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Hölder estimate for non-uniform parabolic equations in highly heterogeneous media

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ABSTRACT

Uniform bound for the solutions of non-uniform parabolic equations in highly heterogeneous media is concerned. The media considered are periodic and they consist of a connected high permeability sub-region and a disconnected matrix block subset with low permeability. Parabolic equations with diffusion depending on the permeability of the media have fast diffusion in the high permeability sub-region and slow diffusion in the low permeability subset, and they form non-uniform parabolic equations. Each medium is associated with a positive number ϵ, denoting the size ratio of matrix blocks to the whole domain of the medium. Let the permeability ratio of the matrix block subset to the connected high permeability sub-region be of the order $\epsilon^{2\tau}$ for $\tau \in (0, 1]$. It is proved that the Hölder norm of the solutions of the above non-uniform parabolic equations in the connected high permeability sub-region are bounded uniformly in ϵ. One example also shows that the Hölder norm of the solutions in the disconnected subset may not be bounded uniformly in ϵ.

1. Introduction

Uniform Hölder estimate for the solutions of non-uniform parabolic equations in highly heterogeneous media is presented. The equations have many applications in multi-phase flows in porous media, the stress in composite materials, and so on (see [1–4] and references therein). The media $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) contain a connected high permeability sub-region and a disconnected matrix block subset with low permeability. Let $\partial\Omega$ denote the boundary of Ω, $\epsilon \in (0, 1)$, $\Omega(2\epsilon) \equiv \{x \in \Omega |\ \mathrm{dist}(x, \partial\Omega) > 2\epsilon\}$, and $Y \equiv (0, 1)^n$ denote a cell consisting of a sub-domain Y_m completely surrounded by another connected sub-domain Y_f ($\equiv Y \setminus \overline{Y}_m$). The disconnected matrix block subset of Ω is $\Omega_m^\epsilon \equiv \{x |\ x \in \epsilon(Y_m + j) \subset \Omega(2\epsilon)$ for some $j \in \mathbb{Z}^n\}$ with boundary $\partial\Omega_m^\epsilon$, and the connected sub-region is $\Omega_f^\epsilon \equiv \Omega \setminus \overline{\Omega}_m^\epsilon$. The non-uniform parabolic equations (see [4]) in $[0, T] \times \Omega$ are

$$\begin{cases} \partial_t U_\epsilon - \nabla \cdot (\Lambda_\tau^\epsilon \nabla U_\epsilon) = F_\epsilon & \text{in } (0, T] \times \Omega, \\ U_\epsilon = 0 & \text{on } (0, T] \times \partial\Omega, \\ U_\epsilon = U_{\epsilon,0} & \text{in } \{0\} \times \Omega, \end{cases} \tag{1.1}$$

where $\tau \in (0, \infty)$, $\Lambda_\tau^\epsilon \equiv \begin{cases} \mathbf{K}_\epsilon & \text{in } \Omega_f^\epsilon \\ \epsilon^{2\tau}\mathbf{k}_\epsilon & \text{in } \Omega_m^\epsilon \end{cases}$ (depending on the permeability of Ω), and both $\mathbf{K}_\epsilon$, $\mathbf{k}_\epsilon$ are positive smooth functions in Ω.

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Since $\epsilon \in (0,1)$, equations in (1.1) are non-uniform parabolic equations with discontinuous coefficients. In [5], existence of solution in $W_p^{2,1}([0,T]\times\Omega)$ space for uniform parabolic equations with discontinuous coefficients can be found. For non-uniform parabolic equations with smooth coefficients, existence of solution in $C^{2,\alpha}([0,T]\times\Omega)$ space was studied in [6]. It is also known that if F_ϵ, $U_{\epsilon,0}$ are smooth, a piecewise regular solution of (1.1) exists uniquely for each ϵ and, by the energy method, the H^1 norm of the parabolic solution of (1.1) in the connected high permeability sub-region is bounded uniformly in ϵ [2,7]. Hölder continuity of the parabolic solution of (1.1) in $[0,T]\times\Omega$ is proved for each ϵ [7], but the Hölder norm of the solution may go to infinity as $\epsilon \searrow 0$. In [4], convergence of solution of (1.1) in $L^\infty([0,T];L^2(\Omega))$ space as $\epsilon \searrow 0$ was obtained. Many studies of the uniform estimate in ϵ for the solutions of the elliptic equations in heterogeneous media had been done [2,3,8–11], but not the case for parabolic equations. The existence of piecewise regular solutions for elliptic diffraction equations in Hilbert space was considered in [2,9]. The uniform Lipschitz estimate in ϵ for a Laplace equation in perforated domains was given in [11], and a uniform L^p estimate in ϵ of the same problem was considered in [10]. A Lipschitz estimate for uniform elliptic equations was studied in [3]. Uniform Hölder and Lipschitz estimates in ϵ for uniform elliptic equations in periodic domains were obtained in [8].

This work is to present a uniform Hölder estimate in ϵ for the solutions of the non-uniform parabolic equations with discontinuous coefficients. More precisely, the Hölder norm of the non-uniform parabolic solutions in the connected high permeability sub-region is shown to be bounded uniformly in ϵ. However, the Hölder norm of the solutions in the disconnected subset may not be bounded uniformly in ϵ. This is due to the non-zero source in the disconnected subset. In Section 2, we present one example to show that. Certainly this is different from usual uniform parabolic equation cases, in which solutions are regular in the whole time–space domains. From the proof, we can see that the results are established for complex-valued solutions. On the other hand, one also notes that a complex-valued solution of (1.1) with complex-valued coefficients may be discontinuous or even unbounded [12]. A similar case could be found in elliptic equations with complex-valued coefficients (see [13]). It seems that the techniques used here could be used to study more general systems of elliptic type and parabolic type, and this will be pursued later. Some related uniform regularity results in the case of elliptic systems can be seen in [14,15].

The rest of the work is organized as follows: Notation and main results are stated in Section 2. The main results are proved in Section 3 based on semigroup theory and on uniform Hölder estimate in ϵ for non-uniform elliptic equations. To apply semigroup theory, an infinitesimal generator of an analytic semigroup from elliptic equations is required. So a $W^{2,p}$ estimate for solutions of elliptic diffraction equations is derived in Section 4. Two convergence results for solutions of non-uniform elliptic equations are shown in Section 5. By results in Section 5, a uniform Hölder estimate in ϵ for non-uniform elliptic solutions is proved in Section 6.

2. Notation and main result

Let $\overline{\Omega}$ be the closure of the domain Ω. Let $L^p(\Omega)$ (resp. $H^k(\Omega)$, $W^{k,p}(\Omega)$) denote a complex Sobolev space with norm $\|\cdot\|_{L^p(\Omega)}$ (resp. $\|\cdot\|_{H^k(\Omega)}$, $\|\cdot\|_{W^{k,p}(\Omega)}$), $W_0^{1,p}(\Omega) \equiv \{\varphi \in W^{1,p}(\Omega) | \varphi|_{\partial\Omega} = 0\}$, $H_0^1(\Omega) \equiv W_0^{1,2}(\Omega)$, $C_0^\infty(\Omega)$ be the set containing all infinite differentiable functions with compact support in Ω, $C(\overline{\Omega})$ consist of all continuous functions in $\overline{\Omega}$ with norm $\|\cdot\|_{C(\overline{\Omega})}$, $C^\sigma(\overline{\Omega})$ (resp. $C^{1,\sigma}(\overline{\Omega})$) denote a Hölder space with norm $\|\cdot\|_{C^\sigma(\overline{\Omega})}$ (resp. $\|\cdot\|_{C^{1,\sigma}(\overline{\Omega})}$), and $[\varphi]_{C^\sigma(\overline{\Omega})}$ (resp. $[\varphi]_{C^{1,\sigma}(\overline{\Omega})}$) denote the Hölder semi-norm of φ (resp. $\nabla\varphi$) for $k \geq -1$, $p \in [1,\infty]$, and $\sigma \in (0,1]$ [16,17]. If φ is a complex function, $\overline{\varphi}$ denotes its complex conjugate. If $\mathbf{B}_1$ and $\mathbf{B}_2$ are two Banach spaces, $\mathcal{L}(\mathbf{B}_1,\mathbf{B}_2)$ is the set of all bounded linear maps from $\mathbf{B}_1$ to $\mathbf{B}_2$ with norm $\|\cdot\|_{\mathcal{L}(\mathbf{B}_1,\mathbf{B}_2)}$. For any Banach space $\mathbf{B}$, define $\|\varphi_1,\varphi_2,\ldots,\varphi_m\|_{\mathbf{B}} \equiv \|\varphi_1\|_{\mathbf{B}} + \|\varphi_2\|_{\mathbf{B}} + \cdots + \|\varphi_m\|_{\mathbf{B}}$, denote its dual space by $\mathbf{B}'$, and denote the pairing between $\mathbf{B}$ and its dual space $\mathbf{B}'$ by $\langle\cdot,\cdot\rangle_{\mathbf{B},\mathbf{B}'}$. $L^\infty(I;\mathbf{B}) \equiv \{\varphi : I \to \mathbf{B} | \sup_{t\in I}\|\varphi(t)\|_{\mathbf{B}} < \infty\}$. The function spaces $C(I;\mathbf{B})$, $C^\sigma(I;\mathbf{B})$ for $\sigma \in (0,1]$ and an interval $I \subset \mathbb{R}$ are defined as those in pages 1, 3 [18]. $B_r(x)$ represents a ball centered at x with radius r. For any domain $\mathbb{D}$, $\overline{\mathbb{D}}$ is the closure of $\mathbb{D}$, $\partial\mathbb{D}$ is the boundary of $\mathbb{D}$, $\mathbb{D}/r \equiv \{x | rx \in \mathbb{D}\}$, $|\mathbb{D}|$ is the volume of $\mathbb{D}$, and $\mathcal{X}_{\mathbb{D}}$ is the characteristic function on $\mathbb{D}$. For any $\varphi \in L^1(B_r(x)\cap\Omega)$,

$$(\varphi)_{x,r} \equiv \fint_{B_r(x)\cap\Omega} \varphi(y)dy \equiv \frac{1}{|B_r(x)\cap\Omega|}\int_{B_r(x)\cap\Omega} \varphi(y)dy.$$

For any $p \in (1,\infty)$, $\tau \in (0,\infty)$, and $\epsilon \in (0,1)$,

$$\begin{cases} \mathcal{A}_\tau^\epsilon\varphi \equiv -\nabla\cdot(\Lambda_\tau^\epsilon\nabla\varphi), \\ \mathbb{B}_p(\mathcal{A}_\tau^\epsilon) \equiv \left\{\varphi \in W_0^{1,p}(\Omega) |\, \varphi \in W^{2,p}(\Omega_f^\epsilon)\cap W^{2,p}(\Omega_m^\epsilon), \mathbf{K}_\epsilon\nabla\varphi\cdot\vec{\mathbf{n}}^\epsilon|_{\partial\Omega_m^\epsilon} = \epsilon^{2\tau}\mathbf{k}_\epsilon\nabla\varphi\cdot\vec{\mathbf{n}}^\epsilon|_{\partial\Omega_m^\epsilon}\right\}, \end{cases}$$

where $\vec{\mathbf{n}}^\epsilon$ is a normal vector on $\partial\Omega_m^\epsilon$. It is not difficult to see that $\mathbb{B}_p(\mathcal{A}_\tau^\epsilon)$ with norm $\|\varphi\|_{\mathbb{B}_p(\mathcal{A}_\tau^\epsilon)} \equiv \|\mathcal{A}_\tau^\epsilon\varphi\|_{L^p(\Omega)}$ is a normed space. Let $\overline{\mathbb{B}_p(\mathcal{A}_\tau^\epsilon)}$ denote the closure of $\mathbb{B}_p(\mathcal{A}_\tau^\epsilon)$ in L^p space (we shall see $\overline{\mathbb{B}_p(\mathcal{A}_\tau^\epsilon)} = L^p(\Omega)$ from Lemma 3.4). For any $\lambda,\nu > 0$, we define

$$\hat{\mathbf{K}}_{\lambda,\nu}(x) \equiv \mathbf{K}_\lambda(\nu x) \quad \text{and} \quad \hat{\mathbf{k}}_{\lambda,\nu}(x) \equiv \mathbf{k}_\lambda(\nu x). \tag{2.1}$$

Let $Y_m \subset \mathbf{D} \subset Y = Y_f \cup \overline{Y}_m$ satisfy

$$\min\{\mathrm{dist}(Y_m,\partial\mathbf{D}), \mathrm{dist}(\mathbf{D},\partial Y)\} > 0. \tag{2.2}$$

We assume that there are $\epsilon, \sigma, \mathbf{e} \in (0,1)$, $\tau \in (0,\infty)$, and $\delta, \alpha, \beta > 0$ such that

A1. Ω and Y_m are $C^{1,\mathbf{e}}$ domains,

A2. $\mathbf{K}_\epsilon, \mathbf{k}_\epsilon \in W^{1,\infty}(\Omega)$, $\mathbf{K}_\epsilon, \mathbf{k}_\epsilon \in (\alpha,\beta)$, $\|\hat{\mathbf{K}}_{\epsilon,\epsilon}\|_{W^{1,\infty}(\Omega/\epsilon)}$ is bounded independent of ϵ, and there is a set $\{\alpha_{\epsilon,j} \in (\alpha,\beta) | \epsilon \in (0,1), j \in \mathbb{Z}^n\}$ satisfying

$$\|\hat{\mathbf{K}}_{\epsilon,\epsilon} - \alpha_{\epsilon,j}\|_{W^{1,\infty}((\mathbf{D}\setminus\overline{Y}_m+j)\cap\Omega/\epsilon)} + \|\hat{\mathbf{k}}_{\epsilon,\epsilon} - \alpha_{\epsilon,j}\|_{W^{1,\infty}((Y_m+j)\cap\Omega/\epsilon)} \le c\alpha_{\epsilon,j}$$

where c is small and depends on Y_m,

A3. $F_\epsilon \in C^\sigma([0,T]; L^{n+\delta}(\Omega))$, $\mathcal{A}^\epsilon_\tau U_{\epsilon,0} - F_\epsilon|_{t=0} \in \overline{\mathbb{B}_{n+\delta}(\mathcal{A}^\epsilon_\tau)}$, $U_{\epsilon,0} \in \mathbb{B}_{n+\delta}(\mathcal{A}^\epsilon_\tau)$.

The main results are:

Theorem 2.1. *Under* A1–A3*, the solution of* (1.1) *satisfies*

$$\|U_\epsilon\|_{C^1([0,T];L^{n+\delta}(\Omega))} + \|U_\epsilon\|_{C([0,T];\mathbb{B}_{n+\delta}(\mathcal{A}^\epsilon_\tau))} \le c\left(\|U_{\epsilon,0}\|_{\mathbb{B}_{n+\delta}(\mathcal{A}^\epsilon_\tau)} + \|F_\epsilon\|_{C^\sigma([0,T];L^{n+\delta}(\Omega))}\right),$$

where c is a constant independent of ϵ, τ.

Theorem 2.2. *Under* A1–A3 *and $\tau \in (0,1]$, the solution of* (1.1) *satisfies*

$$\|U_\epsilon\|_{C^1([0,T];L^{n+\delta}(\Omega))} + \|U_\epsilon\|_{C([0,T];C^\mu(\overline{\Omega}^\epsilon_f))} + \sup_{\substack{j\in\mathbb{Z}^n \\ \epsilon(Y_m+j)\subset\Omega^\epsilon_m}} \epsilon^\tau \|U_\epsilon\|_{C([0,T];C^\mu(\epsilon(\overline{Y}_m+j)))}$$
$$\le c\left(\|U_{\epsilon,0}\|_{\mathbb{B}_{n+\delta}(\mathcal{A}^\epsilon_\tau)} + \|F_\epsilon\|_{C^\sigma([0,T];L^{n+\delta}(\Omega))}\right), \tag{2.3}$$

where c is a constant independent of ϵ. Here $\mu \in (0, \frac{\delta}{2(n+\delta)})$ is a constant depending on $n, \delta, \sigma, \alpha, \beta, Y_f, \Omega$. Besides, there is a $\nu \in (0,\mu)$ such that

$$\|U_\epsilon\|_{C^\nu([0,T]\times\overline{\Omega}^\epsilon_f)} \le c\left(\|U_{\epsilon,0}\|_{\mathbb{B}_{n+\delta}(\mathcal{A}^\epsilon_\tau)} + \|F_\epsilon\|_{C^\sigma([0,T];L^{n+\delta}(\Omega))}\right), \tag{2.4}$$

where c is a constant independent of ϵ.

In (2.3), we do not prove that the Hölder norm of the solution of (1.1) in the disconnected subset is bounded uniformly in ϵ. We now give one example to show that if the source F_ϵ is not zero in the disconnected subset, it is really the case. Suppose $\varphi \in C_0^\infty(\mathbb{R}^n)$ has support in Y_m. Define, for $\epsilon \in (0,1)$,

$$\varphi_\epsilon(x) \equiv \begin{cases} \varphi\left(\dfrac{x}{\epsilon} - j\right) & \text{if } x \in \epsilon(Y_m+j) \subset \Omega(2\epsilon) \text{ for some } j \in \mathbb{Z}^n, \\ 0 & \text{elsewhere,} \end{cases}$$

$$\Phi_\epsilon(t,x) \equiv e^{-t}\varphi_\epsilon(x) \quad \text{in } \mathbb{R}^n.$$

Then we see that $\Phi_\epsilon = 0$ in $[0,T]\times\Omega^\epsilon_f$ and Φ_ϵ has support in $[0,T]\times\Omega^\epsilon_m$. If we set $\tau = \mathbf{K}_\epsilon = \mathbf{k}_\epsilon = 1$ in Λ^ϵ_τ, then Φ_ϵ satisfies

$$\begin{cases} \partial_t\Phi_\epsilon - \nabla\cdot(\Lambda^\epsilon_1\nabla\Phi_\epsilon) = f_\epsilon & \text{in } (0,T]\times\Omega, \\ \Phi_\epsilon = 0 & \text{on } (0,T]\times\partial\Omega, \\ \Phi_\epsilon(t=0) = \varphi_\epsilon & \text{in } \Omega, \end{cases}$$

where

$$f_\epsilon(x) \equiv \begin{cases} -e^{-t}\left(\Delta\varphi\left(\dfrac{x}{\epsilon} - j\right) + \varphi\left(\dfrac{x}{\epsilon} - j\right)\right) & \text{if } x \in \epsilon(Y_m+j) \subset \Omega(2\epsilon) \text{ for } j \in \mathbb{Z}^n, \\ 0 & \text{elsewhere.} \end{cases}$$

Clearly, for any $\delta > 0$ and $\epsilon, \sigma \in (0,1)$, $\|\varphi_\epsilon\|_{\mathbb{B}_{n+\delta}(\mathcal{A}^\epsilon_\tau)} + \|f_\epsilon\|_{C^\sigma([0,T];L^{n+\delta}(\Omega))}$ is bounded uniformly in ϵ. But the Hölder norm of the functions Φ_ϵ in the disconnected subset Ω^ϵ_m is not be bounded uniformly in ϵ if the source function $f_\epsilon \ne 0$ in Ω^ϵ_m.

Remark 2.1. We recall an extension result from [19].

For $1 \le p < \infty$, there is a constant $\gamma(Y_f, p)$ and a linear continuous extension operator $\Pi_\epsilon : W^{1,p}(\Omega^\epsilon_f) \to W^{1,p}(\Omega)$ such that

(1) *If $\varphi \in W^{1,p}(\Omega^\epsilon_f)$, then*

$$\begin{cases} \Pi_\epsilon\varphi = \varphi & \text{in } \Omega^\epsilon_f \text{ almost everywhere}, \\ \|\Pi_\epsilon\varphi\|_{L^p(\Omega)} \le \gamma(Y_f,p)\|\varphi\|_{L^p(\Omega^\epsilon_f)}, & \\ \|\nabla\Pi_\epsilon\varphi\|_{L^p(\Omega)} \le \gamma(Y_f,p)\|\nabla\varphi\|_{L^p(\Omega^\epsilon_f)}, & \\ \|\Pi_\epsilon\varphi\|_{C^\sigma(\overline{\Omega})} \le \gamma(Y_f,p)\|\varphi\|_{C^\sigma(\overline{\Omega}^\epsilon_f)} & \text{if } \varphi \in C^\sigma(\overline{\Omega}^\epsilon_f) \text{ for } \sigma \in (0,1), \\ \Pi_\epsilon\varphi = \zeta & \text{in } \Omega \text{ if } \varphi = \zeta|_{\Omega^\epsilon_f} \text{ for some linear function } \zeta \text{ in } \Omega. \end{cases} \tag{2.5}$$

(2) *If* $\zeta(x) \equiv \varphi(sx)$ *in* $B_1(x_0) \cap \Omega_f^\epsilon/s$ *for any* $x_0 \in \Omega/s$ *and constant* $s > \epsilon$, *then* $\Pi_{\epsilon/s}\zeta(x) = \Pi_\epsilon\varphi(sx)$ *in* $B_{1/2}(x_0) \cap \Omega/s$.

The Hölder estimate $(2.5)_4$ and the statement (2) are not written in [19], but can be seen from its proof.

From [4], we know that the solution U_ϵ of (1.1) with $\tau \in (0, 1)$ converges to a function U in $L^\infty([0, T]; L^2(\Omega))$ as $\epsilon \searrow 0$, and the function U satisfies a heat equation. By Theorem 2.2 and Remark 2.1, $\|\Pi_\epsilon U_\epsilon|_{\Omega_f^\epsilon}\|_{C^\nu([0,T]\times\overline{\Omega})}$ is bounded independent of ϵ. It is not difficult to see that, for the solution U_ϵ of (1.1) with $\tau \in (0, 1)$, $\Pi_\epsilon U_\epsilon|_{\Omega_f^\epsilon}$ also converges to U in $C^\nu([0, T] \times \overline{\Omega})$ norm for some $\nu \in (0, 1)$ as $\epsilon \searrow 0$.

3. Proofs of Theorems 2.1 and 2.2

Proofs of Theorems 2.1 and 2.2 are based on a sequence of lemmas. First we consider an interpolation result.

Lemma 3.1. *If* $\varphi \in L^q(\Omega) \cap C^\mu(\overline{\Omega})$ *for any* $q \in (1, \infty)$ *and* $\mu \in (0, 1)$, *then*

$$\|\varphi\|_{C^\nu(\overline{\Omega})} \le c\|\varphi\|_{L^q(\Omega)}^{1-\theta}\|\varphi\|_{C^\mu(\overline{\Omega})}^{\theta},$$

where $\nu \in (0, \mu)$, $\theta \in (0, 1)$, *and* c *is a constant depending on* $\nu, n, q, \mu, \theta, \Omega$.

