Remarks on Diversity-Multiplexing Tradeoffs for Multiple-Access and Point-to-Point MIMO Channels

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***Abstract*—In this paper, we answer several open questions related to diversity-multiplexing tradeoffs (DMTs) for point-to-point and multiple-access (MAC) MIMO channels. By analyzing the DMT performance of a simple code, we show that the optimal MAC-DMT holds even when the channel remains fixed for less than $Kn_t + n_r - 1$ channel uses, where K is the number of users, n_t is the number of transmit antennas of each user, and n_r is the number of receive antennas at receiver. We also prove that the simple code is MAC-DMT optimal. A general code design criterion for constructing MAC-DMT optimal codes that is much more relaxed than the previously known design criterion is provided. Finally, by changing some design parameters, the simple code is modified for use in point-to-point MIMO channels. We show the modified code achieves the same DMT performance as the Gaussian random code.**

***Index Terms*—Diversity-multiplexing gain tradeoff (DMT), multiple access channel (MAC), multiple-input multiple-output (MIMO) channel, space-time block codes (STBCs).**

I. Introduction

It is known that using multiple antennas at both transmitting and receiving ends in a point-to-point multiple-input-multiple-output (MIMO) channel can increase the transmission rate and simultaneously provide higher diversity gain. Assuming there are n_t transmit antennas and n_r receive antennas, it has been shown that the ergodic channel capacity of such MIMO Rayleigh block fading channel is approximately $\min\{n_t, n_r\}\log_2 \mathrm{SNR}$ in bits per channel use [1], and the maximal achievable diversity gain is $n_t n_r$ [2], [3], provided that the channel remains fixed for at least n_t channel uses. Let $R = r\log_2 \mathrm{SNR}$ be the actual transmission rate, where r is termed the multiplexing gain. Zheng and Tse [4] showed there is a fundamental tradeoff between multiplexing gain r and diversity value d. Such tradeoff is commonly known as the diversity-multiplexing gain tradeoff (DMT) and is reproduced below.

Theorem 1 ([4]): In a MIMO Rayleigh block fading channel with n_t transmit and n_r receive antennas, assuming the transmitter transmits at multiplexing gain r, the maximal diversity gain $d^*(r)$ can be achieved by any coding schemes is a piecewise linear function connecting the points $(r, (n_t - r)(n_r - r))$ for $r = 0, 1, \cdots, \min\{n_t, n_r\}$, when the channel is fixed for at least $T \geq n_t + n_r - 1$ channel uses ■.

If the MIMO channel cannot hold static for at least n_t+n_r-1 channel uses, some lower bounds on DMT based on Gaussian random coding schemes are provided in [4]. By using space-time codes constructed from cyclic division algebra (CDA) [5], Elia *et al.* [6] proved that the same DMT $d^*(r)$ holds whenever the channel is static for at least $T \geq n_t$ channel uses. However, such result cannot be further improved, and the exact DMT for $T < n_t$ is still uncertain.

Beyond the DMT for point-to-point MIMO communication, Tse *et al.* [7] investigated the DMT for MIMO multiple-access (MAC) channel with K users, each having n_t transmit antennas and transmitting at multiplexing gain r. Assuming no cooperation among the users, they showed the following result.

Theorem 2 (MAC-DMT [7]): Let the MIMO-MAC channel be specified as the above; then given multiplexing gain r, the maximal diversity gain can be achieved by any coding schemes is

$$\begin{aligned} d^*_{n_t,n_r,K}(r) &= \min_{1\leq k\leq K} d^*_{kn_t,n_r}(kr) \\ &= \begin{cases} d^*_{n_t,n_r}(r), & \text{if } r \in \left[0, \min\left\{n_t, \frac{n_r}{K+1}\right\}\right] \\ d^*_{Kn_t,n_r}(Kr), & \text{if } r \in \left[\min\left\{n_t, \frac{n_r}{K+1}\right\}, \min\left\{n_t, \frac{n_r}{K}\right\}\right] \end{cases} \end{aligned} \quad (1)$$

whenever the MIMO-MAC channel holds static for at least $T \geq Kn_t + n_r - 1$ channel uses. ■

