+y^2}{(y^2-a^2)(y^2-\tilde{\ell}^2)}\,\mathrm{d}y^2\Bigg\} \end{aligned} \tag{B.1}$$

with $\tilde{\ell} = \ell$. It appears that $a/\tilde{\ell}$ is a physical parameter, while both a and $\tilde{\ell}$ can be scaled with the coordinates freely. However, for $|y| \leqslant a$, the metric is isometric to the whole of AdS_4,

$$\mathrm{d}s^2 = \ell^2 \left\{ -(1+R^2)\,\mathrm{d}T^2 + R^2 \sin^2\Theta^2\,\mathrm{d}\Phi^2 + \frac{\mathrm{d}R^2}{1+R^2} + R^2\,\mathrm{d}\Theta^2 \right\},$$

by using the diffeomorphisms

$$\tilde{t} = \frac{\tilde{\ell}}{\tilde{\ell}^2 - a^2} T - \frac{a}{\tilde{\ell}^2 - a^2}\Phi, \tag{B.2}$$

$$\psi = \frac{1}{a(\tilde{\ell}^2 - a^2)}\Phi - \frac{1}{\tilde{\ell}(\tilde{\ell}^2 - a^2)} T, \tag{B.3}$$

$$R^2 \sin^2\Theta = \frac{(r^2 + a^2)(a^2 - y^2)}{a^2(\tilde{\ell}^2 - a^2)}, \tag{B.4}$$

$$1 + R^2 = \frac{(\tilde{\ell}^2 - y^2)(r^2 + \tilde{\ell}^2)}{\tilde{\ell}^2(\tilde{\ell}^2 - a^2)}. \tag{B.5}$$

For $|y| > \tilde{\ell}$, we simply exchange a for $\tilde{\ell}$. However, in this case, the diffeomorphism covers only half of the two-sphere $\cos\Theta > 0$.

Appendix C. Profiles in de Sitter Kerr

We give here a derivation of the profiles of Δ_y and Δ_r for a positive cosmological constant. These results supplement the numerics of [31]. As with a negative cosmological constant, our tool is the deformation of their graphs by varying M and L.

For $M = 0$, there are always two roots and one bounded region, where Δ_r is positive. Whether there is one or three extrema depends, respectively, on $1 < a^2g^2$ or $1 \geqslant a^2g^2$. As we turn on M, there will be a change in the number of roots of Δ_r when an extremum of Δ_r crosses the horizontal axis: $\Delta_r = \Delta_r' = 0$. Eliminating M from the two equations gives

$$3g^2r^4 - (1 - a^2g^2)r^2 + a^2 = 0,$$

with the solutions of positive r^2 only when $1 - a^2g^2 > 0$. The two solutions are

$$\bar{r}_\pm^2 = \frac{1}{6g^2}(1 - a^2g^2 \pm \sqrt{(1 - a^2g^2)^2 - 12a^2g^2}),$$

and the determinant is non-negative when $|a^2g^2 - 7| \geqslant 4\sqrt{3}$. So there is no change in roots, unless $a^2g^2 \leqslant 7 - 2\sqrt{12}$, in which case there are two transitions at M_- and M_+.

The extremal parameters (a, M) are parametrized in terms of $\bar{r}$ as

$$M = \bar{r}\frac{(1 - g^2\bar{r}^2)^2}{1 + g^2\bar{r}^2}, \qquad a^2 = \bar{r}^2\frac{1 - 3g^2\bar{r}^2}{1 + g^2\bar{r}^2}. \tag{C.1}$$

However, the function $\bar{r} \mapsto a^2$ is not one-to-one; see figure C1. That is, for any a, there are two critical values of M and $M_\pm$, where Δ_r has a double root. The profile of Δ_r is shown in figure C2. The parametric plot of (M^2, a) as a function of $\bar{r}$ is drawn in figure 3.

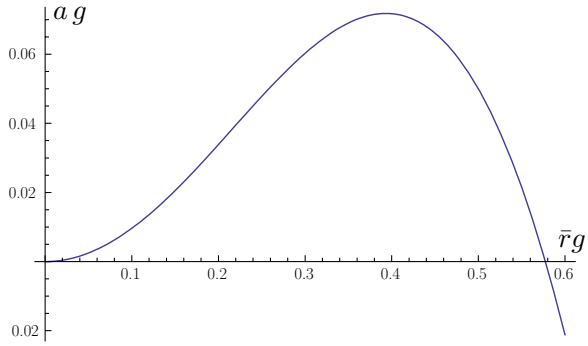


Figure C1. Extremal Δ_r, a as a function of $\bar{r}$.