Proof. By Proposition 1.1.3 [18], φ satisfies

$$\|\varphi\|_{C^\nu(\overline{\Omega})} \le c\|\varphi\|_{C(\overline{\Omega})}^{1-\theta_1}\|\varphi\|_{C^\mu(\overline{\Omega})}^{\theta_1}, \tag{3.1}$$

where $\nu \in (0, \mu)$, $\theta_1 \in (0, 1)$, and c is a constant depending on ν, μ, θ_1. Fix $x \in \Omega$ and $\delta > 0$ to see

$$\begin{aligned}|\varphi(x)| &\le \left|\varphi(x) - \fint_{B_\delta(x)\cap\Omega}\varphi(y)dy\right| + \left|\fint_{B_\delta(x)\cap\Omega}\varphi(y)dy\right| \\ &\le [\varphi]_{C^\mu(\overline{\Omega})}\fint_{B_\delta(x)\cap\Omega}|x-y|^\mu dy + \left|\fint_{B_\delta(x)\cap\Omega}|\varphi(y)|^q dy\right|^{1/q} \\ &\le c_1\delta^\mu[\varphi]_{C^\mu(\overline{\Omega})} + c_2\delta^{-n/q}\|\varphi\|_{L^q(\Omega)},\end{aligned} \tag{3.2}$$

where constants c_1, c_2 depend on domain Ω only. Taking the minimum of the right hand side of (3.2) on δ, we obtain

$$|\varphi(x)| \le c(n, q, \mu, \Omega)\|\varphi\|_{L^q(\Omega)}^{1-\theta_2}[\varphi]_{C^\mu(\overline{\Omega})}^{\theta_2}, \tag{3.3}$$

where $\theta_2 \in (0, 1)$ and c depend on n, q, μ, Ω. (3.1) and (3.3) imply the lemma. □

From the proof of Lemma 3.1, we also have

Lemma 3.2. *If* $\varphi \in L^q(\Omega_f^\epsilon) \cap C^\mu(\overline{\Omega}_f^\epsilon)$ *for any* $q \in (1, \infty)$ *and* $\mu \in (0, 1)$, *then*

$$\|\varphi\|_{L^\infty(\Omega_f^\epsilon)} \le c\|\varphi\|_{L^q(\Omega_f^\epsilon)}^{1-\theta}[\varphi]_{C^\mu(\overline{\Omega}_f^\epsilon)}^{\theta},$$

where $\theta \in (0, 1)$ *and* c *is a constant depending on* n, q, μ, Y_m *but independent of* ϵ.

Consider the following elliptic problem:

$$\begin{cases}-\nabla\cdot(\Lambda_\tau^\epsilon\nabla\varphi_\epsilon) = f_\epsilon & \text{in } \Omega, \\ \varphi_\epsilon = 0 & \text{on } \partial\Omega.\end{cases} \tag{3.4}$$

We have the following uniform a-priori estimates:

Lemma 3.3. *If* A1–A2 *hold, then*

(1) *The solution of* (3.4) *satisfies, for* $p \in (1, \infty)$, $\tau \in (0, \infty)$, *and* $\epsilon \in (0, 1)$,

$$\|\varphi_\epsilon\|_{W^{1,p}(\Omega)} + \|\varphi_\epsilon\|_{W^{2,p}(\Omega_f^\epsilon)} + \|\varphi_\epsilon\|_{W^{2,p}(\Omega_m^\epsilon)} \le c_{\epsilon,p}\|f_\epsilon\|_{L^p(\Omega)}, \tag{3.5}$$

where $c_{\epsilon,p}$ *is a constant independent of* $\varphi_\epsilon, f_\epsilon$ *but depending on* ϵ, p, τ.

(2) *The solution of* (3.4) *satisfies, for any* $\delta > 0$, $\tau \in (0, 1]$, *and* $\epsilon \in (0, 1)$,

$$\|\varphi_\epsilon\|_{C^\mu(\overline{\Omega}_f^\epsilon)} + \sup_{\substack{j\in\mathbb{Z}^n \\ \epsilon(Y_m+j)\subset\Omega_m^\epsilon}}\epsilon^\tau\|\varphi_\epsilon\|_{C^\mu(\epsilon(\overline{Y}_m+j))} \le c\|f_\epsilon\|_{L^{n+\delta}(\Omega)}, \tag{3.6}$$

where c *is a constant independent of* ϵ. *Here* $\mu \in (0, \frac{\delta}{2(n+\delta)})$ *is a constant depending on* $n, \delta, \alpha, \beta, Y_f, \Omega$ *(see* A2*).*

The proof of (3.5) is given in Section 4 and the proof of (3.6) is in Section 6.

Lemma 3.4. *For any $p \in (1,\infty)$, $\tau \in (0,\infty)$, and $\epsilon \in (0,1)$, the set $\mathbb{B}_p(\mathcal{A}^\epsilon_\tau)$ is dense in $L^p(\Omega)$ and $\mathbb{B}_p(\mathcal{A}^\epsilon_\tau)$ with norm $\|\varphi\|_{\mathbb{B}_p(\mathcal{A}^\epsilon_\tau)} \equiv \|\mathcal{A}^\epsilon_\tau\varphi\|_{L^p(\Omega)}$ is a Banach space.*

Proof. Define $\mathcal{O}_\epsilon \equiv \{x \in \Omega|\ \mathrm{dist}(x, \partial\Omega^\epsilon_f) \geq \epsilon^2\}$ and let $\mathcal{X}_{\mathcal{O}_\epsilon}$ be the characteristic function on $\mathcal{O}_\epsilon$. Then $\mathcal{X}_{\mathcal{O}_\epsilon}$ converges to 1 in measure (see page 91 [20]) on domain Ω as $\epsilon \searrow 0$. For any $\varphi \in L^p(\Omega)$, we have $\varphi\mathcal{X}_{\mathcal{O}_\epsilon} \in L^p(\Omega)$ and $\varphi\mathcal{X}_{\mathcal{O}_\epsilon} = 0$ in a neighborhood of $\partial\Omega^\epsilon_f$. By the Lebesque dominant theorem and Proposition in page 92 [20], there is a subsequence of $\varphi\mathcal{X}_{\mathcal{O}_\epsilon}$ (same notation for subsequence) converging to φ in $L^p(\Omega)$ as $\epsilon \searrow 0$. So for any $\delta > 0$, there is a ϵ_0 such that $\|\varphi - \varphi\mathcal{X}_{\mathcal{O}_\epsilon}\|_{L^p(\Omega)} \leq \delta/2$ as $\epsilon < \epsilon_0$. From pages 147–148 [17], there is a mollifier η_δ such that the convolution of η_δ and $\varphi\mathcal{X}_{\mathcal{O}_\epsilon}$ (i.e., $(\varphi\mathcal{X}_{\mathcal{O}_\epsilon}) * \eta_\delta$) for some $\epsilon < \epsilon_0$ satisfies $\|\varphi\mathcal{X}_{\mathcal{O}_\epsilon} - (\varphi\mathcal{X}_{\mathcal{O}_\epsilon}) * \eta_\delta\|_{L^p(\Omega)} \leq \delta/2$ and $(\varphi\mathcal{X}_{\mathcal{O}_\epsilon}) * \eta_\delta = 0$ in some neighborhood of $\partial\Omega^\epsilon_f$. Clearly, $(\varphi\mathcal{X}_{\mathcal{O}_\epsilon}) * \eta_\delta \in \mathbb{B}_p(\mathcal{A}^\epsilon_\tau)$ and $\|\varphi - (\varphi\mathcal{X}_{\mathcal{O}_\epsilon}) * \eta_\delta\|_{L^p(\Omega)} \leq \delta$. So $\mathbb{B}_p(\mathcal{A}^\epsilon_\tau)$ is dense in $L^p(\Omega)$. By (3.5) in Lemma 3.3, we see that $\mathbb{B}_p(\mathcal{A}^\epsilon_\tau)$ with norm $\|\cdot\|_{\mathbb{B}_p(\mathcal{A}^\epsilon_\tau)}$ is a Banach space. □

Lemma 3.5. *For any $p \in (1,\infty)$, $\tau \in (0,\infty)$, and $\epsilon \in (0,1)$, the adjoint operator of $\mathcal{A}^\epsilon_\tau : \mathbb{B}_p(\mathcal{A}^\epsilon_\tau) \subset L^p(\Omega) \to L^p(\Omega)$ is $\mathcal{A}^\epsilon_\tau : \mathbb{B}_q(\mathcal{A}^\epsilon_\tau) \subset L^q(\Omega) \to L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. Fix a $p \in (1,\infty)$, $\tau \in (0,\infty)$, and $\epsilon \in (0,1)$, denote the adjoint of $\mathcal{A}^\epsilon_\tau : \mathbb{B}_p(\mathcal{A}^\epsilon_\tau) \subset L^p(\Omega) \to L^p(\Omega)$ by $\mathcal{A}^{\tau'}_\epsilon$, and assume $\frac{1}{p} + \frac{1}{q} = 1$. Integration by parts yields

$$\langle \mathcal{A}^\epsilon_\tau\zeta, \eta\rangle_{L^p(\Omega),L^q(\Omega)} = \langle \zeta, \mathcal{A}^\epsilon_\tau\eta\rangle_{L^p(\Omega),L^q(\Omega)} \tag{3.7}$$

for every $\zeta \in \mathbb{B}_p(\mathcal{A}^\epsilon_\tau)$ and $\eta \in \mathbb{B}_q(\mathcal{A}^\epsilon_\tau)$. See Section 2 for $\langle\cdot,\cdot\rangle_{L^p(\Omega),L^q(\Omega)}$. Therefore $\mathbb{B}_q(\mathcal{A}^\epsilon_\tau) \subset \mathrm{dom}(\mathcal{A}^{\tau'}_\epsilon)$ (that is, the domain of $\mathcal{A}^{\tau'}_\epsilon$) and $\mathcal{A}^\epsilon_\tau\eta = \mathcal{A}^{\tau'}_\epsilon\eta$ for $\eta \in \mathbb{B}_q(\mathcal{A}^\epsilon_\tau)$.

Let $\eta \in \mathrm{dom}(\mathcal{A}^{\tau'}_\epsilon) \subset L^q(\Omega)$ and $\varphi = \mathcal{A}^{\tau'}_\epsilon(\eta)$. Then, by the definition of the adjoint operator, we have

$$\langle \mathcal{A}^\epsilon_\tau\zeta, \eta\rangle_{L^p(\Omega),L^q(\Omega)} = \langle \zeta, \varphi\rangle_{L^p(\Omega),L^q(\Omega)} \quad \text{for all } \zeta \in \mathbb{B}_p(\mathcal{A}^\epsilon_\tau). \tag{3.8}$$

Since $\mathbb{B}_q(\mathcal{A}^\epsilon_\tau)$ is dense in $L^q(\Omega)$ by Lemma 3.4, there is a sequence $\eta_s \in \mathbb{B}_q(\mathcal{A}^\epsilon_\tau)$ such that $\eta_s \to \eta$ in $L^q(\Omega)$ as $s \to \infty$. By (3.7) and (3.8),

$$\begin{aligned}\lim_{s\to\infty}\langle \zeta, \mathcal{A}^\epsilon_\tau\eta_s\rangle_{L^p(\Omega),L^q(\Omega)} &= \lim_{s\to\infty}\langle \mathcal{A}^\epsilon_\tau\zeta, \eta_s\rangle_{L^p(\Omega),L^q(\Omega)} \\ &= \langle \mathcal{A}^\epsilon_\tau\zeta, \eta\rangle_{L^p(\Omega),L^q(\Omega)} = \langle \zeta, \varphi\rangle_{L^p(\Omega),L^q(\Omega)}.\end{aligned}$$

Since $\mathbb{B}_p(\mathcal{A}^\epsilon_\tau)$ is dense in $L^p(\Omega)$ by Lemma 3.4, $\mathcal{A}^\epsilon_\tau\eta_s$ converges to φ weakly in $L^q(\Omega)$ as $s \to \infty$. By (3.5) in Lemma 3.3, we see $\eta \in \mathbb{B}_q(\mathcal{A}^\epsilon_\tau)$. So $\varphi = \mathcal{A}^\epsilon_\tau(\eta)$. Therefore, $\mathrm{dom}(\mathcal{A}^{\tau'}_\epsilon) \subset \mathbb{B}_q(\mathcal{A}^\epsilon_\tau)$ and $\mathcal{A}^\epsilon_\tau = \mathcal{A}^{\tau'}_\epsilon$. □

Next we want to show $-\mathcal{A}^\epsilon_\tau$ is an infinitesimal generator of an analytic semigroup. If so, by semigroup group theory, we can obtain the existence of the solutions of some time-dependent problems. For this purpose, we shall work on complex-valued functions in the next lemma.

Lemma 3.6. *For any $p \in (1,\infty)$, $\tau \in (0,\infty)$, and $\epsilon \in (0,1)$, the operator $-\mathcal{A}^\epsilon_\tau$ is an infinitesimal generator of an analytic semigroup of contractions on $L^p(\Omega)$ and*

$$\|(\lambda + \mathcal{A}^\epsilon_\tau)^{-1}\|_{\mathcal{L}(L^p(\Omega),L^p(\Omega))} \leq \frac{1}{\lambda} \quad \textit{for any } \lambda > 0. \tag{3.9}$$

Moreover, there is a $\theta \in (0, \pi/2)$ independent of ϵ, τ such that

(1) *The resolvent set $\rho(-\mathcal{A}^\epsilon_\tau)$ of $-\mathcal{A}^\epsilon_\tau$ (see page 8 [21]) satisfies*

$$\rho(-\mathcal{A}^\epsilon_\tau) \supset \Re(\theta) \equiv \{z \in \mathbb{C}|\ |\arg(z)| < \pi - \theta\},$$

where $\arg(z)$ denotes the argument of the complex number z.

(2) $\|(\lambda + \mathcal{A}^\epsilon_\tau)^{-1}\|_{\mathcal{L}(L^p(\Omega),L^p(\Omega))} \leq \frac{1}{c_\theta|\lambda|}$ *for any $\lambda \in \Re(\theta)$, where c_θ is a constant independent of ϵ, τ.*

Proof. We assume $p \in (1,\infty)$, $\tau \in (0,\infty)$, and $\epsilon \in (0,1)$. The proof of this lemma includes three steps.

Step 1. Claim $\lambda + \mathcal{A}^\epsilon_\tau : \mathbb{B}_p(\mathcal{A}^\epsilon_\tau) \subset L^p(\Omega) \to L^p(\Omega)$ is injective for any $\lambda > 0$. Let $q = \frac{p}{p-1}$. If $\varphi \in \mathbb{B}_p(\mathcal{A}^\epsilon_\tau)$, we define $\varphi_* \equiv |\varphi|^{p-2}\overline{\varphi} \in L^q(\Omega)$ ($\overline{\varphi}$ is the complex conjugate of φ). Then $\langle \varphi, \varphi_*\rangle_{L^p(\Omega),L^q(\Omega)} = \|\varphi\|^p_{L^p(\Omega)}$. Integration by parts yields

$$\begin{aligned}\langle \mathcal{A}^\epsilon_\tau\varphi, \varphi_*\rangle_{L^p(\Omega),L^q(\Omega)} &= -\int_\Omega \nabla\cdot(\Lambda^\epsilon_\tau\nabla\varphi)|\varphi|^{p-2}\overline{\varphi}dx = \int_\Omega \Lambda^\epsilon_\tau\nabla\varphi\nabla(|\varphi|^{p-2}\overline{\varphi})dx \\ &= \int_\Omega \Lambda^\epsilon_\tau\big(|\varphi|^{p-2}\nabla\varphi\nabla\overline{\varphi} + \overline{\varphi}\nabla\varphi\nabla|\varphi|^{p-2}\big)dx.\end{aligned}$$

Note $\nabla|\varphi|^{p-2} = \frac{p-2}{2}|\varphi|^{p-4}(\overline{\varphi}\nabla\varphi + \varphi\nabla\overline{\varphi})$. Denote $|\varphi|^{(p-4)/2}\overline{\varphi}\nabla\varphi \equiv \ell + i\omega$. We find

$$\langle \mathcal{A}_\tau^\epsilon \varphi, \varphi_* \rangle_{L^p(\Omega),L^q(\Omega)} = \int_\Omega \Lambda_\tau^\epsilon \big((p-1)|\ell|^2 + |\omega|^2 + i(p-2)\ell\cdot\omega \big) dx,$$

where $|\ell|$ (resp. $|\omega|$) is the length of the vector ℓ (resp. ω). So the real part of $\langle \mathcal{A}_\tau^\epsilon \varphi, \varphi_* \rangle_{L^p(\Omega),L^q(\Omega)}$ satisfies, by A2,

$$\mathrm{Re}\langle \mathcal{A}_\tau^\epsilon \varphi, \varphi_* \rangle_{L^p(\Omega),L^q(\Omega)} \geq c_{p,\alpha} \left| \int_{\Omega_f^\epsilon} |\ell|^2 + |\omega|^2 dx + \epsilon^{2\tau} \int_{\Omega_m^\epsilon} |\ell|^2 + |\omega|^2 dx \right| \geq 0, \tag{3.10}$$

where $c_{p,\alpha}$ is a constant depending on p, α. The ratio of the imaginary part to the real part of $\langle \mathcal{A}_\tau^\epsilon \varphi, \varphi_* \rangle_{L^p(\Omega),L^q(\Omega)}$ then satisfies, by A2,

$$\frac{|\mathrm{Im}\langle \mathcal{A}_\tau^\epsilon \varphi, \varphi_* \rangle_{L^p(\Omega),L^q(\Omega)}|}{|\mathrm{Re}\langle \mathcal{A}_\tau^\epsilon \varphi, \varphi_* \rangle_{L^p(\Omega),L^q(\Omega)}|} \leq \frac{|p-2|\beta \left(\int_{\Omega_f^\epsilon} |\ell|^2 + |\omega|^2 dx + \epsilon^{2\tau} \int_{\Omega_m^\epsilon} |\ell|^2 + |\omega|^2 dx \right)}{2c_{p,\alpha} \left(\int_{\Omega_f^\epsilon} |\ell|^2 + |\omega|^2 dx + \epsilon^{2\tau} \int_{\Omega_m^\epsilon} |\ell|^2 + |\omega|^2 dx \right)} = \frac{|p-2|\beta}{2c_{p,\alpha}}. \tag{3.11}$$

From (3.10) it follows that, for any $\lambda > 0$ and $\varphi \in \mathbb{B}_p(\mathcal{A}_\tau^\epsilon)$,

$$\lambda \|\varphi\|_{L^p(\Omega)} \leq \|(\lambda + \mathcal{A}_\tau^\epsilon)\varphi\|_{L^p(\Omega)}. \tag{3.12}$$

By (3.12), $\lambda + \mathcal{A}_\tau^\epsilon : \mathbb{B}_p(\mathcal{A}_\tau^\epsilon) \subset L^p(\Omega) \to L^p(\Omega)$ is injective. So we prove the claim.

Step 2. Claim $\lambda + \mathcal{A}_\tau^\epsilon : \mathbb{B}_p(\mathcal{A}_\tau^\epsilon) \subset L^p(\Omega) \to L^p(\Omega)$ is bijective for any $\lambda > 0$. If $\eta \in L^q(\Omega)$ for $q = \frac{p}{p-1}$ satisfies $\langle (\lambda + \mathcal{A}_\tau^\epsilon)\varphi, \eta \rangle_{L^p(\Omega),L^q(\Omega)} = 0$ for all $\varphi \in \mathbb{B}_p(\mathcal{A}_\tau^\epsilon)$, then η is in the domain of the adjoint operator $\lambda + \mathcal{A}_\epsilon^{\tau'}$ (here $\mathcal{A}_\epsilon^{\tau'}$ is the adjoint operator of $\mathcal{A}_\tau^\epsilon$) of $\lambda + \mathcal{A}_\tau^\epsilon$. By Lemma 3.5, $\langle \varphi, (\lambda + \mathcal{A}_\tau^\epsilon)\eta \rangle_{L^p(\Omega),L^q(\Omega)} = 0$ and $\eta \in \mathbb{B}_q(\mathcal{A}_\tau^\epsilon)$. Since $\mathbb{B}_p(\mathcal{A}_\tau^\epsilon)$ is dense on $L^p(\Omega)$ by Lemma 3.4, $(\lambda + \mathcal{A}_\tau^\epsilon)\eta = 0$. Then (3.12), with p replaced by q, implies $\eta = 0$. So the range of $\lambda + \mathcal{A}_\tau^\epsilon$ is dense in $L^p(\Omega)$. By (3.5), $\mathcal{A}_\tau^\epsilon : \mathbb{B}_p(\mathcal{A}_\tau^\epsilon) \subset L^p(\Omega) \to L^p(\Omega)$ is a closed linear operator. It is not difficult to see that $\lambda + \mathcal{A}_\tau^\epsilon$ is also a closed linear operator. Thus, the range of $\lambda + \mathcal{A}_\tau^\epsilon$ is a closed set in $L^p(\Omega)$. Since the range of $\lambda + \mathcal{A}_\tau^\epsilon$ is dense and closed in $L^p(\Omega)$, the range of $\lambda + \mathcal{A}_\tau^\epsilon$ is $L^p(\Omega)$. So we prove the claim. Moreover, by (3.12),

$$\|(\lambda + \mathcal{A}_\tau^\epsilon)^{-1}\|_{\mathcal{L}(L^p(\Omega),L^p(\Omega))} \leq \frac{1}{\lambda} \quad \text{for any } \lambda > 0.$$

So we prove (3.9).