In both DMT results, Theorems 1 and 2, the proofs proceed by first establishing an upper bound on DMT based on an outage formulation, and then by using a Gaussian random coding scheme to show the converse based on a union bound argument. It should be noted that in both point-to-point and MAC cases the requirement on the channel coherence time T for the optimal DMT to hold actually comes from the union bound, not the outage. When $T \geq Kn_t + n_t - 1$, Coronel *et al.* [8] presented a criterion for constructing MAC-DMT optimal codes. For any coding schemes, let $\mathcal{E}_k$ denote the error event that only the messages from k users are erroneously decoded. Coronel *et al.* showed that for any k-subsets of users, $1 \leq k \leq K$, if $\Pr\{\mathcal{E}_k\}$ is upper bounded by the probability of the corresponding outage event formulated by these k users, i.e., if one can show

$$\Pr\{\mathcal{E}_k\} \dot{\leq} \Pr\left\{\log\det\left(I_{n_r} + \mathrm{SNR} H_k H_k^\dagger\right) \leq kr\log \mathrm{SNR}\right\} \quad (2)$$

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where $H = [H_{i_1} \cdots H_{i_k}]$ is the overall channel matrix and H_{i_j} is the $(n_r \times n_t)$ channel matrix of the i_jth user, then the code is MAC-DMT optimal. Notions of exponential inequalities $\dot{\geq}$, $\dot{\leq}$, $\dot{>}$, $\dot{<}$, and equality $\doteq$ are defined in [4]. Specifically, in terms of code design, the above criterion (2) means that the $(kn_t \times T)$ matrix obtained by vertically concatenating the signal matrices from k users must be of full row rank and should perhaps satisfy the nonvanishing determinant (NVD) criterion [6], [9]. This full NVD design criterion was explicitly given in [8].

The aim of this paper is to answer the following questions.

1) Is it possible to achieve the optimal MAC-DMT $d^*_{n_t,n_r,K}(r)$ when $T < Kn_t + n_r - 1$?
2) Is design criterion (2) necessary? or is it only sufficient?
3) In order to be MAC-DMT optimal, is it necessary for a code to satisfy the NVD criterion for any $(kn_t \times T)$ submatrix formed by any k-subsets of users?
4) In point-to-point MIMO channel, can one design a nonrandom DMT optimal code for $T < n_t$? Also, will the resulting DMT be the same as $d^*_{n_t,n_r}(r)$? In other words, when $T < n_t$, it relates to the question of whether the outage event will dominate the error performance.

The major contribution of this paper is not to provide constructions of codes having performance better than the previously known DMT optimal codes, for example, the CDA based codes [6], the Golden perfect codes [10], the max-order codes [11], or the multiblock codes [12]. Instead, we aim to address the above four questions that none of these codes can answer.

By analyzing the DMT performance of a very simple code, we will provide answers to all the above questions. We will consider a MIMO-MAC channel with $K = 2$ users, each having only $n_t = 1$ transmit antenna, and we will assume there are $n_r = 2$ receive antennas at receiving end. While Theorem 2 holds for codes with $T \geq Kn_t + n_r - 1 = 3$ channel uses, we will prove this simple code achieves the same optimal MAC-DMT $d^*_{1,2,2}(r)$ with only $T = 1$ or 2 channel uses. Furthermore, from the DMT analysis of this code we will see that criterion (2) is only sufficient, not necessary, and one does not need full NVD in order to achieve the optimal MAC-DMT. By slightly modifying the parameters of this code, we will show in the point-to-point MIMO scenario this simple code achieves the same DMT performance as the Gaussian random code over the fast Rayleigh fading channel, i.e., the case when $T = 1$, which relates to the fourth question in the above list.

This paper is organized as follows. In Section II we will present the simple code as well as the corresponding DMT performance analysis. Inferences from the DMT analysis will be given in Section III and will answer all the above questions of interest.

II. DMT Performance of a Simple Code

For simplicity, we will first present the code for use in a MIMO-MAC channel. For point-to-point MIMO channels, the same code can be easily modified and will be discussed in the next section.

Consider a MIMO-MAC channel with $K = 2$ users, each having $n_t = 1$ transmit antenna and transmitting at multiplexing gain r. Assume there are $n_r = 2$ receive antennas at receiving end. The code to be analyzed is the following:

$$\mathcal{S} = \left\{ S = \kappa \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} : s_{ij} \in \mathcal{A}(\mathrm{SNR}) \right\} \tag{3}$$

where

$$\mathcal{A}(\mathrm{SNR}) = \left\{ a + b\imath : |a|, |b| \leq \mathrm{SNR}^{\frac{r}{2}}, a, b \text{ odd} \right\} \tag{4}$$