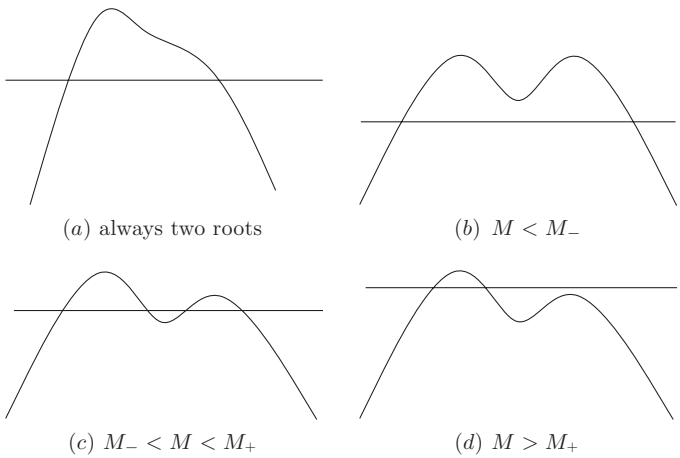


Figure C2. Graph of $\Delta_r(r)$ in Kerr–dS for fixed a. (a) The graph for the case $a^2g^2 > 7-4\sqrt{3}$ and (b)–(d) the graphs for $a^2g^2 < 7-4\sqrt{3}$.

Appendix D. AdS3 coordinates

Anti-de Sitter space in $d = 3$ has isometry algebra

$$\mathfrak{so}(2,2) = \mathfrak{sl}(2,\mathbb{R})_{\mathrm{L}} \oplus \mathfrak{sl}(2,\mathbb{R})_{\mathrm{R}} = \langle l_a\rangle_{a=0,1,2} \oplus \langle r_a\rangle_{a=0,1,2}, \tag{D.1}$$

where we choose a basis such that $[r_a, r_b] = -\epsilon_{ab}{}^c r_c$ and $[l_a, l_b] = -\epsilon_{ab}{}^c l_c$.

The (universal cover of) AdS_3 metric $\mathrm{d}s^2_{\mathrm{AdS}_3}$ can be described in the coordinates (x, u, τ) so that ∂_u and ∂_τ are manifest commuting isometries and x is hypersurface orthogonal. Up to isometries, diffeomorphisms $x \mapsto x'(x)$, parity transformations and $\mathrm{GL}(2,\mathbb{R})$ matrix transformations on (u, τ), the classification of $\mathfrak{sl}(2,\mathbb{R})$ elements results in the following choices.

(1) Global coordinates, for which $\partial_\tau = \frac{1}{2}(r_0 + l_0)$ and $\partial_\phi = \frac{1}{2}(r_0 - l_0)$,

$$\mathrm{d}s^2_{\mathrm{AdS}_3} = -(1+x^2)\,\mathrm{d}\tau^2 + \frac{\mathrm{d}x^2}{x^2+1} + x^2\,\mathrm{d}u^2, \tag{D.2}$$

with $u = u + 2\pi$, $\tau \in \mathbb{R}$ and $x \geqslant 0$.

(2) Spacelike self-dual global coordinates, for which $\partial_u = l_2$ and $\partial_\tau = r_0$,

$$\mathrm{d}s^2_{\mathrm{AdS}_3} = \frac{1}{4}\left(-(x^2+1)\,\mathrm{d}\tau^2 + \frac{\mathrm{d}x^2}{x^2+1} + (\mathrm{d}u + x\,\mathrm{d}\tau)^2\right), \tag{D.3}$$

with $x, u, \tau \in \mathbb{R}$. It covers the space globally.

(3) Spacelike self-dual non-extremal coordinates, for which $\partial_u = l_2$ and $\partial_\tau = r_2$,

$$\mathrm{d}s^2_{\mathrm{AdS}_3} = \frac{1}{4}\left(-(x^2-1)\,\mathrm{d}\tau^2 + \frac{\mathrm{d}x^2}{x^2-1} + (\mathrm{d}u + x\,\mathrm{d}\tau)^2\right), \tag{D.4}$$

with $x, u, \tau \in \mathbb{R}$.

(4) Spacelike self-dual extremal coordinates, for which $\partial_u = l_2$ and $\partial_\tau = r_0 + r_2$,

$$\mathrm{d}s^2_{\mathrm{AdS}_3} = \frac{1}{4}\left(-x^2\,\mathrm{d}\tau^2 + \frac{\mathrm{d}x^2}{x^2} + (\mathrm{d}u + x\,\mathrm{d}\tau)^2\right), \tag{D.5}$$

with $x, u, \tau \in \mathbb{R}$. These are the relevant coordinates that we used in the NHEK.

