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Existence of traveling wave solutions for diffusive predator–prey type systems

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A B S T R A C T

In this work we investigate the existence of traveling wave solutions for a class of diffusive predator–prey type systems whose each nonlinear term can be separated as a product of suitable smooth functions satisfying some monotonic conditions. The profile equations for the above system can be reduced as a four-dimensional ODE system, and the traveling wave solutions which connect two different equilibria or the small amplitude traveling wave train solutions are equivalent to the heteroclinic orbits or small amplitude periodic solutions of the reduced system. Applying the methods of Wazewski Theorem, LaSalle's Invariance Principle and Hopf bifurcation theory, we obtain the existence results. Our results can apply to various kinds of ecological models.

1. Introduction

This work concerns with the existence of traveling wave solutions for the following diffusive predator–prey type system:

$$\begin{cases} u_t = d_1 u_{xx} - h(u)\big(g(w) - p(u)\big), \\ w_t = d_2 w_{xx} - \ell(w) q(u), \end{cases} \tag{1.1}$$

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where $d_1 > 0$, $d_2 > 0$, $p(u)$, $g(w)$, $h(u)$, $\ell(w)$ and $q(u)$ are smooth functions satisfying some monotonic conditions which will be mentioned later. System (1.1) is a general form of the diffusive predator–prey system which contains many known models. Indeed, system (1.1) describes not only the interspecies relations for ecological and social models, but also the base block of more complicated food web, food chain and biochemical network structure. In ecology, the functions $u(x,t)$ and $w(x,t)$ represent the species densities of the prey and predator, respectively; the constants d_1 and d_2 are the spatial diffusion rates of the two species; the function $h(u)p(u)$ is the net growth rate of the prey in the absence of predator; the function $h(u)$ is the predator functional response which describes consumption rate of prey by a unit number of predators; the graphs $g(w) - p(u) = 0$ and $q(u) = 0$ are the prey nullcline and predator nullcline on the phase portrait, respectively. In the sequel, we will illustrate some models where the existence of traveling wave solutions has been studied in the past decades.

In 1983, Dunbar [4,5] considers the existence of traveling wave solutions for the following reaction–diffusion system based on the Lotka–Volterra differential equation model of a predator–prey interaction:

$$\begin{cases} u_t = d_1 u_{xx} + Au\left(1 - \dfrac{u}{K}\right) - Buw, \\ w_t = d_2 w_{xx} - Cw + Duw, \end{cases} \tag{1.2}$$

where d_1, d_2, A, B, C, D, K are positive constants. A is the intrinsic rate of increasing for the prey species; C is the death rate for the predator in the absence of the prey; K is the carrying capacity of the environment; the predator functional response here is the identity function of u. By using the Wazewski Theorem (an extension of shooting argument in higher dimension) together with a Lyapunov function and LaSalle's Invariance Principle, he proves the existence of traveling wave solutions.

Dunbar [6] further considered the existence of traveling wave solutions for system (1.2) but with Holling type II functional response $H_2(u) = \frac{u}{1+Eu}$, i.e.,

$$\begin{cases} u_t = d_1 u_{xx} + Au\left(1 - \dfrac{u}{K}\right) - BH_2(u)w, \\ w_t = d_2 w_{xx} - Cw + DH_2(u)w, \end{cases} \tag{1.3}$$

where $E > 0$. System (1.3) includes the effects of predation satiation: the consumption rate of prey by a unit number of predators cannot continue to grow linearly with the number of prey available but must "saturate" at some value (see [8,9]). The parameter $1/E$ here is the *satiation rate of predation*. Assume $d_1 = 0$, Dunbar uses the method similar to that in [4,5] and the invariant manifold theory to prove the existence of traveling wave train and traveling front solutions for system (1.3). The case for $d_1 \neq 0$ is then considered by Huang, Lu and Ruan [12]. Using the same shooting argument and the Hopf bifurcation theory, they establish the existence of the traveling wave solutions connecting two rest states as well as the existence of small amplitude traveling wave train solutions.

Later, Li and Wu [16] also consider the system (1.3) but with Holling type III functional response $H_3(u) = \frac{u^2}{1+Eu^2}$, i.e.,

$$\begin{cases} u_t = d_1 u_{xx} + Au\left(1 - \dfrac{u}{K}\right) - BH_3(u)w, \\ w_t = d_2 w_{xx} - Cw + DH_3(u)w. \end{cases} \tag{1.4}$$

By using the similar methods of [4,5], they establish the existence of traveling wave solutions of (1.4) for the case $d_1 = 0$. In this work, we generalize the results of [16] to the case $d_1 \neq 0$.

In addition to the previous Holling types of functional responses, Ivlev in 1961 [14] introduces another functional response $H_4(u) = E(1 - e^{-Mu})$ where E represents the maximum rate of predation and M is a constant representing the decrease in motivation to hunt. The diffusive predator–prey model with logistic growth rate of prey and Ivlev type functional response is described by

$$\begin{cases} u_t = d_1 u_{xx} + Au\left(1 - \dfrac{u}{K}\right) - BH_4(u)w, \\ w_t = d_2 w_{xx} - Cw + DH_4(u)w. \end{cases} \tag{1.5}$$

If $d_1 = d_2 = 0$, system (1.5) is studied by many authors, see [1,2,15,17,18,20,22,24,25]. Most of these papers concentrate on the existence and stability of limit cycle. Recently, in [23], Wang, Shi and Wei also study the global bifurcation of a class of more general predator–prey models with a strong Allee effect in prey population. On the other hand, if $d_1 \neq 0$ and $d_2 \neq 0$, there seems no results for the existence of traveling wave solution of system (1.5). In Section 5.4 of this work, we will apply our main theorem to obtain the new existence results for traveling wave solutions of system (1.5).

For other examples, Owen and Lewis [19] consider the following general system

$$\begin{cases} u_t = \varepsilon\alpha_0 u_{xx} + \alpha_1 u f_1(u) - \alpha_2 w f_2(u), \\ w_t = \alpha_0 w_{xx} + \alpha_3 w f_2(u) - \alpha_4 w, \end{cases} \tag{1.6}$$

where $\varepsilon \approx 0$ and α_i's are positive constants. They study the mechanism for which predation pressure can slow, stall or reverse a spatial invasion of prey. Some numerical results of traveling wave solutions are demonstrated in [19] for specific f_i's described below. The function f_1 is given by $f_1(u) = (1-u)$ or $f_1(u) = k(1-u)(u-a)$ for some constants k and a; while f_2 is given by Holling type I ($f_2(u) = u$), type II, or type III functional response. However there is no theoretical proof for their numerical results.

Motivated by the above models, throughout this article, we consider $p(u)$, $g(w)$, $h(u)$, $\ell(w)$ and $q(u)$ to be C^1 functions satisfying the following assumptions:

(A1) $p'(u) < 0$ for $u > 0$, and $p(K) = 0$ for some $u = K > 0$.
(A2) $q'(u) < 0$ for $u > 0$, and $q(u_*) = 0$ for some $u_* \in (0, K)$.
(A3) $g'(w) > 0$, $\ell'(w) > 0$, $\ell''(w) \leqslant 0$ for $w \in \mathbb{R}$, $g(0) = \ell(0) = 0$ and $g(\infty) = \ell(\infty) = \infty$.
(A4) $h(0) = 0$ and $h'(u) > 0$ for $u \in \mathbb{R}$.

Note that (A1)–(A4) hold for the systems (1.2)–(1.6) provided the corresponding parameters lying in suitable regions. For example, let $p(u) = A(1 - u/K)$, $g(w) = Bw$, $h(u) = u$, $\ell(w) = w$ and $q(u) = C - Du$ for (1.2), then (A1)–(A4) hold if $C/D < K$.

For further simplification, we introduce the parameter $d = d_1/d_2$ and rescale the spatial variable x by $\tilde{x} = x/\sqrt{d_2}$. Then system (1.1) is recast as (still using x instead of $\tilde{x}$)

$$\begin{cases} u_t = du_{xx} - h(u)\big(g(w) - p(u)\big), \\ w_t = w_{xx} - \ell(w)q(u). \end{cases} \tag{1.7}$$

According to assumptions (A1)–(A4), it is easy to see that system (1.7) has three spatially uniform equilibria: $E_0 = (0, 0)$, $E_1 = (K, 0)$, and $E_2 = (u_*, w_*)$ where

$$w_* = g^{-1} \circ p(u_*) > 0.$$

Note that E_0 corresponds to the absence of both species; E_1 corresponds to the prey being at the environment carrying capacity in the absence of the predator; and E_2 corresponds to the coexistence of the two species. The purpose of this work is to establish the traveling wave solutions of system (1.7) connecting the equilibria E_1 and E_2, which is called the "wave of invasion", cf. [3].

A traveling wave solution of (1.7) is a solution of the form

$$u(x,t) = u(x+ct) = u(s) \quad \text{and} \quad w(x,t) = w(x+ct) = w(s), \tag{1.8}$$

where the constant $c > 0$ is the wave speed; $s = x + ct$ is called the moving coordinate. Substituting (1.8) into (1.7), we have the following profile equations:

$$\begin{cases} cu' = du'' - h(u)\big(g(w) - p(u)\big), \\ cw' = w'' - \ell(w)q(u), \end{cases} \tag{1.9}$$

where $'$ denotes the differentiation with respect to s. It is required that u and w of system (1.7) are nonnegative for natural ecological restriction. Then we look for the nonnegative solutions of (1.9) connecting the equilibria E_1 and E_2, i.e., satisfying the following boundary conditions:

$$u(-\infty) = K, \qquad w(-\infty) = 0, \qquad u(\infty) = u_*, \quad \text{and} \quad w(\infty) = w_*. \tag{1.10}$$

Our main results are stated as follows.

Theorem 1.1. *Assume* (A1)–(A4) *hold, and let* $d < 1$, $c^* := \sqrt{-4\ell'(0)q(K)}$.

(i) *If* $0 < c < c^*$, *then there is no nonnegative traveling wave solution of system* (1.7) *connecting the equilibria* E_1 *and* E_2.
(ii) *If* $c > c^*$, $\ell(w) = \alpha g(w)$ *and* $q(u) = \beta(h(u) - h(u_*))$ *for some* $\alpha > 0$ *and* $\beta < 0$, *then there is a nonnegative traveling wave solution of* (1.7) *connecting the equilibria* E_1 *and* E_2.
Furthermore, there exists a $\sigma^* > 0$ *such that*
   (1) *if* $|\alpha\beta| < \sigma^*$, *then the traveling wave solutions approach* E_2 *monotonically for large* s;
   (2) *if* $|\alpha\beta| > \sigma^*$, *then the traveling wave solutions have exponentially damped oscillations about* E_2 *for large* s.

Extending the ideas of [4,5], we apply the Wazewski Theorem (see Theorem 2.3) together with LaSalle's Invariance Principle (see [11]) to prove Theorem 1.1. Note that although we apply the techniques similar to those of [4,5], there are some differences. First, the model that we consider is more general, and our results contain (or extend) all the results of [4,5,12,19] and some other models, e.g., the predator–prey model with Ivlev's functional response (1.5) and some typical S.I.R. models, such as Kermack–McKendrick model (cf. [7]). Second, due to the general setting of system (1.1), the construction of Wazewski set is more complicated than those of [4,5], and it's more difficult to find an invariant orbit of system (1.9) in the Wazewski set. Third, we construct the Lyapunov function for system (1.1) more generally to prove the existence results.

According to Theorem 1.1, we know that

$$c^* = 2\sqrt{DK - C}, 2\sqrt{DH_2(K) - C}, 2\sqrt{DH_3(K) - C}$$

for systems (1.2), (1.3) and (1.4) respectively. Note that for specific form of system (1.2), Dunbar [4] pointed out that c^* is a distinguished speed dividing the positive traveling wave solutions into two types: wave of speed $c < c^*$ being one type connecting E_0 and E_2, wave of speed $c \geqslant c^*$ being of the other type connecting E_1 and E_2. In our case, the existence of positive traveling wave solutions connecting E_0 and E_2 is still open, and will be in our further study.

This paper is organized as follows. In Section 2, we recall a variant of Wazewski Theorem and construct the Wazewski set. Then we use the standard Stable Manifold Theorem to investigate the behavior of solutions for system (1.9) in the 4-dimensional phase space and prove that there is an invariant solution orbit in the Wazewski set. In Section 3, we construct the Lyapunov function for the invariant orbit. In Section 4, we prove Theorem 1.1 by using LaSalle's Invariance Principle. In Section 5, we apply our main theorem to systems (1.2)–(1.5). We further investigate the existence of traveling wave train solutions for these systems by using the Hopf bifurcation theory. The technical proofs for Proposition 2.4 and Lemma 2.18 are given in Appendices A and B respectively.

2. Construction of Wazewski set and invariant orbit

In this section, we will apply the Wazewski Theorem to prove that there is an orbit invariant in a bounded region containing E_1 and E_2. First, let's rewrite system (1.9) as a system of first order ODEs in $\mathbb{R}^4$,

$$\begin{cases} u' = v, \\ dv' = cv + h(u)\big(g(w) - p(u)\big), \\ w' = z, \\ z' = cz + \ell(w)q(u). \end{cases} \tag{2.1}$$

Then the boundary conditions (1.10) yield

$$\begin{cases} u(-\infty) = K, \quad v(-\infty) = 0, \quad w(-\infty) = 0, \quad z(-\infty) = 0, \\ u(\infty) = u_*, \quad v(\infty) = 0, \quad w(\infty) = w_*, \quad z(\infty) = 0. \end{cases} \tag{2.2}$$

It's obvious that

$$\mathcal{H} := \big\{(u, v, w, z)\colon u = v = 0\big\} \quad \text{and} \quad \mathcal{V} := \big\{(u, v, w, z)\colon w = z = 0\big\}$$

are invariant manifolds of (2.1). The eigenvalues of the linearization of (2.1) at $(K, 0, 0, 0)$ are

$$\lambda_1 = \frac{c + \sqrt{c^2 - 4dh(K)p'(K)}}{2d} > 0, \qquad \lambda_2 = \frac{c + \sqrt{c^2 + 4\ell'(0)q(K)}}{2},$$
$$\lambda_3 = \frac{c - \sqrt{c^2 + 4\ell'(0)q(K)}}{2}, \qquad \lambda_4 = \frac{c - \sqrt{c^2 - 4dh(K)p'(K)}}{2d} < 0.$$

The corresponding eigenvectors are given by

$$\begin{aligned} \mathbf{e}_1 &= (-1, -\lambda_1, 0, 0), \\ \mathbf{e}_2 &= \big(-1, -\lambda_2, -\psi(\lambda_2), -\lambda_2\psi(\lambda_2)\big), \\ \mathbf{e}_3 &= \big(-1, -\lambda_3, -\psi(\lambda_3), -\lambda_3\psi(\lambda_3)\big), \\ \mathbf{e}_4 &= (-1, -\lambda_4, 0, 0), \end{aligned} \tag{2.3}$$

where

$$\psi(\lambda) = \frac{1}{g'(0)h(K)}\big(d\lambda^2 - c\lambda + h(K)p'(K)\big). \tag{2.4}$$

Note that λ_1 and λ_4 satisfy the equation

$$d\lambda^2 - c\lambda + h(K)p'(K) = 0; \tag{2.5}$$

λ_2 and λ_3 satisfy the equation

$$\lambda^2 - c\lambda - \ell'(0)q(K) = 0.$$

Let $d < 1$. If $c^2 < -4\ell'(0)q(K)$, then λ_2 and λ_3 are complex conjugate eigenvalues and $\lambda_1 > \mathrm{Re}\lambda_2 = \mathrm{Re}\lambda_3 > 0 > \lambda_4$. Thus there is a 1-dimensional strongest unstable manifold, which is tangent to $\mathbf{e}_1$ at $(K, 0, 0, 0)$. This manifold is actually contained in the invariant manifold $\mathcal{V}$. Therefore a solution of

(2.1)–(2.2) cannot lie in the strongest unstable manifold. It follows that a solution of (2.1)–(2.2) must tend spirally to $(K,0,0,0)$. Hence $w(s)<0$ for some s. Therefore, there is no nonnegative solution of (2.1)–(2.2). The part (i) of Theorem 1.1 is then proved.

On the other hand, if $c^2>-4\ell'(0)q(K)$, then it's obvious that $\lambda_1>\lambda_2>\lambda_3>0>\lambda_4$. Note that $\psi(\lambda_2)<0$ and $\psi(\lambda_3)<0$.