$\imath = \sqrt{-1}$, and where

$$\kappa^2 \doteq \mathrm{SNR}^{1-r}. \tag{5}$$

Entries s_{ij} are independently drawn from the QAM set $\mathcal{A}(\mathrm{SNR})$. During transmission, the first user transmits the first row of S while the second user sends the second row. Clearly, the two users do not cooperate. Given $S \in \mathcal{S}$, the received signal matrix is

$$Y = HS + W \tag{6}$$

where $H = [\underline{h}_1 \underline{h}_2]$ is the overall (2×2) channel matrix whose entries are modeled as i.i.d. complex Gaussian random variables $\mathbb{C}\mathcal{N}(0,1)$ and where W is the (2×2) noise matrix. $\underline{h}_i$ is the channel vector associated with the $\imath$th user. We assume H is known to the receiver but is unknown to either of the users.

Obviously, the code $\mathcal{S}$ of (3) is uncoded since the entries are just plain QAM symbols with some scaling factor κ that is chosen to satisfy the power constraint $\mathbb{E}|\kappa s_{ij}|^2 \dot{\leq} \mathrm{SNR}$. Nevertheless, below we will show that this uncoded scheme $\mathcal{S}$ achieves the optimal MAC-DMT (cf. (1))

$$d^*_{1,2,2}(r) = \min\left\{ d^*_{1,2}(r), d^*_{2,2}(2r) \right\}$$

over the two-user MIMO-MAC channel.

To prove the claim, we will partition the error event $\mathcal{E}$ into several subevents $\mathcal{E}_1, \cdots, \mathcal{E}_n$ for some n, and analyze the probability of each. Then we will apply the union bound

$$\Pr\left\{ \mathcal{E} = \bigcup_{i=1}^{n} \mathcal{E}_i \right\} \leq \sum_{i=1}^{n} \Pr\{\mathcal{E}_i\}$$

to establish the claim. Although during the analysis some subevents can be combined, in order to be extra cautious we will analyze separately the error probabilities of these events. Below we distinguish five different kinds of error events.

a) Type-I Error Event: The first-type error event $\mathcal{E}_1$ corresponds to the case when only one entry in S is erroneously decoded. Without loss of generality, below we focus on a specific subevent of $\mathcal{E}_1$: for any $S \neq S' \in \mathcal{S}$

$$\mathcal{E}_{1,1} := \left\{ S - S' = \kappa \begin{bmatrix} d_{11} & 0 \\ 0 & 0 \end{bmatrix} : 0 \neq d_{11} = s_{11} - s'_{11} \right\}. \tag{7}$$

That is, events with only one $d_{ij} \neq 0$ can be considered the same as $\mathcal{E}_{1,1}$ and we have $\Pr\{\mathcal{E}_1\} \leq 4\Pr\{\mathcal{E}_{1,1}\}$.

To find out the error probability of $\mathcal{E}_{1,1}$, it suffices to note that the subcode $\mathcal{S}_{11} = \{\kappa s_{11} : s_{11} \in \mathcal{A}(\text{SNR})\}$ is exactly the CDA-based code proposed by Elia *et al.* [6] with $n_t = 1$ and $T = 1$. Hence we have

$$\Pr\{\mathcal{E}_{1,1}\} \dot{\leq} \text{SNR}^{-d^*_{1,2}(r)}. \tag{8}$$

To make the present paper self-contained, below we briefly highlight some key steps in proving (8). The proof actually follows from the fact that the bounded-distance decoder would make an error if the noise vector $\underline{w}_1$ has norm larger than half the minimum Euclidean distance, i.e., if

$$\|\underline{w}_1\|^2 \geq \min_{S-S' \in \mathcal{E}_1} \|H(S-S')\|^2 = \min_{d_{11}\neq 0} \|\kappa \underline{h}_1 d_{11}\|^2$$

where we have set the noise matrix $W = [\underline{w}_1 \underline{w}_2]$. By $\|A\|$ we mean the Frobenius norm of matrix A.