Step 3. Claim $-\mathcal{A}_\tau^\epsilon$ is an infinitesimal generator of an analytic semigroup on $L^p(\Omega)$. By Step 2, (3.5) in Lemmas 3.3 and 3.4, the Hille–Yosida Theorem [21] implies that $-\mathcal{A}_\tau^\epsilon$ is an infinitesimal generator of a C_0-semigroup of contractions on $L^p(\Omega)$. To prove that the semigroup generated by $-\mathcal{A}_\tau^\epsilon$ is analytic, we observe that, by (3.10) and (3.11), the numerical range $\mathcal{N}(-\mathcal{A}_\tau^\epsilon)$ of $-\mathcal{A}_\tau^\epsilon$ (see page 12 [21] and Remark 3.2 in page 25 [22]) is contained in the set

$$\mathcal{N}_{\theta_1} \equiv \{z \in \mathbb{C} \mid |\arg(z)| > \pi - \theta_1\},$$

where $\theta_1 = \tan^{-1}(\frac{|p-2|\beta}{2c_{p,\alpha}}) \in (0, \pi/2)$. Choosing $\theta_1 < \theta < \pi/2$ and denoting

$$\Re(\theta) \equiv \{z \in \mathbb{C} \mid |\arg(z)| < \pi - \theta\},$$

there is a constant $c_\theta > 0$ independent of ϵ, τ such that the distance from $z \in \Re(\theta)$ to $\overline{\mathcal{N}(-\mathcal{A}_\tau^\epsilon)}$ (i.e., $\mathrm{dist}(z, \overline{\mathcal{N}(-\mathcal{A}_\tau^\epsilon)})$) satisfies

$$\mathrm{dist}(z, \overline{\mathcal{N}(-\mathcal{A}_\tau^\epsilon)}) \geq c_\theta |z|.$$

Since $\lambda > 0$ is in the resolvent set $\rho(-\mathcal{A}_\tau^\epsilon)$ of $-\mathcal{A}_\tau^\epsilon$ by Step 2, Theorem 3.9 in page 12 [21] then implies $\Re(\theta) \subset \rho(-\mathcal{A}_\tau^\epsilon)$ and

$$\|(\lambda + \mathcal{A}_\tau^\epsilon)^{-1}\|_{\mathcal{L}(L^p(\Omega),L^p(\Omega))} \leq \frac{1}{c_\theta|\lambda|} \quad \text{for } \lambda \in \Re(\theta).$$

By (3.5) of Lemma 3.3 and the energy method, $0 \in \rho(-\mathcal{A}_\tau^\epsilon)$. By Theorem 5.2(c) in page 61 [21], $-\mathcal{A}_\tau^\epsilon$ is an infinitesimal generator of an analytic semigroup on $L^p(\Omega)$. □

Proof of Theorem 2.1. Tracing the proofs of Proposition 2.1.1, Eq. (4.0.3), and Theorem 4.3.1 [18], and employing Lemma 3.6, we know

Let $\delta, \tau > 0, \sigma \in (0,1), F_\epsilon \in C^\sigma([0,T]; L^{n+\delta}(\Omega)), U_{\epsilon,0} \in \mathbb{B}_{n+\delta}(\mathcal{A}_\tau^\epsilon)$, and $\mathcal{A}_\tau^\epsilon U_{\epsilon,0} - F_\epsilon(t=0) \in \overline{\mathbb{B}_{n+\delta}(\mathcal{A}_\tau^\epsilon)}$. A strict solution U_ϵ of (1.1) exists and there is a constant c independent of ϵ, τ such that

$$\|U_\epsilon\|_{C^1([0,T];L^{n+\delta}(\Omega))} + \|U_\epsilon\|_{C([0,T];\mathbb{B}_{n+\delta}(\mathcal{A}_\tau^\epsilon))} \le c\big(\|U_{\epsilon,0}\|_{\mathbb{B}_{n+\delta}(\mathcal{A}_\tau^\epsilon)} + \|F_\epsilon\|_{C^\sigma([0,T];L^{n+\delta}(\Omega))}\big). \tag{3.13}$$

So we prove Theorem 2.1. □

Proof of Theorem 2.2. By (1.1) and for each fixed $t \in (0,T]$,

$$\begin{cases} -\nabla\cdot(\Lambda_\tau^\epsilon \nabla U_\epsilon(t,\cdot)) = F_\epsilon(t,\cdot) - \partial_t U_\epsilon(t,\cdot) & \text{in } \Omega, \\ U_\epsilon(t,\cdot) = 0 & \text{on } \partial\Omega. \end{cases}$$

(3.6) in Lemma 3.3 and Theorem 2.1 then imply (2.3). By Remark 2.1, we can extend the function $U_\epsilon|_{\Omega_f^\epsilon}(t,\cdot)$ to Ω. The extended function $\Pi_\epsilon U_\epsilon$ satisfies, by (2.3), (3.13), and Remark 2.1,

$$\|\Pi_\epsilon U_\epsilon\|_{C^1([0,T];L^{n+\delta}(\Omega))} + \|\Pi_\epsilon U_\epsilon\|_{C([0,T];C^\mu(\overline{\Omega}))} \le c\big(\|U_{\epsilon,0}\|_{\mathbb{B}_{n+\delta}(\mathcal{A}_\tau^\epsilon)} + \|F_\epsilon\|_{C^\sigma([0,T];L^{n+\delta}(\Omega))}\big), \tag{3.14}$$

where $\mu \in (0,1)$ and c is independent of ϵ. (2.4) follows from Proposition 1.1.4 [18], (3.14), and Lemma 3.1. So we prove Theorem 2.2. □

4. Proof of (3.5) of Lemma 3.3

Let $\Gamma(x-y)$ denote the fundamental solution of the Laplace's equation (see Section 6.2 [23]). Define the single-layer and the double-layer potentials as, for any smooth function φ on the boundary $\partial\mathbb{D}$ of a bounded $C^{1,\mathbf{e}}$ domain $\mathbb{D}$,

$$\begin{cases} \mathcal{E}_{\partial\mathbb{D}}(\varphi)(x) \equiv \int_{\partial\mathbb{D}} \Gamma(x-y)\varphi(y)d\sigma_y \\ \mathcal{T}_{\partial\mathbb{D}}(\varphi)(x) \equiv \int_{\partial\mathbb{D}} \nabla_y\Gamma(x-y)\cdot\vec{\mathbf{n}}_y\,\varphi(y)d\sigma_y & \text{for } x\in\partial\mathbb{D}, \\ \mathcal{T}^*_{\partial\mathbb{D}}(\varphi)(x) \equiv \int_{\partial\mathbb{D}} \nabla_x\Gamma(x-y)\cdot\vec{\mathbf{n}}_x\,\varphi(y)d\sigma_y \end{cases}$$

where $\mathbf{e} \in (0,1)$ and $\vec{\mathbf{n}}_y$ (resp. $\vec{\mathbf{n}}_x$) is the unit vector outward normal to $\partial\mathbb{D}$ at point $y \in \partial\mathbb{D}$ (resp. $x \in \partial\mathbb{D}$).

Lemma 4.1. *For any $p \in (1,\infty)$, the linear operators*

$$\begin{cases} \mathcal{E}_{\partial\mathbb{D}} : W^{1-\frac{1}{p},p}(\partial\mathbb{D}) \to W^{2-\frac{1}{p},p}(\partial\mathbb{D}) \\ \mathcal{T}_{\partial\mathbb{D}} : W^{1-\frac{1}{p},p}(\partial\mathbb{D}) \to W^{2-\frac{1}{p},p}(\partial\mathbb{D}) \end{cases} \tag{4.1}$$

are bounded. The operator $I - \lambda\mathcal{T}_{\partial\mathbb{D}}$ is continuously invertible in $W^{2-\frac{1}{p},p}(\partial\mathbb{D})$ for any $p \in (1,\infty)$ and $\lambda \in [-2,2]$, where I is the identity operator. Furthermore, there is a constant c independent of $\lambda \in [-2,2]$ so that

$$\|\varphi\|_{W^{2-\frac{1}{p},p}(\partial\mathbb{D})} \le c\|(I-\lambda\mathcal{T}_{\partial\mathbb{D}})(\varphi)\|_{W^{2-\frac{1}{p},p}(\partial\mathbb{D})} \quad \text{for } \varphi \in W^{2-\frac{1}{p},p}(\partial\mathbb{D}). \tag{4.2}$$

Proof. Denote by $OPS^{-1}_{1,0}$ the pseudo-differential operator of order -1 (see page 38 [24]). Tracing the proof of Theorem 2.5 Chapter XI [24], we see that if $\mathcal{G} \in OPS^{-1}_{1,0}(\partial\mathbb{D})$, then $\mathcal{G}$ is a bounded linear operator from $W^{1-\frac{1}{p},p}(\partial\mathbb{D})$ to $W^{2-\frac{1}{p},p}(\partial\mathbb{D})$. Since $\mathcal{E}_{\partial\mathbb{D}}, \mathcal{T}_{\partial\mathbb{D}} \in OPS^{-1}_{1,0}(\partial\mathbb{D})$ (see pages 87–93 [23]), we know that $\mathcal{E}_{\partial\mathbb{D}}, \mathcal{T}_{\partial\mathbb{D}}$ are bounded operators from $W^{1-\frac{1}{p},p}(\partial\mathbb{D})$ to $W^{2-\frac{1}{p},p}(\partial\mathbb{D})$.

Since $\mathbb{D}$ is a $C^{1,\mathbf{e}}$ domain, both $\mathcal{T}_{\partial\mathbb{D}}, \mathcal{T}^*_{\partial\mathbb{D}}$ are compact operators in $L^p(\partial\mathbb{D})$ for $p \in (1,\infty)$ (see Corollary 2.2.14 [25]). For any $\lambda \in \mathbb{R}$, the dimensions of the kernels of $I-\lambda\mathcal{T}_{\partial\mathbb{D}}$ and $I-\lambda\mathcal{T}^*_{\partial\mathbb{D}}$ are same by Theorem 4.12 [26]. From Theorem 2.2.21 [25] and Section 3.4 [27], there is a $p_0 \in (2,\infty)$ such that $I-\lambda\mathcal{T}^*_{\partial\mathbb{D}}$ is continuously invertible in $L^p(\partial\mathbb{D})$ for any $p \in (1,p_0)$ and $\lambda \in [-2,2]$. Since $L^p(\partial\mathbb{D}) \subset L^2(\partial\mathbb{D})$ for $p \in [2,\infty)$, $I-\lambda\mathcal{T}^*_{\partial\mathbb{D}}$ is injective for any $p \in [2,\infty)$ and $\lambda \in [-2,2]$. By Theorem 4.12 [26], $I-\lambda\mathcal{T}^*_{\partial\mathbb{D}}$ is continuously invertible for any $p \in [2,\infty)$ and $\lambda \in [-2,2]$. Again by Theorem 4.12 [26], we see that $I-\lambda\mathcal{T}_{\partial\mathbb{D}}$ is also continuously invertible in $L^p(\partial\mathbb{D})$ for $p \in (1,\infty)$ and $\lambda \in [-2,2]$. By (4.1) and inverse mapping theorem [28], $I-\lambda\mathcal{T}_{\partial\mathbb{D}}$ is continuously invertible in $W^{2-\frac{1}{p},p}(\partial\mathbb{D})$ for $p \in (1,\infty)$ and $\lambda \in [-2,2]$.

(4.2) is proved as follows. From above, we know that $\mathcal{T}_{\partial\mathbb{D}}$ is a bounded linear operator in $W^{2-\frac{1}{p},p}(\partial\mathbb{D})$ and $I-\lambda\mathcal{T}_{\partial\mathbb{D}}$ is continuously invertible in $W^{2-\frac{1}{p},p}(\partial\mathbb{D})$ for any $\lambda \in [-2,2]$ and $p \in (1,\infty)$. So for each $\lambda \in [-2,2]$, there is a set

$\{c_\lambda, d_\lambda, B_{d_\lambda}(\lambda)\}$ (depending on λ) satisfying

$$\begin{cases} c_\lambda, d_\lambda > 0, \\ \|(I-\lambda\mathcal{T}_{\partial\mathbb{D}})(\varphi)\|_{W^{2-\frac{1}{p},p}(\partial\mathbb{D})} \geq c_\lambda \|\varphi\|_{W^{2-\frac{1}{p},p}(\partial\mathbb{D})}, \\ \|(I-s\mathcal{T}_{\partial\mathbb{D}})(\varphi)\|_{W^{2-\frac{1}{p},p}(\partial\mathbb{D})} \geq \|(I-\lambda\mathcal{T}_{\partial\mathbb{D}})(\varphi)\|_{W^{2-\frac{1}{p},p}(\partial\mathbb{D})} - |s-\lambda|\, \|\mathcal{T}_{\partial\mathbb{D}}(\varphi)\|_{W^{2-\frac{1}{p},p}(\partial\mathbb{D})} \\ \quad \geq \dfrac{c_\lambda}{2}\|\varphi\|_{W^{2-\frac{1}{p},p}(\partial\mathbb{D})} \quad \text{if } s \in B_{d_\lambda}(\lambda) \subset \mathbb{R}. \end{cases}$$

Now we consider the open covering $\{B_{d_\lambda}(\lambda)\}_{\lambda\in[-2,2]}$ of $[-2,2]$. Since $[-2,2]$ is a compact set, we can find a finite set $\mathcal{Z} \subset [-2,2]$ so that $\{B_{d_\lambda}(\lambda)\}_{\lambda\in\mathcal{Z}}$ is also a covering of $[-2,2]$. Based on the finite sets $\{c_\lambda, d_\lambda, B_{d_\lambda}(\lambda)\}_{\lambda\in\mathcal{Z}}$, we define

$$c^* = \min_{\{c_\lambda, d_\lambda, B_{d_\lambda}(\lambda)\}_{\lambda\in\mathcal{Z}}} \frac{c_\lambda}{2}.$$

That is, c^* is the minimum value of $\frac{c_\lambda}{2}$ for λ in the finite set $\mathcal{Z}$. If the c in (4.2) is taken to be $c = 1/c^*$, we obtain (4.2). □

Now we consider the following problem

$$\begin{cases} -\nabla\cdot(\mathbf{K}\nabla\Psi_\epsilon) = G_\epsilon & \text{in } Y_f, \\ -\epsilon^{2\tau}\nabla\cdot(\mathbf{k}\nabla\psi_\epsilon) = \epsilon^\tau g_\epsilon & \text{in } Y_m, \\ \mathbf{K}\nabla\Psi_\epsilon\cdot\vec{\mathbf{n}}_y = \epsilon^{2\tau}\mathbf{k}\nabla\psi_\epsilon\cdot\vec{\mathbf{n}}_y & \text{on } \partial Y_m, \\ \Psi_\epsilon = \psi_\epsilon & \text{on } \partial Y_m, \end{cases} \tag{4.3}$$

where $\tau\in(0,\infty)$, $\epsilon\in(0,1)$, and $\vec{\mathbf{n}}_y$ is the unit vector outward normal to ∂Y_m. By $\mathbf{D}$ in (2.2), we define

$$\mathbf{D}_1 \equiv \left\{ x\in Y_f \,|\, \mathrm{dist}(x,\partial Y_f) > \frac{1}{4}\min\{\mathrm{dist}(Y_m,\partial\mathbf{D}), \mathrm{dist}(\mathbf{D},\partial Y)\} \right\}.$$

Then $\partial\mathbf{D}\subset\mathbf{D}_1$.

Lemma 4.2. *Suppose*

(1) $\mathbf{K}, \mathbf{k}$ *in* Y *satisfy* $\|\mathbf{K}-\mathbf{d}\|_{W^{1,\infty}(Y_f)} + \|\mathbf{k}-\mathbf{d}\|_{W^{1,\infty}(Y_m)} \leq c_0\mathbf{d}$ *where* $\mathbf{d}>0$ *is a constant and* $c_0 < \frac{1}{2}$ *is a small number depending on* Y_m,

(2) $\tau>0$, $\varpi\equiv\min\{2,p\}$ *for* $p\in(1,\infty)$, $\|\Psi_\epsilon\|_{L^\varpi(Y_f)} + \|G_\epsilon\|_{L^p(Y_f)} + \|g_\epsilon\|_{L^p(Y_m)}$ *is bounded independently of* ϵ,

then any solution of (4.3) *satisfies*

$$\|\Psi_\epsilon\|_{W^{2,p}(\mathbf{D}\setminus\overline{Y}_m)} + \epsilon^\tau\|\psi_\epsilon\|_{W^{2,p}(Y_m)} \leq c, \tag{4.4}$$

where c *is a constant independent of* ϵ, τ.

Proof. Denote by c a constant independent of $\epsilon, \tau, \mathbf{d}$. Consider $(4.3)_1$ in Y_f. Theorem 8.8 and Theorem 9.11 [17] implies

$$\mathbf{d}\|\Psi_\epsilon\|_{W^{2,p}(\mathbf{D}_1)} \leq c. \tag{4.5}$$

Let $\widehat{\psi}_\epsilon$ be a solution of

$$\begin{cases} -\nabla\cdot(\epsilon^{2\tau}\mathbf{d}\nabla\widehat{\psi}_\epsilon + \epsilon^{2\tau}(\mathbf{k}-\mathbf{d})\nabla\psi_\epsilon) = \epsilon^\tau g_\epsilon & \text{in } Y_m, \\ \widehat{\psi}_\epsilon|_{\partial Y_m} = 0, \end{cases} \tag{4.6}$$

and $\widehat{\Psi}_\epsilon$ a solution of

$$\begin{cases} -\nabla\cdot(\mathbf{d}\nabla\widehat{\Psi}_\epsilon + (\mathbf{K}-\mathbf{d})\nabla\Psi_\epsilon) = G_\epsilon & \text{in } \mathbf{D}\setminus\overline{Y}_m, \\ \widehat{\Psi}_\epsilon|_{\partial Y_m} = 0, \\ \widehat{\Psi}_\epsilon - \Psi_\epsilon|_{\partial\mathbf{D}} = 0. \end{cases} \tag{4.7}$$

Then, by (4.5) and Theorem 9.15 of [17],

$$\begin{cases} \mathbf{d}\|\widehat{\psi}_\epsilon\|_{W^{2,p}(Y_m)} \leq c\big(\epsilon^{-\tau} + \|(\mathbf{k}-\mathbf{d})\nabla\psi_\epsilon\|_{W^{1,p}(Y_m)}\big), \\ \mathbf{d}\|\widehat{\Psi}_\epsilon\|_{W^{2,p}(\mathbf{D}\setminus\overline{Y}_m)} \leq c\big(1 + \|(\mathbf{K}-\mathbf{d})\nabla\Psi_\epsilon\|_{W^{1,p}(\mathbf{D}\setminus\overline{Y}_m)}\big). \end{cases} \tag{4.8}$$

Define $\breve{\psi}_\epsilon \equiv \psi_\epsilon - \widehat{\psi}_\epsilon$ in Y_m and $\breve{\Psi}_\epsilon \equiv \Psi_\epsilon - \widehat{\Psi}_\epsilon$ in $\mathbf{D}\setminus\overline{Y}_m$. (4.3) and (4.6)–(4.7) imply

$$\begin{cases} -\epsilon^{2\tau}\Delta\breve{\psi}_\epsilon = 0 & \text{in } Y_m, \\ -\Delta\breve{\Psi}_\epsilon = 0 & \text{in } \mathbf{D}\setminus\overline{Y}_m, \\ \breve{\Psi}_\epsilon|_{\partial Y_m} = \breve{\psi}_\epsilon|_{\partial Y_m}, \\ \nabla\breve{\Psi}_\epsilon\cdot\vec{\mathbf{n}}_y|_{\partial Y_m} - \epsilon^{2\tau}\nabla\breve{\psi}_\epsilon\cdot\vec{\mathbf{n}}_y|_{\partial Y_m} = \mathcal{F}_\epsilon\cdot\vec{\mathbf{n}}_y/\mathbf{d}, \\ \breve{\Psi}_\epsilon|_{\partial\mathbf{D}} = 0, \end{cases} \tag{4.9}$$

where $\mathcal{F}_\epsilon = (\mathbf{d}-\mathbf{K})\nabla\Psi_\epsilon - \epsilon^{2\tau}(\mathbf{d}-\mathbf{k})\nabla\psi_\epsilon - \mathbf{d}\nabla\widehat{\Psi}_\epsilon + \epsilon^{2\tau}\mathbf{d}\nabla\widehat{\psi}_\epsilon$. By (4.5), (4.8), and trace theorems in pages 240–241 [16],

$$\|\mathcal{F}_\epsilon\|_{W^{1-1/p,p}(\partial Y_m)} \le c\big(1+\epsilon^{2\tau}\|(\mathbf{k}-\mathbf{d})\nabla\psi_\epsilon\|_{W^{1,p}(Y_m)} + \|(\mathbf{K}-\mathbf{d})\nabla\Psi_\epsilon\|_{W^{1,p}(\mathbf{D}\setminus\overline{Y}_m)}\big). \tag{4.10}$$

By Green's formula, (4.9), and Theorem 6.5.1 [23], we see that

$$\begin{cases} \breve{\psi}_\epsilon/2 + \mathcal{T}_{\partial Y_m}(\breve{\psi}_\epsilon) = \mathcal{E}_{\partial Y_m}(\partial_{\mathbf{n}_y}\breve{\psi}_\epsilon) \\ \breve{\Psi}_\epsilon/2 - \mathcal{T}_{\partial Y_m}(\breve{\Psi}_\epsilon) = -\mathcal{E}_{\partial Y_m}(\partial_{\mathbf{n}_y}\breve{\Psi}_\epsilon) + \mathcal{E}_{\partial\mathbf{D}}(\partial_{\mathbf{n}_y}\breve{\Psi}_\epsilon|_{\partial\mathbf{D}}) \end{cases} \quad \text{on } \partial Y_m,$$

where $\partial_{\mathbf{n}_y}\breve{\Psi}_\epsilon|_{\partial\mathbf{D}}$ is the normal derivative of $\breve{\Psi}_\epsilon$ on $\partial\mathbf{D}$. Therefore, by $(4.9)_4$,

$$\frac{\epsilon^{2\tau}+1}{2(1-\epsilon^{2\tau})}\breve{\psi}_\epsilon - \mathcal{T}_{\partial Y_m}(\breve{\psi}_\epsilon) = \frac{\mathcal{E}_{\partial\mathbf{D}}(\partial_{\mathbf{n}_y}\breve{\Psi}_\epsilon|_{\partial\mathbf{D}})}{1-\epsilon^{2\tau}} - \frac{\mathcal{E}_{\partial Y_m}(\mathcal{F}_\epsilon\cdot\vec{\mathbf{n}}_y)}{(1-\epsilon^{2\tau})\mathbf{d}} \quad \text{on } \partial Y_m. \tag{4.11}$$

By (4.5), (4.8), and trace theorems in pages 240–241 [16],

$$\mathbf{d}\|\partial_{\mathbf{n}_y}\breve{\Psi}_\epsilon\|_{W^{1-1/p,p}(\partial\mathbf{D})} \le c\big(1+\|(\mathbf{K}-\mathbf{d})\nabla\Psi_\epsilon\|_{W^{1,p}(\mathbf{D}\setminus\overline{Y}_m)}\big). \tag{4.12}$$

By (4.11) and Lemma 4.1, we have

$$\|\breve{\psi}_\epsilon\|_{W^{2-\frac{1}{p},p}(\partial Y_m)} \le c\Big(\mathbf{d}^{-1}\|\mathcal{F}_\epsilon\|_{W^{1-\frac{1}{p},p}(\partial Y_m)} + \|\partial_{\mathbf{n}_y}\breve{\Psi}_\epsilon\|_{W^{1-\frac{1}{p},p}(\partial\mathbf{D})}\Big). \tag{4.13}$$