To investigate the structure of the eigenvalues at $(u_*,0,w_*,0)$, we recall the Routh–Hurwitz Stability Criterion. Consider the polynomial equation

$$a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=0.$$

The Routh array for the above equation is defined by

$$\begin{pmatrix} a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\ a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\ b_1 & b_2 & b_3 & b_4 & \cdots \\ c_1 & c_2 & c_3 & c_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

where

$$b_k=-\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2k} \\ a_{n-1} & a_{n-2k-1}\end{vmatrix},\qquad c_k=-\frac{1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-2k-1} \\ b_1 & b_{k+1}\end{vmatrix}$$

and so on. For example, the Routh array for a four-degree polynomial ($n=4$) is given by

$$\begin{pmatrix} a_4 & a_2 & a_0 & 0 \\ a_3 & a_1 & 0 & 0 \\ b_1 & b_2 & 0 & 0 \\ c_1 & c_2 & 0 & 0 \\ d_1 & 0 & 0 & 0 \end{pmatrix}$$

where

$$b_1=-\frac{1}{a_3}\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1\end{vmatrix},\qquad b_2=-\frac{1}{a_3}\begin{vmatrix} a_4 & a_0 \\ a_3 & 0\end{vmatrix},$$

$$c_1=-\frac{1}{b_1}\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2\end{vmatrix},\qquad c_2=-\frac{1}{b_1}\begin{vmatrix} a_3 & 0 \\ b_1 & 0\end{vmatrix}=0,\qquad d_1=-\frac{1}{c_1}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2\end{vmatrix}.$$

With the Routh array, the Routh–Hurwitz Stability Criterion [10,13,21] tells us how many roots having positive real parts.

Proposition 2.1. *The number of sign changes in the first column of the Routh array equals to the number of roots with positive real parts.*

Now we consider the characteristic equation of the linearization of (2.1) at $(u_*,0,w_*,0)$, i.e.,

$$\lambda^4-\left(c+\frac{c}{d}\right)\lambda^3+\frac{c^2-\xi_*}{d}\lambda^2+\frac{c\xi_*}{d}\lambda+\frac{\zeta_*}{d}=0, \tag{2.6}$$

where $\xi_*=-h(u_*)p'(u_*)>0$ and $\zeta_*=-\ell(w_*)h(u_*)q'(u_*)g'(w_*)>0$. Applying Proposition 2.1, we have the following lemma.

Lemma 2.2. *Eq.* (2.6) *has two eigenvalues with positive real parts and two eigenvalues with negative real parts.*

Proof. After simple computation, we have the following Routh array for Eq. (2.6)

$$\begin{pmatrix} 1 & (c^2-\xi_*)/d & \zeta_*/d & 0 \\ -c-c/d & c\xi_*/d & 0 & 0 \\ (c^2-\xi_*)/d-\xi_*/(d+1) & \zeta_*/d & 0 & 0 \\ c\xi_*/d+c(1+d)\zeta_*/(b_1 d^2) & 0 & 0 & 0 \\ \zeta_*/d & 0 & 0 & 0 \end{pmatrix},$$

where $b_1=(c^2-\xi_*)/d-\xi_*/(d+1)$. It can be verified that the signs of first column always change twice. Hence Eq. (2.6) has two roots with positive real parts. On the other hand, if we replace λ by $i\omega$ in Eq. (2.6) then we have

$$\omega^2=-\xi_*/(1+d)<0,$$

which is a contradiction. Hence there is no pure imaginary roots. The proof is complete. □

2.1. Wazewski Theorem

We now recall a variant of Wazewski Theorem which is a formalization and extension of the shooting method in higher dimension (see Proposition 2 of [5]).

Let us consider the differential equation:

$$y'(s)=f\big(y(s)\big), \tag{2.7}$$

where $f:\mathbb{R}^n\to\mathbb{R}^n$ is a Lipschitz continuous function. Denote $y(s;y_0)$ as the unique solution of (2.7) with initial value $y(0)=y_0$. For convenience, the notation $y_0\cdot s$ stands for $y(s;y_0)$ and $y_0\cdot S$ for the set of points $y\cdot s$ with $s\in S\subset\mathbb{R}$. Now we define the following sets.

- Given $W\subseteq\mathbb{R}^n$, we define the immediate exit set W^- of W by

$$W^-=\big\{y_0\in W\colon\ \forall s>0,\ y_0\cdot[0,s)\nsubseteq W\big\}.$$

- Given $\Sigma\subseteq W$, we set $\Sigma^0=\{y_0\in\Sigma\colon\ \exists s_0>0 \text{ such that } y_0\cdot s_0\notin W\}$.
- For $y_0\in\Sigma^0$, we define the exit time $T(y_0)$ of y_0 by

$$T(y_0)=\sup\big\{s\colon\ y_0\cdot[0,s]\subset W\big\}.$$

Note that $y_0\cdot T(y_0)\in W^-$ and $T(y_0)=0$ if and only if $y_0\in W^-$. The Wazewski Theorem is stated as the following.

Theorem 2.3. *Consider Eq.* (2.7)*. Suppose that*

(i) *if $y_0\in\Sigma$ and $y_0\cdot[0,s]\subseteq c\ell(W)$, then $y_0\cdot[0,s]\subseteq W$;*
(ii) *if $y_0\in\Sigma$, $y_0\cdot s\in W$ and $y_0\cdot s\notin W^-$, then there is an open set V_s about $y_0\cdot s$ disjoint from W^-;*
(iii) *$\Sigma=\Sigma^0$, Σ is a compact set and intersects a trajectory of $y'=f(y)$ only once.*

Then the mapping $F(y_0)=y_0\cdot T(y_0)$ is a homeomorphism from Σ to its image on W^-.

A set $W\subseteq\mathbb{R}^n$ satisfying the conditions (i) and (ii) of Theorem 2.3 is called a Wazewski set.

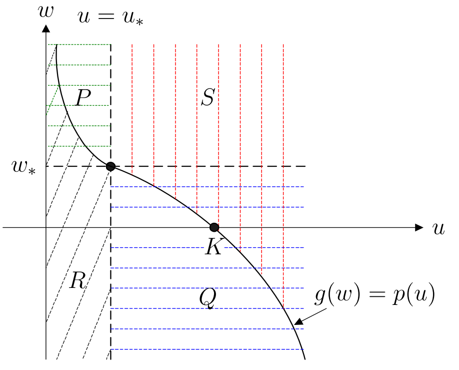

Fig. 1. The projection of P, Q, R, and S in the uw-plane.

2.2. The exit set W^-

According to Theorem 2.3, the idea for choosing a Wazewski set for (2.1) is to exclude the region where the trajectories will go to infinity. The vector field of system (2.1) leads us to exclude the region where v and v' (or z and z', resp.) are both positive or negative. Thus, we set W (see Fig. 1) by

$$W = \overline{\mathbb{R}^+} \oplus \mathbb{R}^3 \setminus (P \cup Q \cup R \cup S), \tag{2.8}$$

where

$$\begin{aligned}
P &= \big\{(u, v, w, z)\colon\ 0 < u < u_*,\ w > w_*,\ z > 0\big\},\\
Q &= \big\{(u, v, w, z)\colon\ u > u_*,\ w < w_*,\ z < 0\big\},\\
R &= \big\{(u, v, w, z)\colon\ 0 < u < u_*,\ g(w) - p(u) < 0,\ v < 0\big\},\\
S &= \big\{(u, v, w, z)\colon\ u > u_*,\ g(w) - p(u) > 0,\ v > 0\big\}.
\end{aligned}$$

Note that in the block P (or $Q \cap \{w > 0\}$, resp.) $z \to \infty$ (or $z \to -\infty$, resp.); in the block S (or R, resp.) $v \to \infty$ (or $v \to -\infty$, resp.); the set W is the complement of the four blocks P, Q, R, S in $\overline{\mathbb{R}^+} \oplus \mathbb{R}^3$. It is easy to see that

$$\partial W = (\partial P \setminus R) \cup (\partial Q \setminus S) \cup (\partial S \setminus Q) \cup (\partial R \setminus P),$$

since $P \cap R \neq \emptyset$, and $Q \cap S \neq \emptyset$. Using the phase space analysis, the structure of W^- is described in the following proposition.

Proposition 2.4. *The exit set W^- is given by*

$$W^- = \partial W \setminus \big((u_*, 0, w_*, 0) \cup (K, 0, 0, 0) \cup J_1 \cup J_2\big),$$

where

$$\begin{aligned}
J_1 &= J_{10} \cup J_{11} \cup J_{12} \cup J_{13},\\
J_{10} &= \big\{(u, v, w, z)\colon\ u \geqslant u_*,\ v > 0,\ w = z = 0\big\},\\
J_{11} &= \big\{(u, v, w, z)\colon\ u = u_*,\ v > 0,\ w < 0,\ z = 0\big\},
\end{aligned}$$

$$J_{12} = \{(u, v, w, z): u > u_*,\ v < 0,\ w < 0,\ z = 0\},$$
$$J_{13} = \{(u, v, w, z): u > u_*,\ v \geqslant 0,\ w < 0,\ z = 0,\ g(w) - p(u) < 0\},$$
$$J_2 = \{(u, v, w, z): u = v = 0,\ w \in \mathbb{R},\ z \in \mathbb{R}\}.$$

Proof. The proof is tedious and illustrated in Appendix A. □

2.3. Construction of Σ

By the standard Stable Manifold Theorem, there is a 1-dimensional strongest unstable manifold Ω_1 tangent to $\mathbf{e}_1$ at $(K, 0, 0, 0)$, and a parametric representation for this manifold in a small neighborhood of $(K, 0, 0, 0)$ given by

$$F_1(\varepsilon_1) = (K, 0, 0, 0) + \varepsilon_1 \mathbf{e}_1 + O\left(|\varepsilon_1|^2\right).$$

There is also a 2-dimensional strongly unstable manifold Ω_2 tangent to the linear subspace spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ at $(K, 0, 0, 0)$, and a parametric representation for this manifold in a small neighborhood of $(K, 0, 0, 0)$ given by

$$F_2(\varepsilon_1, \varepsilon_2) = (K, 0, 0, 0) + \varepsilon_1 \mathbf{e}_1 + \varepsilon_2 \mathbf{e}_2 + O\left(|\varepsilon_1|^2 + |\varepsilon_2|^2\right).$$

Finally, the 3-dimensional unstable manifold Ω_3 at $(K, 0, 0, 0)$ has a parametric representation in a small neighborhood of $(K, 0, 0, 0)$ given by

$$F_3(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (K, 0, 0, 0) + \varepsilon_1 \mathbf{e}_1 + \varepsilon_2 \mathbf{e}_2 + \varepsilon_3 \mathbf{e}_3 + O\left(|\varepsilon_1|^2 + |\varepsilon_2|^2 + |\varepsilon_3|^2\right).$$

Throughout the rest of this article, $\mathbf{y}(s; \mathbf{y}_0)$ stands for the solution of (2.1) with initial value $\mathbf{y}_0 = (u_0, v_0, w_0, z_0)$; $u(s; \mathbf{y}_0)$ stands for the u coordinate of $\mathbf{y}(s; \mathbf{y}_0)$, and similarly for the other three coordinates of $\mathbf{y}$.

For $\mathbf{y}_0 \in \Omega_1$, we have the following properties.

Lemma 2.5. *Let $\mathbf{y}(s; \mathbf{y}_0)$ be the solution of (2.1) with $\mathbf{y}_0 \in \Omega_1$ and $0 < u_0 < K$. Then there is a finite $s_0 > 0$ such that $u(s_0; \mathbf{y}_0) < u_*$ and $v(s; \mathbf{y}_0) < 0$ for $s \in [0, s_0]$. That is, the solution enters region R.*

Proof. Since $\mathbf{e}_1 \in \mathcal{V}$ is an invariant manifold, it follows that $\Omega_1 \subset \mathcal{V}$. Thus, to investigate the dynamics of solutions on Ω_1, we may let $w = z = 0$ in (2.1). Let us fix a $\mathbf{y}_0 \in \Omega_1$ closed to $(K, 0, 0, 0)$. The parametrization F_1 of Ω_1 implies that there exists $m > n > 0$ such that $\mathbf{y}_0$ lies between the two curves: $v = m(h(u) - h(K))$ and $v = n(h(u) - h(K))$. If m and n are large and small enough respectively, then we claim that $\mathbf{y}(s; \mathbf{y}_0)$ always lies between those two curves until $u = u_*$. We prove the claim by contradiction. Suppose that there is an $s_1 > 0$ such that $v = m(h(u) - h(K))$ and $(v - m(h(u) - h(K)))' \leqslant 0$ at $s = s_1$ (where $u_* < u(s_1) < K$), then we have

$$\begin{aligned}
0 &\geqslant v'(s_1) - mh'\big(u(s_1)\big)v(s_1) \\
&= cv(s_1) - h\big(u(s_1)\big)p\big(u(s_1)\big) - mh'\big(u(s_1)\big)v(s_1) \\
&= -h'\big(u(s_1)\big)\big(h\big(u(s_1)\big) - h(K)\big)m^2 + c\big(h\big(u(s_1)\big) - h(K)\big)m - p\big(u(s_1)\big)h\big(u(s_1)\big).
\end{aligned}$$

However, the above inequality cannot hold when m is large enough. Therefore, the trajectory $\mathbf{y}(s; \mathbf{y}_0)$ with $s > 0$ cannot lie below the curve $v = m(h(u) - h(K))$ whenever $u_* < u(s; \mathbf{y}_0) < K$.

Similarly, suppose that there is an $s_2 > 0$ such that $v = n(h(u) - h(K))$ and $(v - n(h(u) - h(K)))' \geqslant 0$ at $s = s_2$ (where $u_* < u(s_2) < K$), then we have

$$\begin{aligned}0 &\leqslant v'(s_2) - nh'\big(u(s_2)\big)v(s_2)\\ &= cv(s_2) - h\big(u(s_2)\big)p\big(u(s_2)\big) - nh'\big(u(s_2)\big)v(s_2)\\ &= -h'\big(u(s_2)\big)\big(h\big(u(s_2)\big) - h(K)\big)n^2 + c\big(h\big(u(s_2)\big) - h(K)\big)n - h\big(u(s_2)\big)p\big(u(s_2)\big).\end{aligned}$$

The above inequality also cannot hold when n is small enough. Therefore, $\mathbf{y}(s;\mathbf{y}_0)$ with $s > 0$ cannot lie above the curve $v = n(h(u) - h(K))$ whenever $u_* < u(s;\mathbf{y}_0) < K$.

Since $\mathbf{y}(s;\mathbf{y}_0)$ is bounded by the curves $v = m(h(u) - h(K))$ and $v = n(h(u) - h(K))$, it follows that $v(s;\mathbf{y}_0) < 0$ and $u(s;\mathbf{y}_0)$ decreases until $u(s;\mathbf{y}_0) < u_*$. The proof is complete. □

Since the invariant manifold Ω_1 has $w = 0$ and $z = 0$, we immediately have the following lemma.

Lemma 2.6. *Any trajectory $\mathbf{y}(s;\mathbf{y}_0)$ with $\mathbf{y}_0 \in \Omega_1$, $u_0 > K$ and $v_0 > 0$ will stay in the region $\{u > K,\ v > 0\}$ for $s > 0$.*

Proof. Let $\mathbf{y}_0 \in \Omega_1$ be near $(K, 0, 0, 0)$, then $w(s;\mathbf{y}_0) = 0$ for all s. Since $u_0 > K$ and $v_0 > 0$, we have $v'(s;\mathbf{y}_0) > 0$ for $s > 0$. Hence the assertion follows. □

Lemma 2.7. *Any trajectory $\mathbf{y}(s;\mathbf{y}_0)$ with $0 < u_0 < K$, $w_0 > 0$, and $z_0 > \frac{c}{2}w_0$ will stay in the region $\{w > 0,\ z > \frac{c}{2}w\}$ whenever $0 < u(s;\mathbf{y}_0) < K$.*

Proof. Assume the assertion of this lemma is false. Let $s_1 > 0$ be the first time that $\mathbf{y}(s;\mathbf{y}_0)$ leaves the region $\{w > 0,\ z > \frac{c}{2}w\}$ with $0 < u(s_1,\mathbf{y}_0) < K$. Then for $s \in [0, s_1)$, we have

$$w'(s) = z(s) > \frac{c}{2}w(s) \quad \text{with } w(0) > 0,$$

which implies $w(s_1) > 0$. Since

$$z(s_1) = cw(s_1)/2 \quad \text{and} \quad z'(s_1) - (c/2)w'(s_1) \leqslant 0,$$

we have

$$\begin{aligned}0 &\geqslant cz(s_1) + \ell\big(w(s_1)\big)q\big(u(s_1)\big) - \frac{c}{2}z(s_1)\\ &\geqslant \frac{c^2}{4}w(s_1) + \ell\big(w(s_1)\big)q(K) \geqslant \left(\frac{c^2}{4} + \ell'(0)q(K)\right)w(s_1).\end{aligned}$$

This contradicts the assumption $c > c^*$. The proof is complete. □

On Ω_2, let's parameterize a small circle centered at $(K, 0, 0, 0)$ by

$$G(\theta) = F_2\big(\varepsilon\cos(\theta + \psi_0), \varepsilon\sin(\theta + \psi_0)\big), \tag{2.9}$$

where $\theta \in [0, 2\pi]$ and the constant phase ψ_0 is chosen such that $G(0)$ lies in Ω_1 with $u < K$. Set

$$A := \big\{\theta \in [0, 2\pi)\colon \exists s_0 > 0 \text{ satisfying } u\big(s_0; G(\theta)\big) = u_* \text{ and } v\big(s; G(\theta)\big) < 0 \text{ on } s \in (0, s_0]\big\}.$$

By Lemma 2.5, A is nonempty since $\theta = 0 \in A$. Denote

$$\theta_1 := \sup\big\{\theta \in A\colon [0, \theta) \subset A\big\} \quad \text{and} \quad \mathbf{y}_1 := G(\theta_1).$$

Remark 2.8.