Given the channel vector $\underline{h}_1$, if

$$\min_{d_{11}\neq 0} \|\kappa \underline{h}_1 d_{11}\|^2 \dot{>} \text{SNR}^0$$

then it can be shown that $\Pr\left\{\|\underline{w}_1\|^2 \dot{>} \text{SNR}^0\right\} \doteq 0$. Hence

$$\begin{aligned}\Pr\{\mathcal{E}_{1,1}\} &\dot{\leq} \Pr\left\{\|\underline{w}_1\|^2 \dot{\leq} \text{SNR}^0 \Big| \min_{d_{11}\neq 0} \|\kappa \underline{h}_1 d_{11}\|^2 \dot{\leq} \text{SNR}^0\right\} \times \\ &\qquad \Pr\left\{\min_{d_{11}\neq 0} \|\kappa \underline{h}_1 d_{11}\|^2 \dot{\leq} \text{SNR}^0\right\} \\ &\leq \Pr\left\{\min_{d_{11}\neq 0} \|\kappa \underline{h}_1 d_{11}\|^2 \dot{\leq} \text{SNR}^0\right\} \\ &\overset{(a)}{\leq} \Pr\left\{\|\kappa \underline{h}_1\|^2 \dot{\leq} \text{SNR}^0\right\} \\ &\doteq \text{SNR}^{-(2-2r)^+} = \text{SNR}^{-d^*_{1,2}(r)}\end{aligned}$$

where (a) follows from $0 \neq d_{11} \in \mathbb{Z}[\imath]$ and $\|d_{11}\|^2 \geq 1$. The notation $(x)^+$ is defined as $(x)^+ = \max\{x, 0\}$. Thus, we conclude $\Pr\{\mathcal{E}_1\} \leq 4\Pr\{\mathcal{E}_{1,1}\} \dot{\leq} \text{SNR}^{-d^*_{1,2}(r)}$.

b) Type-II Error Event: The second-type is the event when only the messages from exactly one of the two users are erroneously decoded in both channel uses, i.e., the case when s_{i1} and s_{i2} are both erroneously decoded for $i = 1$ or 2. Clearly for this specific code $\mathcal{S}$ we have

$$\Pr\{\mathcal{E}_2\} \leq 4\Pr\{\mathcal{E}_1\} \dot{\leq} \text{SNR}^{-d^*_{1,2}(r)}. \tag{9}$$

The factor of 4 comes from that the probability of both s_{11} and s_{12} are erroneously decoded is at most twice of $\Pr\{\mathcal{E}_1\}$ and the same holds for s_{21} and s_{22} in error.

The previous two types of error events concerns the case when only one user is in error. The remaining ones will deal with situations when both users are in error.

c) Type-III Error Event: The third-type error event is the case when both users are in error but only one of the two transmissions is erroneously decoded. Again, without loss of generality, we focus on the case when the first transmission is erroneously decoded, i.e., it is of the following form:

$$\mathcal{E}_{3,1} := \left\{S - S' = \kappa \begin{bmatrix} d_{11} & 0 \\ d_{21} & 0 \end{bmatrix} : 0 \neq d_{i1} = s_{i1} - s'_{i1}\right\}. \tag{10}$$

and we have $\Pr\{\mathcal{E}_3\} \leq 2\Pr\{\mathcal{E}_{3,1}\}$.

We remark that in this case the difference matrix $\Delta S = S - S'$ is of rank 1 and does not satisfy the full NVD criterion given in [8] [and cf. (2)]. Furthermore, if one applies the conventional mismatched bound on product of eigenvalues [13], which is subsequently used as a key ingredient for proving the DMT optimality of CDA-based codes [6], the resulting bound on the DMT of present event $\mathcal{E}_{3,1}$ would be too loose to become any useful. Thus, below we will use a novel technique to analyze the DMT performance of this case.

Let $\underline{s}_1 = [s_{11}\ s_{21}]^\top$, $\underline{s}'_1 = [s'_{11}\ s'_{21}]^\top$, $s_{i1} \neq s'_{i1} \in \mathcal{A}(\text{SNR})$, where by $\underline{a}^\top$ we mean the usual transpose of vector $\underline{a}$. Set $\underline{d}_1 = \underline{s}_1 - \underline{s}'_1$. Then from the pairwise error probability analysis [2], [3], the probability of erroneously decoding $\underline{s}_1$ as $\underline{s}'_1$ is given by

$$\Pr\{\underline{s}_1 \to \underline{s}'_1\} \doteq \left[1 + \kappa^2 \|\underline{d}_1\|^2\right]^{-2}. \tag{11}$$