(5) Timelike self-dual coordinates, for which $\partial_\tau = r_0 + l_2$ and $\partial_u = l_0$,

$$\mathrm{d}s^2_{\mathrm{AdS}_3} = \frac{1}{4}\left(x^2\,\mathrm{d}\tau^2 + \frac{\mathrm{d}x^2}{x^2} - (\mathrm{d}u + x\,\mathrm{d}\tau)^2\right), \tag{D.6}$$

with $x, u, \tau \in \mathbb{R}$. These are the relevant coordinates we used in the polar limit.

(6) The Poincaré coordinates, for which $\partial_\tau = r_0 + r_2$ and $\partial_u = l_0 + l_2$,

$$\mathrm{d}s^2_{\mathrm{AdS}_3} = \frac{\mathrm{d}x^2}{x^2} + x^2\,\mathrm{d}u\,\mathrm{d}\tau, \tag{D.7}$$

with $u, \tau \in \mathbb{R}$ and $x > 0$.

In the above, we have set the cosmological length $R = 1$.

A parametrization of the quadric

$$(X^{-1})^2 + (X^0)^2 - (X^1)^2 - (X^2)^2 = R^2$$

in terms of the extremal spacelike self-dual coordinates, $b = 0$, can be achieved by $(-\infty < \tau < \infty, -\infty < u < \infty, 0 < x)$

$$\begin{aligned}
A^+ &\equiv X^{-1} + X^1 = R\sqrt{x}\sinh\frac{u}{2},\\
A^- &\equiv X^{-1} - X^1 = -R\left(\tau\sqrt{x}\cosh\frac{u}{2} + \frac{1}{\sqrt{x}}\sinh\frac{u}{2}\right),\\
B^+ &\equiv X^0 + X^2 \;\; = R\left(\tau\sqrt{x}\sinh\frac{u}{2} + \frac{1}{\sqrt{x}}\cosh\frac{u}{2}\right),\\
B^- &\equiv X^0 - X^2 \;\; = R\sqrt{x}\cosh\frac{u}{2},
\end{aligned}$$

and the four-dimensional metric $g_{MN} = \mathrm{diag}\,(-1, -1, +1, +1)$. Note that this parametrization covers only the region with $B^+ > 0$ and $B^+ > A^+$. A parametrization of the global spacelike self-dual coordinates, $b = 1$ in (D.5), was given in [29]. This can be related to the above embedding after an infinite boost $b \to 0$; see for instance [28]. The right-invariant 1-forms θ^a and the left-invariant 1-forms $\hat{\theta}^a$ are expressed as

$$\begin{aligned}
\theta^0 &= -\frac{2}{R^2}[X^{-1}\,\mathrm{d}X^0 - X^0\,\mathrm{d}X^{-1} + X^1\,\mathrm{d}X^2 - X^2\,\mathrm{d}X^1],\\
\theta^1 &= -\frac{2}{R^2}[X^{-1}\,\mathrm{d}X^1 - X^1\,\mathrm{d}X^{-1} + X^0\,\mathrm{d}X^2 - X^2\,\mathrm{d}X^0],\\
\theta^2 &= -\frac{2}{R^2}[X^{-1}\,\mathrm{d}X^2 - X^2\,\mathrm{d}X^{-1} - X^0\,\mathrm{d}X^1 + X^1\,\mathrm{d}X^0],
\end{aligned}$$

$$\tilde{\theta}^0 = \frac{2}{R^2}[X^{-1}\,\mathrm{d}X^0 - X^0\,\mathrm{d}X^{-1} - X^1\,\mathrm{d}X^2 + X^2\,\mathrm{d}X^1],$$
$$\tilde{\theta}^1 = \frac{2}{R^2}[X^{-1}\,\mathrm{d}X^1 - X^1\,\mathrm{d}X^{-1} - X^0\,\mathrm{d}X^2 + X^2\,\mathrm{d}X^0],$$
$$\tilde{\theta}^2 = \frac{2}{R^2}[X^{-1}\,\mathrm{d}X^2 - X^2\,\mathrm{d}X^{-1} + X^0\,\mathrm{d}X^1 - X^1\,\mathrm{d}X^0],$$

regardless of what metric we choose in 2+2 dimensions.