Eqs. $(4.3)_4$, (4.8), (4.10), (4.12) and (4.13) imply

$$\mathbf{d}\|\Psi_\epsilon\|_{W^{2,p}(\mathbf{D}\setminus\overline{Y}_m)} + \epsilon^\tau\mathbf{d}\|\psi_\epsilon\|_{W^{2,p}(Y_m)} \le c\big(1+\epsilon^\tau\|(\mathbf{k}-\mathbf{d})\nabla\psi_\epsilon\|_{W^{1,p}(Y_m)} + \|(\mathbf{K}-\mathbf{d})\nabla\Psi_\epsilon\|_{W^{1,p}(\mathbf{D}\setminus\overline{Y}_m)}\big).$$

By assumption on $\mathbf{K}$ and $\mathbf{k}$, we obtain (4.4). □

Denote a portion of the boundary of Y by $\partial_1 Y \equiv \{y \in \partial Y | y = (0, y_2, \ldots, y_n)\}$, and consider the following problem

$$\begin{cases} -\nabla\cdot(\mathbf{K}\nabla\Psi_\epsilon) = G_\epsilon & \text{in } Y_f, \\ -\epsilon^{2\tau}\nabla\cdot(\mathbf{k}\nabla\psi_\epsilon) = \epsilon^\tau g_\epsilon & \text{in } Y_m, \\ \mathbf{K}\nabla\Psi_\epsilon\cdot\vec{\mathbf{n}}_y = \epsilon^{2\tau}\mathbf{k}\nabla\psi_\epsilon\cdot\vec{\mathbf{n}}_y & \text{on } \partial Y_m, \\ \Psi_\epsilon = \psi_\epsilon & \text{on } \partial Y_m, \\ \Psi_\epsilon = \Psi_{b_\epsilon} & \text{on } \partial_1 Y, \end{cases} \tag{4.14}$$

where $\tau \in (0,\infty)$ and $\vec{\mathbf{n}}_y$ is the unit vector outward normal to ∂Y_m. Let $Y_m \subset \widetilde{\mathbf{D}} \subset Y$ satisfy

$$\min\{\text{dist}(Y_m, \partial\widetilde{\mathbf{D}}), \text{dist}(\widetilde{\mathbf{D}}, \partial Y\setminus\partial_1 Y)\} > 0 \quad \text{and} \quad \partial\widetilde{\mathbf{D}}\cap\partial_1 Y \ne \emptyset.$$

By an analogous argument as Lemma 4.2, we also have

Lemma 4.3. *Let $\tau \in (0,\infty)$ and $\|\mathbf{K}-\mathbf{d}\|_{W^{1,\infty}(Y_f)} + \|\mathbf{k}-\mathbf{d}\|_{W^{1,\infty}(Y_m)} \le c_0\mathbf{d}$ where $\mathbf{d} > 0$ and $c_0 < \frac{1}{2}$ is a small number depending on Y_m. Any solution of (4.14) satisfies*

$$\|\Psi_\epsilon\|_{W^{2,p}(\widetilde{\mathbf{D}}\setminus\overline{Y}_m)} + \epsilon^\tau\|\psi_\epsilon\|_{W^{2,p}(Y_m)} \le c\big(\|\Psi_\epsilon\|_{L^\varpi(Y_f)} + \|G_\epsilon\|_{L^p(Y_f)} + \|g_\epsilon\|_{L^p(Y_m)} + \|\Psi_{b_\epsilon}\|_{W^{2,p}(Y_f)}\big),$$

where $p \in (1,\infty)$, $\varpi \equiv \min\{2,p\}$, and c is a constant independent of ϵ, τ.

Now we give the proof of (3.5) of Lemma 3.3. By partition of unity, A2, Theorem 8.8 and Theorem 9.11 [17], Lemmas 4.2 and 4.3, we see that the solution of (3.4) satisfies, for fixed $p \in (1,\infty)$, $\tau \in (0,\infty)$, and $\epsilon \in (0,1)$,

$$\|\varphi_\epsilon\|_{W^{1,p}(\Omega)} + \|\varphi_\epsilon\|_{W^{2,p}(\Omega_f^\epsilon)} + \|\varphi_\epsilon\|_{W^{2,p}(\Omega_m^\epsilon)} \le c\big(\|f_\epsilon\|_{L^p(\Omega)} + \|\varphi_\epsilon\|_{L^\varpi(\Omega_f^\epsilon)}\big), \tag{4.15}$$

where $\varpi \equiv \min\{2,p\}$ and c is a constant.

Now we consider the case $p \in [2,\infty)$. The solution of (3.4) satisfies, by the energy method,

$$\|\varphi_\epsilon\|_{H^1(\Omega)} \le c\|f_\epsilon\|_{L^2(\Omega)},$$

where c is a constant. Together with (4.15), we see that (3.5) of Lemma 3.3 holds for $p \in [2,\infty)$.

For any function $\zeta \in L^r(\Omega)$ with $r \in [2,\infty)$, we obtain η_ϵ by solving

$$\begin{cases} -\nabla\cdot(\Lambda_\tau^\epsilon\nabla\eta_\epsilon) = \zeta & \text{in } \Omega, \\ \eta_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.16}$$

We have proved that if $r \in [2,\infty)$, the solution of (4.16) satisfies

$$\|\eta_\epsilon\|_{W^{1,r}(\Omega)} + \|\eta_\epsilon\|_{W^{2,r}(\Omega_f^\epsilon)} + \|\eta_\epsilon\|_{W^{2,r}(\Omega_m^\epsilon)} \le c\|\zeta\|_{L^r(\Omega)}, \tag{4.17}$$

where c is a constant. Multiply (3.4) by η_ϵ and use Green's theorem, (4.17), and the Hölder inequality to obtain

$$\int_\Omega \varphi_\epsilon \zeta\, dx = -\int_\Omega \varphi_\epsilon \nabla\cdot(\Lambda^\epsilon_\tau \nabla\eta_\epsilon)dx = \int_\Omega f_\epsilon \eta_\epsilon dx \le c\|f_\epsilon\|_{L^p(\Omega)}\|\zeta\|_{L^r(\Omega)},$$

for $p\in(1,2]$ and $1/p+1/r=1$. So

$$\|\varphi_\epsilon\|_{L^p(\Omega)} \le c\|f_\epsilon\|_{L^p(\Omega)} \quad \text{for } p\in(1,2]. \tag{4.18}$$

(4.15) and (4.18) imply that the solution of (3.4) satisfies (3.5) for $p\in(1,2]$ case. Therefore (3.5) of Lemma 3.3 holds for $p\in(1,\infty)$ case.

5. Two convergence results

Before the proof of (3.6) of Lemma 3.3, we present two convergence results: Lemmas 5.3 and 5.5. The two lemmas allow us to derive the estimate (3.6) under general permeability fields $\mathbf{K}_\epsilon \mathcal{X}_{\Omega^\epsilon_f} + \epsilon^{2\tau}\mathbf{k}_\epsilon \mathcal{X}_{\Omega^\epsilon_m}$. Lemma 5.3 is used in the interior estimate in Section 6.1 and Lemma 5.5 is used in the boundary estimate in Section 6.2. Define $\mathcal{V}_\epsilon \equiv \{\varphi\in H^1(\Omega^\epsilon_f)|\varphi|_{\partial\Omega}=0\}$ and denote $\mathcal{V}'_\epsilon$ the dual space of $\mathcal{V}_\epsilon$. By Remark 2.1, $\Pi_\epsilon : \mathcal{V}_\epsilon \to H^1_0(\Omega)$ is a linear continuous extension operator. We denote $\Pi'_\epsilon : H^{-1}(\Omega)\to\mathcal{V}'_\epsilon$ the adjoint of Π_ϵ and it is a linear continuous map satisfying

$$\langle \Pi'_\epsilon\varphi,\zeta\rangle_{\mathcal{V}'_\epsilon,\mathcal{V}_\epsilon} = \langle\varphi,\Pi_\epsilon\zeta\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} \quad \text{for } \varphi\in H^{-1}(\Omega),\ \zeta\in\mathcal{V}_\epsilon.$$

For any φ in Ω^σ_f for $\sigma\in(0,1)$, we define a 0-extension function $\mathcal{Q}^\sigma(\varphi):\mathbb{R}^n\to\mathbb{R}$ by

$$\mathcal{Q}^\sigma(\varphi)(x) \equiv \begin{cases}\varphi(x) & \text{if } x\in\Omega^\sigma_f,\\ 0 & \text{if } x\in\mathbb{R}^n\setminus\Omega^\sigma_f.\end{cases}$$

Lemma 5.1. *Assume $\overline{B_1(0)}\subset\Omega$, $\tau\in(0,\infty)$, and $\epsilon,\nu<1$. Consider the following problem*

$$\begin{cases}-\nabla\cdot(\breve{\mathbf{K}}_\nu\nabla\Psi_{\epsilon,\nu}) = G_{\epsilon,\nu} & \text{in } B_1(0)\cap\Omega^\nu_f,\\ -\epsilon^{2\tau}\nabla\cdot(\breve{\mathbf{k}}_\nu\nabla\psi_{\epsilon,\nu}) = \epsilon^\tau g_{\epsilon,\nu} & \text{in } B_1(0)\cap\Omega^\nu_m,\\ \breve{\mathbf{K}}_\nu\nabla\Psi_{\epsilon,\nu}\cdot\vec{\mathbf{n}}^\nu = \epsilon^{2\tau}\breve{\mathbf{k}}_\epsilon\nabla\psi_{\epsilon,\nu}\cdot\vec{\mathbf{n}}^\nu & \text{on } B_1(0)\cap\partial\Omega^\nu_m,\\ \Psi_{\epsilon,\nu}=\psi_{\epsilon,\nu} & \text{on } B_1(0)\cap\partial\Omega^\nu_m,\end{cases} \tag{5.1}$$

where $\vec{\mathbf{n}}^\nu$ is a unit vector normal to $\partial\Omega^\nu_m$. If

$$\begin{cases}\breve{\mathbf{K}}_\nu,\breve{\mathbf{k}}_\nu\in(\alpha,\beta) \quad \text{and} \quad \alpha,\beta>0,\\ \|\Psi_{\epsilon,\nu}\|_{L^2(B_1(0)\cap\Omega^\nu_f)},\ \|\epsilon^\tau\psi_{\epsilon,\nu}\|_{L^2(B_1(0)\cap\Omega^\nu_m)}\le 1,\\ \lim\limits_{\epsilon,\nu\to0}\|G_{\epsilon,\nu}\|_{L^2(B_1(0)\cap\Omega^\nu_f)} + \max\{\epsilon^\tau,\nu\}\|g_{\epsilon,\nu}\|_{L^2(B_1(0)\cap\Omega^\nu_m)} = 0,\end{cases} \tag{5.2}$$

then

(1) $\|\Pi_\nu\Psi_{\epsilon,\nu}\|_{H^1(B_{3/4}(0))}$ *is bounded independent of* ϵ,ν,τ,

(2) *A subsequence of* $\mathcal{Q}^\nu(\breve{\mathbf{K}}_\nu\nabla\Psi_{\epsilon,\nu})$ *converges weakly to* $\xi\in[L^2(B_{3/4}(0))]^n$ *as* $\epsilon,\nu\to0$ *and* $\nabla\cdot\xi=0$ *in* $B_{3/4}(0)$,

(3) $\nabla\cdot\mathcal{Q}^\nu(\breve{\mathbf{K}}_\nu\nabla\Psi_{\epsilon,\nu})$ *converges to* 0 *in* $H^{-1}(B_{1/2}(0))$ *as* $\epsilon,\nu\to0$.

Proof. By the energy method, Remark 2.1, and (5.2), we see

$$\|\nabla\Psi_{\epsilon,\nu}\|_{L^2(B_{3/4}(0)\cap\Omega^\nu_f)} + \epsilon^\tau\|\nabla\psi_{\epsilon,\nu}\|_{L^2(B_{3/4}(0)\cap\Omega^\nu_m)} \le c, \tag{5.3}$$

where c is a constant independent of ϵ,ν,τ. Remark 2.1 and (5.3) imply statement (1).

By (5.3) and the compactness principle, a subsequence of $\mathcal{Q}^\nu(\breve{\mathbf{K}}_\nu\nabla\Psi_{\epsilon,\nu})$ converges weakly to ξ in $[L^2(B_{3/4}(0))]^n$ as $\epsilon,\nu\to0$ (the same notation for subsequence). Multiply (5.1) by a function $\zeta\in H^1_0(B_{3/4}(0))$ to see

$$\int_{B_{3/4}(0)}(\breve{\mathbf{K}}_\nu\nabla\Psi_{\epsilon,\nu}\mathcal{X}_{\Omega^\nu_f} + \epsilon^{2\tau}\breve{\mathbf{k}}_\nu\nabla\psi_{\epsilon,\nu}\mathcal{X}_{\Omega^\nu_m})\nabla\zeta\, dx = \int_{B_{3/4}(0)}(G_{\epsilon,\nu}\mathcal{X}_{\Omega^\nu_f} + \epsilon^\tau g_{\epsilon,\nu}\mathcal{X}_{\Omega^\nu_m})\zeta\, dx.$$

As $\epsilon,\nu\to0$, we see, by (5.2)–(5.3),

$$\nabla\cdot\xi = 0 \quad \text{in } B_{3/4}(0). \tag{5.4}$$

So we prove statement (2).

Let $\eta \in C_0^\infty(B_{3/4}(0))$ be a bell-shaped function satisfying $\eta \in [0, 1]$ and $\eta = 1$ in $B_{1/2}(0)$. From (5.1), we have

$$\begin{cases} -\nabla \cdot (\eta \breve{\mathbf{K}}_\nu \nabla \Psi_{\epsilon,\nu}) = \eta G_{\epsilon,\nu} - \breve{\mathbf{K}}_\nu \nabla \Psi_{\epsilon,\nu} \nabla \eta & \text{in } B_{3/4}(0) \cap \Omega_f^\nu, \\ -\epsilon^{2\tau} \nabla \cdot (\eta \breve{\mathbf{k}}_\nu \nabla \psi_{\epsilon,\nu}) = \eta \epsilon^\tau g_{\epsilon,\nu} - \epsilon^{2\tau} \breve{\mathbf{k}}_\nu \nabla \psi_{\epsilon,\nu} \nabla \eta & \text{in } B_{3/4}(0) \cap \Omega_m^\nu, \\ \eta \breve{\mathbf{K}}_\nu \nabla \Psi_{\epsilon,\nu} \cdot \vec{\mathbf{n}}^\nu = \epsilon^{2\tau} \eta \breve{\mathbf{k}}_\nu \nabla \psi_{\epsilon,\nu} \cdot \vec{\mathbf{n}}^\nu & \text{on } B_{3/4}(0) \cap \partial \Omega_m^\nu, \\ \eta \Psi_{\epsilon,\nu} = \eta \psi_{\epsilon,\nu} & \text{on } B_{3/4}(0) \cap \partial \Omega_m^\nu, \\ \eta \Psi_{\epsilon,\nu} \mathcal{X}_{\Omega_f^\nu} + \eta \psi_{\epsilon,\nu} \mathcal{X}_{\Omega_m^\nu} = 0 & \text{on } \partial B_{3/4}(0). \end{cases} \tag{5.5}$$

Claim that $\nabla \cdot (\eta \mathcal{Q}^\nu (\breve{\mathbf{K}}_\nu \nabla \Psi_{\epsilon,\nu}))$ is in a compact subset of $H^{-1}(B_{3/4}(0))$. Multiply (5.5) by $\zeta_{\epsilon,\nu} \in H_0^1(B_{3/4}(0))$ to obtain

$$\begin{aligned} &\langle -\nabla \cdot (\eta \mathcal{Q}^\nu (\breve{\mathbf{K}}_\nu \nabla \Psi_{\epsilon,\nu})), \zeta_{\epsilon,\nu} \rangle_{H^{-1}(B_{3/4}(0)), H_0^1(B_{3/4}(0))} \\ &= \int_{B_{3/4}(0)} \eta \mathcal{Q}^\nu (\breve{\mathbf{K}}_\nu \nabla \Psi_{\epsilon,\nu}) \nabla \zeta_{\epsilon,\nu} dx = \int_{B_{3/4}(0) \cap \Omega_f^\nu} \eta \breve{\mathbf{K}}_\nu \nabla \Psi_{\epsilon,\nu} \nabla \zeta_{\epsilon,\nu} dx \\ &= -\epsilon^{2\tau} \int_{B_{3/4}(0) \cap \Omega_m^\nu} \eta \breve{\mathbf{k}}_\nu \nabla \psi_{\epsilon,\nu} \nabla \zeta_{\epsilon,\nu} dx + \int_{B_{3/4}(0)} \eta (G_{\epsilon,\nu} \mathcal{X}_{\Omega_f^\nu} + \epsilon^\tau g_{\epsilon,\nu} \mathcal{X}_{\Omega_m^\nu}) \zeta_{\epsilon,\nu} dx \\ &\quad - \int_{B_{3/4}(0)} \nabla \eta (\breve{\mathbf{K}}_\nu \nabla \Psi_{\epsilon,\nu} \mathcal{X}_{\Omega_f^\nu} + \epsilon^{2\tau} \breve{\mathbf{k}}_\nu \nabla \psi_{\epsilon,\nu} \mathcal{X}_{\Omega_m^\nu}) \zeta_{\epsilon,\nu} dx. \end{aligned} \tag{5.6}$$

We choose $\zeta_{\epsilon,\nu}$ in (5.6) in such a way that it satisfies

$$\begin{cases} \Delta \zeta_{\epsilon,\nu} = \nabla \cdot (\eta \mathcal{Q}^\nu (\breve{\mathbf{K}}_\nu \nabla \Psi_{\epsilon,\nu})) & \text{in } B_{3/4}(0), \\ \zeta_{\epsilon,\nu} = 0 & \text{on } \partial B_{3/4}(0). \end{cases} \tag{5.7}$$

(5.7) is solvable uniquely by the Lax–Milgram theorem [17] and $\|\zeta_{\epsilon,\nu}\|_{H^1(B_{3/4}(0))}$ is bounded by a constant independent of ϵ, ν by (5.3). By the compactness principle, $\zeta_{\epsilon,\nu}$ weakly converges to ζ in $H_0^1(B_{3/4}(0))$ as $\epsilon, \nu \to 0$, and ζ satisfies, by statement (2),

$$\begin{cases} \Delta \zeta = \nabla \cdot (\eta \xi) & \text{in } B_{3/4}(0), \\ \zeta = 0 & \text{on } \partial B_{3/4}(0). \end{cases}$$

By (5.2)–(5.3), (5.6)–(5.7), and Lemma 6.1 [29], $\|\nabla \cdot (\eta \mathcal{Q}^\nu (\breve{\mathbf{K}}_\nu \nabla \Psi_{\epsilon,\nu}))\|^2_{H^{-1}(B_{3/4}(0))}$ converges to $\langle -\xi \nabla \eta, \zeta \rangle_{L^2(B_{3/4}(0)), L^2(B_{3/4}(0))}$ as $\epsilon, \nu \to 0$. Since $\nabla \cdot \xi = 0$ in $B_{3/4}(0)$ by (5.4),

$$\langle -\xi \nabla \eta, \zeta \rangle_{L^2(B_{3/4}(0)), L^2(B_{3/4}(0))} = \|\nabla \cdot (\xi \eta)\|^2_{H^{-1}(B_{3/4}(0))}.$$

Since $\nabla \cdot (\eta \mathcal{Q}^\nu (\breve{\mathbf{K}}_\nu \nabla \Psi_{\epsilon,\nu}))$ converges weakly to $\nabla \cdot (\xi \eta)$ in $H^{-1}(B_{3/4}(0))$ as $\epsilon, \nu \to 0$, $\nabla \cdot (\eta \mathcal{Q}^\nu (\breve{\mathbf{K}}_\nu \nabla \Psi_{\epsilon,\nu}))$ converges to $\nabla \cdot (\xi \eta)$ in $H^{-1}(B_{3/4}(0))$ by Remark 1.16 and Proposition 1.17 [30]. So we prove the claim. Moreover, by (5.4), we see that $\nabla \cdot (\eta \mathcal{Q}^\nu (\breve{\mathbf{K}}_\nu \nabla \Psi_{\epsilon,\nu}))$ converges to 0 in $H^{-1}(B_{1/2}(0))$.

The above conclusion is true for any subsequence of $\Psi_{\epsilon,\nu}$, so we prove statement (3). □

Let us define $\mathcal{M}(\ell_1, \ell_2; \mathbb{D})$ as a set containing positive definite matrices, that is,

$$\mathcal{M}(\ell_1, \ell_2; \mathbb{D}) \equiv \{\varphi : \mathbb{D} \to \mathbb{R}^{n \times n} | \ell_1 I \le \varphi \le \ell_2 I,\ \ell_1, \ell_2 > 0, I \text{ is the identity matrix}\}.$$

Lemma 5.2. *For any $\nu < 1$, consider the following problem*

$$\begin{cases} -\nabla \cdot (\breve{\mathbf{K}}_\nu \nabla \Phi_\nu) = \Pi'_\nu G & \text{in } \Omega_f^\nu, \\ \breve{\mathbf{K}}_\nu \nabla \Phi_\nu \cdot \vec{\mathbf{n}}^\nu = 0 & \text{on } \partial \Omega_m^\nu, \\ \Phi_\nu = 0 & \text{on } \partial \Omega, \end{cases} \tag{5.8}$$

where $\breve{\mathbf{K}}_\nu \in (\alpha, \beta)$, $\alpha, \beta > 0$, and $G \in H^{-1}(\Omega)$. There is an element $\mathbf{K}_* \in \mathcal{M}(\gamma^{-2}\alpha, \beta; \Omega)$ and a subsequence of the solutions Φ_ν of (5.8) (same notation for subsequence) such that

(1) *$\Pi_\nu \Phi_\nu$ converges to Φ weakly in $H_0^1(\Omega)$ as $\nu \to 0$,*
(2) *$\mathcal{Q}^\nu (\breve{\mathbf{K}}_\nu \nabla \Phi_\nu)$ converges to $\mathbf{K}_* \nabla \Phi$ weakly in $[L^2(\Omega)]^n$ as $\nu \to 0$,*
(3) *$-\nabla \cdot (\mathbf{K}_* \nabla \Phi) = G$ in Ω,*
(4) *$\nabla \cdot \mathcal{Q}^\nu (\breve{\mathbf{K}}_\nu \nabla \Phi_\nu)$ is in a compact subset of $H^{-1}(\Omega)$.*

Note: Constant γ here is the $\gamma(Y_f, p)$ in Remark 2.1.