(i) ψ_0 is close to zero provided $\varepsilon \approx 0$.
(ii) According to Lemma 2.5, there exists an $s_0 > 0$ such that $u(s_0; G(0)) < u_*$ and $v(s; G(0)) < 0$ for $s \in [0, s_0]$. The continuous dependence of a solution on initial condition implies that $\theta_1 > 0$.
(iii) Since $v(0; G(\theta)) \leqslant 0$ for $\theta \in A$, we have $A \subset [0, 3\pi/4 - \psi_0)$. If $\theta \in [0, 3\pi/4 - \psi_0)$, then the components u and w of $G(\theta)$ satisfy $0 < u < K$ and $w > 0$. Thus, we have $w(0; \mathbf{y}_1) > 0$.

Lemma 2.9. *Let $\varepsilon > 0$ be small enough. If $\theta \in [0, 3\pi/4 - \psi_0)$, then the trajectory $\mathbf{y}(s; G(\theta))$ with $s \geqslant 0$ will stay in the region $\{w > 0,\ z > cw/2\}$ whenever $0 < u(s; G(\theta)) < K$.*

Proof. Let $\mathbf{y}_0 = G(\theta) \in \Omega_2$. From (2.9), the w and z coordinates of $\mathbf{y}_0$ satisfy $w > 0$ and $z \approx \lambda_2 w > cw/2$. Then the assertion follows by Lemma 2.7. □

Lemma 2.10. *Suppose $\mathbf{y}_0 = G(\theta)$ for some $\theta \in (0, \theta_1)$. Then $\mathbf{y}(s; \mathbf{y}_0)$ will leave W and enter the region R or P.*

Proof. Fix a $\theta \in (0, \theta_1)$, then there exists s_0 such that

$$u\big(s_0; G(\theta)\big) = u_* \quad \text{and} \quad v\big(s; G(\theta)\big) < 0 \quad \text{for } s \in (0, s_0].$$

If $(g(w) - p(u))_{s=s_0} < 0$, we have

$$dv'(s_0) = \big(cv + h(u)\big(g(w) - p(u)\big)\big)_{s=s_0} < 0,$$

which implies $v(s_0^+) < 0$ and $u(s_0^+) < u_*$. That is, the trajectory enters region R.

If $(g(w) - p(u))_{s=s_0} \geqslant 0$, then $w(s_0) \geqslant w_*$ by $u(s_0) = u_*$. Since $v(s_0) < 0$, we have $u(s_0^+) < u_*$. By Lemma 2.9, we have $w(s_0) > 0$ and $z(s_0) > \frac{c}{2} w(s_0) > 0$. Thus $w(s_0^+) > w_*$. That is, the trajectory enters region P. The proof is complete. □

The next lemma shows that there is a “last” trajectory on Ω_2 such that $u(s)$ decreases to the value $u = u_*$.

Lemma 2.11. *There exists an $s_0 > 0$ such that $u(s_0; \mathbf{y}_1) = u_*$ and $v(s_0; \mathbf{y}_1) = 0$, see Fig. 2. Moreover, we have*

$$g\big(w(s_0; \mathbf{y}_1)\big) - p\big(u(s_0; \mathbf{y}_1)\big) > 0 \quad \textit{and} \quad w(s_0; \mathbf{y}_1) > w_*.$$

Proof. Recall that $u_* < u(0; \mathbf{y}_1) < K$, $v(0; \mathbf{y}_1) \leqslant 0$ and $w(0; \mathbf{y}_1) > 0$. The proof consists of several steps as follows.

(1) We claim that $u(s; \mathbf{y}_1) \leqslant u_*$ or $v(s; \mathbf{y}_1) \geqslant 0$ for some $s > 0$.

Suppose the claim is false, i.e., $u(s; \mathbf{y}_1) > u_*$ and $v(s; \mathbf{y}_1) < 0$ for all $s > 0$. Then $u(s; \mathbf{y}_1)$ decreases monotonically to $u(\infty; \mathbf{y}_1) \geqslant u_*$ and $v(\infty; \mathbf{y}_1) = 0$. By Lemma 2.9, we have

$$w'(s; \mathbf{y}_1) = z(s; \mathbf{y}_1) > c/2 w(s; \mathbf{y}_1),$$

which implies $w(\infty; \mathbf{y}_1) = \infty$. Then it follows that

$$dv'(s; \mathbf{y}_1) = cv(s; \mathbf{y}_1) + h\big(u(s; \mathbf{y}_1)\big)\big(g\big(w(s; \mathbf{y}_1)\big) - p\big(u(s; \mathbf{y}_1)\big)\big) \to \infty,$$

as $s \to \infty$. However, this fact contradicts $v(\infty; \mathbf{y}_1) = 0$. Hence the claim follows.

(2) Let s_0 be the first time that $u(s; \mathbf{y}_1) = u_*$ or $v(s; \mathbf{y}_1) = 0$. We claim that $v(s_0; \mathbf{y}_1) = 0$, $v(s; \mathbf{y}_1) < 0$ for $s \in (0, s_0)$, and $u(s_0; \mathbf{y}_1) \geqslant u_*$.

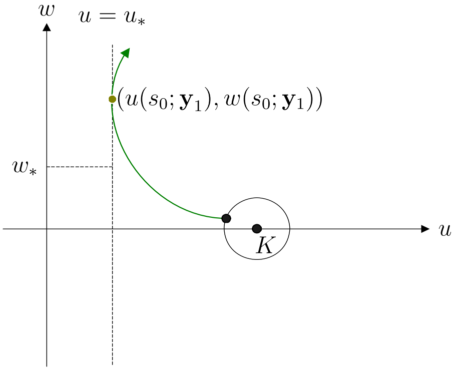

Fig. 2. Projection of the trajectory $\mathbf{y}(s;\mathbf{y}_1)$ in the uw-plane.

Suppose the claim is false, i.e.,

$$v(s;\mathbf{y}_1)<0 \quad \text{for } s\in(0,s_0] \quad \text{and} \quad u(s_0;\mathbf{y}_1)=u_*.$$

Then, by the Implicit Function Theorem, there exists a function $s_0(\theta)$ with $\theta\approx\theta_1$ such that

$$u\big(s_0(\theta);G(\theta)\big)=u_*.$$

By the continuous dependence of the solution on θ, we have for $\theta\approx\theta_1$

$$v\big(s;G(\theta)\big)<0 \quad \text{on } s\in\big(0,s_0(\theta_1)+\delta\big].$$

Also, by continuity of the function $s_0(\theta)$, we have $s_0(\theta)\in(0,s_0(\theta_1)+\delta]$ for $\theta\approx\theta_1$. Therefore, there are $\theta\gtrsim\theta_1$ satisfying

$$v\big(s_0;G(\theta)\big)<0 \quad \text{on } \big(0,s_0(\theta)\big] \quad \text{and} \quad u\big(s_0(\theta);G(\theta)\big)=u_*.$$

This fact contradicts the definition of θ_1. Thus the claim follows.

(3) We claim that $g(w(s_0;\mathbf{y}_1))-p(u(s_0;\mathbf{y}_1))>0$ and $v'(s_0;\mathbf{y}_1)>0$.

Indeed, since $v(s_0;\mathbf{y}_1)=0$ and $v(s;\mathbf{y}_1)<0$ on $s\in(0,s_0)$, we have $v'(s_0;\mathbf{y}_1)\geqslant 0$ and

$$dv'(s_0;\mathbf{y}_1)=h\big(u(s_0;\mathbf{y}_1)\big)\big(g\big(w(s_0;\mathbf{y}_1)\big)-p\big(u(s_0;\mathbf{y}_1)\big)\big)\geqslant 0.$$

Thus $g(w(s_0;\mathbf{y}_1))-p(u(s_0;\mathbf{y}_1))\geqslant 0$. Suppose $g(w(s_0;\mathbf{y}_1))-p(u(s_0;\mathbf{y}_1))=0$, then

$$dv''(s_0;\mathbf{y}_1)\geqslant h\big(u(s_0;\mathbf{y}_1)\big)g'\big(w(s_0;\mathbf{y}_1)\big)z(s_0;\mathbf{y}_1),$$

which leads to

$$dv''(s_0;\mathbf{y}_1)>h\big(u(s_0;\mathbf{y}_1)\big)g'\big(w(s_0;\mathbf{y}_1)\big)cw(s_0;\mathbf{y}_1)/2>0$$

by Lemma 2.9. This implies that $v(s;\mathbf{y}_1)\geqslant 0$ for $s\approx s_0$, which contradicts the definition of s_0. Therefore $g(w(s_0;\mathbf{y}_1))-p(u(s_0;\mathbf{y}_1))>0$ and $v'(s_0;\mathbf{y}_1)>0$.

(4) We claim that $u(s_0;\mathbf{y}_1)=u_*$.

Since $v(s_0; \mathbf{y}_1) = 0$ and $v'(s_0; \mathbf{y}_1) > 0$, by the Implicit Function Theorem, there exists a function $s_0(\theta)$ for $\theta \approx \theta_1$ such that $v(s_0(\theta); G(\theta)) = 0$. Suppose $u(s_0; \mathbf{y}_1) > u_*$. Then, the continuous dependence of the solution on θ implies

$$v\big(s_0(\theta); G(\theta)\big) = 0, \qquad v'\big(s_0(\theta); G(\theta)\big) > 0; \qquad v\big(s; G(\theta)\big) < 0 \quad \text{on } s \in \big(0, s_0(\theta)\big);$$

and $u(s_0(\theta), G(\theta)) > u_*$ for $\theta \approx \theta_1$. Thus $\theta \notin A$ for $\theta \approx \theta_1$, a contradiction. Hence $u(s_0; \mathbf{y}_1) = u_*$. It follows from $g(w(s_0; \mathbf{y}_1)) - p(u(s_0; \mathbf{y}_1)) > 0$ that $w(s_0; \mathbf{y}_1) > w_*$. The proof is complete. □

Lemma 2.12. *There exists a $\theta_2 \in [\theta_1, 3\pi/4 - \psi_0)$ such that the v coordinate of $\mathbf{y}_2 := G(\theta_2)$ is equal to zero.*

Proof. By (2.9), the v coordinate of $G(\theta)$ is given by

$$v = -\varepsilon\sqrt{\lambda_1^2 + \lambda_2^2}\,\sin(\theta + \psi_0 + \psi_1) + O\big(\varepsilon^2\big),$$

where $\sin\psi_1 = \lambda_1/\sqrt{\lambda_1^2 + \lambda_2^2}$ and $\psi_1 \in (\pi/4, \pi/2)$. Obviously $v = 0$ at

$$\theta_2 := \pi - \psi_0 - \psi_1 + O(\varepsilon) \in (0, 3\pi/4 - \psi_0).$$

Recall that the v coordinate of $G(\theta_1)$ is non-positive. It follows that $\theta_2 \geqslant \theta_1$. The proof is complete. □

On Ω_3, we consider a small sphere centered at $(K, 0, 0, 0)$ with radius ε, which is parameterized by

$$U(\theta, \phi) = F_3\big(\varepsilon\cos(\theta + \psi_0)\sin\phi, \varepsilon\sin(\theta + \psi_0)\sin\phi, \varepsilon\cos\phi\big), \tag{2.10}$$

where $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$. The constant phase ψ_0 is the one in (2.9). This sphere contains the arc $G(\theta) = U(\theta, \pi/2)$. According to Lemma 2.12 we know that the sphere intersects the hyperplane $v = 0$ at least one point $U(\theta_2, \pi/2)$. The next lemma shows that the intersection is a smooth closed curve.

Lemma 2.13. *The intersection of the sphere defined by* (2.10) *and the hyperplane $v = 0$ is a smooth closed curve.*

Proof. The equation for the intersection of the sphere with $v = 0$ is given by

$$M(\theta, \phi) := \lambda_1\cos(\theta + \psi_0)\sin\phi + \lambda_2\sin(\theta + \psi_0)\sin\phi + \lambda_3\cos\phi + O(\varepsilon) = 0.$$

Since the v coordinate of $G(\theta_2)$ is zero, we have $M(\theta_2, \pi/2) = 0$. Furthermore,

$$\left.\frac{\partial M}{\partial\phi}\right|_{(\theta_2, \pi/2)} = -\lambda_3 + O(\varepsilon) \neq 0,$$

when ε is small enough. By the Implicit Function Theorem, there exists a C^1 function $\phi(\theta)$, θ near $\pi/2$ solving $M(\theta, \phi) = 0$. The points solving $M(\theta, \phi) = 0$ in a neighborhood of the curve can be defined by

$$\cot\phi = -\frac{1}{\lambda_3}\big(\lambda_1\cos(\theta + \psi_0) + \lambda_2\sin(\theta + \psi_0)\big).$$

Moreover, the points where $\frac{\partial M}{\partial \phi} = 0$ in a neighborhood of the curve are defined by

$$\tan\phi = \frac{1}{\lambda_3}\big(\lambda_1 \cos(\theta + \psi_0) + \lambda_2 \sin(\theta + \psi_0)\big).$$

Since the two curves are disjoint, by the Implicit Function Theorem, the function $\phi(\theta)$ can be extend to $\theta \in [0, 2\pi]$. The proof is complete. □

Lemma 2.14. *The intersection of the sphere defined by* (2.10) *and the hyperplane* $z = 0$ *is a smooth closed curve.*

Proof. The equation for the intersection of the sphere with $z = 0$ is given by

$$N(\theta, \phi) := \lambda_2 \psi(\lambda_2) \sin(\theta + \psi_0) \sin\phi + \lambda_3 \psi(\lambda_3) \cos\phi + O(\varepsilon) = 0. \tag{2.11}$$

Since the z coordinate of $G(0)$ is zero, we have $N(0, \pi/2) = 0$. Furthermore,

$$\left.\frac{\partial N}{\partial \phi}\right|_{(0,\pi/2)} = -\lambda_3 \psi(\lambda_3) + O(\varepsilon) \neq 0,$$

when ε is small enough. By the Implicit Function Theorem, there exists a C^1 function $\phi(\theta)$, with θ near 0 solving $N(\theta, \phi) = 0$. The points solving $N(\theta, \phi) = 0$ in a neighborhood of the curve can be defined by

$$\cot\phi = -\frac{1}{\lambda_3 \psi(\lambda_3)} \lambda_2 \psi(\lambda_2) \sin(\theta + \psi_0).$$

Furthermore, the points where $\frac{\partial N}{\partial \phi} = 0$ in a neighborhood of the curve are defined by

$$\tan\phi = \frac{1}{\lambda_3 \psi(\lambda_3)} \lambda_2 \psi(\lambda_2) \sin(\theta + \psi_0).$$

Since the two curves are disjoint, by the Implicit Function Theorem, we can extend the domain of $\phi(\theta)$ to $\theta \in [0, 2\pi]$. The proof is complete. □

Lemma 2.15. *There exists a point* $\mathbf{y}_3$ *lying on the sphere defined by* (2.10) *such that the* v *and* z *coordinates of* $\mathbf{y}_3$ *are both zero.*

Proof. Let $\theta(\phi)$ be the function solving (2.11), which defines the smooth curve of the intersection of the sphere with $\{z = 0\}$. It follows that

$$\theta(\pi/2) + \psi_0 = 0 + O(\varepsilon) \quad \text{or} \quad \pi + O(\varepsilon).$$

Substituting $\theta(\phi)$ into $N(\theta(\pi/2), \pi/2) = 0$ gives the v coordinate of the smooth curve of the intersection of the sphere with $\{z = 0\}$ at $\phi = \pi/2$. Indeed, we have $N(0 + O(\varepsilon), \pi/2) > 0$ and $N(\pi + O(\varepsilon), \pi/2) < 0$. Therefore the v coordinate takes both positive and negative values on the close curve of the intersection of the sphere with $\{z = 0\}$. The proof is complete. □

Now we are ready to give the definition of Σ. First, the range of ϕ is restricted to make $\cos\phi \geqslant 0$ so that the hemisphere of the sphere defined by (2.10) is under our consideration. Then we will define Σ as a subset of the hemisphere. The following notations and Fig. 3 can help us to understand the set Σ.