Fixing $\underline{s}_1$, we see that the number of $\underline{s}'_1$ such that $\|\underline{d}_1\|^2 \dot{\leq} \text{SNR}^z$ for some $0 \leq z \leq r$ can be upper bounded by

$$\left|\left\{\underline{s}'_1 : \|\underline{s}_1 - \underline{s}'_1\|^2 \dot{\leq} \text{SNR}^z\right\}\right| \leq \text{SNR}^{2z} \tag{12}$$

due to the choice of $\mathcal{A}(\text{SNR})$. The exponent $2z$ comes from the independent choices of s'_{11} and s'_{21}. Now from the union bound we see

$$\begin{aligned}\Pr\{\mathcal{E}_{3,1}\} &\leq \sum_{\underline{s}'_1 : \substack{s'_{11}\neq s_{11} \\ s'_{21}\neq s_{21}}} \Pr\{\underline{s}_1 \to \underline{s}'_1\} \\ &\doteq \sup_{0\leq z\leq r} \left[1 + \kappa^2 \text{SNR}^z\right]^{-2} \text{SNR}^{2z} \\ &\doteq \kappa^{-4} = \text{SNR}^{-(2-2r)^+} = \text{SNR}^{-d^*_{1,2}(r)}.\end{aligned}$$

Thus we conclude that the diversity gain achieved by $\mathcal{S}$ in $\mathcal{E}_3$ equals $d^*_{1,2}(r)$.

d) Type-IV Error Event: The fourth error event concerns the case when the messages from both users are erroneously decoded in both transmissions, but the difference matrix has only rank 1. That is, $\mathcal{E}_4$ can be formulated as

$$\mathcal{E}_4 := \left\{S - S' = \kappa \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} : \begin{matrix} 0 \neq d_{ij} = s_{ij} - s'_{ij}, \\ \text{rank}(S - S') = 1 \end{matrix}\right\}. \tag{13}$$

The conditions of $d_{ij} \neq 0$ for all $i, j = 1, 2$ and $\text{rank}(S - S') = 1$ distinguish this case from the remaining one, the type-V error event, the case when $\text{rank}(S - S') = 2$. Specifically, if $d_{ij} = 0$ for only one pair of i and j, then the difference matrix would have rank equal to 2. The same also applies to the cases of $d_{12} = d_{21} = 0$ or $d_{11} = d_{22} = 0$. On the other hand, if $d_{11} = d_{21} = 0$ then it is equivalent to $\mathcal{E}_3$. Similarly, the case of $d_{11} = d_{12} = 0$ reduces to type-II.

Analyzing the probability of $\mathcal{E}_4$ might be the most troublesome as neither the weighted pairwise error probability technique used in analyzing type-II nor the conventional techniques [6] used for analyzing the DMT performance of CDA-based codes would work in this case. Furthermore, same as $\mathcal{E}_3$, this event again belongs to the situation when both users are in error but the difference matrix is singular, a situation violating the full NVD design criterion (2).

Nevertheless, following the same bounded distance argument as in type-I it can be shown that event $\mathcal{E}_4$ would occur if the noise matrix W has norm larger than half the minimum Euclidean distance $\min_{S\neq S':\mathcal{E}_4}\|H(S-S')\|$. Next, let $\underline{d}_1=[d_{11}d_{21}]^\top$ and $\underline{d}_2=[d_{12}\,d_{22}]^\top$; then note

$$\min_{S\neq S':\mathcal{E}_4}\|H(S-S')\|^2=\min_{\underline{d}_1,\underline{d}_2:\mathcal{E}_4}\left[\|\kappa H\underline{d}_1\|^2+\|\kappa H\underline{d}_2\|^2\right] \geq \min_{\underline{d}_1,\underline{d}_2:\mathcal{E}_4}\|\kappa H\underline{d}_1\|^2.$$

Thus, we see

$$\begin{aligned}\Pr\{\mathcal{E}_4\}&\dot{\leq}\Pr\left\{H:\min_{\underline{d}_1,\underline{d}_2:\mathcal{E}_4}\left[\|\kappa H\underline{d}_1\|^2+\|\kappa H\underline{d}_2\|^2\right]\dot{\leq}\mathrm{SNR}^0\right\}\\&\leq\Pr\left\{H:\min_{\underline{d}_1,\underline{d}_2:\mathcal{E}_4}\|\kappa H\underline{d}_1\|^2\dot{\leq}\mathrm{SNR}^0\right\}\\&\doteq\Pr\{\mathcal{E}_{3,1}\}\dot{\leq}\mathrm{SNR}^{-d^*_{1,2}(r)}.\end{aligned}$$

Remark 1: Another quick-and-dirty way to show the above is to note the relation between events $\mathcal{E}_3$ and $\mathcal{E}_4$, and it can be seen that

$$\Pr\{\mathcal{E}_4\}\leq 1-(1-\Pr\{\mathcal{E}_{3,1}\})^2\leq 2\Pr\{\mathcal{E}_{3,1}\}\dot{\leq}\mathrm{SNR}^{-d^*_{1,2}(r)}.$$