Proof. Statements (1)–(3) are from Definition 1.3 and Theorem 1.8 [31]. So we only prove statement (4). By the energy method and Remark 2.1, we know

$$\|\nabla \Phi_\nu\|_{L^2(\Omega_f^\nu)} \le c, \tag{5.9}$$

where c is independent of ν. We claim that every weakly convergent subsequence of $\nabla \cdot \mathcal{Q}^\nu(\breve{\mathbf{K}}_\nu \nabla \Phi_\nu)$ in $H^{-1}(\Omega)$ is strongly convergent as $\nu \to 0$. By (5.9), $\nabla \cdot \mathcal{Q}^\nu(\breve{\mathbf{K}}_\nu \nabla \Phi_\nu)$ is bounded in $H^{-1}(\Omega)$. Let ξ denote the weak limit of $\mathcal{Q}^\nu(\breve{\mathbf{K}}_\nu \nabla \Phi_\nu)$, that is, $\xi = \mathbf{K}_* \nabla \Phi$. Statement (3) implies

$$G = -\nabla \cdot \xi. \tag{5.10}$$

If $\eta_\nu \in H_0^1(\Omega)$, (5.8) implies

$$\begin{aligned}\langle -\nabla \cdot \mathcal{Q}^\nu(\breve{\mathbf{K}}_\nu \nabla \Phi_\nu), \eta_\nu\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} &= \int_{\Omega_f^\nu} \breve{\mathbf{K}}_\nu \nabla \Phi_\nu \nabla \eta_\nu dx \\ &= \langle G, \Pi_\nu \eta_\nu|_{\Omega_f^\nu}\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.\end{aligned} \tag{5.11}$$

In (5.11), we take $\eta_\nu \in H_0^1(\Omega)$ satisfying

$$\begin{cases}\Delta \eta_\nu = \nabla \cdot \mathcal{Q}^\nu(\breve{\mathbf{K}}_\nu \nabla \Phi_\nu) & \text{in } \Omega, \\ \eta_\nu = 0 & \text{on } \partial\Omega.\end{cases} \tag{5.12}$$

The existence of (5.12) is from the Lax–Milgram theorem [17] and $\|\eta_\nu\|_{H_0^1(\Omega)}$ is bounded independent of ν. If η_ν weakly converges to η in $H_0^1(\Omega)$, then η satisfies

$$\begin{cases}\Delta \eta = \nabla \cdot \xi & \text{in } \Omega, \\ \eta = 0 & \text{on } \partial\Omega.\end{cases} \tag{5.13}$$

By (5.10)–(5.13), Lemma 2.1 [31], and Lemma 6.1 [29],

$$\lim_{\nu\to 0} \|\nabla \cdot \mathcal{Q}^\nu(\breve{\mathbf{K}}_\nu \nabla \Phi_\nu)\|^2_{H^{-1}(\Omega)} = \langle -\nabla \cdot \xi, \eta\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \|\nabla \cdot \xi\|^2_{H^{-1}(\Omega)}.$$

By Remark 1.16 and Proposition 1.17 [30], we prove statement (4). □

Lemma 5.3. *Under the same assumptions in Lemma 5.1, there is an element $\mathbf{K}^* \in \mathcal{M}(\gamma^{-2}\alpha, \beta; \Omega)$ and a subsequence of the solutions $\Psi_{\epsilon,\nu}$ of (5.1) (same notation for subsequence) such that, as $\epsilon, \nu \to 0$,*

(1) *$\Pi_\nu \Psi_{\epsilon,\nu}$ converges to Ψ weakly in $H^1(B_{1/2}(0))$,*
(2) *$\mathcal{Q}^\nu(\breve{\mathbf{K}}_\nu \nabla \Psi_{\epsilon,\nu})$ converges to $\mathbf{K}^* \nabla \Psi$ weakly in $[L^2(B_{1/2}(0))]^n$,*
(3) *$-\nabla \cdot (\mathbf{K}^* \nabla \Psi) = 0$ in $B_{1/2}(0)$.*

Note: γ here is the $\gamma(Y_f, p)$ in Remark 2.1.

Proof. By Lemma 5.1, there is a subsequence of the solutions $\Psi_{\epsilon,\nu}$ of (5.1) (same notation for subsequence) satisfying, as $\epsilon, \nu \to 0$,

(1) $\Pi_\nu \Psi_{\epsilon,\nu}$ converges to Ψ weakly in $H^1(B_{3/4}(0))$,
(2) $\mathcal{Q}^\nu(\breve{\mathbf{K}}_\nu \nabla \Psi_{\epsilon,\nu})$ converges to ξ weakly in $[L^2(B_{3/4}(0))]^n$,
(3) $\nabla \cdot \mathcal{Q}^\nu(\breve{\mathbf{K}}_\nu \nabla \Psi_{\epsilon,\nu})$ converges to 0 in $H^{-1}(B_{1/2}(0))$,
(4) $-\nabla \cdot \xi = 0$ in $B_{3/4}(0)$.

Let $\mathbf{K}_*$ be the one in Lemma 5.2. For any $\Phi \in H_0^1(\Omega)$, we define $G \equiv -\nabla \cdot (\mathbf{K}_* \nabla \Phi) \in H^{-1}(\Omega)$ and use the defined G to obtain Φ_ν by solving (5.8). By Lemma 5.2 and the Lax–Milgram Theorem [17], we see that function $\Pi_\nu \Phi_\nu$ converges to Φ weakly in $H_0^1(\Omega)$ as $\nu \to 0$. Clearly

$$\nabla \Pi_\nu \Phi_\nu \cdot \mathcal{Q}^\nu(\breve{\mathbf{K}}_\nu \nabla \Psi_{\epsilon,\nu}) = \nabla \Pi_\nu \Psi_{\epsilon,\nu} \cdot \mathcal{Q}^\nu(\breve{\mathbf{K}}_\nu \nabla \Phi_\nu) \quad \text{in } B_{1/2}(0).$$

As $\epsilon, \nu \to 0$, by Lemma 5.2 and the divergence-curl lemma (see Lemma 1.1 [32]),

$$\nabla \Phi \cdot \xi = \nabla \Psi \cdot \mathbf{K}_* \nabla \Phi \quad \text{almost everywhere in } B_{1/2}(0).$$

Since $\Phi \in H_0^1(\Omega)$ is arbitrary, we see $\xi = \mathbf{K}^* \nabla \Psi$ (here $\mathbf{K}^*$ is the transpose of $\mathbf{K}_*$). So we prove the lemma. □

Let $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a function satisfying

$$\phi(0) = |\nabla\phi(0)| = 0 \quad \text{and} \quad \|\phi\|_{C^{1,\mathbf{e}}(\mathbb{R}^{n-1})} \le c_{\mathbf{e}} \quad \text{for some } \mathbf{e} \in (0,1). \tag{5.14}$$

By A1, Ω is a $C^{1,\mathbf{e}}$ domain. We assume

$$\begin{cases} 0 \in \partial\Omega, \\ B_1(0) \cap \Omega/\nu = \begin{cases} B_1(0) \cap \{(x', x_n) \in \mathbb{R}^n |\ \nu x_n > \phi(\nu x')\} & \text{if } \nu \in (0,1], \\ B_1(0) \cap \{(x', x_n) \in \mathbb{R}^n |\ x_n > 0\} & \text{if } \nu = 0. \end{cases} \end{cases} \tag{5.15}$$

Tracing the proof of Lemma 5.1, we have

Lemma 5.4. *Let $\tau \in (0,\infty)$ and $\epsilon, \lambda, \nu \in (0,1)$. Consider the following problem*

$$\begin{cases} -\nabla \cdot (\hat{\mathbf{K}}_{\lambda,\nu} \nabla \Psi_{\epsilon,\lambda,\nu}) = G_{\epsilon,\lambda,\nu} & \text{in } B_1(0) \cap \Omega_f^\lambda/\nu, \\ -\epsilon^{2\tau} \nabla \cdot (\hat{\mathbf{k}}_{\lambda,\nu} \nabla \psi_{\epsilon,\lambda,\nu}) = \epsilon^\tau g_{\epsilon,\lambda,\nu} & \text{in } B_1(0) \cap \Omega_m^\lambda/\nu, \\ \hat{\mathbf{K}}_{\lambda,\nu} \nabla \Psi_{\epsilon,\lambda,\nu} \cdot \vec{\mathbf{n}}^{\lambda/\nu} = \epsilon^{2\tau} \hat{\mathbf{k}}_{\lambda,\nu} \nabla \psi_{\epsilon,\lambda,\nu} \cdot \vec{\mathbf{n}}^{\lambda/\nu} & \text{on } B_1(0) \cap \partial\Omega_m^\lambda/\nu, \\ \Psi_{\epsilon,\lambda,\nu} = \psi_{\epsilon,\lambda,\nu} & \text{on } B_1(0) \cap \partial\Omega_m^\lambda/\nu, \\ \Psi_{\epsilon,\lambda,\nu} = 0 & \text{on } B_1(0) \cap \partial\Omega/\nu, \end{cases} \tag{5.16}$$

where $\vec{\mathbf{n}}^{\lambda/\nu}$ is a unit vector normal to $\partial\Omega_m^\lambda/\nu$ (see (2.1) for $\hat{\mathbf{K}}_{\lambda,\nu}, \hat{\mathbf{k}}_{\lambda,\nu}$). If

$$\begin{cases} \nu \to \nu_* \in [0,1], \\ \hat{\mathbf{K}}_{\lambda,\nu}, \hat{\mathbf{k}}_{\lambda,\nu} \in (\alpha,\beta) \quad \text{and} \quad \alpha,\beta > 0, \\ \|\Psi_{\epsilon,\lambda,\nu}\|_{L^2(B_1(0)\cap\Omega_f^\lambda/\nu)}, \epsilon^\tau \|\psi_{\epsilon,\lambda,\nu}\|_{L^2(B_1(0)\cap\Omega_m^\lambda/\nu)} \le 1, \\ \lim\limits_{\epsilon,\lambda/\nu\to 0} \|G_{\epsilon,\lambda,\nu}\|_{L^2(B_1(0)\cap\Omega_f^\lambda/\nu)} + \max\{\epsilon^\tau, \lambda/\nu\} \|g_{\epsilon,\lambda,\nu}\|_{L^2(B_1(0)\cap\Omega_m^\lambda/\nu)} = 0, \end{cases} \tag{5.17}$$

then there is a subsequence of $\Psi_{\epsilon,\lambda,\nu}$ (same notation for subsequence) satisfying

(1) *$\|\Pi_{\lambda/\nu}\Psi_{\epsilon,\lambda,\nu}\|_{H^1(B_{3/4}(0)\cap\Omega/\nu)}$ is bounded independent of $\epsilon, \lambda, \nu, \tau$,*

(2) *$\mathcal{Q}^{\lambda/\nu}(\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu})$ converges weakly to $\xi \in [L^2(B_{3/4}(0))]^n$ as $\epsilon, \lambda/\nu \to 0$ and $\nabla \cdot \xi = 0$ in $B_{3/4}(0) \cap \Omega/\nu_*$,*

(3) *$\nabla \cdot (\mathcal{Q}^{\lambda/\nu}(\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu}))$ converges to 0 in $H^{-1}(B_{1/2}(0) \cap \mathbb{D})$ as $\epsilon, \lambda/\nu \to 0$ for any compact subset $\mathbb{D} \subset B_{3/4}(0) \cap \Omega/\nu_*$.*

Proof. By the energy method, (5.17), $\lambda/\nu < 1$, and Remark 2.1, we see

$$\|\Pi_{\lambda/\nu}\Psi_{\epsilon,\lambda,\nu}\|_{H^1(B_{3/4}(0)\cap\Omega/\nu)} + \epsilon^\tau \|\psi_{\epsilon,\lambda,\nu}\|_{H^1(B_{3/4}(0)\cap\Omega_m^\lambda/\nu)} \le c, \tag{5.18}$$

where c is independent of $\epsilon, \lambda, \nu, \tau$. That is statement (1).

Note $\mathcal{Q}^{\lambda/\nu}(\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu})$ is bounded independent of $\epsilon, \lambda, \nu, \tau$ in $[L^2(B_{3/4}(0))]^n$ and there is a subsequence converging weakly to $\xi \in [L^2(B_{3/4}(0))]^n$ as $\epsilon, \lambda/\nu \to 0$. Let $\mathbb{D}$ be any compact subset in $B_{3/4}(0) \cap \Omega/\nu_*$. So if ν is close to ν_*, then $\mathbb{D} \subset B_{3/4}(0) \cap \Omega/\nu$. Multiply (5.16) for ν close to ν_* by any function $\zeta \in C_0^\infty(\mathbb{D})$ to see

$$\int_{\mathbb{D}} (\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu}\mathcal{X}_{\Omega_f^\lambda/\nu} + \epsilon^{2\tau}\hat{\mathbf{k}}_{\lambda,\nu}\nabla\psi_{\epsilon,\lambda,\nu}\mathcal{X}_{\Omega_m^\lambda/\nu})\nabla\zeta\, dx = \int_{\mathbb{D}} (G_{\epsilon,\lambda,\nu}\mathcal{X}_{\Omega_f^\lambda/\nu} + \epsilon^\tau g_{\epsilon,\lambda,\nu}\mathcal{X}_{\Omega_m^\lambda/\nu})\zeta\, dx.$$

As $\epsilon, \lambda/\nu \to 0$, we see, by (5.17)–(5.18),

$$\nabla \cdot \xi = 0 \quad \text{in } \mathbb{D}.$$

Since $\mathbb{D}$ is any compact subset in $B_{3/4}(0) \cap \Omega/\nu_*$, we prove statement (2).

Let $\eta \in C_0^\infty(B_{3/4}(0))$ be a bell-shaped function satisfying $\eta \in [0,1]$ and $\eta = 1$ in $B_{1/2}(0)$. From (5.16), we have

$$\begin{cases} -\nabla \cdot (\eta\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu}) = \eta G_{\epsilon,\lambda,\nu} - \hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu}\nabla\eta & \text{in } B_{\frac{3}{4}}(0) \cap \Omega_f^\lambda/\nu, \\ -\epsilon^{2\tau}\nabla \cdot (\eta\hat{\mathbf{k}}_{\lambda,\nu}\nabla\psi_{\epsilon,\lambda,\nu}) = \eta\epsilon^\tau g_{\epsilon,\lambda,\nu} - \epsilon^{2\tau}\hat{\mathbf{k}}_{\lambda,\nu}\nabla\psi_{\epsilon,\lambda,\nu}\nabla\eta & \text{in } B_{\frac{3}{4}}(0) \cap \Omega_m^\lambda/\nu, \\ \eta\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu} \cdot \vec{\mathbf{n}}^{\lambda/\nu} = \eta\epsilon^{2\tau}\hat{\mathbf{k}}_{\lambda,\nu}\nabla\psi_{\epsilon,\lambda,\nu} \cdot \vec{\mathbf{n}}^{\lambda/\nu} & \text{on } B_{\frac{3}{4}}(0) \cap \partial\Omega_m^\lambda/\nu, \\ \eta\Psi_{\epsilon,\lambda,\nu} = \eta\psi_{\epsilon,\lambda,\nu} & \text{on } B_{\frac{3}{4}}(0) \cap \partial\Omega_m^\lambda/\nu, \\ \eta\Psi_{\epsilon,\lambda,\nu}\mathcal{X}_{\Omega_f^\lambda/\nu} + \eta\psi_{\epsilon,\lambda,\nu}\mathcal{X}_{\Omega_m^\lambda/\nu} = 0 & \text{on } \partial(B_{\frac{3}{4}}(0) \cap \Omega/\nu). \end{cases} \tag{5.19}$$

Claim that $\nabla\cdot(\eta\mathcal{Q}^{\lambda/\nu}(\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu}))$ is in a compact subset of $H^{-1}(\mathbb{D})$, where $\mathbb{D}$ is any compact subset in $B_{3/4}(0)\cap\Omega/\nu_*$. Multiply (5.19) for ν close to ν_* by any $\zeta_{\epsilon,\nu}\in H_0^1(\mathbb{D})$ to obtain

$$\begin{aligned}&\langle-\nabla\cdot(\eta\mathcal{Q}^{\lambda/\nu}(\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu})),\zeta_{\epsilon,\nu}\rangle_{H^{-1}(B_{3/4}(0)),H_0^1(B_{3/4}(0))}\\&=\int_{B_{3/4}(0)}\eta\mathcal{Q}^{\lambda/\nu}(\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu})\nabla\zeta_{\epsilon,\nu}dx=\int_{B_{3/4}(0)\cap\Omega_f^\lambda/\nu}\eta\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu}\nabla\zeta_{\epsilon,\nu}dx\\&=-\epsilon^{2\tau}\int_{B_{3/4}(0)\cap\Omega_m^\lambda/\nu}\eta\hat{\mathbf{k}}_{\lambda,\nu}\nabla\psi_{\epsilon,\lambda,\nu}\nabla\zeta_{\epsilon,\nu}dx+\int_{B_{3/4}(0)}\eta(G_{\epsilon,\lambda,\nu}\mathcal{X}_{\Omega_f^\lambda/\nu}+\epsilon^\tau g_{\epsilon,\lambda,\nu}\mathcal{X}_{\Omega_m^\lambda/\nu})\zeta_{\epsilon,\nu}dx\\&\quad-\int_{B_{3/4}(0)}\nabla\eta(\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu}\mathcal{X}_{\Omega_f^\lambda/\nu}+\epsilon^{2\tau}\hat{\mathbf{k}}_{\lambda,\nu}\nabla\psi_{\epsilon,\lambda,\nu}\mathcal{X}_{\Omega_m^\lambda/\nu})\zeta_{\epsilon,\nu}dx.\end{aligned}\tag{5.20}$$

We choose $\zeta_{\epsilon,\nu}\in H_0^1(\mathbb{D})$ in (5.20) satisfying

$$\begin{cases}\Delta\zeta_{\epsilon,\nu}=\nabla\cdot(\eta\mathcal{Q}^{\lambda/\nu}(\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu})) & \text{in }\mathbb{D},\\ \zeta_{\epsilon,\nu}=0 & \text{on }\partial\mathbb{D}.\end{cases}\tag{5.21}$$

(5.21) is solvable uniquely by the Lax–Milgram theorem [17] and $\|\zeta_{\epsilon,\nu}\|_{H^1(\mathbb{D})}$ is bounded by a constant independent of ϵ,λ,ν by (5.18). By the compactness principle, $\zeta_{\epsilon,\nu}$ weakly converges to ζ in $H_0^1(\mathbb{D})$ as $\epsilon,\lambda/\nu\to0$, and ζ satisfies, by statement (2),

$$\begin{cases}\Delta\zeta=\nabla\cdot(\eta\xi) & \text{in }\mathbb{D},\\ \zeta=0 & \text{on }\partial\mathbb{D}.\end{cases}$$

By (5.17), (5.18), (5.20), (5.21), and Lemma 6.1 [29],

$$\|\nabla\cdot(\eta\mathcal{Q}^{\lambda/\nu}(\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu}))\|^2_{H^{-1}(\mathbb{D})}\to\langle-\xi\nabla\eta,\zeta\rangle_{L^2(B_{3/4}(0)),L^2(B_{3/4}(0))}$$

as $\epsilon,\lambda/\nu\to0$. Since $\nabla\cdot\xi=0$ in $B_{3/4}(0)\cap\Omega/\nu_*$ by statement (2),

$$\langle-\xi\nabla\eta,\zeta\rangle_{L^2(B_{3/4}(0)),L^2(B_{3/4}(0))}=\|\nabla\cdot(\xi\eta)\|^2_{H^{-1}(\mathbb{D})}.$$

Since $\nabla\cdot(\eta\mathcal{Q}^{\lambda/\nu}(\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu}))$ converges weakly to $\nabla\cdot(\xi\eta)$ in $H^{-1}(B_{3/4}(0))$ as $\epsilon,\lambda/\nu\to0$, we know that $\nabla\cdot(\eta\mathcal{Q}^{\lambda/\nu}(\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu}))$ converges to $\nabla\cdot(\xi\eta)$ in $H^{-1}(\mathbb{D})$ by Remark 1.16 and Proposition 1.17 [30]. The above convergence is true for any compact subset $\mathbb{D}$ in $B_{3/4}(0)\cap\Omega/\nu_*$. The claim then follows by a diagonal process.

Also note that, by statement (2), $\nabla\cdot(\mathcal{Q}^{\lambda/\nu}(\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu}))$ converges to 0 in $H^{-1}(B_{1/2}(0)\cap\mathbb{D})$ as $\epsilon,\lambda/\nu\to0$. So we prove statement (3). □

For any solution $\Psi_{\epsilon,\lambda,\nu}$ in (5.16), we define

$$\mathcal{S}_\nu(\Pi_{\lambda/\nu}\Psi_{\epsilon,\lambda,\nu})\equiv\begin{cases}\Pi_{\lambda/\nu}\Psi_{\epsilon,\lambda,\nu} & \text{if }x\in B_{3/4}(0)\cap\Omega/\nu,\\ 0 & \text{if }x\in B_{3/4}(0)\setminus\Omega/\nu.\end{cases}$$

Modifying of the proofs of Lemmas 5.2 and 5.3, we also have

Lemma 5.5. *Under the same assumptions of Lemma 5.4, there is an element $\mathbf{K}^*\in\mathcal{M}(\gamma^{-2}\alpha,\beta;B_{1/2}(0)\cap\Omega/\nu_*)$ and a subsequence of $\Psi_{\epsilon,\lambda,\nu}$ (same notation for subsequence) such that, as $\epsilon,\lambda/\nu\to0$ and $\nu\to\nu_*$,*

(1) *$\mathcal{S}_\nu(\Pi_{\lambda/\nu}\Psi_{\epsilon,\lambda,\nu})$ converges to Ψ weakly in $H^1(B_{1/2}(0)\cap\Omega/\nu_*)$,*
(2) *$\mathcal{Q}^{\lambda/\nu}(\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu})$ converges to $\mathbf{K}^*\nabla\Psi$ weakly in $[L^2(B_{1/2}(0)\cap\Omega/\nu_*)]^n$,*
(3) $\begin{cases}-\nabla\cdot(\mathbf{K}^*\nabla\Psi)=0 & \text{in }B_{1/2}(0)\cap\Omega/\nu_*,\\ \Psi=0 & \text{on }B_{1/2}(0)\cap\partial\Omega/\nu_*.\end{cases}$

Note: γ here is the $\gamma(Y_f,p)$ in Remark 2.1.