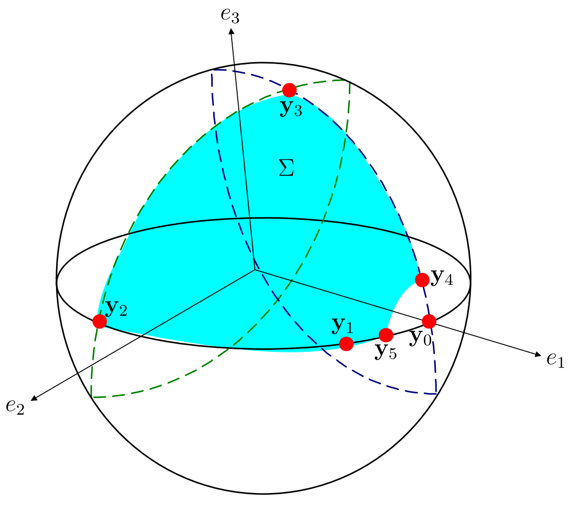

Fig. 3. The topological quadrangle Σ.

Notation 2.16.

(1) Let $\mathbf{y}_0 := G(\theta_0)$ be the intersection of the sphere with Ω_1 in the region $0 < u < K$.
(2) Denote by $\widehat{\mathbf{y}_0\mathbf{y}_i}$, $i = 1, 2$, the portion of the circle defined in (2.9) with $\theta \in (0, \theta_i)$.
(3) Denote by $\widehat{\mathbf{y}_2\mathbf{y}_3}$ the portion of the intersection of the hemisphere with $\{v = 0\}$ lying between (not including) $\mathbf{y}_2$ and $\mathbf{y}_3$.
(4) Denote by $\widehat{\mathbf{y}_3\mathbf{y}_0}$ the portion of the intersection of hemisphere with $\{z = 0\}$ lying between (not including) $\mathbf{y}_3$ and $\mathbf{y}_0$.
(5) Let B be a small ball around $\mathbf{y}_0$ in the space spanned by $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. Let $\mathbf{y}_4$ and $\mathbf{y}_5$ be the intersection points of B with $\widehat{\mathbf{y}_3\mathbf{y}_0}$ and $\widehat{\mathbf{y}_2\mathbf{y}_2}$ respectively. Denote by $\widehat{\mathbf{y}_4\mathbf{y}_5}$ the portion of intersection of the hemisphere with B (not including $\mathbf{y}_4$, $\mathbf{y}_5$).

Now we define the set Σ as the closed topological quadrangle in the hemisphere, whose sides consist of the closure of the arcs $\widehat{\mathbf{y}_i\mathbf{y}_{i+1}}$, $i = 1, 2, 3, 4$, and $\widehat{\mathbf{y}_5\mathbf{y}_1}$.

2.4. Existence of an invariant orbit

According to previous construction, the set Σ is obviously simply connected. Under the hypothesis $\Sigma^0 = \Sigma$, it can be shown that the image of Σ under the mapping F defined in Theorem 2.3 is not simply connected.

Lemma 2.17. *If $\Sigma^0 = \Sigma$, then the set $F(\Sigma)$ is not simply connected, where $F(\cdot)$ is defined in Theorem* 2.3.

Proof. The result can be proved by following the ideas of [5] with slight modifications. Since the detail is tedious, we illustrate it in Appendix B. □

Next, we prove the existence of an invariant orbit in the set W by using the Wazewski Theorem.

Lemma 2.18. *There exists a $\mathbf{y} \in \Sigma$ such that $\mathbf{y} \cdot s \in W$ for $s \geqslant 0$.*

Proof. Suppose that no such $\mathbf{y}$ exists in Σ, i.e., $\Sigma^0 = \Sigma$; we will show that it contradicts the result of Theorem 2.3.

Obviously the condition (i) of Theorem 2.3 holds since W is closed. The condition (iii) of Theorem 2.3 also holds by the construction of Σ. Next, we show that the condition (ii) in Theorem 2.3 holds.

First, we claim that if $\mathbf{y} \in \Sigma$ then

$$\mathbf{y} \cdot s \notin J_1 \cup J_2 \cup \{(u_*, 0, w_*, 0), (K, 0, 0, 0)\}, \quad \text{for all } s \geqslant 0. \tag{2.12}$$

Obviously, $\mathbf{y} \cdot s \neq (u_*, 0, w_*, 0), (K, 0, 0, 0)$ since they are equilibria. Furthermore, $\mathbf{y} \cdot s \notin J_2$ since $J_2 = \mathcal{H}$ is an invariant manifold, and $\mathbf{y}$ cannot be in J_2 ($\mathbf{y}$ is close to $(K, 0, 0, 0)$). Also $\mathbf{y} \cdot s \notin J_{10}$ since J_{10} is a subset of the invariant manifold $\mathcal{V}$ while $\mathbf{y}$ cannot be in $\mathcal{V}$.

If $\mathbf{y} \cdot s_1 \in J_{12} \cup J_{13}$ for some $s_1 < T(\mathbf{y})$, then

$$z'(s_1) = \ell\big(w(s_1)\big)q\big(u(s_1)\big) > 0,$$

which implies $z(s_1^-) < 0$ and $\mathbf{y} \cdot s_1^- \in Q$. This fact contradicts $s_1 < T(\mathbf{y})$. Thus $\mathbf{y} \cdot s \notin J_{12} \cup J_{13}$.

If $\mathbf{y} \cdot s_1 \in J_{11}$ for some $s_1 < T(\mathbf{y})$, then

$$u(s_1) = u_*, \qquad v(s_1) > 0, \qquad w(s_1) < 0, \qquad z(s_1) = 0,$$

which implies $z'(s_1) = 0$ and $z''(s_1) > 0$. Therefore,

$$u\big(s_1^-\big) < u_*, \qquad v\big(s_1^-\big) > 0, \qquad w\big(s_1^-\big) < 0 \quad \text{and} \quad z\big(s_1^-\big) > 0.$$

Let $s_2 := \inf\{s\colon\ u(t) < u_*,\ w(t) < 0, \text{ on } (s, s_1)\}$. Then $s_2 > 0$ since $u(0) > u_*$. It follows that

$$dv' \leqslant cv \quad \text{and} \quad z' \leqslant cz, \quad \text{for } s \in (s_2, s_1). \tag{2.13}$$

At time s_2, there are two possibilities:

$$\text{(a)} \quad u(s_2) = u_*, \quad v(s_2) \leqslant 0 \quad \text{or} \quad \text{(b)} \quad w(s_2) = 0, \quad z(s_2) \leqslant 0.$$

By (2.13), case (a) yields $v(s) \leqslant 0$ on (s_2, s_1) and which contradicts $v(s_1^-) > 0$. Similarly, by (2.13), case (b) yields $z(s) \leqslant 0$ on (s_2, s_1) and which contradicts $z(s_1^-) > 0$. Thus we conclude that $\mathbf{y} \cdot s \notin J_{11}$. Hence the assertion of the claim (2.12) follows.

Now we verify the condition (ii) in Theorem 2.3. Let $\mathbf{y} \in \Sigma$, $\mathbf{y} \cdot s \in W$ and $\mathbf{y} \cdot s \notin W^-$. According to (2.12), $\mathbf{y} \cdot s$ must be in the interior of W. Hence there is an open set about $\mathbf{y} \cdot s$ disjoint from W^-.

Since all the conditions in Theorem 2.3 hold, it follows that $F(\Sigma)$ is homeomorphic to Σ. Then we have a contradiction since $F(\Sigma)$ is not simply connected. The proof is complete. □

3. Lyapunov function for the invariant orbit

Let $\bar{\mathbf{y}}(s)$ be the orbit which is positively invariant in W. Our purpose is to construct a Lyapunov function for $\bar{\mathbf{y}}(s)$. Some prior estimations for $\bar{\mathbf{y}}(s)$ are needed for the construction of Lyapunov function.

Lemma 3.1. *The coordinate functions $\bar{u}(s)$ and $\bar{w}(s)$ of $\bar{\mathbf{y}}(s)$ are positive for all s.*

Proof. Since $\bar{\mathbf{y}}(0) \in \Omega_3 \setminus \Omega_2$ (see Claim II of Appendix B) and $0 < \lambda_3 < \lambda_2 < \lambda_1$, $\bar{\mathbf{y}}(s)$ will approach $(K, 0, 0, 0)$ along the direction of $\mathbf{e}_3$ as $s \to -\infty$. The construction of Σ implies that the components of $\bar{\mathbf{y}}(s)$ satisfy $0 < \bar{u}(s) < K$ and $\bar{w}(s) > 0$ for $s \to -\infty$.

Suppose $s_1 := \sup\{s\colon\ \bar{u}(s) > 0\} < \infty$. Then $\bar{u}(s_1) = 0$ and $\bar{v}(s_1) \leqslant 0$. Since $\bar{\mathbf{y}}(s) \notin \mathcal{H}$ and $\bar{v}(s_1) < 0$, then $\bar{u}(s_1^+) < 0$ and $\bar{\mathbf{y}}(s_1^+) \notin W$, which leads to a contradiction. Thus s_1 must be ∞, i.e., $\bar{u}(s) > 0$ for all s.

Suppose $s_2 := \sup\{s\colon\ \bar{w}(s) > 0\} < \infty$. Then $\bar{w}(s_2) = 0$ and $\bar{z}(s_2) \leqslant 0$. Since $\bar{\mathbf{y}}(s) \notin \mathcal{V}$, Q and R, we have $\bar{z}(s_2) < 0$, $\bar{u}(s_2) \in (0, u_*]$ and $\bar{v}(s_2) \geqslant 0$ respectively. If $\bar{v}(s_2) = 0$ then

$$d\bar{v}'(s_2) < -h\big(\bar{u}(s_2)\big)p\big(\bar{u}(s_2)\big) < 0,$$

which implies $\bar{v}(s_2^+) < 0$, $\bar{u}(s_2^+) < u_*$ and $\bar{\mathbf{y}}(s_2^+) \in R$, a contradiction. Hence, $\bar{v}(s_2) > 0$ and $\bar{u}(s_2^+) > u_*$, which leads to $\bar{\mathbf{y}}(s_2^+) \in Q$, also a contradiction. Thus, s_2 must be ∞, i.e., $\bar{w}(s) > 0$ for all s. The proof is complete. □

Lemma 3.2. *The coordinate functions $\bar{u}(s)$ and $\bar{w}(s)$ of $\bar{\mathbf{y}}(s)$ are bounded above. In fact, $\bar{u}(s) < K$ and $\bar{w}(s) < M$ for some constant $M > 0$.*

Proof. The proof consists of the following six steps.

(1) We prove that $\bar{u}(s) < K$ for all $s > 0$.

Suppose $s_1 := \sup\{s > 0\colon\ \bar{u}(s) < K\} < \infty$, then $\bar{v}(s_1) \geqslant 0$. Since $\bar{w}(s) > 0$ for all s, we have

$$\bar{u}(s_1) = K > u_*, \qquad g\big(\bar{w}(s_1)\big) - p\big(\bar{u}(s_1)\big) > 0 \quad \text{and} \quad \bar{v}(s_1) \geqslant 0.$$

Then $\bar{v}'(s_1) > 0$, $\bar{v}(s_1^+) > 0$, $\bar{u}(s_1^+) = K > u_*$, $g(\bar{w}(s_1^+)) - p(\bar{u}(s_1^+)) > 0$ and $\bar{\mathbf{y}}(s_1^+) \in S$, a contradiction. Thus, s_1 must be ∞, i.e., $\bar{u}(s) < K$ for all $s > 0$.

(2) Let's study the behavior of $\bar{\mathbf{y}}(s)$ projected in uw-plane, cf. Fig. 1.

First, we have the following observations:

- since $\bar{\mathbf{y}}(s)$ does not enter region P, we have $\bar{z}(s) < 0$ whenever $0 < \bar{u}(s) < u_*$ and $\bar{w}(s) > w_*$;
- since $\bar{\mathbf{y}}(s)$ does not enter region R, we have $\bar{v}(s) > 0$ whenever $0 < \bar{u}(s) < u_*$ and $0 < \bar{w}(s) < w_*$;
- since $\bar{\mathbf{y}}(s)$ does not enter region Q, we have $\bar{z}(s) > 0$ whenever $\bar{u}(s) > u_*$ and $0 < \bar{w}(s) < w_*$;
- since $\bar{\mathbf{y}}(s)$ does not enter region S, we have $\bar{v}(s) < 0$ whenever $\bar{u}(s) > u_*$ and $\bar{w}(s) > w_*$.

Therefore, we know that

- $\bar{w}(s)$ is decreasing in the region $\{0 < u < u_*,\ w > w_*\}$ and increasing in the region $\{u > u_*,\ 0 < w < w_*\}$;
- $\bar{u}(s)$ is increasing in the region $\{0 < u < u_*,\ 0 < w < w_*\}$ and decreasing in the region $\{u > u_*,\ w > w_*\}$.

Thus, to prove that $\bar{w}(s)$ is bounded above, it suffices to prove that $\bar{w}(s)$ is bounded above in the region $\{u > u_*\}$.

(3) Now we prove that it is impossible that $\bar{u}(s) > u_*$ for all sufficiently large s and $\lim_{s\to\infty} \bar{w}(s) = \infty$.

Indeed, since $\bar{\mathbf{y}}(s)$ does not enter region S, we have $\bar{v}(s) \leqslant 0$ and which implies that $\bar{u}(s)$ is monotonically decreasing to $\bar{u}(\infty) \geqslant u_*$ and $\lim_{s\to\infty} \bar{v}(s) = 0$. Then it follows that

$$d\bar{v}(s)' \geqslant c\bar{v}(s) + h\big(\bar{u}(s)\big)\big(g\big(\bar{w}(s)\big) - p\big(\bar{u}(s)\big)\big) \to \infty,$$

which contradicts $\lim_{s\to\infty} \bar{v}(s) = 0$.

(4) We claim that if $\bar{\mathbf{y}}(s)$ enters the region

$$\Gamma := \big\{u > u_*,\ w > 0,\ g(w) - p(u) < 0\big\},$$

then $\bar{z}(s) \leqslant cw_*$ whenever $\bar{\mathbf{y}}(s)$ remains in region Γ.

If $\bar{\mathbf{y}}(s) \in \Gamma$ then $\bar{z}'(s) \leqslant c\bar{z}(s)$ and $\bar{z}(s) \geqslant 0$ (since $\bar{\mathbf{y}}(s) \notin Q$). This yields $\bar{w}'(s) = \bar{z}(s) \geqslant z'(s)/c$ and $d\bar{z}/d\bar{w} \leqslant c$. Integrating $d\bar{z}/d\bar{w} \leqslant c$ with respect to $\bar{w}$ from 0 to w_* gives $\bar{z}(s) \leqslant cw_*$. Hence the claim follows.

(5) We claim that if $\bar{\mathbf{y}}(s)$ enters into the region Γ and crosses the boundary $g(w) - p(u) = 0$ into the region $\{u > u_*,\ w > 0,\ g(w) - p(u) > 0\}$, then the component $\bar{v}$ is uniformly bounded below as it reaches the boundary $g(w) - p(u) = 0$.