The reason for this method being dirty is that the error probability calculation does not capture the fact that the channel remains static for two consecutive channel uses. It relies rather on the ergodicity of channel variation. ■

e) Type-V Error Event: Finally, the last error event addresses the case when both users are in error but the difference matrix is of full rank, i.e., it is of the following form:

$$\mathcal{E}_5:=\left\{S-S'=\kappa\begin{bmatrix}d_{11}&d_{12}\\d_{21}&d_{22}\end{bmatrix}:\mathrm{rank}(S-S')=2\right\}. \tag{14}$$

Analyzing the probability of $\mathcal{E}_5$ is relatively easy since the matrix

$$D=\begin{bmatrix}d_{11}&d_{12}\\d_{21}&d_{22}\end{bmatrix}$$

has determinant in $\mathbb{Z}[\imath]$. Therefore, the code satisfies the full NVD criterion in $\mathcal{E}_5$. It can be shown along similar lines as in [6] that

$$\begin{aligned}\Pr\{\mathcal{E}_5\}&\dot{\leq}\Pr\left\{\log\det\left(I_2+\mathrm{SNR}HH^\dagger\right)\leq 2r\log\mathrm{SNR}\right\}\\&\doteq\mathrm{SNR}^{-d^*_{2,2}(2r)}.\end{aligned}$$

Overall, we have proved the following result.

Theorem 3: The error probability of the simple code $\mathcal{S}$ is

$$\Pr\{\mathcal{E}\}\leq\sum_{i=1}^{5}\Pr\{\mathcal{E}_i\}\doteq\max\left\{\mathrm{SNR}^{-d^*_{1,2}(r)},\mathrm{SNR}^{-d^*_{2,2}(2r)}\right\}$$

and the diversity gain is

$$d(r)=\min\left\{d^*_{1,2}(r),d^*_{2,2}(2r)\right\}.$$

Hence the simple code $\mathcal{S}$ is MAC-DMT optimal. ■

III. Inferences From DMT Analysis of Code $\mathcal{S}$

In this section we will take a closer look at the results presented in the previous section and then address the four open questions posed in Section I.

A. Alternative Design Criterion for MAC-DMT Optimal Codes

Recall that among the five types of error events analyzed in the previous section, only $\mathcal{E}_4$ and $\mathcal{E}_5$ belong to the case when both users are in error. However, it was proved that $\mathcal{E}_4$ achieves diversity gain $d^*_{1,2}(r)$ rather than $d^*_{2,2}(2r)$, which was required by the design criterion (2). We also note that events $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, and $\mathcal{E}_4$ all correspond to the case when the difference matrix $(S-S')$ is of rank 1, and all achieve the same diversity order $d^*_{1,2}(r)$. Thus, below we summarize this observation and provide an alternative, yet much relaxed, design criterion for constructing MAC-DMT optimal codes.

Theorem 4 (Relaxed Design Criterion): In a MIMO-MAC channel with K users, each having n_t transmit antennas and transmitting at multiplexing gain r, let $\mathcal{S}_i$ be the space-time code of the ith user, and let $\mathcal{S}=\mathcal{S}_1\times\cdots\mathcal{S}_K$ be the overall code obtained by vertically concatenating the code matrices from all users. Let $\mathcal{E}_{n,m}$ denote the error event that n users are in error but the difference matrix $(S-S')$ has only rank mn_t with $1\leq m\leq n$. If for all m and n

$$\Pr\{\mathcal{E}_{n,m}\}\dot{\leq}\mathrm{SNR}^{-d^*_{mn_t,n_r}(mr)} \tag{15}$$

then $\mathcal{S}$ is optimal in MAC-DMT. ■

The above design criterion is weaker than (2) since (2) excludes the possibility of having error event $\mathcal{E}_{n,m}$ when $m<n$, that is, (2) requires whenever n users are in error, the difference matrix must be of rank nn_t. Codes satisfying (15) must be MAC-DMT optimal since it follows from [7] that $\mathrm{SNR}^{-d^*_{mn_t,n_r}(mr)}\leq\mathrm{SNR}^{-d^*_{n_t,n_r}(r)}$ and $\mathrm{SNR}^{-d^*_{mn_t,n_r}(mr)}\leq\mathrm{SNR}^{-d^*_{Kn_t,n_r}(Kr)}$ for any $1\leq m\leq K$. Hence the events $\mathcal{E}_{n,m}$ with $1\leq m<n\leq K$ are not dominant in the union bound, and the requirement on the error performance of these error events can be much relaxed without worsening the overall DMT performance.