6. Proof of (3.6) of Lemma 3.3

This section includes two Sections 6.1 and 6.2. The Hölder estimate in the interior region is derived in Section 6.1, and the Hölder estimate around the boundary is in Section 6.2. The idea of the proof for the Hölder estimate is from the three-step compactness argument in [8]. A1–A2 are assumed in this section.

6.1. Interior estimate

For convenience we assume $\overline{B_1(0)}\subset\Omega$.

Lemma 6.1. *For $\delta, \tau > 0$, there are $\mu, \theta_1, \theta_2 \in (0,1)$ (depending on $n, \delta, \alpha, \beta, Y_f$) satisfying $\theta_1 < \theta_2^2$ and there is a $\epsilon_0 \in (0,1)$ (depending on $\theta_1, \theta_2, n, \delta, \tau, \alpha, \beta, Y_f$) such that if*

$$\begin{cases} -\nabla\cdot(\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu}) = G_{\epsilon,\lambda,\nu} & \text{in } B_1(0)\cap\Omega_f^\lambda/\nu,\\ -\epsilon^{2\tau}\nabla\cdot(\hat{\mathbf{k}}_{\lambda,\nu}\nabla\psi_{\epsilon,\lambda,\nu}) = \epsilon^\tau g_{\epsilon,\lambda,\nu} & \text{in } B_1(0)\cap\Omega_m^\lambda/\nu,\\ \hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu}\cdot\vec{\mathbf{n}}^{\lambda/\nu} = \epsilon^{2\tau}\hat{\mathbf{k}}_{\lambda,\nu}\nabla\psi_{\epsilon,\lambda,\nu}\cdot\vec{\mathbf{n}}^{\lambda/\nu} & \text{on } B_1(0)\cap\partial\Omega_m^\lambda/\nu,\\ \Psi_{\epsilon,\lambda,\nu} = \psi_{\epsilon,\lambda,\nu} & \text{on } B_1(0)\cap\partial\Omega_m^\lambda/\nu, \end{cases} \tag{6.1}$$

if

$$\begin{cases} \|\Psi_{\epsilon,\lambda,\nu}\|_{L^2(B_1(0)\cap\Omega_f^\lambda/\nu)},\ \epsilon^\tau\|\psi_{\epsilon,\lambda,\nu}\|_{L^2(B_1(0)\cap\Omega_m^\lambda/\nu)} \le 1,\\ \epsilon_0^{-1}\|G_{\epsilon,\lambda,\nu}\mathcal{X}_{\Omega_f^\lambda/\nu} + \max\{\epsilon^\tau,\lambda/\nu\}g_{\epsilon,\lambda,\nu}\mathcal{X}_{\Omega_m^\lambda/\nu}\|_{L^{n+\delta}(B_1(0))} \le 1, \end{cases} \tag{6.2}$$

and if $\epsilon, \lambda/\nu \le \epsilon_0$, $\nu \in (0,1]$, and $\theta \in [\theta_1,\theta_2]$, then

$$\begin{cases} \displaystyle\fint_{B_\theta(0)} |\Pi_{\lambda/\nu}\Psi_{\epsilon,\lambda,\nu} - (\Pi_{\lambda/\nu}\Psi_{\epsilon,\lambda,\nu})_{0,\theta}|^2dx \le \theta^{2\mu},\\ \displaystyle\fint_{B_\theta(0)\cap\Omega_m^\lambda/\nu} \epsilon^{2\tau}|\psi_{\epsilon,\lambda,\nu} - (\Pi_{\lambda/\nu}\Psi_{\epsilon,\lambda,\nu})_{0,\theta}|^2dx \le \theta^{2\mu}. \end{cases} \tag{6.3}$$

See Section 2 for $\hat{\mathbf{K}}_{\lambda,\nu}$, $\hat{\mathbf{k}}_{\lambda,\nu}$, $(\Pi_{\lambda/\nu}\Psi_{\epsilon,\lambda,\nu})_{0,\theta}$.

Proof. Assume $\mathbf{K}^* \in \mathcal{M}(\gamma^{-2}\alpha,\beta;\Omega)$ and Ψ is a solution of the uniform elliptic equation $-\nabla\cdot(\mathbf{K}^*\nabla\Psi) = 0$ in $B_{1/2}(0)$. Then, by Theorem 8.24 [17],

$$\|\Psi\|_{C^s(B_{1/8}(0))} \le c\|\Psi\|_{L^2(B_{1/2}(0))},$$

where $s(<1)$, c are constants depending on n, α, β, Y_f. Define $\mu \equiv \frac{1}{2}\min\{s, \frac{\delta}{n+\delta}\}$. If μ' satisfies $\mu < \mu' < 2\mu$, then, by Theorem 1.2 in page 70 [33],

$$\fint_{B_\theta(0)} |\Psi - (\Psi)_{0,\theta}|^2dx \le \theta^{2\mu'}\fint_{B_{1/2}(0)} |\Psi|^2dx \tag{6.4}$$

for θ (depending on $\mu, n, \alpha, \beta, Y_f$) sufficiently small. Fix two values $\theta_1, \theta_2 \le 1/8$ such that (1) $\theta_1 < \theta_2^2$ and (2) Inequality (6.4) holds for any $\theta \in [\theta_1,\theta_2]$.

With μ, θ_1, θ_2 above, we claim $(6.3)_1$. If not, there is a sequence $\{\epsilon_\lambda, \lambda, \nu_\lambda, \theta_{\epsilon_\lambda,\lambda,\nu_\lambda}, \Psi_{\epsilon_\lambda,\lambda,\nu_\lambda}, \psi_{\epsilon_\lambda,\lambda,\nu_\lambda}, G_{\epsilon_\lambda,\lambda,\nu_\lambda}, g_{\epsilon_\lambda,\lambda,\nu_\lambda}\}$ satisfying (6.1) and

$$\begin{cases} \epsilon_\lambda, \lambda/\nu_\lambda \to 0,\\ \nu_\lambda \in (0,1], \quad \theta_{\epsilon_\lambda,\lambda,\nu_\lambda} \in [\theta_1,\theta_2],\\ \max\{\|\Psi_{\epsilon_\lambda,\lambda,\nu_\lambda}\|_{L^2(B_1(0)\cap\Omega_f^\lambda/\nu_\lambda)}, \epsilon_\lambda^\tau\|\psi_{\epsilon_\lambda,\lambda,\nu_\lambda}\|_{L^2(B_1(0)\cap\Omega_m^\lambda/\nu_\lambda)}\} \le 1,\\ \displaystyle\lim_{\epsilon_\lambda,\lambda/\nu_\lambda\to 0}\|G_{\epsilon_\lambda,\lambda,\nu_\lambda}\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\lambda/\nu_\lambda)} + \max\{\epsilon^\tau,\lambda/\nu_\lambda\}\|g_{\epsilon_\lambda,\lambda,\nu_\lambda}\|_{L^{n+\delta}(B_1(0)\cap\Omega_m^\lambda/\nu_\lambda)} = 0,\\ \displaystyle\fint_{B_{\theta_{\epsilon_\lambda,\lambda,\nu_\lambda}}(0)} |\Pi_{\lambda/\nu_\lambda}\Psi_{\epsilon_\lambda,\lambda,\nu_\lambda} - (\Pi_{\lambda/\nu_\lambda}\Psi_{\epsilon_\lambda,\lambda,\nu_\lambda})_{0,\theta_{\epsilon_\lambda,\lambda,\nu_\lambda}}|^2dx > \theta_{\epsilon_\lambda,\lambda,\nu_\lambda}^{2\mu}. \end{cases} \tag{6.5}$$

By Lemma 5.3, there is a subsequence (same notation for subsequence) such that, as $\epsilon_\lambda, \lambda/\nu_\lambda \to 0$,

$$\begin{cases} \theta_{\epsilon_\lambda,\lambda,\nu_\lambda} \to \theta_* \in [\theta_1,\theta_2],\\ \Pi_{\lambda/\nu_\lambda}\Psi_{\epsilon_\lambda,\lambda,\nu_\lambda} \to \Psi & \text{in } L^2(B_{1/2}(0)) \text{ strongly},\\ \hat{\mathbf{K}}_{\lambda,\nu_\lambda}\nabla\Psi_{\epsilon_\lambda,\lambda,\nu_\lambda}\mathcal{X}_{\Omega_f^\lambda/\nu_\lambda} \to \mathbf{K}^*\nabla\Psi & \text{in } [L^2(B_{1/2}(0))]^n \text{ weakly},\\ \epsilon_\lambda^{2\tau}\hat{\mathbf{k}}_{\lambda,\nu_\lambda}\nabla\psi_{\epsilon_\lambda,\lambda,\nu_\lambda}\mathcal{X}_{\Omega_m^\lambda/\nu_\lambda} \to 0 & \text{in } [L^2(B_{1/2}(0))]^n \text{ strongly}, \end{cases} \tag{6.6}$$

where $\mathbf{K}^* \in \mathcal{M}(\gamma^{-2}\alpha,\beta;\Omega)$ and Ψ is a solution of the uniform elliptic equation $-\nabla\cdot(\mathbf{K}^*\nabla\Psi) = 0$ in $B_{1/2}(0)$. By (6.4)–(6.6),

$$\begin{aligned} \theta_*^{2\mu} &= \lim_{\epsilon_\lambda,\lambda/\nu_\lambda\to 0}\theta_{\epsilon_\lambda,\lambda,\nu_\lambda}^{2\mu}\\ &\le \lim_{\epsilon_\lambda,\lambda/\nu_\lambda\to 0}\fint_{B_{\theta_{\epsilon_\lambda,\lambda,\nu_\lambda}}(0)} |\Pi_{\lambda/\nu_\lambda}\Psi_{\epsilon_\lambda,\lambda,\nu_\lambda} - (\Pi_{\lambda/\nu_\lambda}\Psi_{\epsilon_\lambda,\lambda,\nu_\lambda})_{0,\theta_{\epsilon_\lambda,\lambda,\nu_\lambda}}|^2dx\\ &= \fint_{B_{\theta_*}(0)} |\Psi|^2dx - \left|\fint_{B_{\theta_*}(0)} \Psi dx\right|^2 = \fint_{B_{\theta_*}(0)} |\Psi - (\Psi)_{0,\theta_*}|^2dx \le \theta_*^{2\mu'}\fint_{B_{1/2}(0)} |\Psi|^2dx. \end{aligned}$$

If θ_2 is small enough, then the right hand side of the above equation is less than $\theta_*^{2\mu''}$ for some $\mu'' \in (\mu, \mu')$. So we get $\theta_*^{2\mu} \le \theta_*^{2\mu''}$ for $\mu'' \in (\mu, \mu')$. But this is impossible. Therefore we prove $(6.3)_1$.

Let us define

$$\begin{cases} \hat{\Psi}_{\epsilon,\lambda,\nu} \equiv \theta^{-\mu}(\Pi_{\lambda/\nu}\Psi_{\epsilon,\lambda,\nu} - (\Pi_{\lambda/\nu}\Psi_{\epsilon,\lambda,\nu})_{0,\theta}), \\ \hat{\psi}_{\epsilon,\lambda,\nu} \equiv \theta^{-\mu}(\psi_{\epsilon,\lambda,\nu} - (\Pi_{\lambda/\nu}\Psi_{\epsilon,\lambda,\nu})_{0,\theta}). \end{cases}$$

Then (6.1) implies, for any smooth function η with support in $\lambda\nu^{-1}(Y_m + j) \subset B_\theta(0) \cap \Omega_m^\lambda/\nu$ for some $j \in \mathbb{Z}^n$,

$$\epsilon^{2\tau}\int_{\lambda\nu^{-1}(Y_m+j)} (\hat{\psi}_{\epsilon,\lambda,\nu} - \hat{\Psi}_{\epsilon,\lambda,\nu})\nabla\cdot(\hat{\mathbf{k}}_{\lambda,\nu}\nabla\eta)dx = \int_{\lambda\nu^{-1}(Y_m+j)} \epsilon^{2\tau}\hat{\mathbf{k}}_{\lambda,\nu}\nabla\hat{\Psi}_{\epsilon,\lambda,\nu}\nabla\eta - \theta^{-\mu}\epsilon^{\tau}\eta\, g_{\epsilon,\lambda,\nu}\, dx.$$

If η is the solution of

$$\begin{cases} \nabla\cdot(\hat{\mathbf{k}}_{\lambda,\nu}\nabla\eta) = \hat{\psi}_{\epsilon,\lambda,\nu} - \hat{\Psi}_{\epsilon,\lambda,\nu} & \text{in } \lambda\nu^{-1}(Y_m + j), \\ \eta = 0 & \text{on } \lambda\nu^{-1}(\partial Y_m + j), \end{cases}$$

then

$$c_1\frac{\nu}{\lambda}\|\eta\|_{L^2(\lambda\nu^{-1}(Y_m+j))} \le \|\nabla\eta\|_{L^2(\lambda\nu^{-1}(Y_m+j))} \le c_2\frac{\lambda}{\nu}\|\hat{\psi}_{\epsilon,\lambda,\nu} - \hat{\Psi}_{\epsilon,\lambda,\nu}\|_{L^2(\lambda\nu^{-1}(Y_m+j))},$$

where c_1, c_2 are independent of $\epsilon, \lambda/\nu$. Inequality $(6.3)_2$ follows from above estimates if ϵ_0 is small enough. □

Lemma 6.2. *Let $\delta, \tau, \mu(< \frac{\delta}{2(n+\delta)}), \theta_1, \theta_2, \epsilon_0$ be same as those in Lemma 6.1. If*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\lambda\nabla\Psi_{\epsilon,\lambda}) = G_{\epsilon,\lambda} & \text{in } B_1(0)\cap\Omega_f^\lambda, \\ -\epsilon^{2\tau}\nabla\cdot(\mathbf{k}_\lambda\nabla\psi_{\epsilon,\lambda}) = \epsilon^\tau g_{\epsilon,\lambda} & \text{in } B_1(0)\cap\Omega_m^\lambda, \\ \mathbf{K}_\lambda\nabla\Psi_{\epsilon,\lambda}\cdot\vec{\mathbf{n}}^\lambda = \epsilon^{2\tau}\mathbf{k}_\lambda\nabla\psi_{\epsilon,\lambda}\cdot\vec{\mathbf{n}}^\lambda & \text{on } B_1(0)\cap\partial\Omega_m^\lambda, \\ \Psi_{\epsilon,\lambda} = \psi_{\epsilon,\lambda} & \text{on } B_1(0)\cap\partial\Omega_m^\lambda, \end{cases} \tag{6.7}$$

then, for any $\epsilon, \lambda \le \epsilon_0$, $\theta \in [\theta_1, \theta_2]$, and k satisfying $\lambda/\theta^k \le \epsilon_0$,

$$\begin{cases} \displaystyle\fint_{B_{\theta^k}(0)} |\Pi_\lambda\Psi_{\epsilon,\lambda} - (\Pi_\lambda\Psi_{\epsilon,\lambda})_{0,\theta^k}|^2dx \le \theta^{2k\mu}J_{\epsilon,\lambda}^2, \\ \displaystyle\fint_{B_{\theta^k}(0)\cap\Omega_m^\lambda} \epsilon^{2\tau}|\psi_{\epsilon,\lambda} - (\Pi_\lambda\Psi_{\epsilon,\lambda})_{0,\theta^k}|^2dx \le \theta^{2k\mu}J_{\epsilon,\lambda}^2, \end{cases} \tag{6.8}$$

where $J_{\epsilon,\lambda} \equiv \|\Psi_{\epsilon,\lambda}\mathcal{X}_{\Omega_f^\lambda} + \epsilon^\tau\psi_{\epsilon,\lambda}\mathcal{X}_{\Omega_m^\lambda}\|_{L^2(B_1(0))} + \epsilon_0^{-1}\|G_{\epsilon,\lambda}\mathcal{X}_{\Omega_f^\lambda} + \max\{\epsilon^\tau, \lambda\}g_{\epsilon,\lambda}\mathcal{X}_{\Omega_m^\lambda}\|_{L^{n+\delta}(B_1(0))}$.

Proof. For $k = 1$, we define $\hat{\Psi}_\epsilon \equiv \frac{\Psi_{\epsilon,\lambda}}{J_{\epsilon,\lambda}}$, $\hat{\psi}_\epsilon \equiv \frac{\psi_{\epsilon,\lambda}}{J_{\epsilon,\lambda}}$, $\hat{G}_\epsilon \equiv \frac{G_{\epsilon,\lambda}}{J_{\epsilon,\lambda}}$, $\hat{g}_\epsilon \equiv \frac{g_{\epsilon,\lambda}}{J_{\epsilon,\lambda}}$. Then these functions satisfy (6.1) and (6.2) with $\nu = 1$. By Lemma 6.1,

$$\begin{cases} \displaystyle\fint_{B_\theta(0)} |\Pi_\lambda\hat{\Psi}_\epsilon - (\Pi_\lambda\hat{\Psi}_\epsilon)_{0,\theta}|^2dx \le \theta^{2\mu}, \\ \displaystyle\fint_{B_\theta(0)\cap\Omega_m^\lambda} \epsilon^{2\tau}|\hat{\psi}_\epsilon - (\Pi_\lambda\hat{\Psi}_\epsilon)_{0,\theta}|^2dx \le \theta^{2\mu}. \end{cases}$$

This implies (6.8) for $k = 1$. If (6.8) holds for some k satisfying $\lambda/\theta^k \le \epsilon_0$, we define

$$\begin{cases} \hat{\Psi}_\epsilon(x) \equiv J_{\epsilon,\lambda}^{-1}\theta^{-k\mu}\left(\Psi_{\epsilon,\lambda}(\theta^k x) - (\Pi_\lambda\Psi_{\epsilon,\lambda})_{0,\theta^k}\right) \\ \hat{G}_\epsilon(x) \equiv J_{\epsilon,\lambda}^{-1}\theta^{k(2-\mu)}G_{\epsilon,\lambda}(\theta^k x) \end{cases} \quad \text{in } B_1(0)\cap\Omega_f^\lambda/\theta^k,$$

$$\begin{cases} \hat{\psi}_\epsilon(x) \equiv J_{\epsilon,\lambda}^{-1}\theta^{-k\mu}\left(\psi_{\epsilon,\lambda}(\theta^k x) - (\Pi_\lambda\Psi_{\epsilon,\lambda})_{0,\theta^k}\right) \\ \hat{g}_\epsilon(x) \equiv J_{\epsilon,\lambda}^{-1}\theta^{k(2-\mu)}g_{\epsilon,\lambda}(\theta^k x) \end{cases} \quad \text{in } B_1(0)\cap\Omega_m^\lambda/\theta^k.$$

Then these functions satisfy

$$\begin{cases} -\nabla\cdot(\hat{\mathbf{K}}_{\lambda,\theta^k}\nabla\hat{\Psi}_\epsilon) = \hat{G}_\epsilon & \text{in } B_1(0)\cap\Omega_f^\lambda/\theta^k, \\ -\epsilon^{2\tau}\nabla\cdot(\hat{\mathbf{k}}_{\lambda,\theta^k}\nabla\hat{\psi}_\epsilon) = \epsilon^\tau\hat{g}_\epsilon & \text{in } B_1(0)\cap\Omega_m^\lambda/\theta^k, \\ \hat{\mathbf{K}}_{\lambda,\theta^k}\nabla\hat{\Psi}_\epsilon\cdot\vec{\mathbf{n}}^{\lambda/\theta^k} = \epsilon^{2\tau}\hat{\mathbf{k}}_{\lambda,\theta^k}\nabla\hat{\psi}_\epsilon\cdot\vec{\mathbf{n}}^{\lambda/\theta^k} & \text{on } B_1(0)\cap\partial\Omega_m^\lambda/\theta^k, \\ \hat{\Psi}_\epsilon = \hat{\psi}_\epsilon & \text{on } B_1(0)\cap\partial\Omega_m^\lambda/\theta^k, \end{cases}$$

where $\vec{\mathbf{n}}^{\lambda/\theta^k}$ is a unit vector normal to $\partial\Omega_m^\lambda/\theta^k$. See (2.1) for $\hat{\mathbf{K}}_{\lambda,\theta^k}$, $\hat{\mathbf{k}}_{\lambda,\theta^k}$. By induction,

$$\begin{cases} \max\{\|\hat{\Psi}_\epsilon\|_{L^2(B_1(0)\cap\Omega_f^\lambda/\theta^k)}, \epsilon^\tau\|\hat{\psi}_\epsilon\|_{L^2(B_1(0)\cap\Omega_m^\lambda/\theta^k)}\} \le 1, \\ \epsilon_0^{-1}\|\hat{G}_\epsilon\mathcal{X}_{\Omega_f^\lambda/\theta^k} + \max\{\epsilon^\tau, \lambda\theta^{-k}\}\hat{g}_\epsilon\mathcal{X}_{\Omega_m^\lambda/\theta^k}\|_{L^{n+\delta}(B_1(0))} \le 1. \end{cases}$$

By Lemma 6.1 (take $\nu = \theta^k$), we obtain

$$\begin{cases} \fint_{B_\theta(0)} \left|\Pi_{\lambda/\theta^k}\hat{\Psi}_\epsilon - (\Pi_{\lambda/\theta^k}\hat{\Psi}_\epsilon)_{0,\theta}\right|^2 dx \le \theta^{2\mu}, \\ \fint_{B_\theta(0)\cap\Omega_m^\lambda/\theta^k} \epsilon^{2\tau}\left|\hat{\psi}_\epsilon - (\Pi_{\lambda/\theta^k}\hat{\Psi}_\epsilon)_{0,\theta}\right|^2 dx \le \theta^{2\mu}. \end{cases} \tag{6.9}$$

Note, by Remark 2.1,

$$\fint_{B_\theta(0)} \left|\Pi_{\lambda/\theta^k}\hat{\Psi}_\epsilon - (\Pi_{\lambda/\theta^k}\hat{\Psi}_\epsilon)_{0,\theta}\right|^2 dx = \fint_{B_{\theta^{k+1}}(0)} \frac{\left|\Pi_\lambda\Psi_{\epsilon,\lambda} - (\Pi_\lambda\Psi_{\epsilon,\lambda})_{0,\theta^{k+1}}\right|^2}{J_{\epsilon,\lambda}^2\theta^{2k\mu}} dx, \tag{6.10}$$

$$\fint_{B_\theta(0)\cap\Omega_m^\lambda/\theta^k} \left|\hat{\psi}_\epsilon - (\Pi_{\lambda/\theta^k}\hat{\Psi}_\epsilon)_{0,\theta}\right|^2 dx = \fint_{B_{\theta^{k+1}}(0)\cap\Omega_m^\lambda} \frac{\left|\psi_{\epsilon,\lambda} - (\Pi_\lambda\Psi_{\epsilon,\lambda})_{0,\theta^{k+1}}\right|^2}{J_{\epsilon,\lambda}^2\theta^{2k\mu}} dx. \tag{6.11}$$

Eqs. (6.9)–(6.11) imply (6.8) for $k+1$ case. □

Lemma 6.3. *For any $\delta, \tau > 0$, there are $\mu, \epsilon_* \in (0,1)$ (depending on $n, \delta, \tau, \alpha, \beta, Y_f$) such that if $\epsilon, \lambda \le \epsilon_*$, any solution of (6.7) satisfies*

$$[\Psi_{\epsilon,\lambda}]_{C^\mu(B_{1/2}(0)\cap\overline{\Omega}_f^\lambda)} + \sup_{\substack{j\in\mathbb{Z}^n \\ \lambda(Y_m+j)\subset B_{1/2}(0)\cap\Omega_m^\lambda}} \epsilon^\tau[\psi_{\epsilon,\lambda}]_{C^\mu(\lambda(\overline{Y}_m+j))} \le cJ_{\epsilon,\lambda},$$

where c is a constant independent of ϵ, λ. See Lemma 6.2 for $J_{\epsilon,\lambda}$ and $\mu < \frac{\delta}{2(n+\delta)}$ is from Lemma 6.2.