Suppose s_1 is the first time that $g(\bar{w}(s_1)) - p(\bar{u}(s_1)) = 0$, then $(g(\bar{w}(s_1)) - p(\bar{u}(s_1)))' \geqslant 0$, which leads to

$$g'\big(\bar{w}(s_1)\big)z(s_1) - p'\big(\bar{u}(s_1)\big)\bar{v}(s_1) \geqslant 0$$

or

$$\bar{v}(s_1) \geqslant \frac{\bar{z}(s_1)g'(\bar{w}(s_1))}{p'(\bar{u}(s_1))} \geqslant \frac{cw_* g'(\bar{w}(s_1))}{p'(\bar{u}(s_1))}.$$

The above last term is uniformly bounded below since $g(w)$ and $p(u)$ are C^1 functions and the closure of Γ is a compact set. Then the claim follows.

(6) We prove that $M := \sup\{\bar{w}(s):\ u_* < \bar{u}(s) < K\} < \infty$.

Take $\gamma_n \in \mathbb{R}$ such that $\lim_{n\to\infty} \gamma_n = \infty$. If $M = \infty$, then there exists a sequence s_n such that $g(\bar{w}(s_n)) - p(\bar{u}(s_n)) = \gamma_n$. Let

$$\tau_n := \max\big\{t < s_n:\ g\big(\bar{w}(t)\big) - p\big(\bar{u}(t)\big) = 0\big\}.$$

By step (2), $v(\tau_n) \leqslant 0$ is uniformly bounded below for all n. Steps (2) and (3) also imply that $\bar{\mathbf{y}}(s)$ must enter the regions $\{0 < u < u_*,\ w > w_*\}$ and $\{u > u_*,\ 0 < w < w_*\}$ infinitely many times. It follows that

$$t_n := \min\big\{t > s_n:\ \bar{z}(t) = 0\big\} < \infty \quad \text{and} \quad \bar{u}(t_n) \geqslant u_*.$$

Without loss of generality, we may assume $\lim(\bar{w}(t_n) - \bar{w}(s_n)) \neq 0$ (by suitable selecting γ_n). Since $\bar{w}'(s) = \bar{z}(t_n^+) = 0 \neq \infty$, we have $\lim_{n\to\infty}(t_n - s_n) \neq 0$. Integrating

$$dv'(s) = cv + h\big(u(s)\big)\big(g\big(w(s)\big) - p\big(u(s)\big)\big)$$

from $s = \tau_n$ to $s = t_n$ gives

$$\begin{aligned} e^{-\frac{c}{d}t_n}v(t_n) &= e^{-\frac{c}{d}\tau_n}v(\tau_n) + \int_{\tau_n}^{t_n} e^{-\frac{c}{d}s}h\big(u(s)\big)\big(g\big(w(s)\big) - p\big(u(s)\big)\big)\,ds \\ &\geqslant v(\tau_n) + \int_{s_n}^{t_n} e^{-\frac{c}{d}s}h\big(u(s)\big)\big(g\big(w(s)\big) - p\big(u(s)\big)\big)\,ds \\ &\geqslant v(\tau_n) + \frac{c}{d}\big(e^{-\frac{c}{d}s_n} - e^{-\frac{c}{d}t_n}\big)h(u_*)\gamma_n. \end{aligned}$$

The last term of the above inequalities is positive for sufficiently large n since $\lim_{n\to\infty}\gamma_n = \infty$ and $\lim_{n\to\infty}(t_n - s_n) \neq 0$. It follows that $v(t_n) > 0$, and which contradicts that $\bar{\mathbf{y}}(s)$ does not enter region S, see Fig. 4. Thus $M < \infty$ and the proof is complete. □

Next, we show that the coordinate functions $\bar{v}(s)$ and $\bar{w}(s)$ of $\bar{\mathbf{y}}(s)$ are also bounded.

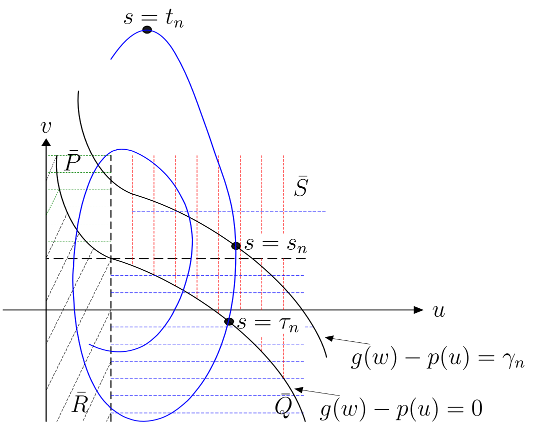

Fig. 4. Phase plane for the proof of step (6) of Lemma 3.2.

Lemma 3.3. *There exist positive constants K_i, $i = 1, 2, 3, 4$, such that the coordinate functions of $\bar{\mathbf{y}}(s)$ satisfying*

$$-K_1 h\big(\bar{u}(s)\big) < \bar{v}(s) < K_2 \bar{u}(s) \quad \text{and} \quad -K_3 \ell\big(\bar{w}(s)\big) < \bar{z}(s) < K_4 \bar{w}(s) \tag{3.1}$$

for all $s \geqslant 0$.

Proof. Since $\bar{\mathbf{y}}(s)$ approaches $(K, 0, 0, 0)$ along the direction of $\mathbf{e}_3$ as $s \to -\infty$, the construction of Σ implies that the components of $\bar{\mathbf{y}}(s)$ satisfy

$$0 < \bar{u}(s) < K, \qquad \bar{v}(s) < 0, \qquad \bar{w}(s) > 0 \quad \text{and} \quad \bar{z}(s) > 0, \quad \text{for } s \to -\infty.$$

Therefore, we may assume $0 < \bar{u}(0) < K$, $\bar{v}(0) < 0$, $\bar{w}(0) > 0$ and $\bar{z}(0) > 0$. Then (3.1) holds for $s = 0$ provided each K_i is sufficiently large. In the following four steps, we prove that (3.1) holds for $s > 0$.

(1) We claim that there is a $K_1 > 0$ such that $-K_1 h(\bar{u}(s)) < \bar{v}(s)$ for $s > 0$.

Suppose the claim is false, then for any $K_1 > 0$ there is an $s_1 > 0$ such that

$$\bar{v}(s_1) = -K_1 h\big(\bar{u}(s_1)\big) \quad \text{and} \quad \bar{v}'(s_1) \leqslant -K_1 h'\big(\bar{u}(s_1)\big)\bar{u}'(s_1).$$

If $\bar{v}(s) < -K_1 h(\bar{u}(s))$ for $s > s_1$ and K_1 is large enough, then the boundedness of $\bar{u}(s)$ and $\bar{w}(s)$ implies that

$$d\bar{v}' = c\bar{v} + h(\bar{u})\big(g(\bar{w}) - p(\bar{u})\big) \leqslant h(\bar{u})\big(-cK_1 + g(\bar{w}) - p(\bar{u})\big) < 0$$

for $s > s_1$. This yields

$$\bar{u}'(s) = \bar{v}(s) < \bar{v}(s_1) = -K_1 h\big(\bar{u}(s_1)\big) < 0$$

for $s > s_1$, which contradicts the positivity of $\bar{u}(s)$. Therefore, there exists an $s_2 > s_1$ such that

$$\bar{v}(s_2) = -K_1 h\big(\bar{u}(s_2)\big) \quad \text{and} \quad \bar{v}'(s_2) \geqslant -K_1 h\big(\bar{u}(s_2)\big)u'(s_2).$$

However, this fact also leads to the following contradiction:

$$
\begin{aligned}
0 &\leqslant c\bar{v}(s_2) + h\big(\bar{u}(s_2)\big)\big(g\big(\bar{w}(s_2)\big) - p\big(\bar{u}(s_2)\big)\big) + dK_1 h'\big(\bar{u}(s_2)\big)\bar{v}(s_2) \\
&\leqslant h\big(\bar{u}(s_2)\big)\big(-cK_1 + g\big(\bar{w}(s_2)\big) - p\big(\bar{u}(s_2)\big) - dK_1^2 h'\big(\bar{u}(s_2)\big)\big) < 0,
\end{aligned}
$$

provided K_1 is large enough. Hence the claim follows.

(2) We claim that there is a $K_2 > 0$ such that $\bar{v}(s) < K_2\bar{u}(s)$ for $s > 0$.

Suppose the claim is false, then for any $K_2 > 0$ there is an $s_1 > 0$ such that

$$
\bar{v}(s_1) = K_2\bar{u}(s_1) \quad \text{and} \quad \bar{v}'(s_1) \geqslant K_2\bar{u}'(s_1).
$$

Then, at $s = s_1$, we have the following contradiction:

$$
0 \leqslant c\bar{v} + h(\bar{u})\big[g(\bar{w}) - p(\bar{u})\big] - dK_2\bar{v} \leqslant \big(cK_2 - dK_2^2\big)\bar{u} + h(\bar{u})\big(g(\bar{w}) - p(\bar{u})\big) < 0,
$$

provided K_2 is large enough. Hence the claim follows.

(3) We claim that there is a $K_3 > 0$ such that $\bar{z}(s) > -K_3\bar{w}(s)$ for $s > 0$.

Suppose the claim is false, then for any $K_3 > 0$ there is an $s_1 > 0$ such that

$$
\bar{z}(s_1) = -K_3\ell\big(\bar{w}(s_1)\big) \quad \text{and} \quad \bar{z}'(s_1) \leqslant -K_3\ell'\big(\bar{w}(s_1)\big).
$$

If $\bar{z}(s) < -K_3\ell(\bar{w}(s))$ for all $s > s_1$, then the boundedness of $\bar{u}(s)$ and $\bar{w}(s)$ implies that

$$
\bar{z}' = c\bar{z} + \ell(\bar{w})q(\bar{u}) < \ell(\bar{w})\big(-cK_3 + q(\bar{u})\big) < 0 \quad \text{for } s > s_1,
$$

provided $K_3 > 0$ is large enough. This yields

$$
\bar{w}'(s) = \bar{z}(s) < \bar{z}(s_1) < 0,
$$

which contradicts the positivity of $\bar{w}$. Hence, there is an $s_2 > s_1$ such that

$$
\bar{z}(s_2) = -K_3\ell\big(\bar{w}(s_2)\big) \quad \text{and} \quad \bar{z}'(s_2) \geqslant -K_3\ell'\big(\bar{w}(s_2)\big)\bar{w}'(s_2).
$$

However, this fact also leads to the following contradiction:

$$
\begin{aligned}
0 &\leqslant c\bar{z}(s_2) + \ell\big(\bar{w}(s_2)\big)q\big(\bar{u}(s_2)\big) + K_3\ell'\big(\bar{w}(s_2)\big)\bar{z}(s_2) \\
&\leqslant \ell\big(\bar{w}(s_2)\big)\big(-cK_3 - K_3^2\ell'\big(\bar{w}(s_2)\big) + q\big(\bar{u}(s_2)\big)\big) < 0,
\end{aligned}
$$

provided $K_3 > 0$ is large enough. Hence the claim follows.

(4) We claim that there is a $K_4 > 0$ such that $\bar{z}(s) < K_4\bar{w}(s)$ for $s > 0$.

Suppose the claim is false, then for any $K_4 > 0$ there is an $s_1 > 0$ such that

$$
\bar{z}(s_1) = K_4\bar{w}(s_1) \quad \text{and} \quad \bar{z}'(s) \geqslant K_4\bar{w}'(s).
$$

Then, at $s = s_1$, we have the following contradiction:

$$
0 \leqslant c\bar{z} + \ell(\bar{w})q(\bar{u}) - K_4\bar{z} = \bar{w}\big(cK_4 - K_4^2\big) + \ell(\bar{w})q(\bar{u}) < 0,
$$

provided $K_4 > 0$ is large enough. Hence the claim follows, and the proof is complete. $\square$

According to Lemmas 3.1–3.3, $\bar{\mathbf{y}}(s)$ is positively invariant in the set $\mathscr{D}$ defined by

$$
\mathscr{D} := \big\{0 < u < K,\ 0 < w < M,\ -K_1h(u) < v < K_2u,\ -K_3\ell(w) < z < K_4w\big\}.
$$

Now we define the Lyapunov function $V(u,v,w,z)$ on $\mathscr{D}$ by

$$V(u,v,w,z) = \alpha\beta\left(dv - cu - h(u_*)\frac{dv}{h(u)} + ch(u_*)H(u)\right) - \left(z - cw - \ell(w_*)\frac{z}{\ell(w)} + c\ell(w_*)L(w)\right), \tag{3.2}$$

where $\alpha > 0$, $\beta < 0$,

$$H(u) := \int_{u_*}^{u} \frac{dx}{h(x)} \quad \text{and} \quad L(w) := \int_{w_*}^{w} \frac{dx}{\ell(x)}.$$

It is easy to verify that $V(\mathbf{y})$ is bounded below on $\mathscr{D}$. Moreover, the derivative of V along any trajectory $\mathbf{y}(s)$ of (2.1) lying in $\mathscr{D}$ is equal to

$$\frac{dV}{ds} = -\alpha\beta\big(h(u) - h(u_*)\big)\big(p(u) - p(u_*)\big) + \alpha\beta h(u_*)\frac{dv^2 h'(u)}{h^2(u)} - \ell(w_*)\frac{z^2\ell'(w)}{\ell^2(w)}.$$

4. Proof of the main results

First, we recall the following LaSalle's Invariance Principle.

Proposition 4.1 *(LaSalle's Invariance Principle). (Cf.* [11]*.) Consider the following initial value problem*:

$$\mathbf{y}' = f(\mathbf{y}), \quad y \in \mathbb{R}^n. \tag{4.1}$$

Let $\mathscr{D} \subseteq \mathbb{R}^n$ be an open set in $\mathbb{R}^n$. Suppose $\mathbf{y}(s)$ is a solution of (4.1) *which is positively invariant in $\mathscr{D}$. If there is a continuous and bounded below function $V : \mathscr{D} \to \mathbb{R}$ such that the orbital derivative of V along $\mathbf{y}(s)$ is non-positive, i.e.,*

$$\frac{dV}{ds} = \nabla V(\mathbf{y}) \cdot f(\mathbf{y}) \leqslant 0,$$

then the ω-limit set of $\mathbf{y}(s)$ is contained in $\mathcal{I}$, where $\mathcal{I}$ be the largest invariant set in $\{\mathbf{y} \in \mathscr{D}\colon dV/ds = 0\}$.

Proof of Theorem 1.1. According to the discussion in Section 2, we only need to prove part (ii) of the theorem. By Lemma 3.3, $\bar{\mathbf{y}}(s)$ is positively invariant in $\mathscr{D}$. It's obvious that the Lyapunov function $V(y)$ defined by (3.2) is continuous, bounded below and has non-positive orbital derivative along $\bar{\mathbf{y}}(s)$. By the LaSalle's Invariance Principle, the ω-limit set of $\bar{\mathbf{y}}(s)$ is contained in the largest invariant subset of $\{\mathbf{y} \in \mathscr{D}\colon dV/ds = 0\}$, which is the singleton $(u_*, 0, w_*, 0)$. It follows that $\bar{\mathbf{y}}(\infty) = (u_*, 0, w_*, 0)$. On the other hand, since $\bar{\mathbf{y}}(0)$ lies in the unstable manifold of $(K, 0, 0, 0)$, we have $\bar{\mathbf{y}}(-\infty) = (K, 0, 0, 0)$. By Lemma 3.1, $\bar{u}(s)$ and $\bar{w}(s)$ are positive for all s. Thus, there is a nonnegative traveling wave solution of (1.1) connecting the equilibria E_1 and E_2.

Next, according to Eq. (2.6), the characteristic equation of the linearization of (2.1) at $(u_*, 0, w_*, 0)$ is given by

$$\begin{aligned} P(\lambda) &= d\lambda^4 - (dc + c)\lambda^3 + \big(c^2 - \xi_*\big)\lambda^2 + c\xi_*\lambda + \zeta_* \\ &= \lambda^2(\lambda - c)(d\lambda - c) - \xi_*\lambda(\lambda - c) - \alpha\beta g(w_*)h(u_*)h'(u_*)g'(w_*). \end{aligned}$$

By Proposition 2.1, $P(\lambda)$ always has two roots with positive real parts and two roots with negative real parts. Regard $P(\lambda)$ as the constant shift of the polynomial $\lambda^2(\lambda - c)(d\lambda - c) - \xi_*\lambda(\lambda - c)$ which has two distinct positive real roots and two distinct negative real roots. Since $-\alpha\beta g(w_*)h(u_*)h'(u_*)g'(w_*) > 0$, it's easy to see that there exists a $\sigma^* > 0$ such that $P(\lambda)$ has two distinct negative real eigenvalues when $|\alpha\beta| < \sigma^*$; repeated negative real eigenvalues when $|\alpha\beta| = \sigma^*$; and a complex conjugate pair of eigenvalues with negative real part when $|\alpha\beta| > \sigma^*$. Hence, by the Stable Manifold Theorem, we prove the assertion of the theorem for the behavior of traveling wave solutions for large s. The proof is complete. □

5. Applications and Hopf bifurcation

In this section, we will apply our main theorem to systems (1.2)–(1.5). We further investigate the existence of traveling wave train solutions for systems (1.3) and (1.4) via the mechanism of Hopf bifurcation. These results generalize the works [12,16].