Thus, in this section we have answered the second and the third questions posed in Section I. We showed that criterion (2) is only sufficient, not necessary, and it is unnecessary to design codes to meet the full NVD criterion. Moreover, we have provided in Theorem 4 an alternative, yet much relaxed, code design criterion for constructing MAC-DMT optimal codes.

B. Requirement on Minimal Channel Coherence Time

In Theorem 2 it was shown that in the MIMO-MAC channel with $K = 2$ users, $n_t = 1$ and $n_r = 2$, the MAC-DMT $d^*_{n_t,n_r,K}(r)$ holds whenever the channel remains fixed for $T \geq Kn_t + n_r - 1 = 3$ channel uses. The requirement on T was improved by the simple code $\mathcal{S}$ analyzed in Section II. We proved that $\mathcal{S}$ achieves the same MAC-DMT optimality with only $T = 2$ channel uses, and hence improves the result on minimal channel coherence time required by Theorem 2. In particular, we note that in this specific channel we actually have

$$d^*_{1,2}(r) \leq d^*_{2,2}(2r), \text{ for all } 0 \leq r \leq 1.$$

In other words, the single-user performance dominates the entire region of $r \in [0,1]$, and there is no region of antenna-pooling [7] in this case.

From the analyses presented in the previous section, we can further strengthen the MAC-DMT result to the following. The vector code

$$\mathcal{S}_{\text{vec}} = \left\{ \kappa \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} : s_i \in \mathcal{A}(\text{SNR}) \right\} \tag{16}$$

that is a subcode of $\mathcal{S}$ given in Section II and is obtained by taking only the first column of code matrices in $\mathcal{S}$, is in fact MAC-DMT optimal. To see this, from the error events $\mathcal{E}_1$ and $\mathcal{E}_3$, the error probability of $\mathcal{S}_{\text{vec}}$ is

$$\Pr\{\mathcal{E}(\mathcal{S}_{\text{vec}})\} \leq \Pr\{\mathcal{E}_1\} + \Pr\{\mathcal{E}_3\} \dot{\leq} \text{SNR}^{-(2-2r)^+}. \tag{17}$$

It then implies that the MAC-DMT $d^*_{1,2,2}(r)$ holds even for fast fading channel, i.e., the case when $T = 1$. This answers the first question posed in Section I.

C. Point-To-Point MIMO Channel

The vector code $\mathcal{S}_{\text{vec}}$ of (16) can be easily modified for use in a point-to-point MIMO channel. To this end, let

$$\mathcal{A}_{SU}(\text{SNR}) = \left\{ a + b\imath : |a|, |b| \leq \text{SNR}^{\frac{r}{4}}, a, b \text{ odd} \right\} \tag{18}$$

and set

$$\mathcal{S}_{\text{SU}} = \left\{ \kappa_{SU} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} : s_i \in \mathcal{A}_{SU}(\text{SNR}) \right\} \tag{19}$$

where $\kappa^2_{SU} = \text{SNR}^{1-\frac{r}{2}}$. In other words, $\mathcal{S}_{\text{SU}}$ can be obtained from $\mathcal{S}_{\text{vec}}$ when both users transmit at multiplexing gain $\frac{r}{2}$ such that the overall multiplexing gain achieved by $\mathcal{S}_{\text{SU}}$ equals r. Because of this, the error probability of $\mathcal{S}_{\text{SU}}$ is upper bounded by

$$\Pr\{\mathcal{E}(\mathcal{S}_{\text{SU}})\} \dot{\leq} \text{SNR}^{-(2-r)^+} \tag{20}$$

and the diversity gain equals $d(r) = 2 - r$ for $r \in [0,2]$. The maximal multiplexing gain $r_{\max} = 2$, same as that indicated by the ergodic channel capacity [1], [4] of this channel. However, the resulting DMT is only $d(r) = 2 - r$, much worse than the optimal DMT $d^*_{2,2}(r)$. It is understandable since the latter requires the channel to be fixed for at least two channel uses, while the former changes from one channel use to another, and there is no coding across independent channel uses.