Proof. Let $\theta_1, \theta_2, \epsilon_0, \mu(< \frac{\delta}{2(n+\delta)})$ be same as those in Lemma 6.2, define $\epsilon_* \equiv \epsilon_0\theta_2/2$, and let $\epsilon, \lambda \le \epsilon_*$. Denote by c a constant independent of ϵ, λ. Because of $\theta_1 < \theta_2^2$, for any $r \in [\lambda/\epsilon_0, \theta_2]$, there are $\theta \in [\theta_1, \theta_2]$ and $k \in \mathbb{N}$ satisfying $r = \theta^k$. Lemma 6.2 implies, for any $r \in [\lambda/\epsilon_0, \theta_2]$,

$$\begin{cases} \fint_{B_r(0)} \left|\Pi_\lambda\Psi_{\epsilon,\lambda} - (\Pi_\lambda\Psi_{\epsilon,\lambda})_{0,r}\right|^2 dx \le cr^{2\mu}J_{\epsilon,\lambda}^2, \\ \fint_{B_r(0)\cap\Omega_m^\lambda} \epsilon^{2\tau}\left|\psi_{\epsilon,\lambda} - (\Pi_\lambda\Psi_{\epsilon,\lambda})_{0,r}\right|^2 dx \le cr^{2\mu}J_{\epsilon,\lambda}^2. \end{cases} \tag{6.12}$$

Define

$$\begin{cases} \hat{\Psi}_\epsilon(x) \equiv J_{\epsilon,\lambda}^{-1}\lambda^{-\mu}\left(\Psi_{\epsilon,\lambda}(\lambda x) - (\Pi_\lambda\Psi_{\epsilon,\lambda})_{0,2\lambda/\epsilon_0}\right) \\ \hat{G}_\epsilon(x) \equiv J_{\epsilon,\lambda}^{-1}\lambda^{2-\mu}G_{\epsilon,\lambda}(\lambda x) \end{cases} \quad \text{in } B_{\frac{2}{\epsilon_0}}(0)\cap\Omega_f^\lambda/\lambda,$$

$$\begin{cases} \hat{\psi}_\epsilon(x) \equiv J_{\epsilon,\lambda}^{-1}\lambda^{-\mu}\left(\psi_{\epsilon,\lambda}(\lambda x) - (\Pi_\lambda\Psi_{\epsilon,\lambda})_{0,2\lambda/\epsilon_0}\right) \\ \hat{g}_\epsilon(x) \equiv J_{\epsilon,\lambda}^{-1}\lambda^{2-\mu}g_{\epsilon,\lambda}(\lambda x) \end{cases} \quad \text{in } B_{\frac{2}{\epsilon_0}}(0)\cap\Omega_m^\lambda/\lambda.$$

Then those functions satisfy

$$\begin{cases} -\nabla\cdot(\hat{\mathbf{K}}_{\lambda,\lambda}\nabla\hat{\Psi}_\epsilon) = \hat{G}_\epsilon & \text{in } B_{\frac{2}{\epsilon_0}}(0)\cap\Omega_f^\lambda/\lambda, \\ -\epsilon^{2\tau}\nabla\cdot(\hat{\mathbf{k}}_{\lambda,\lambda}\nabla\hat{\psi}_\epsilon) = \epsilon^\tau\hat{g}_\epsilon & \text{in } B_{\frac{2}{\epsilon_0}}(0)\cap\Omega_m^\lambda/\lambda, \\ \hat{\mathbf{K}}_{\lambda,\lambda}\nabla\hat{\Psi}_\epsilon\cdot\vec{\mathbf{n}}^{\lambda/\lambda} = \epsilon^{2\tau}\hat{\mathbf{k}}_{\lambda,\lambda}\nabla\hat{\psi}_\epsilon\cdot\vec{\mathbf{n}}^{\lambda/\lambda} & \text{on } B_{\frac{2}{\epsilon_0}}(0)\cap\partial\Omega_m^\lambda/\lambda \\ \hat{\Psi}_\epsilon = \hat{\psi}_\epsilon & \text{on } B_{\frac{2}{\epsilon_0}}(0)\cap\partial\Omega_m^\lambda/\lambda, \end{cases}$$

where $\vec{\mathbf{n}}^{\lambda/\lambda}$ is a unit vector normal to $\partial\Omega_m^\lambda/\lambda$. See (2.1) for $\hat{\mathbf{K}}_{\lambda,\lambda}$, $\hat{\mathbf{k}}_{\lambda,\lambda}$. Take $r = \frac{2\lambda}{\epsilon_0}$ in (6.12) to get

$$\|\hat{\Psi}_\epsilon \mathcal{X}_{\Omega_f^\lambda/\lambda} + \epsilon^\tau \hat{\psi}_\epsilon \mathcal{X}_{\Omega_m^\lambda/\lambda}\|_{L^2\left(B_{\frac{2}{\epsilon_0}}(0)\right)} + \|\hat{G}_\epsilon \mathcal{X}_{\Omega_f^\lambda/\lambda} + \hat{g}_\epsilon \mathcal{X}_{\Omega_m^\lambda/\lambda}\|_{L^{n+\delta}\left(B_{\frac{2}{\epsilon_0}}(0)\right)} \le c.$$

By A1–A2 and Lemma 4.2,

$$[\hat{\Psi}_\epsilon]_{C^\mu\left(B_{\frac{1}{\epsilon_0}}(0)\cap\overline{\Omega}_f^\lambda/\lambda\right)} + \epsilon^\tau[\hat{\psi}_\epsilon]_{C^\mu\left(B_{\frac{1}{\epsilon_0}}(0)\cap\overline{\Omega}_m^\lambda/\lambda\right)} \le c. \tag{6.13}$$

Remark 2.1, (6.13), and Theorem 1.2 in page 70 [33] imply

$$⨍_{B_r(0)} \left|\Pi_\lambda \Psi_{\epsilon,\lambda} - (\Pi_\lambda \Psi_{\epsilon,\lambda})_{0,r}\right|^2 dx \le c r^{2\mu} J_{\epsilon,\lambda}^2 \quad \text{for } r \le \lambda/\epsilon_0.$$

Then we shift the origin of the coordinate system to any point $z \in B_{1/2}(0)$ and repeat above argument to see that $(6.12)_1$ with 0 replaced by z also holds for $r \in (0, \theta_2)$. Together with Theorem 1.2 in page 70 [33], we obtain the Hölder estimate of $\Pi_\lambda \Psi_{\epsilon,\lambda}$ in $B_{1/2}(0)$. Hölder estimate of $\psi_{\epsilon,\lambda}$ in $\lambda(\overline{Y}_m + j) \subset B_{1/2}(0) \cap \overline{\Omega}_m^\lambda$ is from (6.13). □

6.2. Boundary estimate

Assume (5.14)–(5.15). So $0 \in \partial\Omega$.

Lemma 6.4. *If $\delta, \tau > 0$, there are $\mu, \tilde{\theta}_1, \tilde{\theta}_2 \in (0, 1)$ (depending on $n, \delta, \alpha, \beta, Y_f, \Omega$) satisfying $\tilde{\theta}_1 < \tilde{\theta}_2^2$ and there is a $\tilde{\epsilon}_0 \in (0, 1)$ (depending on $\tilde{\theta}_1, \tilde{\theta}_2, n, \delta, \tau, \alpha, \beta, Y_f, \Omega$) satisfying $\tilde{\epsilon}_0 < \min\{\frac{2}{3}, \epsilon_0\}$ (ϵ_0 is that in Lemma 6.1) such that if*

$$\begin{cases} -\nabla\cdot(\hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu}) = G_{\epsilon,\lambda,\nu} & \text{in } B_1(0)\cap\Omega_f^\lambda/\nu, \\ -\epsilon^{2\tau}\nabla\cdot(\hat{\mathbf{k}}_{\lambda,\nu}\nabla\psi_{\epsilon,\lambda,\nu}) = \epsilon^\tau g_{\epsilon,\lambda,\nu} & \text{in } B_1(0)\cap\Omega_m^\lambda/\nu, \\ \hat{\mathbf{K}}_{\lambda,\nu}\nabla\Psi_{\epsilon,\lambda,\nu}\cdot\vec{\mathbf{n}}^{\lambda/\nu} = \epsilon^{2\tau}\hat{\mathbf{k}}_{\lambda,\nu}\nabla\psi_{\epsilon,\lambda,\nu}\cdot\vec{\mathbf{n}}^{\lambda/\nu} & \text{on } B_1(0)\cap\partial\Omega_m^\lambda/\nu, \\ \Psi_{\epsilon,\lambda,\nu} = \psi_{\epsilon,\lambda,\nu} & \text{on } B_1(0)\cap\partial\Omega_m^\lambda/\nu, \\ \Psi_{\epsilon,\lambda,\nu} = 0 & \text{on } B_1(0)\cap\partial\Omega/\nu, \end{cases} \tag{6.14}$$

and if

$$\max\{\|\Psi_{\epsilon,\lambda,\nu}\|_{L^2(B_1(0)\cap\Omega_f^\lambda/\nu)}, \epsilon^\tau\|\psi_{\epsilon,\lambda,\nu}\|_{L^2(B_1(0)\cap\Omega_m^\lambda/\nu)}, \tilde{\epsilon}_0^{-1}\|G_{\epsilon,\lambda,\nu}\mathcal{X}_{\Omega_f^\lambda/\nu} + \max\{\epsilon^\tau, \lambda/\nu\}g_{\epsilon,\lambda,\nu}\mathcal{X}_{\Omega_m^\lambda/\nu}\|_{L^{n+\delta}(B_1(0))}\} \le 1,$$

then, for any $\epsilon, \lambda/\nu \le \tilde{\epsilon}_0$, $\nu \in (0, 1]$, and $\tilde{\theta} \in [\tilde{\theta}_1, \tilde{\theta}_2]$,

$$\begin{cases} ⨍_{B_{\tilde{\theta}}(0)\cap\Omega/\nu} |\Pi_{\lambda/\nu}\Psi_{\epsilon,\lambda,\nu}|^2 dx \le \tilde{\theta}^{2\mu}, \\ ⨍_{B_{\tilde{\theta}}(0)\cap\Omega_m^\lambda/\nu} \epsilon^{2\tau}|\psi_{\epsilon,\lambda,\nu}|^2 dx \le \tilde{\theta}^{2\mu}. \end{cases} \tag{6.15}$$

See (2.1) for $\hat{\mathbf{K}}_{\lambda,\nu}$, $\hat{\mathbf{k}}_{\lambda,\nu}$.

Proof. Let $\mathbf{K}^* \in \mathcal{M}(\gamma^{-2}\alpha, \beta; B_1(0)\cap\Omega/\nu_*)$ for $\nu_* \in [0, 1]$ and assume Ψ is a solution of the uniform elliptic equation

$$\begin{cases} -\nabla\cdot(\mathbf{K}^*\nabla\Psi) = 0 & \text{in } B_{1/2}(0)\cap\Omega/\nu_*, \\ \Psi = 0 & \text{on } B_{1/2}(0)\cap\partial\Omega/\nu_*. \end{cases} \tag{6.16}$$

By Theorem 8.25 and Theorem 8.29 [17] and (5.15), we have

$$\|\Psi\|_{C^s(B_{1/4}(0)\cap\Omega/\nu_*)} \le c\|\Psi\|_{L^2(B_{1/2}(0)\cap\Omega/\nu_*)}, \tag{6.17}$$

where $s(< 1)$, c are constants depending on $n, \alpha, \beta, Y_f, \Omega$. Define $\mu \equiv \frac{1}{2}\min\{s, \frac{\delta}{n+\delta}\}$. If $\tilde{\theta}$ is small enough (depending on $\mu, n, \alpha, \beta, Y_f, \Omega$ but independent of ν_*), then, by (6.17),

$$⨍_{B_{\tilde{\theta}}(0)\cap\Omega/\nu_*} |\Psi|^2 dx \le \tilde{\theta}^{2\mu'} ⨍_{B_{1/2}(0)\cap\Omega/\nu_*} |\Psi|^2 dx \tag{6.18}$$

holds for some $\mu' \in (\mu, 2\mu)$. Fix two values $\tilde{\theta}_1, \tilde{\theta}_2 \le 1/4$ such that (1) $\tilde{\theta}_1 < \tilde{\theta}_2^2$ and (2) Inequality (6.18) holds for $\tilde{\theta} \in [\tilde{\theta}_1, \tilde{\theta}_2]$.

With $\mu, \tilde{\theta}_1, \tilde{\theta}_2$ above, we claim (6.15)$_1$. If not, there is a sequence $\{\epsilon_\lambda, \lambda, \nu_\lambda, \tilde{\theta}_{\epsilon_\lambda,\lambda,\nu_\lambda}, \Psi_{\epsilon_\lambda,\lambda,\nu_\lambda}, \psi_{\epsilon_\lambda,\lambda,\nu_\lambda}, G_{\epsilon_\lambda,\lambda,\nu_\lambda}, g_{\epsilon_\lambda,\lambda,\nu_\lambda}\}$ satisfying (6.14) and

$$\begin{cases} \epsilon_\lambda, \lambda/\nu_\lambda \to 0, \\ \nu_\lambda \in (0,1], \quad \tilde{\theta}_{\epsilon_\lambda,\lambda,\nu_\lambda} \in [\tilde{\theta}_1, \tilde{\theta}_2], \\ \max\{\|\Psi_{\epsilon_\lambda,\lambda,\nu_\lambda}\|_{L^2(B_1(0)\cap\Omega_f^\lambda/\nu_\lambda)}, \epsilon_\lambda^\tau \|\psi_{\epsilon_\lambda,\lambda,\nu_\lambda}\|_{L^2(B_1(0)\cap\Omega_m^\lambda/\nu_\lambda)}\} \le 1, \\ \lim\limits_{\epsilon_\lambda,\lambda/\nu_\lambda\to 0} \|G_{\epsilon_\lambda,\lambda,\nu_\lambda}\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\lambda/\nu_\lambda)} + \max\{\epsilon^\tau, \lambda/\nu_\lambda\}\|g_{\epsilon_\lambda,\lambda,\nu_\lambda}\|_{L^{n+\delta}(B_1(0)\cap\Omega_m^\lambda/\nu_\lambda)} = 0, \\ \displaystyle\fint_{B_{\tilde{\theta}_{\epsilon_\lambda,\lambda,\nu_\lambda}}(0)\cap\Omega/\nu_\lambda} \left|\Pi_{\lambda/\nu_\lambda}\Psi_{\epsilon_\lambda,\lambda,\nu_\lambda}\right|^2 dx > \tilde{\theta}_{\epsilon_\lambda,\lambda,\nu_\lambda}^{2\mu}. \end{cases} \tag{6.19}$$

By Lemma 5.5, there is a subsequence (same notation for subsequence) such that, as $\epsilon_\lambda, \lambda/\nu_\lambda \to 0$,

$$\begin{cases} \nu_\lambda \to \nu_* \in [0,1], \\ \tilde{\theta}_{\epsilon_\lambda,\lambda,\nu_\lambda} \to \tilde{\theta}_* \in [\tilde{\theta}_1, \tilde{\theta}_2], \\ \mathcal{S}_{\nu_\lambda}(\Pi_{\lambda/\nu_\lambda}\Psi_{\epsilon_\lambda,\lambda,\nu_\lambda}) \to \Psi & \text{in } L^2(B_{1/2}(0)\cap\Omega/\nu_*) \text{ strongly}, \\ \mathcal{Q}^{\lambda/\nu_\lambda}(\hat{\mathbf{K}}_{\lambda,\nu_\lambda}\nabla\Psi_{\epsilon_\lambda,\lambda,\nu_\lambda}) \to \mathbf{K}^*\nabla\Psi & \text{in } [L^2(B_{1/2}(0)\cap\Omega/\nu_*)]^n \text{ weakly}, \\ \epsilon_\lambda^{2\tau}\hat{\mathbf{k}}_{\lambda,\nu_\lambda}\nabla\psi_{\epsilon_\lambda,\lambda,\nu_\lambda}\mathcal{X}_{\Omega_m^\lambda/\nu_\lambda} \to 0 & \text{in } [L^2(B_{1/2}(0)\cap\Omega/\nu_*)]^n \text{ strongly}, \end{cases} \tag{6.20}$$

and Ψ is a solution of (6.16). By (6.18)–(6.20), we conclude

$$\begin{aligned} \tilde{\theta}_*^{2\mu} &= \lim_{\epsilon_\lambda,\lambda/\nu_\lambda\to 0} \tilde{\theta}_{\epsilon_\lambda,\lambda,\nu_\lambda}^{2\mu} \le \lim_{\epsilon_\lambda,\lambda/\nu_\lambda\to 0} \fint_{B_{\tilde{\theta}_{\epsilon_\lambda,\lambda,\nu_\lambda}}(0)\cap\Omega/\nu_\lambda} \left|\Pi_{\lambda/\nu_\lambda}\Psi_{\epsilon_\lambda,\lambda,\nu_\lambda}\right|^2 dx \\ &= \fint_{B_{\tilde{\theta}_*}(0)\cap\Omega/\nu_*} |\Psi|^2 dx \le \tilde{\theta}_*^{2\mu'} \fint_{B_{1/2}(0)\cap\Omega/\nu_*} |\Psi|^2 dx. \end{aligned} \tag{6.21}$$

If $\tilde{\theta}_2$ is small enough, the right hand side of (6.21) is less than $\tilde{\theta}_*^{2\mu''}$ for $\mu'' \in (\mu, \mu')$. Which means $\tilde{\theta}_*^{2\mu} \le \tilde{\theta}_*^{2\mu''}$ for $\mu'' \in (\mu, \mu')$ and this is impossible. Therefore, we prove (6.15)$_1$. Clearly, $\tilde{\epsilon}_0$ can be chosen so that $\tilde{\epsilon}_0 < \epsilon_0$. The proof of (6.15)$_2$ is similar to that of (6.3)$_2$, so we skip it. □

Lemma 6.5. *Let $\delta, \tau, \mu(< \frac{\delta}{2(n+\delta)}), \tilde{\theta}_1, \tilde{\theta}_2, \tilde{\epsilon}_0$ be same as those in Lemma 6.4. If*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\lambda\nabla\Psi_{\epsilon,\lambda}) = G_{\epsilon,\lambda} & \text{in } B_1(0)\cap\Omega_f^\lambda, \\ -\epsilon^{2\tau}\nabla\cdot(\mathbf{k}_\lambda\nabla\psi_{\epsilon,\lambda}) = \epsilon^\tau g_{\epsilon,\lambda} & \text{in } B_1(0)\cap\Omega_m^\lambda, \\ \mathbf{K}_\lambda\nabla\Psi_{\epsilon,\lambda}\cdot\vec{\mathbf{n}}^\lambda = \epsilon^{2\tau}\mathbf{k}_\lambda\nabla\psi_{\epsilon,\lambda}\cdot\vec{\mathbf{n}}^\lambda & \text{on } B_1(0)\cap\partial\Omega_m^\lambda, \\ \Psi_{\epsilon,\lambda} = \psi_{\epsilon,\lambda} & \text{on } B_1(0)\cap\partial\Omega_m^\lambda, \\ \Psi_{\epsilon,\lambda} = 0 & \text{on } B_1(0)\cap\partial\Omega, \end{cases} \tag{6.22}$$

then, for any $\epsilon, \lambda \le \tilde{\epsilon}_0$, $\tilde{\theta} \in [\tilde{\theta}_1, \tilde{\theta}_2]$, and k satisfying $\lambda/\tilde{\theta}^k \le \tilde{\epsilon}_0$,

$$\begin{cases} \displaystyle\fint_{B_{\tilde{\theta}^k}(0)\cap\Omega} \left|\Pi_\lambda\Psi_{\epsilon,\lambda}\right|^2 dx \le \tilde{\theta}^{2k\mu}\tilde{J}_{\epsilon,\lambda}^2, \\ \displaystyle\fint_{B_{\tilde{\theta}^k}(0)\cap\Omega_m^\lambda} \epsilon^{2\tau}\left|\psi_{\epsilon,\lambda}\right|^2 dx \le \tilde{\theta}^{2k\mu}\tilde{J}_{\epsilon,\lambda}^2, \end{cases} \tag{6.23}$$

where $\tilde{J}_{\epsilon,\lambda} \equiv \|\Psi_{\epsilon,\lambda}\mathcal{X}_{\Omega_f^\lambda} + \epsilon^\tau\psi_{\epsilon,\lambda}\mathcal{X}_{\Omega_m^\lambda}\|_{L^2(B_1(0))} + \frac{1}{\tilde{\epsilon}_0}\|G_{\epsilon,\lambda}\mathcal{X}_{\Omega_f^\lambda} + \max\{\epsilon^\tau, \lambda\}g_{\epsilon,\lambda}\mathcal{X}_{\Omega_m^\lambda}\|_{L^{n+\delta}(B_1(0))}$.