5.1. Applications to system (1.2)

After rescaling (see [4]), we may consider system (1.2) in the form

$$\begin{cases} u_t = du_{xx} - u\big(w - (1-u)\big), \\ w_t = w_{xx} - \alpha w(b-u), \end{cases} \tag{5.1}$$

where $\alpha > 0$ and $b > 0$ are positive constants. Then

$$h(u) = u, \qquad g(w) = w, \qquad \ell(w) = \alpha w, \qquad p(u) = 1-u, \qquad q(u) = b-u,$$
$$K = 1, \qquad u_* = b, \qquad w_* = 1-b.$$

Therefore, the assumptions (A1)–(A4) hold if $b < 1$. By Theorem 1.1, we have the following results.

Theorem 5.1. *Assume* $0 < b < 1$. *If* $c > 2\sqrt{\alpha(1-b)}$, *then there is a nonnegative traveling wave solution of* (5.1) *connecting the equilibria* $(1, 0)$ *and* (u_*, w_*).

Note that the result of Theorem 5.1 is consistent with the work of [4].

5.2. Applications to system (1.3)

After rescaling (see [12]), we may consider system (1.3) in the form

$$\begin{cases} u_t = du_{xx} - \dfrac{u}{1+u}\big(w - a(b-u)(1+u)\big), \\ w_t = w_{xx} - w\left(1 - \dfrac{ru}{1+u}\right), \end{cases} \tag{5.2}$$

where $b > 0$, $a > 0$ and $r > 0$ are positive constants. Then

$$h(u) = \frac{u}{1+u}, \qquad g(w) = w, \qquad \ell(w) = w,$$
$$p(u) = a(b-u)(1+u), \qquad q(u) = 1 - \frac{ru}{1+u},$$
$$K = b, \qquad u_* = \frac{1}{r-1}, \qquad w_* = a\left(b - \frac{1}{r-1}\right)\left(1 + \frac{1}{r-1}\right).$$

Therefore, the assumptions (A1)–(A4) hold if

$$b<1 \quad \text{and} \quad \frac{b+1}{b}<r. \tag{5.3}$$

By Theorem 1.1, we have the following results.

Theorem 5.2. *Assume* (5.3)*. If* $c>2\sqrt{(rb-1-b)/(1+b)}$*, then there is a nonnegative traveling wave solution of* (5.2) *connecting the equilibria* $(b,0)$ *and* (u_*,w_*)*.*

Next, we investigate the phenomena of Hopf bifurcation for the following reduced system

$$\begin{cases} u'=v, \\ dv'=cv+\dfrac{u}{1+u}\big(w-a(b-u)(1+u)\big), \\ w'=z, \\ z'=cz+w-\dfrac{ru}{1+u}. \end{cases} \tag{5.4}$$

According to Eq. (2.6), we know that

$$\xi_*=-h(u_*)p'(u_*)=-\frac{a}{r}\left(b-1-\frac{2}{r-1}\right),$$
$$\zeta_*=-\ell(w_*)h(u_*)q'(u_*)g'(w_*)=\frac{ab(r-1)-a}{r}.$$

If $b>1$ and $r>(b+1)/(b-1)$ then $\xi_*<0$ and $\zeta_*>0$. Substituting $\lambda=ki$ into Eq. (2.6), we have

$$k^4-\frac{c^2-\xi_*}{d}k^2+\frac{\zeta_*}{d}=0 \quad \text{and} \quad k^2=-\frac{\xi_*}{1+d}.$$

Then, a pair of pure imaginary eigenvalues of (2.6) exists if the parameters satisfy the following condition:

$$c^2=\frac{\xi_*}{1+d}-(1+d)\frac{\zeta_*}{\xi_*}. \tag{5.5}$$

Let us fix the parameters d, a, b and consider λ, ξ_* and ζ_* as functions of r. Then, differentiating Eq. (2.6) with respect to r gives

$$\frac{d\lambda}{dr}=\frac{\dot{\xi}_*\lambda^2-c\dot{\xi}_*\lambda-\dot{\zeta}_*}{4d\lambda^3-3\lambda^2(cd+c)+2(c^2-\xi_*)\lambda+c\xi_*}. \tag{5.6}$$

Substituting $\lambda=ki$ into Eq. (5.6), we obtain

$$\left.\frac{d\lambda}{dr}\right|_{\lambda=ki}=-\frac{(\dot{\xi}_*k^2+\dot{\zeta}_*)+c\dot{\xi}_*ki}{(3k^2(cd+c)+c\xi_*)+(2k(c^2-\xi_*)-4dk^3)i},$$
$$\operatorname{Re}\left.\frac{d\lambda}{dr}\right|_{\lambda=ki}=-c\xi_*\left(\frac{\xi_*\dot{\xi}_*(3-d)}{(1+d)^2}-2\dot{\zeta}_*+\frac{(\xi_*-2c^2)\dot{\xi}_*}{1+d}\right)$$
$$=-c\xi_*\left(\frac{4\xi_*-2c^2(1+d)}{(1+d)^2}\dot{\xi}_*-2\dot{\zeta}_*\right). \tag{5.7}$$

Since $\dot{\zeta}_* = a(b+1)/r^2 > 0$, $\xi_* < 0$, if $\dot{\xi}_* > 0$, i.e.,

$$\dot{\xi}_* = \frac{a}{r^2}\left(b - 1 - \frac{2(2r-1)}{(r-1)^2}\right) > 0 \quad \text{or} \quad r > \frac{b+1+\sqrt{2(b+1)}}{b-1},$$

then $\operatorname{Re} \frac{d\lambda}{dr}|_{\lambda=ki} < 0$. Therefore, we obtain the following results.

Theorem 5.3. *Assume $b > 1$. If $r > \frac{b+1+\sqrt{2(b+1)}}{b-1}$, then as r crosses the curve*

$$c^2 = \frac{\xi_*(r)}{1+d} - (1+d)\frac{\zeta_*(r)}{\xi_*(r)}$$

in the (r, c)-plane, the system (5.4) undergoes a Hopf bifurcation at the equilibrium $(u_*, 0, w_*, 0)$ and there is a small amplitude periodic solution, which corresponds to a small traveling wave train solution of system (5.2).

Remark 5.4.

(1) Since we construct the Lyapunov function more generally, the result of Theorem 5.2 extends the result of Theorem 2.2 of [12].
(2) Here we point out the difference between our result of Theorem 5.3 with Theorem 2.3 of [12]. In [12], there is a typing error for $r(\beta)$ (see p. 149). Hence our result of Theorem 5.3 provides the correct region of parameters for Hopf bifurcation.

5.3. Applications to system (1.4)

After rescaling (see [16]), we may consider system (1.4) in the form

$$\begin{cases} u_t = du_{xx} - au(b-u) - \dfrac{u^2 w}{1+u^2}, \\ w_t = w_{xx} - w\left(1 - \dfrac{ru^2}{1+u^2}\right), \end{cases} \tag{5.8}$$

where $a > 0$, $r > 0$ and $b > 0$ are positive constants. Then

$$h(u) = \frac{u^2}{1+u^2}, \qquad g(w) = w, \qquad \ell(w) = w,$$

$$p(u) = a(b-u)\frac{1+u^2}{u}, \qquad q(u) = 1 - \frac{ru^2}{1+u^2},$$

$$K = b, \qquad u_* = \frac{1}{\sqrt{r-1}}, \qquad w_* = a\left(b - \frac{1}{\sqrt{r-1}}\right)\frac{r}{\sqrt{r-1}}.$$

Therefore, the assumptions (A1)–(A4) hold if

$$b < 3\sqrt{3} \quad \text{and} \quad \frac{b^2+1}{b^2} < r. \tag{5.9}$$

By Theorem 1.1, we have the following results.

Theorem 5.5. *Assume* (5.9). *If* $c > 2\sqrt{(rb^2 - 1 - b^2)/(1 + b^2)}$, *then there is a nonnegative traveling wave solution of* (5.8) *connecting the equilibria* $(b, 0)$ *and* (u_*, w_*).

Next, we investigate the phenomena of Hopf bifurcation for the following reduced system

$$\begin{cases} u' = v, \\ dv' = cv - au(b-u) + \dfrac{u^2 w}{1+u^2}, \\ w' = z, \\ z' = cz + w\left(1 - \dfrac{ru^2}{1+u^2}\right). \end{cases} \tag{5.10}$$

According to Eq. (2.6), we know that

$$\xi_* = -h(u_*)p'(u_*) = -\frac{a}{r}\left((2-r)b - \frac{2}{\sqrt{r-1}}\right),$$
$$\zeta_* = -\ell(w_*)h(u_*)q'(u_*)g'(w_*) = 2\frac{a}{r}(r-1)\left(b - \frac{1}{\sqrt{r-1}}\right).$$

By elementary computation, we have

$$\xi_* < 0 \quad \Leftrightarrow \quad (r-2)b + \frac{2}{\sqrt{r-1}} < 0, \tag{5.11}$$

$$\dot{\xi}_* > 0 \quad \Leftrightarrow \quad 2b > \frac{3r-2}{(r-1)\sqrt{r-1}}, \tag{5.12}$$

$$\dot{\zeta}_* > 0 \quad \Leftrightarrow \quad 2b\sqrt{r-1} + r - 2 > 0 \quad \Leftrightarrow \quad \sqrt{r-1} > \sqrt{b^2+1} - b. \tag{5.13}$$

If $r > (b^2+1)/b^2$ then $\dot{\zeta}_* > 0$. Assume $b > 3\sqrt{3}$, then there exists

$$\big(b^2+1\big)/b^2 < r_1(b) < r_2(b) < 2$$

such that $\xi_* < 0$ holds if $0 < r_1(b) < r < r_2(b)$. Furthermore, there exists $0 < r_3(b) < r_2(b)$ such that $\dot{\xi}_* > 0$ when $r > r_3(b)$. Similarly, let us fix the parameters d, a, b and consider λ, ξ_* and ζ_* as functions of r. According to (5.7), if

$$r_\ell(b) := \max\big\{r_1(b), r_3(b)\big\} < r < r_r(b) := r_2(b) \tag{5.14}$$

then $\operatorname{Re} \frac{d\lambda}{dr}|_{\lambda=ki} < 0$. Therefore, we obtain the following results.

Theorem 5.6. *Assume* $b > 3\sqrt{3}$. *If* $r_\ell(b) < r < r_r(b)$, *then as* r *crosses the curve*

$$c^2 = \frac{1}{1+d} - (1+d)\frac{\zeta_*(r)}{\xi_*(r)}$$

in the (r, c)*-plane, the system* (5.10) *undergoes a Hopf bifurcation at the equilibrium* $(u_*, 0, w_*, 0)$ *and there is a small amplitude periodic solution, which corresponds to a small traveling wave train solution of system* (5.8).

Remark 5.7. In [16], the authors only consider system (1.4) in special case: $d_1 = 0$. Therefore, in Theorems 5.5 and 5.6, we provide the new results for the case $d_1 \neq 0$.

5.4. Applications to system (1.5)

After rescaling, we may consider system (1.5) in the simple form

$$\begin{cases} u_t = du_{xx} - au(b-u) - w\big(1-e^{-mu}\big), \\ w_t = w_{xx} - w\big(1 - r\big(1-e^{-mu}\big)\big), \end{cases} \tag{5.15}$$

where $a > 0$, $b > 0$, $r > 0$ and $m > 0$ are positive constants. Then

$$h(u) = 1 - e^{-mu}, \qquad g(w) = w, \qquad \ell(w) = w,$$
$$p(u) = \frac{au(b-u)}{1-e^{-mu}}, \qquad q(u) = 1 - r\big(1-e^{-mu}\big),$$
$$K = b, \qquad u_* = \frac{-1}{m}\ln\left(1 - \frac{1}{r}\right), \qquad w_* = rau_*(b-u_*).$$

It can be verified that if $mb < 2$ and $r(1-e^{-mb}) > 1$ then assumptions (A1)–(A4) hold. By Theorem 1.1, we have the following results.

Theorem 5.8. *Assume $mb < 2$ and $r(1-e^{-mb}) > 1$. If $c > 2\sqrt{r(1-e^{-mb})-1}$, then there is a nonnegative traveling wave solution of (5.15) connecting the equilibria $(1,0)$ and (u_*, w_*).*

Appendix A. Proof of Proposition 2.4

To start with the proof of Proposition 2.4, we first illustrate the following claim which holds obviously and will be used in the proof.

Claim I.

(1) *If $z = 0$, $u \neq u_*$, then $z' = \ell(w)q(u)$ has the same sign with $-w \cdot (u - u_*)$.*
(2) *If $z = 0$, $u = u_*$, then $z' = 0$ and $z'' = \ell(w)q'(u)v$ has the same sign with $-v$.*
(3) *If $v = 0$, $g(w) - p(u) \neq 0$, $u \neq 0$, then $dv' = h(u)(g(w) - p(u))$ has the same sign with $g(w) - p(u)$.*
(4) *If $v = 0$, $g(w) - p(u) = 0$, $u \neq 0$, then $v' = 0$ and $dv'' = h(u)g'(w)z$ has the same sign with z.*
(5) *If $v \neq 0$, $g(w) - p(u) = 0$, then*

$$\big(g(w) - p(u)\big)' = g'(w)z - p'(u)v > 0, \quad \text{if } z \geqslant 0,\ v \geqslant 0,\ (z,v) \neq (0,0);$$
$$\big(g(w) - p(u)\big)' = g'(w)z - p'(u)v < 0, \quad \text{if } z \leqslant 0,\ v \leqslant 0,\ (z,v) \neq (0,0).$$

Now we establish the exit set W^-. Since W^- is a subset of ∂W, it's required to analyze the dynamics of (2.1) on each portion of ∂W.

For the portion $\partial R \setminus P$, let's set $\partial R = R_0 \cup R_1 \cup R_2 \cup R_3$ with

$$R_0 = \big\{(u,v,w,z)\colon\ u = 0,\ g(w) - p(u) \leqslant 0,\ v \leqslant 0\big\},$$
$$R_1 = \big\{(u,v,w,z)\colon\ u = u_*,\ g(w) - p(u) \leqslant 0,\ v \leqslant 0\big\},$$
$$R_2 = \big\{(u,v,w,z)\colon\ u < u_*,\ g(w) - p(u) = 0,\ v \leqslant 0\big\},$$
$$R_3 = \big\{(u,v,w,z)\colon\ u < u_*,\ g(w) - p(u) < 0,\ v = 0\big\}.$$

Then, we investigate the behavior of solutions on each R_i in the sequel.

On region R_0, we consider the following two subsets.

(1) Assume $u = 0$ and $v < 0$. Since $v < 0$, we know that $u^+ < 0$ and this implies this set belongs to W^-.
(2) Assume $u = 0$ and $v = 0$. In this case, $(0, 0, w, z)$ will stay at W for any $(w, z) \in \mathbb{R}^2$. Thus, $(0, 0, w, z) \in J_2$.

On region R_1, we consider the following four subsets.

(1) Assume $u = u_*$, $w = w_*$ and $v < 0$. If $z \leqslant 0$, we have

$$\big(g(w) - p(u)\big)' = g'(w)z - p'(u)v \leqslant -p'(u)v < 0.$$

Then any trajectory of solutions will enter the region R. On the other hand, if $z > 0$ then $w^+ > w_*$ and this implies that any trajectory of solutions will enter the region P.
(2) Assume $u = u_*$, $w < w_*$ and $v < 0$. In this case, it's easy to see that any trajectory of solutions will enter region R.
(3) Assume $u = u_*$, $w = w_*$ and $v = 0$. If $z = 0$ then it's obvious that $(u_*, 0, w_*, 0) \notin W^-$. If $z > 0$ then $w^+ > w_*$, $v' = 0$, $(g(w) - p(u))' = g'(w)z > 0$ and

$$\begin{aligned} dv'' &= cv' + h'(u)v\big(g(w) - p(u)\big) + h(u)\big(g(w) - p(u)\big)' \\ &= h(u)\big(g(w) - p(u)\big)' > 0. \end{aligned}$$

Thus, $v^+ > 0$, $u' > 0$ and $u^+ > u_*$. Therefore, any trajectory of solutions will enter the region S. If $z < 0$, similar to case of $z > 0$, we can obtain $w^+ < w_*$ and $u^+ < u_*$. Hence, any trajectory of solutions will enter the region R.
(4) Assume $u = u_*$, $w < w_*$ and $v = 0$. In this case, we have

$$dv' = cv + h(u_*)\big(g(w) - p(u_*)\big) < cv + h(u_*)\big(g(w_*) - p(u_*)\big) < 0.$$

Thus $v^+ < 0$. Since $u' = v < 0$, then $u^+ < u_*$ and $g(w) - p(u_*) < g(w_*) - p(u_*) = 0$. Hence, any trajectory of solutions will enter the region R.