Appendix E. Einstein solutions of NHEK type

Let us assume a metric of NHEK type

$$\mathrm{d}s^2 = \mathrm{e}^{2\omega(y)}\left[-x^2\,\mathrm{d}\tau^2 + \frac{\mathrm{d}x^2}{x^2} + \mathrm{e}^{2\lambda(y)}\,(\mathrm{d}u + x\,\mathrm{d}\tau)^2\right] + \mathrm{e}^{2f(y)}\,\mathrm{d}y^2. \tag{E.1}$$

We can make use of the computation of the curvature tensor in section 5 in order to derive its Ricci tensor. The non-trivial components of the Einstein equation $R_{AB} = -\frac{3}{\ell^2}\eta_{AB}$ come from the diagonal flat components, which are

$$\mathrm{e}^{-2\omega} - \frac{1}{2}\,\mathrm{e}^{-2\omega+2\lambda} + (\ddot{\omega} + 3\dot{\omega}^2 + \dot{\omega}(\dot{\lambda} - \dot{f}))\,\mathrm{e}^{-2f} = \frac{3}{\ell^2}, \tag{E.2}$$

$$-\,\mathrm{e}^{-2\omega} + \frac{1}{2}\,\mathrm{e}^{-2\omega+2\lambda} - (\ddot{\omega} + 3\dot{\omega}^2 + \dot{\omega}(\dot{\lambda} - \dot{f}))\,\mathrm{e}^{-2f} = -\frac{3}{\ell^2}, \tag{E.3}$$

$$-\,\frac{1}{2}\,\mathrm{e}^{-2\omega+2\lambda} - (\ddot{\omega} + 3\dot{\omega}^2 + \dot{\omega}(4\dot{\lambda} - \dot{f}) + \ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{f})\,\mathrm{e}^{-2f} = -\frac{3}{\ell^2}, \tag{E.4}$$

$$-\,(3\ddot{\omega} + 3\dot{\omega}^2 + \dot{\omega}(2\dot{\lambda} - 3\dot{f}) + \ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{f})\,\mathrm{e}^{-2f} = -\frac{3}{\ell^2}. \tag{E.5}$$

The first and the second equations are the same. A further simplification is made by taking the gauge $f\,(y) = 0$. Thus, the equations of motion are reduced to the following three equations:

$$\mathrm{e}^{-2\omega} - \frac{1}{2}\,\mathrm{e}^{-2\omega+2\lambda} + \left(\ddot{\omega} + 3\dot{\omega}^2 + \dot{\omega}\dot{\lambda}\right) = \frac{3}{\ell^2}, \tag{E.6}$$

$$\frac{1}{2}\,\mathrm{e}^{-2\omega+2\lambda} + (\ddot{\omega} + 3\dot{\omega}^2 + 4\dot{\omega}\dot{\lambda} + \ddot{\lambda} + \dot{\lambda}^2) = \frac{3}{\ell^2}, \tag{E.7}$$
$$3\ddot{\omega} + 3\dot{\omega}^2 + 2\dot{\omega}\dot{\lambda} + \ddot{\lambda} + \dot{\lambda}^2 = \frac{3}{\ell^2}.$$

The triplet $(\omega, \dot{\omega}, \ddot{\omega})$ can be algebraically solved in terms of λ and its derivatives:

$$\omega = f_1(\lambda, \dot{\lambda}, \ddot{\lambda}), \tag{E.8}$$

$$\dot{\omega} = f_2(\lambda, \dot{\lambda}, \ddot{\lambda}), \tag{E.9}$$

$$\ddot{\omega} = f_3(\lambda, \dot{\lambda}, \ddot{\lambda}). \tag{E.10}$$

Since $\dot{\omega}$ appears quadratically, there are two branches of this solution. However, the gauge $f = 0$ is preserved by $y \mapsto -y$, and in fact, the two branches are seen to be equivalent. Whence ω is given in terms of λ and its derivatives, provided that the solution for λ is consistent with the three aforementioned equations. That is, we have $\dot{f}_1 = f_2$ and $\dot{f}_2 = f_3$. By computer algebra, we find that these two differential equations are identical.

At this point, the solution to (E.1) is given by a third-order highly nonlinear differential equation $\dddot{\lambda} = f(\lambda, \dot{\lambda}, \ddot{\lambda})$. It is difficult to solve even more due to the gauge we chose. We make the observation that the gauge $f = 0$ is also preserved by $y \mapsto y + b$ for any constant b, and thus, the differential equation has only two rather than three gauge-invariant degrees of freedom. This is the same number of parameters as for the NHEK geometry, $\bar{r}^2$ and L. Therefore, up to subtleties on the range of the parameters, the NHEK solution is the most generic Einstein solution to (E.1).

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