The maximal diversity gain $d_{\max}$ achieves by $\mathcal{S}_{\text{SU}}$ is given by $d(0) = 2$, which is the same for any such vector codes. This can be easily seen from the pairwise error probability argument. Taking any fixed vector coding schemes that do not vary with SNR, the resulting multiplexing gain equals 0 and the maximal possible rank distance between any pairs of distinct code vectors equals 1. Hence the resulting diversity order is 2 since there are two receive antennas. Therefore, we conclude that for $T < n_t$ the maximal diversity order is $n_r T$, and the resulting DMT can never be the same as the optimal one $d^*_{n_t,n_r}(r)$, where the maximal diversity order equals $n_t n_r$. Furthermore, it means that the outage event does not dominate the error performance when $T < n_t$. These answer the fourth question posed in Section I.

While the code $\mathcal{S}_{\text{SU}}$ is not optimal in terms of $d^*_{2,2}(r)$, in [4] Zheng and Tse proved the following result.

Theorem 5 ([4]): For a point-to-point MIMO channel with n_t transmit antennas, n_r receive antennas, and $T < n_t + n_r - 1$, the Gaussian random coding scheme achieves the following DMT:

$$d_G(r) = \inf_{\underline{\alpha} \in \mathcal{G}} \left\{ \left[\sum_{i=1}^{M} (2i - 1 + |n_t - n_r|)\alpha_i \right] + T \left(\sum_{i=1}^{M} (1 - \alpha_i) - r \right) \right\} \tag{21}$$

where $M = \min\{n_t, n_r\}$, $\underline{\alpha} = [\alpha_1 \cdots \alpha_M]^\top$, and the constraint $\mathcal{G}$ is given by

$$\mathcal{G} := \left\{ \underline{\alpha} \in [0,1]^M : \alpha_1 \geq \cdots \geq \alpha_M, \sum_{i=1}^{M} (1 - \alpha_i) > r \right\}.$$

■

Substituting $n_t = n_r = 2$ and $T = 1$ into (21) gives

$$d_G(r) = \inf_{\underline{\alpha} \in \mathcal{G}} \{2 - r + 2\alpha_2\} = 2 - r.$$

Thus, we see that the DMT achieved by Gaussian random coding scheme is the same as that achieved by the deterministic code $\mathcal{S}_{\text{SU}}$. Hence $\mathcal{S}_{\text{SU}}$ is DMT optimal in the case of $T = 1$.

Finally, we remark that the well-known Alamouti scheme of orthogonal space-time codes [14] was shown to achieve DMT at $d_A(r) = 4(1-r)^+$ for $r \in [0,2]$ by Zheng and Tse [4]. Thus we see for multiplexing gain $r \geq \frac{2}{3}$ the Alamouti code would perform worse than the uncoded $\mathcal{S}_{\text{SU}}$ in the DMT sense.

IV. Conclusion

In this paper, we have answered all the four open questions posed in Section I. We showed it is possible to achieve the optimal MAC-DMT with $T < Kn_t + n_r - 1$, and previously known full NVD design criterion for MAC-DMT optimal codes [8] is only sufficient. A simple code not satisfying this full NVD criterion is provided, and we proved it is still MAC-DMT optimal. In view of this, we have provided an alternative, yet much more relaxed, criterion for constructing MAC-DMT optimal codes. This simple code is also modified for use in point-to-point MIMO channels. We showed the modified code is optimal in DMT in the sense that it achieves the same DMT performance as the Gaussian random coding schemes.

Below we state without proof a generalization of the results in this paper.

Theorem 6: Consider a MIMO-MAC channel with n users, each having $n_t = 1$ transmit antenna and transmitting at multiplexing gain r. Assume there are n receive antennas at receiver. Then the following overall code:

$$\mathcal{S}_n = \left\{ \kappa \begin{bmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & \ddots & \vdots \\ s_{n1} & \cdots & s_{nn} \end{bmatrix} : s_{ij} \in \mathcal{A}(\mathrm{SNR}) \right\}$$

achieves the optimal MAC-DMT $d^*_{1,n,n}(r)$ with $T = n$ channel uses, where κ and $\mathcal{A}(\mathrm{SNR})$ are defined as before (cf. Section II). Furthermore, the same result holds for the vector code $\mathcal{S}_{n,\mathrm{vec}}$ obtained by taking the first column of code matrices in $\mathcal{S}_n$. Hence $d^*_{1,n,n}(r)$ holds for $T = n_t = 1$ as well. Finally, by setting the multiplexing gain at $\frac{r}{n}$ in $\mathcal{S}_{n,\mathrm{vec}}$ the resulting code achieves DMT $d(r) = n - r$ in the point-to-point MIMO channel with $n_t = n_r = n$ and $T = 1$. It is the same DMT performance achieved by Gaussian random coding schemes. ■

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