On region R_2, we consider the following two subsets.

(1) Assume $0 < u < u_*$, $g(w) - p(u) = 0$ and $v < 0$. If $z > 0$ then $u < u_*$ and $g(w) - p(u) = 0$ imply that $w > w_*$. Hence, any trajectory of solutions will enter the region P. If $z = 0$ then

$$z' = cz + l(w)q(u) > 0 + l(w)q(u_*) = 0,$$

and this implies $z^+ > 0$ and $w^+ > w_*$. Hence, any trajectory of solutions will enter the region P. If $z < 0$, it is easy to check that

$$\big(g(w) - p(u)\big)' = g'(w)z - p'(u)v < g'(w)z < 0.$$

Hence $(g(w) - p(u))^+ < 0$ and any trajectory of solutions will enter the region R.
(2) Assume $0 < u < u_*$, $g(w) - p(u) = 0$ and $v = 0$. If $z \geqslant 0$, by the same arguments as (1), any trajectory of solutions will enter the region P. If $z < 0$, we have

$$\begin{gathered} \big(g(w) - p(u)\big)' = g'(w)z < 0 \quad \Rightarrow \quad \big(g(w) - p(u)\big)^+ < 0, \\ dv' = cv + g(w) - p(u) = 0, \end{gathered}$$

$$dv'' = cv' + h'(u)v\big(g(w) - p(u)\big) + h(u)\big(g'(w)z - p'(u)v\big)$$
$$= 0 + 0 + h(u)g'(w)z + 0 < 0.$$

Hence $v^+ < 0$, and any trajectory of solutions will enter the region R.

On region R_3, we have $0 < u < u_*$, $g(w) - p(u) < 0$ and $v = 0$. Thus,

$$dv' = cv + h(u)\big(g(w) - p(u)\big) = 0 + h(u)\big(g(w) - p(u)\big) < 0.$$

Hence, any trajectory of solutions will enter the region R.

For the portion $\partial S \setminus Q$, let's set $\partial S = S_1 \cup S_2 \cup S_3$ with

$$S_1 = \big\{(u, v, w, z) \,\big|\, u = u_*,\ g(w) - p(u) \geqslant 0,\ v \geqslant 0\big\},$$
$$S_2 = \big\{(u, v, w, z) \,\big|\, u > u_*,\ g(w) - p(u) = 0,\ v \geqslant 0\big\},$$
$$S_3 = \big\{(u, v, w, z) \,\big|\, u > u_*,\ g(w) - p(u) > 0,\ v = 0\big\}.$$

Then we investigate the behavior of solutions on each S_i.

On region S_1, we consider the following four subsets.

(1) Assume $u = u_*$, $w = w_*$ and $v > 0$. If $z \geqslant 0$ then

$$\big(g(w) - p(u)\big)' = g'(w)z - p'(u)v \geqslant -p'(u)v > 0.$$

Hence, any trajectory of solutions will enter the region S. On the other hand, if $z < 0$ then $w^+ < w_*$ and this implies that any trajectory of solutions will enter the region Q.

(2) Assume $u = u_*$, $w > w_*$ and $v > 0$. In this case, it's easy to see that any trajectory of solutions will enter region S.

(3) Assume $u = u_*$, $w = w_*$ and $v = 0$. If $z = 0$, it's obvious that $(u_*, 0, w_*, 0) \notin W^-$. If $z > 0$ then $w^+ > w_*$, $v' = 0$, $(g(w) - p(u))' = g'(w)z > 0$ and

$$dv'' = h(u)\big(g(w) - p(u)\big)' > 0.$$

Thus, $v^+ > 0$, $u' > 0$ and $u^+ > u_*$. Then, any trajectory of solutions will enter the region S. If $z < 0$, similar to case of $z > 0$, we can obtain $w^+ < w_*$ and $u^+ < u_*$. Hence, any trajectory of solutions will enter the region R.

(4) Assume $u = u_*$, $w > w_*$ and $v = 0$. In this case, we have

$$dv' = cv + h(u_*)\big(g(w) - p(u_*)\big) > cv + h(u_*)\big(g(w_*) - p(u_*)\big) = 0,$$

and this implies $v^+ > 0$. Since $u' = v < 0$, then $u^+ > u_*$ and $g(w) - p(u_*) > g(w_*) - p(u_*) = 0$. Hence, any trajectory of solutions will enter the region S.

On region S_2, we consider the following two subsets.

(1) Assume $u > u_*$, $g(w) - p(u) = 0$ and $v > 0$. If $z < 0$ then $u > u_*$ and $g(w) - p(u) = 0$. Thus $w < w_*$, and any trajectory of solutions will enter the region Q. If $z = 0$ then

$$z' = cz + l(w)q(u) < 0 + l(w)q(u_*) = 0,$$

and this implies $z^+ < 0$ and $w^+ < w_*$. Then any trajectory of solutions will enter the region Q. If $z > 0$, it is easy to check that

$$\big(g(w) - p(u)\big)' = g'(w)z - p'(u)v > g'(w)z > 0.$$

Hence $(g(w) - p(u))^+ > 0$ and any trajectory of solutions will enter the region S.

(2) Assume $u > u_*$, $g(w) - p(u) = 0$ and $v = 0$. If $z \leqslant 0$, by the same arguments as (1), any trajectory of solutions will enter the region Q. If $z > 0$, we have

$$\big(g(w) - p(u)\big)' = g'(w)z > 0 \quad \Rightarrow \quad \big(g(w) - p(u)\big)^+ > 0,$$
$$dv' = cv + g(w) - p(u) = 0 \quad \text{and} \quad dv'' = h(u)g'(w)z > 0.$$

Hence $v^+ > 0$, and any trajectory of solutions will enter the region S.

On region S_3, we have $u > u_*$, $g(w) - p(u) > 0$ and $v = 0$. Thus,

$$dv' = cv + h(u)\big(g(w) - p(u)\big) = 0 + h(u)\big(g(w) - p(u)\big) > 0.$$

Hence, any trajectory of solutions will enter the region S.

For the portion $\partial P \setminus R$, let's set

$$\partial P = P_0 \cup P_1 \cup P_{12} \cup P_2 \cup P_3 \cup P_{13} \cup P_{123},$$

where

$$P_0 = \big\{(u, v, w, z) \,\big|\, u = 0,\ w \geqslant w_*,\ z \geqslant 0\big\},$$
$$P_1 = \big\{(u, v, w, z) \,\big|\, u = u_*,\ w > w_*,\ z > 0\big\},$$
$$P_{12} = \big\{(u, v, w, z) \,\big|\, u = u_*,\ w = w_*,\ z > 0\big\},$$
$$P_2 = \big\{(u, v, w, z) \,\big|\, u \in (0, u_*),\ w = w_*,\ z > 0\big\},$$
$$P_3 = \big\{(u, v, w, z) \,\big|\, u \in (0, u_*),\ w \geqslant w_*,\ z = 0\big\},$$
$$P_{13} = \big\{(u, v, w, z) \,\big|\, u = u_*,\ w > w_*,\ z = 0\big\},$$
$$P_{123} = \big\{(u, v, w, z) \,\big|\, u = u_*,\ w = w_*,\ z = 0\big\}.$$

Now we investigate the behavior of solutions on each P_i.

On region P_0, we consider the following three subsets.

(1) Assume $v < 0$. Since $v < 0$, we have $u^+ < 0$ and this implies this set belongs to W^-.
(2) Assume $v = 0$. In this case, $(0, 0, w, z)$ will stay at W for any $(w, z) \in \mathbb{R}^2$. Thus, $(0, 0, w, z) \in J_2$.
(3) Assume $v > 0$. Since $v > 0$, we have $u^+ > 0$. Then part (1) of Claim I implies $z^+ > 0$ and $w^+ > w_*$. Thus, the trajectory of solutions will enter the region P.

On region P_1, we have $g(w) - p(u) > 0$. Then we consider the following three subsets.

(1) Assume $v < 0$. Since $v < 0$, we have $u^+ < u_*$ and this implies the trajectory of solutions will enter the region P.
(2) Assume $v = 0$. Then part (3) of Claim I implies $v^+ > 0$ and $u^+ > u_*$. Thus, the trajectory of solutions will enter the region S.

(3) Assume $v > 0$. Since $v > 0$, we have $u^+ > u_*$. Thus, the trajectory of solutions will enter the region S.

On region P_{12}, we have $g(w) - p(u) = 0$ and $w^+ > w_*$. Then we consider the following three subsets.

(1) Assume $v < 0$. Since $v < 0$, we have $u^+ < u_*$ and this implies the trajectory of solutions will enter the region P.
(2) Assume $v = 0$. Then part (4) of Claim I implies $v^+ > 0$ and $u^+ > u_*$, and part (5) of Claim I implies $[g(w) - p(u)]^+ > 0$. Thus, the trajectory of solutions will enter the region S.
(3) Assume $v > 0$. Since $v > 0$, we have $u^+ > u_*$. Then part (5) of Claim I implies $[g(w) - p(u)]^+ > 0$. Thus, the trajectory of solutions will enter the region S.

On region P_2, we have $w^+ > w_*$. Hence, the trajectory of solutions will enter the region P.

On region P_3, part (1) of Claim I implies $z^+ > 0$ and $w^+ > w_*$. Hence, the trajectory of solutions will enter the region P.

On region P_{13}, we have $g(w) - p(u) > 0$. Then we consider the following three subsets.

(1) Assume $v < 0$. Since $v < 0$, we have $u^+ < u_*$. Then part (2) of Claim I implies $z^+ > 0$. Hence, the trajectory of solutions will enter the region P.
(2) Assume $v = 0$. Then part (3) of Claim I implies $v^+ > 0$ and $u^+ > u_*$. Thus, the trajectory of solutions will enter the region S.
(3) Assume $v > 0$. Since $v > 0$, we have $u^+ > u_*$. Thus, the trajectory of solutions will enter the region S.

For the portion $\partial Q \setminus S$, let's set

$$\partial Q = Q_0 \cup Q_1 \cup Q_{12} \cup Q_2 \cup Q_{3+} \cup Q_{3-} \cup Q_{13+} \cup Q_{13-} \cup P_{123},$$

where

$$\begin{aligned}
Q_0 &= \{(u, v, w, z) \mid u \leqslant u_*,\ w = 0,\ z = 0\},\\
Q_1 &= \{(u, v, w, z) \mid u = u_*,\ w < w_*,\ z < 0\},\\
Q_{12} &= \{(u, v, w, z) \mid u = u_*,\ w = w_*,\ z < 0\},\\
Q_2 &= \{(u, v, w, z) \mid u > u_*,\ w = w_*,\ z < 0\},\\
Q_{3+} &= \{(u, v, w, z) \mid u > u_*,\ 0 < w < w_*,\ z = 0\},\\
Q_{3-} &= \{(u, v, w, z) \mid u > u_*,\ w < 0,\ z = 0\},\\
Q_{13+} &= \{(u, v, w, z) \mid u = u_*,\ 0 < w < w_*,\ z = 0\},\\
Q_{13-} &= \{(u, v, w, z) \mid u = u_*,\ w < 0,\ z = 0\}.
\end{aligned}$$

Now we investigate the behavior of solutions on each P_i.

On region Q_0, the trajectory of solutions are invariant in V and included in J_{10}.

On region Q_1, we have $g(w) - p(u) < 0$. Then we consider the following three subsets.

(1) Assume $v > 0$. Since $v > 0$, we have $u^+ > u_*$ and this implies the trajectory of solutions will enter the region Q.
(2) Assume $v = 0$. Then part (3) of Claim I implies $v^+ < 0$ and $u^+ < u_*$. Thus, the trajectory of solutions will enter the region R.

(3) Assume $v < 0$. Since $v < 0$, we have $u^+ < u_*$. Thus, the trajectory of solutions will enter the region R.

On region Q_{12}, we have $g(w) - p(u) = 0$, $w^+ < w_*$. Then we consider the following three subsets.

(1) Assume $v > 0$. Since $v > 0$, we have $u^+ > u_*$ and this implies the trajectory of solutions will enter the region Q.
(2) Assume $v = 0$. Then part (4) of Claim I implies $v^+ < 0$ and $u^+ < u_*$, and part (5) of Claim I implies $(g(w) - p(u))^+ < 0$. Thus, the trajectory of solutions will enter the region R.
(3) Assume $v < 0$. Since $v < 0$, we have $u^+ < u_*$. Then part (5) of Claim I implies $(g(w) - p(u))^+ < 0$. Thus, the trajectory of solutions will enter the region R.

On region Q_2, we have $w^+ < w_*$. Hence, the trajectory of solutions will enter the region Q.

On region Q_{3+}, part (1) of Claim I implies $z^+ < 0$ and $w^+ < w_*$. Hence, the trajectory of solutions will enter the region Q.

On region Q_{3-}, part (1) of Claim I implies $z^+ > 0$. Then the trajectory will not enter Q immediately. Furthermore, the trajectory will not enter P or R immediately since $u > u_*$. Thus we consider the following four subsets.

(1) Assume $v < 0$. Since $v > 0$, the trajectory cannot enter S immediately. Thus, the trajectory of solutions cannot exit W immediately, and included in J_{12}.
(2) Assume $v \geqslant 0$ and $g(w) - p(u) < 0$. Since $g(w) - p(u) < 0$, the trajectory cannot enter S immediately. Thus, the trajectory of solutions cannot exit W immediately, and included in J_{12}.
(3) Assume $v = 0$ and $g(w) - p(u) > 0$. Then part (3) of Claim I implies $v^+ > 0$. Thus, the trajectory of solutions will enter the region S.
(4) Assume $v = 0$ and $g(w) - p(u) = 0$. We have $(g(w) - p(u))' = 0$ and $(g(w) - p(u))'' = g'(w)z'$. By part (1) of Claim I, $(g(w) - p(u))^+ > 0$. On the other hand, $dv' = dv'' = 0$ and $dv''' = h(u)g'(w)z' > 0$. Then $v^+ > 0$, and the trajectory of solutions will enter the region S.

On region Q_{13+}, we have $g(w) - p(u) < 0$. Then we consider the following three subsets.

(1) Assume $v > 0$. Since $v > 0$, we have $u^+ > u_*$. Then part (2) of Claim I implies $z^+ < 0$. Thus, the trajectory of solutions will enter the region Q.
(2) Assume $v = 0$. Then part (3) of Claim I implies $v^+ < 0$ and $u^+ < u_*$. Thus, the trajectory of solutions will enter the region R.
(3) Assume $v < 0$. Since $v < 0$, we have $u^+ < u_*$. Thus, the trajectory of solutions will enter the region R.

On region Q_{13-}, we have $g(w) - p(u) < 0$. Then we consider the following three subsets.

(1) Assume $v > 0$. It's obvious $g(w) - p(u) < 0$. Thus the trajectory cannot enter S immediately. Since $v > 0$, then $u^+ > u_*$ and the trajectory cannot enter P or R immediately. Furthermore, the part (2) of Claim I implies $z^+ > 0$ and the trajectory cannot enter Q immediately. Therefore, the trajectory of solutions cannot exit W immediately and included in J_{11}.
(2) Assume $v = 0$. Then part (3) of Claim I implies $v^+ < 0$ and $u^+ < u_*$. Thus, the trajectory of solutions will enter the region R.
(3) Assume $v < 0$. Since $v < 0$, we have $u^+ < u_*$. Thus, the trajectory of solutions will enter the region R.

According to previous results, the proof of this appendix is complete.

Appendix B. Proof of Lemma 2.17

First, we prove the following claim.

Claim II.

(1) *If* $\mathbf{y} \in \widehat{\mathbf{y}_5\mathbf{y}_1}$*, then* $\mathbf{y} \cdot s$ *will exit* W *from the boundary of* R *or* P*.*
(2) *If* $\mathbf{y} \in \widehat{\mathbf{y}_1\mathbf{y}_2}$*, then* $\mathbf{y} \cdot s$ *will exit* W *from the boundary of region* R*,* P*, or* S*.*
(3) *If* $\mathbf{y} \in \widehat{\mathbf{y}_2\mathbf{y}_3}$*, then* $\mathbf{y} \cdot s$ *will exit* W *from the boundary of region* S*.*
(4) *If* $\mathbf{y} \in \widehat{\mathbf{y}_3\mathbf{y}_4}$*, then* $\mathbf{y} \cdot s$ *will exit* W *from the boundary of region* Q*.*
(5) *If* $\mathbf{y} \in \widehat{\mathbf{y}_4\mathbf{y}_5}$*, then* $\mathbf{y} \cdot s$ *will exit* W *from the boundary of region* R*.*

Proof. (1) The assertion of (1) follows directly by Lemma 2.10.

(2) Let $\mathbf{y}(0) = G(\theta) \in \widehat{\mathbf{y}_1\mathbf{y}_2}$ with $\theta \in (\theta_1, \theta_2)$, then $u(0) > u_*$, $v(0) < 0$, $w(0) > 0$ and $z(0) > 0$. For $\theta \in (\theta_1, \theta_2)$, let $s_1(\theta)$ be the first time that $u(s; G(\theta)) = u_*$ and $s_2(\theta)$ be the first time that $v(s; G(\theta)) = 0$. By Lemma 2.11, $s_1(\theta)$ and $s_2(\theta)$ are finite.

If $s_1(\theta) < s_2(\theta)$, similar to Lemma 2.10, $\mathbf{y}(s; G(\theta))$ will enter region R or P.

If $s_1(\theta) \geqslant s_2(\theta)$ then $v(s) < 0$ for $s \in (0, s_2)$, $v(s_2) = 0$ and $u(s_2) \geqslant u_*$. Thus $v'(s_2) \geqslant 0$. If $v'(s_2) > 0$, then $g(u(s_2)) - p(w(s_2)) > 0$. It follows that $u(s_2^+) > u_*$, $v(s_2^+) > 0$, $g(w(s_2^+)) - p(u(s_2^+)) > 0$ and $\mathbf{y}(s_2^+)$ entering region S. If $v'(s_2) = 0$, then $g(w(s_2)) - p(u(s_2)) = 0$, $v''(s_2) = h(u(s_2))g(w(s_2))z(s_2)$ and $(g(w) - p(u))'(s_2) = g'(w(s_2))z(s_2)$. By Lemma 2.9, $w(s_2) > 0$ and $z(s_2) > 0$. It follows that $v''(s_2) > 0$ and $(g(w) - p(u))'(s_2) > 0$. Thus we also have $u(s_2^+) > u_*$, $v(s_2^+) > 0$, $g(w(s_2^+)) - p(u(s_2^+)) > 0$ and $\mathbf{y}(s_2^+)$ entering region S.

(3) We are going to show that $g(w) - p(u) > 0$ for $\mathbf{y} \in \widehat{\mathbf{y}_2\mathbf{y}_3}$. By Mean Value Theorem, we have

$$\begin{aligned} \frac{g(w) - p(u)}{\varepsilon} &= g'(w_0)w - p'(u_0)(u - K) \\ &= g'(w_0)\big(-\psi(\lambda_2)c_2 - \psi(\lambda_3)c_3\big) - p'(u_0)(-c_1 - c_2 - c_3) + O(\varepsilon), \end{aligned} \tag{B.1}$$

for some u_0, w_0, where (u_0, w_0) tends to $(K, 0)$ as ε tends to 0. Since $v = 0$,

$$-c_1 = \frac{\lambda_3}{\lambda_1}c_3 + \frac{\lambda_2}{\lambda_1}c_2 + O(\varepsilon). \tag{B.2}$$

Substituting Eq. (B.2) into Eq. (B.1), we have

$$\begin{aligned} \frac{g(w) - p(u)}{\varepsilon} &= \left(-g'(w_0)\psi(\lambda_2) - p'(u_0)\frac{\lambda_2 - \lambda_1}{\lambda_1}\right)c_2 \\ &\quad + \left(-g'(w_0)\psi(\lambda_3) - p'(u_0)\frac{\lambda_3 - \lambda_1}{\lambda_1}\right)c_3 + O(\varepsilon). \end{aligned} \tag{B.3}$$

By (2.5) and (2.4), we have

$$\begin{aligned} -g'(0)\psi(\lambda_2) - p'(K)\frac{\lambda_2 - \lambda_1}{\lambda_1} &= -\lambda_2\left(d\lambda_2 - c + h(K)p'(K)\frac{1}{\lambda_1}\right)/h(K) \\ &> -\lambda_2\left(d\lambda_1 - c + h(K)p'(K)\frac{1}{\lambda_1}\right)/h(K) = 0, \end{aligned}$$

which implies that

$$-g'(w_0)\psi(\lambda_2) - p'(u_0)\frac{\lambda_2 - \lambda_1}{\lambda_1} > 0$$

for ε sufficiently small. Since $z > 0$, $\lambda_3\psi(\lambda_3)c_3 < -\lambda_2\psi(\lambda_2)c_2 + O(\varepsilon)$, which leads to

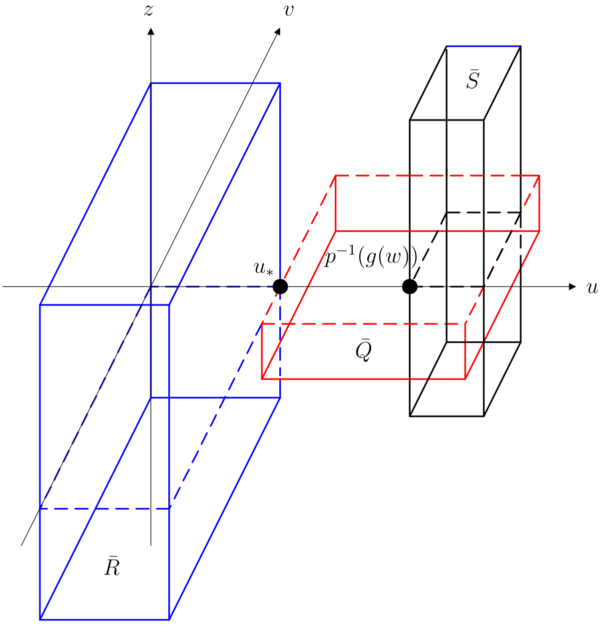

Fig. 5. The projection of Λ on the uvz-space for the case $w < w_*$. All the dash lines lie on the plane $z = 0$.

$$\frac{g(w) - p(u)}{\varepsilon} > c_3\left\{\left(-g'(w_0)\psi(\lambda_2) - p'(u_0)\frac{\lambda_2 - \lambda_1}{\lambda_1}\right)\frac{\lambda_3\psi(\lambda_3)}{-\lambda_2\psi(\lambda_2)} + \left(-g'(w_0)\psi(\lambda_3) - p'(u_0)\frac{\lambda_3 - \lambda_1}{\lambda_1}\right)\right\} + O(\varepsilon). \tag{B.4}$$

From (2.5) and (2.4), we have

$$\lambda_3\psi(\lambda_3)\left(-g'(0)\psi(\lambda_2) - p'(K)\frac{\lambda_2 - \lambda_1}{\lambda_1}\right) - \lambda_2\psi(\lambda_2)\left(-g'(0)\psi(\lambda_3) - p'(K)\frac{\lambda_3 - \lambda_1}{\lambda_1}\right)$$
$$= \frac{d^2\lambda_2\lambda_3}{h(K)}(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3) > 0,$$

which implies the following term

$$\lambda_3\psi(\lambda_3)\left(-g'(w_0)\psi(\lambda_2) - p'(u_0)\frac{\lambda_2 - \lambda_1}{\lambda_1}\right) - \lambda_2\psi(\lambda_2)\left(-g'(w_0)\psi(\lambda_3) - p'(u_0)\frac{\lambda_3 - \lambda_1}{\lambda_1}\right)$$

is positive for sufficiently small ε. Thus, $g(w) - p(u) > 0$ for $\mathbf{y} \in \widehat{\mathbf{y}_2\mathbf{y}_3}$.

If $\mathbf{y}(0) \in \widehat{\mathbf{y}_2\mathbf{y}_3}$ then $u(0) > u_*$, $v(0) = 0$ and $g(w(0)) - p(u(0)) > 0$. Hence, $v'(0) > 0$, $v(0^+) > 0$, and $\mathbf{y}(s)$ enters region S immediately.

(4) Let $\mathbf{y}(0) \in \widehat{\mathbf{y}_3\mathbf{y}_4}$. Obviously, $u(0) > u_*$. Denote by c_i the coefficient of $\mathbf{e}_i$ in (2.10) then $z(0) = -\lambda_2 c_2\psi_2 - \lambda_3 c_3\psi_3$ and $w(0) = -c_2\psi_2 - c_3\psi_3$. Since $z(0) = 0$, $c_3 > 0$ and $\lambda_3 < \lambda_2$, we have $w(0) > 0$. It follows that $z'(0) < 0$ and $z(0^+) < 0$. Thus $\mathbf{y}(s)$ enters region Q immediately.

(5) Let $\mathbf{y}(0) \in \widehat{\mathbf{y}_4\mathbf{y}_5}$. According to Lemma 2.5, $\mathbf{y}_0 \in \Omega$ implies that $\mathbf{y}_0 \cdot s$ will enter region R, which is an open set. Since the whole $\widehat{\mathbf{y}_4\mathbf{y}_5}$ is very close to $\mathbf{y}_0$, it follows that $\mathbf{y}(s)$ will also enter region R.

The proof of Claim II is complete. □

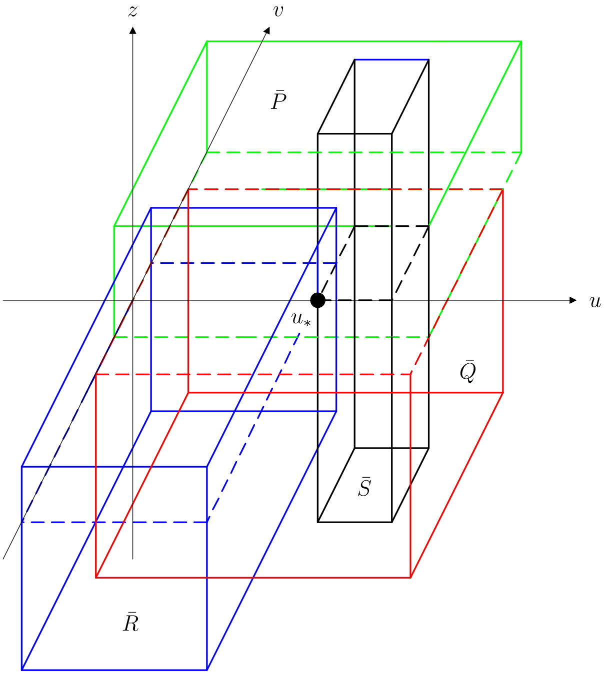

Fig. 6. The projection of Λ on the uvz-space for the case $w = w_*$. All the dash lines lie on the plane $z = 0$.

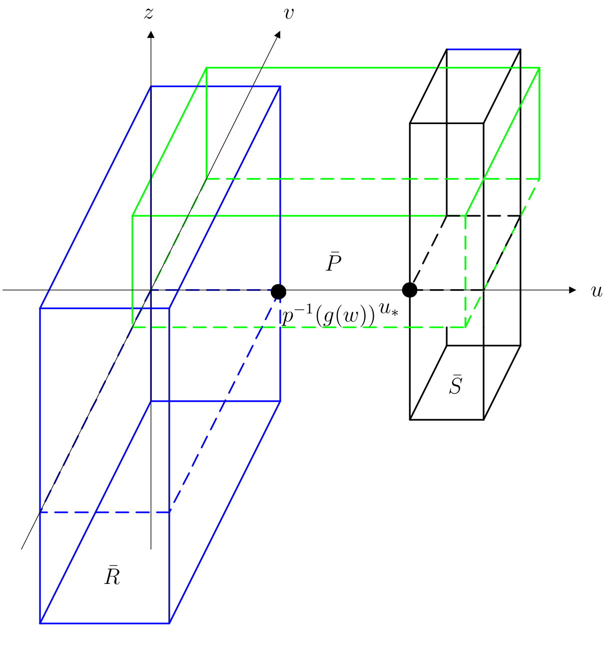

Fig. 7. The projection of Λ on the uvz-space for the case $w > w_*$. All the dash lines lie on the plane $z = 0$.

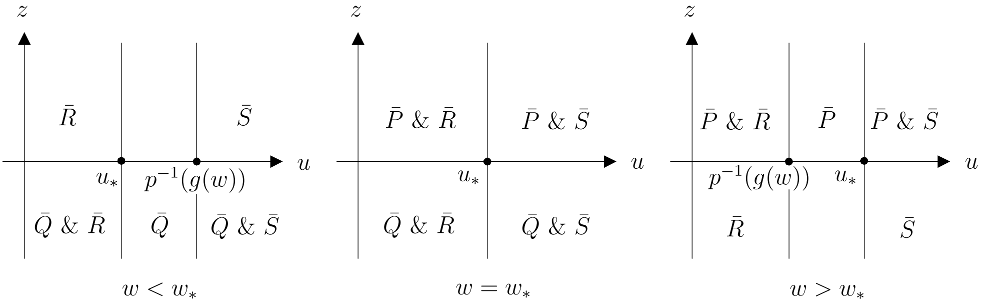

Fig. 8. Compression of Figs. 5, 6 and 7 in the subspace $v = 0$.

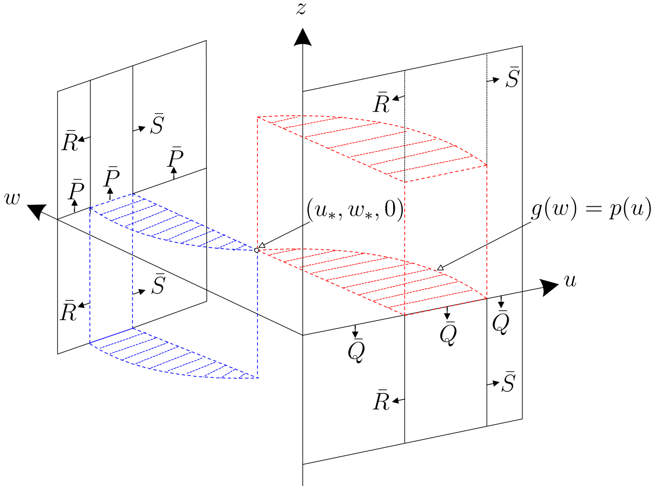

Fig. 9. The deformation retrace of Λ in the uwz-space.

Basing on the results of Claim II, now we proof that the set $F(\Sigma)$ is not simply connected.

Proof of Lemma 2.17. Following the idea of Dunbar (see Appendix II of [5]) with a slight modification, we prove the results as follows.

Let $\Lambda =: \overline{P} \cup \overline{Q} \cup \overline{R} \cup \overline{S}$. For any fixed w, the projection of the sets of Λ on the uvz-space is shown in Fig. 5 for $w < w_*$, Fig. 6 for $w = w_*$, and Fig. 7 for $w > w_*$. The coordinate v of each figure is then "compressed" in the subspace $v = 0$ respectively by a strong deformation retraction, as shown in Fig. 8. Synthesizing the three cases of Fig. 8 in uwz-space yields Fig. 9, where the deformation retraction of $\partial\Lambda$ is the boundary of the two wedges

$$\{u > u_*,\ g(w) - p(u) < 0,\ z > 0\} \quad \text{and} \quad \{0 < u < u_*,\ g(w) - p(u) > 0,\ z < 0\}.$$

The deformation retraction of $F(\partial\Sigma)$ must lie in the boundary of the two wedges. The results of Claim II imply that the boundary $\partial\Sigma$ will be mapped to a closed curve visiting R, P, S, and Q in turns at least once. It follows that the deformation retraction of $F(\partial\Sigma)$ surrounds the straight line $\{u = u_*,\ w = w_*,\ z \in \mathbb{R}\}$ in uwz-space, and cannot be homotopic to a point in W^- since W^- does not contain the point $(u = u_*,\ w = w_*,\ v = 0,\ z = 0)$. Hence, $F(\Sigma)$ is not simply connected. The proof is complete. □

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