Proof. The proof is similar to that of Lemma 6.2 and is done by induction on k. For $k = 1$, (6.23) is deduced from Lemma 6.4 with $\nu = 1$. Suppose (6.23) holds for some k satisfying $\lambda/\tilde{\theta}^k \le \tilde{\epsilon}_0$, then we define

$$\begin{cases} \hat{\Psi}_\epsilon(x) \equiv \tilde{J}_{\epsilon,\lambda}^{-1}\tilde{\theta}^{-k\mu}\Psi_{\epsilon,\lambda}(\tilde{\theta}^k x) \\ \hat{G}_\epsilon(x) \equiv \tilde{J}_{\epsilon,\lambda}^{-1}\tilde{\theta}^{k(2-\mu)}G_{\epsilon,\lambda}(\tilde{\theta}^k x) \end{cases} \quad \text{in } B_1(0)\cap\Omega_f^\lambda/\tilde{\theta}^k,$$

$$\begin{cases} \hat{\psi}_\epsilon(x) \equiv \tilde{J}_{\epsilon,\lambda}^{-1}\tilde{\theta}^{-k\mu}\psi_{\epsilon,\lambda}(\tilde{\theta}^k x) \\ \hat{g}_\epsilon(x) \equiv \tilde{J}_{\epsilon,\lambda}^{-1}\tilde{\theta}^{k(2-\mu)}g_{\epsilon,\lambda}(\tilde{\theta}^k x) \end{cases} \quad \text{in } B_1(0)\cap\Omega_m^\lambda/\tilde{\theta}^k,$$

$$\hat{\Psi}_{b_\epsilon}(x) \equiv 0 \quad \text{in } B_1(0)\cap\Omega/\tilde{\theta}^k.$$

Then these functions satisfy

$$\begin{cases} -\nabla\cdot(\hat{\mathbf{K}}_{\lambda,\tilde\theta^k}\nabla\hat\Psi_\epsilon)=\hat G_\epsilon & \text{in } B_1(0)\cap\Omega_f^\lambda/\tilde\theta^k,\\ -\epsilon^{2\tau}\nabla\cdot(\hat{\mathbf{k}}_{\lambda,\tilde\theta^k}\nabla\hat\psi_\epsilon)=\epsilon^\tau\hat g_\epsilon & \text{in } B_1(0)\cap\Omega_m^\lambda/\tilde\theta^k,\\ \hat{\mathbf{K}}_{\lambda,\tilde\theta^k}\nabla\hat\Psi_\epsilon\cdot\vec{\mathbf{n}}^{\epsilon/\tilde\theta^k}=\epsilon^{2\tau}\hat{\mathbf{k}}_{\lambda,\tilde\theta^k}\nabla\hat\psi_\epsilon\cdot\vec{\mathbf{n}}^{\epsilon/\tilde\theta^k} & \text{on } B_1(0)\cap\partial\Omega_m^\lambda/\tilde\theta^k,\\ \hat\Psi_\epsilon=\hat\psi_\epsilon & \text{on } B_1(0)\cap\partial\Omega_m^\lambda/\tilde\theta^k,\\ \hat\Psi_\epsilon(x)=0 & \text{in } B_1(0)\cap\Omega/\tilde\theta^k, \end{cases}$$

where $\vec{\mathbf{n}}^{\epsilon/\tilde\theta^k}$ is a unit vector normal to $\partial\Omega_m^\lambda/\tilde\theta^k$. See (2.1) for $\hat{\mathbf{K}}_{\lambda,\tilde\theta^k}$, $\hat{\mathbf{k}}_{\lambda,\tilde\theta^k}$. By induction,

$$\begin{cases} \max\{\|\hat\Psi_\epsilon\|_{L^2(B_1(0)\cap\Omega_f^\lambda/\tilde\theta^k)},\epsilon^\tau\|\hat\psi_\epsilon\|_{L^2(B_1(0)\cap\Omega_m^\lambda/\tilde\theta^k)}\}\le 1,\\ \epsilon_0^{-1}\|\hat G_\epsilon\mathcal{X}_{\Omega_f^\lambda/\tilde\theta^k}+\max\{\epsilon^\tau,\lambda\tilde\theta^{-k}\}\hat g_\epsilon\mathcal{X}_{\Omega_m^\lambda/\tilde\theta^k}\|_{L^{n+\delta}(B_1(0))}\le 1. \end{cases}$$

By Lemma 6.4 (take $\nu=\tilde\theta^k$), we obtain

$$\begin{cases} \fint_{B_{\tilde\theta}(0)\cap\Omega/\tilde\theta^k}\left|\Pi_{\lambda/\tilde\theta^k}\hat\Psi_\epsilon\right|^2dx\le\tilde\theta^{2\mu},\\ \fint_{B_{\tilde\theta}(0)\cap\Omega_m^\lambda/\tilde\theta^k}\epsilon^{2\tau}\left|\hat\psi_\epsilon\right|^2dx\le\tilde\theta^{2\mu}. \end{cases}\tag{6.24}$$

Note, by Remark 2.1,

$$\fint_{B_{\tilde\theta}(0)\cap\Omega/\tilde\theta^k}\left|\Pi_{\lambda/\tilde\theta^k}\hat\Psi_\epsilon\right|^2dx=\fint_{B_{\tilde\theta^{k+1}}(0)\cap\Omega}\frac{\left|\Pi_\lambda\Psi_{\epsilon,\lambda}\right|^2}{\tilde J_{\epsilon,\lambda}^2\,\tilde\theta^{2k\mu}}dx,\tag{6.25}$$

$$\fint_{B_{\tilde\theta}(0)\cap\Omega_m^\lambda/\tilde\theta^k}\left|\hat\psi_\epsilon\right|^2dx=\fint_{B_{\tilde\theta^{k+1}}(0)\cap\Omega_m^\lambda}\frac{\left|\psi_{\epsilon,\lambda}\right|^2}{\tilde J_{\epsilon,\lambda}^2\,\tilde\theta^{2k\mu}}dx.\tag{6.26}$$

Eqs. (6.24)–(6.26) imply (6.23) for $k+1$ case. □

Lemma 6.6. *For any $\delta,\tau>0$, there are $\mu,\tilde\epsilon_*\in(0,1)$ (depending on $n,\delta,\tau,\alpha,\beta,Y_f,\Omega$) such that if $\epsilon,\lambda\le\tilde\epsilon_*$, any solution of (6.22) satisfies*

$$[\Psi_{\epsilon,\lambda}]_{C^\mu(B_{1/2}(0)\cap\overline{\Omega}_f^\lambda)}+\sup_{\substack{j\in\mathbb{Z}^n\\ \lambda(Y_m+j)\subset B_{1/2}(0)\cap\Omega_m^\lambda}}\epsilon^\tau[\psi_{\epsilon,\lambda}]_{C^\mu(\lambda(\overline{Y}_m+j))}\le c\tilde J_{\epsilon,\lambda},\tag{6.27}$$

where c is a constant independent of ϵ,λ. See Lemma 6.5 for $\tilde J_{\epsilon,\lambda}$ and $\mu<\frac{\delta}{2(n+\delta)}$ is from Lemma 6.5.

Proof. Let $\tilde\theta_1,\tilde\theta_2,\tilde\epsilon_0,\mu(<\frac{\delta}{2(n+\delta)})$ be same as those in Lemma 6.5, define $\tilde\epsilon_*\equiv\min\{\tilde\epsilon_0\tilde\theta_2/3,\epsilon_*\}$ where ϵ_* is the one in Lemma 6.3, and let $\epsilon,\lambda\le\tilde\epsilon_*$. Denote by c a constant independent of ϵ,λ. For any $x\in B_{\tilde\theta_2/3}(0)\cap\Omega_f^\lambda$, define $\eta(x)\equiv|x-x_0|$ where $x_0\in\partial\Omega$ satisfying $|x-x_0|=\min_{y\in\partial\Omega}|x-y|$. Then we have either case (1) $\eta(x)>\frac{2\lambda}{3\tilde\epsilon_0}$ or case (2) $\eta(x)\le\frac{2\lambda}{3\tilde\epsilon_0}$.

Let us consider case (1). Because of $\tilde\theta_1<\tilde\theta_2^2$, for any $r\in[\lambda/\tilde\epsilon_0,\tilde\theta_2]$, there are $\tilde\theta\in[\tilde\theta_1,\tilde\theta_2]$ and $k\in\mathbb{N}$ satisfying $r=\tilde\theta^k$. Since $\eta(x)\in[\frac{2\lambda}{3\tilde\epsilon_0},\frac{\tilde\theta_2}{3}]$, by Lemma 6.5,

$$\begin{cases} \fint_{B_r(x_0)\cap\Omega}\left|\Pi_\lambda\Psi_{\epsilon,\lambda}\right|^2dy\le r^{2\mu}\tilde J_{\epsilon,\lambda}^2\\ \fint_{B_r(x_0)\cap\Omega_m^\lambda}\epsilon^{2\tau}\left|\psi_{\epsilon,\lambda}\right|^2dy\le r^{2\mu}\tilde J_{\epsilon,\lambda}^2 \end{cases}\quad\text{for } r\in\left[\frac{3}{2}\eta(x),\tilde\theta_2\right].$$

So

$$\begin{cases} \fint_{B_s(x)\cap\Omega}\left|\Pi_\lambda\Psi_{\epsilon,\lambda}-(\Pi_\lambda\Psi_{\epsilon,\lambda})_{x,s}\right|^2dy\le cs^{2\mu}\tilde J_{\epsilon,\lambda}^2\\ \fint_{B_s(x)\cap\Omega_m^\lambda}\epsilon^{2\tau}\left|\psi_{\epsilon,\lambda}-(\Pi_\lambda\Psi_{\epsilon,\lambda})_{x,s}\right|^2dy\le cs^{2\mu}\tilde J_{\epsilon,\lambda}^2 \end{cases}\quad\text{for } s\in\left[\frac{\eta(x)}{2},\frac{\tilde\theta_2}{3}\right].\tag{6.28}$$

Shift the coordinate system so that the origin is at x and define

$$\begin{cases}\hat{\Psi}_\epsilon(y) \equiv \tilde{J}_{\epsilon,\lambda}^{-1}\eta^{-\mu}(x)\big(\Psi_{\epsilon,\lambda}(\eta(x)y) - (\Pi_\lambda \Psi_{\epsilon,\lambda})_{x,\eta(x)}\big) \\ \hat{G}_\epsilon(y) \equiv \tilde{J}_{\epsilon,\lambda}^{-1}\eta^{2-\mu}(x)G_{\epsilon,\lambda}(\eta(x)y)\end{cases} \quad \text{in } B_1(x)\cap \Omega_f^\lambda/\eta(x),$$

$$\begin{cases}\hat{\psi}_\epsilon(y) \equiv \tilde{J}_{\epsilon,\lambda}^{-1}\eta^{-\mu}(x)\big(\psi_{\epsilon,\lambda}(\eta(x)y) - (\Pi_\lambda \Psi_{\epsilon,\lambda})_{x,\eta(x)}\big) \\ \hat{g}_\epsilon(y) \equiv \tilde{J}_{\epsilon,\lambda}^{-1}\eta^{2-\mu}(x)g_{\epsilon,\lambda}(\eta(x)y)\end{cases} \quad \text{in } B_1(x)\cap \Omega_m^\lambda/\eta(x).$$

Then these functions satisfy

$$\begin{cases}-\nabla\cdot(\hat{\mathbf{K}}_{\lambda,\eta(x)}\nabla\hat{\Psi}_\epsilon) = \hat{G}_\epsilon & \text{in } B_1(x)\cap\Omega_f^\lambda/\eta(x),\\ -\epsilon^{2\tau}\nabla\cdot(\hat{\mathbf{k}}_{\lambda,\eta(x)}\nabla\hat{\psi}_\epsilon) = \epsilon^\tau \hat{g}_\epsilon & \text{in } B_1(x)\cap\Omega_m^\lambda/\eta(x),\\ \hat{\mathbf{K}}_{\lambda,\eta(x)}\nabla\hat{\Psi}_\epsilon\cdot\vec{\mathbf{n}}^{\lambda/\eta(x)} = \epsilon^{2\tau}\hat{\mathbf{k}}_{\lambda,\eta(x)}\nabla\hat{\psi}_\epsilon\cdot\vec{\mathbf{n}}^{\lambda/\eta(x)} & \text{on } B_1(x)\cap\partial\Omega_m^\lambda/\eta(x),\\ \hat{\Psi}_\epsilon = \hat{\psi}_\epsilon & \text{on } B_1(x)\cap\partial\Omega_m^\lambda/\eta(x),\end{cases} \tag{6.29}$$

where $\vec{\mathbf{n}}^{\lambda/\eta(x)}$ is a unit vector normal to $\partial\Omega_m^\lambda/\eta(x)$. See (2.1) for $\hat{\mathbf{K}}_{\lambda,\eta(x)}, \hat{\mathbf{k}}_{\lambda,\eta(x)}$. Take $s = \eta(x)$ in (6.28) to see

$$\|\hat{\Psi}_\epsilon \mathcal{X}_{\Omega_f^\lambda/\eta(x)} + \epsilon^\tau \hat{\psi}_\epsilon \mathcal{X}_{\Omega_m^\lambda/\eta(x)}\|_{L^2(B_1(x))} + \tilde{\epsilon}_0^{-1}\|\hat{G}_\epsilon\|_{L^{n+\delta}(B_1(x)\cap\Omega_f^\lambda/\eta(x))}$$
$$+\tilde{\epsilon}_0^{-1}\max\{\epsilon^\tau, \lambda\eta^{-1}(x)\}\|\hat{g}_\epsilon\|_{L^{n+\delta}(B_1(x)\cap\Omega_m^\lambda/\eta(x))} \le c.$$

Apply Lemma 6.3 to (6.29) to obtain

$$[\hat{\Psi}_\epsilon]_{C^\mu(B_{1/2}(x)\cap\overline{\Omega}_f^\lambda/\eta(x))} + \sup_{\substack{j\in\mathbb{Z}^n \\ \frac{\lambda}{\eta(x)}(Y_m+j)\subset B_{1/2}(0)\cap\Omega_m^\lambda/\eta(x)}} \epsilon^\tau[\hat{\psi}_\epsilon]_{C^\mu\left(\frac{\lambda}{\eta(x)}(\overline{Y}_m+j)\right)} \le c \tag{6.30}$$

which implies, by Remark 2.1 and Theorem 1.2 in page 70 [33],

$$\fint_{B_s(x)\cap\Omega}\left|\Pi_\lambda\Psi_{\epsilon,\lambda} - (\Pi_\lambda\Psi_{\epsilon,\lambda})_{x,s}\right|^2 dy \le cs^{2\mu}\tilde{J}_{\epsilon,\lambda}^2 \quad \text{for } s < \eta(x)/2. \tag{6.31}$$

Now we consider case (2). Because of $\tilde{\theta}_1 < \tilde{\theta}_2^2$, for any $r \in [\lambda/\tilde{\epsilon}_0, \tilde{\theta}_2]$, there are $\tilde{\theta} \in [\tilde{\theta}_1, \tilde{\theta}_2]$ and $k \in \mathbb{N}$ satisfying $r = \tilde{\theta}^k$. By Lemma 6.5,

$$\begin{cases}\fint_{B_r(x_0)\cap\Omega}\left|\Pi_\lambda\Psi_{\epsilon,\lambda}\right|^2 dy \le cr^{2\mu}\tilde{J}_{\epsilon,\lambda}^2\\ \fint_{B_r(x_0)\cap\Omega_m^\lambda}\epsilon^{2\tau}\left|\psi_{\epsilon,\lambda}\right|^2 dy \le cr^{2\mu}\tilde{J}_{\epsilon,\lambda}^2\end{cases} \quad \text{for } r\in[\lambda/\tilde{\epsilon}_0, \tilde{\theta}_2]. \tag{6.32}$$

This implies, for $s \in [\frac{\lambda}{3\tilde{\epsilon}_0}, \frac{\tilde{\theta}_2}{3}]$,

$$\begin{cases}\fint_{B_s(x)\cap\Omega}\left|\Pi_\lambda\Psi_{\epsilon,\lambda} - (\Pi_\lambda\Psi_{\epsilon,\lambda})_{x,s}\right|^2 dy \le cs^{2\mu}\tilde{J}_{\epsilon,\lambda}^2,\\ \fint_{B_s(x)\cap\Omega_m^\lambda}\epsilon^{2\tau}\left|\psi_{\epsilon,\lambda} - (\Pi_\lambda\Psi_{\epsilon,\lambda})_{x,s}\right|^2 dy \le cs^{2\mu}\tilde{J}_{\epsilon,\lambda}^2.\end{cases} \tag{6.33}$$

Again we shift the origin to x and define

$$\begin{cases}\hat{\Psi}_\epsilon(y) \equiv \tilde{J}_{\epsilon,\lambda}^{-1}\lambda^{-\mu}\left(\Psi_{\epsilon,\lambda}(\lambda y) - (\Pi_\lambda\Psi_{\epsilon,\lambda})_{x,\lambda/\tilde{\epsilon}_0}\right)\\ \hat{G}_\epsilon(y) \equiv \tilde{J}_{\epsilon,\lambda}^{-1}\lambda^{2-\mu}G_{\epsilon,\lambda}(\lambda y)\end{cases} \quad \text{in } B_{\frac{1}{\tilde{\epsilon}_0}}(x)\cap\Omega_f^\lambda/\lambda,$$

$$\begin{cases}\hat{\psi}_\epsilon(y) \equiv \tilde{J}_{\epsilon,\lambda}^{-1}\lambda^{-\mu}\left(\psi_{\epsilon,\lambda}(\lambda y) - (\Pi_\lambda\Psi_{\epsilon,\lambda})_{x,\lambda/\tilde{\epsilon}_0}\right)\\ \hat{g}_\epsilon(y) \equiv \tilde{J}_{\epsilon,\lambda}^{-1}\lambda^{2-\mu}g_{\epsilon,\lambda}(\lambda y)\end{cases} \quad \text{in } B_{\frac{1}{\tilde{\epsilon}_0}}(x)\cap\Omega_m^\lambda/\lambda,$$

$$\hat{\Psi}_{b_\epsilon} \equiv -\tilde{J}_{\epsilon,\lambda}^{-1}\lambda^{-\mu}(\Pi_\lambda\Psi_{\epsilon,\lambda})_{x,\lambda/\tilde{\epsilon}_0} \quad \text{in } B_{\frac{1}{\tilde{\epsilon}_0}}(x)\cap\Omega/\lambda.$$

From (6.32) $_1$, $\hat{\Psi}_{b_\epsilon}$ is a constant independent of ϵ, λ. Then these functions satisfy

$$\begin{cases} -\nabla\cdot(\hat{\mathbf{K}}_{\lambda,\lambda}\nabla\hat{\Psi}_\epsilon) = \hat{G}_\epsilon & \text{in } B_{\frac{1}{\tilde{\epsilon}_0}}(x)\cap\Omega_f^\lambda/\lambda, \\ -\epsilon^{2\tau}\nabla\cdot(\hat{\mathbf{k}}_{\lambda,\lambda}\nabla\hat{\psi}_\epsilon) = \epsilon^\tau\hat{g}_\epsilon & \text{in } B_{\frac{1}{\tilde{\epsilon}_0}}(x)\cap\Omega_m^\lambda/\lambda, \\ \hat{\mathbf{K}}_{\lambda,\lambda}\nabla\hat{\Psi}_\epsilon\cdot\vec{\mathbf{n}}^{\lambda/\lambda} = \epsilon^{2\tau}\hat{\mathbf{k}}_{\lambda,\lambda}\nabla\hat{\psi}_\epsilon\cdot\vec{\mathbf{n}}^{\lambda/\lambda} & \text{on } B_{\frac{1}{\tilde{\epsilon}_0}}(x)\cap\partial\Omega_m^\lambda/\lambda, \\ \hat{\Psi}_\epsilon = \hat{\psi}_\epsilon & \text{on } B_{\frac{1}{\tilde{\epsilon}_0}}(x)\cap\partial\Omega_m^\lambda/\lambda, \\ \hat{\Psi}_\epsilon = \hat{\Psi}_{b_\epsilon} & \text{on } B_{\frac{1}{\tilde{\epsilon}_0}}(x)\cap\partial\Omega/\lambda, \end{cases}$$

where $\vec{\mathbf{n}}^{\lambda/\lambda}$ is a unit vector normal to $\partial\Omega_m^\lambda/\lambda$. See (2.1) for $\hat{\mathbf{K}}_{\lambda,\lambda}$, $\hat{\mathbf{k}}_{\lambda,\lambda}$. Take $s = \frac{\lambda}{\tilde{\epsilon}_0}$ in (6.33) to see

$$\|\hat{\Psi}_\epsilon\mathcal{X}_{\Omega_f^\lambda/\lambda} + \epsilon^\tau\hat{\psi}_\epsilon\mathcal{X}_{\Omega_m^\lambda/\lambda}\|_{L^2\left(B_{\frac{1}{\tilde{\epsilon}_0}}(x)\right)} + \|\hat{G}_\epsilon\mathcal{X}_{\Omega_f^\lambda/\lambda} + \hat{g}_\epsilon\mathcal{X}_{\Omega_m^\lambda/\lambda}\|_{L^{n+\delta}\left(B_{\frac{1}{\tilde{\epsilon}_0}}(x)\right)} + \|\hat{\Psi}_{b_\epsilon}\|_{W^{2,n+\delta}\left(B_{\frac{1}{\tilde{\epsilon}_0}}(x)\cap\Omega/\lambda\right)} \le c.$$

A1–A2 and Lemma 4.3 imply

$$[\hat{\Psi}_\epsilon]_{C^\mu\left(B_{\frac{1}{2\tilde{\epsilon}_0}}(x)\cap\overline{\Omega}_f^\lambda/\lambda\right)} + \epsilon^\tau[\hat{\psi}_\epsilon]_{C^\mu\left(B_{\frac{1}{2\tilde{\epsilon}_0}}(x)\cap\overline{\Omega}_m^\lambda/\lambda\right)} \le c. \tag{6.34}$$

Remark 2.1 and (6.34) imply $(6.33)_1$ holds for $s \le \frac{\lambda}{2\tilde{\epsilon}_0}$.

The Hölder estimate of $\Pi_\lambda\Psi_{\epsilon,\lambda}$ follows from $(6.28)_1$, (6.31), $(6.33)_1$, (6.34), and Theorem 1.2 in page 70 [33]. The Hölder estimate of $\psi_{\epsilon,\lambda}$ in $\lambda(\overline{Y}_m + j) \subset B_{1/2}(0)\cap\overline{\Omega}_m^\lambda$ is from (6.30) and (6.34). □

Clearly, if $\epsilon < 1$ and $\tau \in (0, 1]$, then $\epsilon \le \epsilon^\tau$. If we take $\lambda = \epsilon$ in (6.7) and (6.22), then we have, by the energy method, partition of unity, and Lemmas 3.2, 6.3 and 6.6,

For any $\tau \in (0, 1]$, $\delta > 0$, there are $\mu, \tilde{\epsilon}_* \in (0, 1)$ such that, for any $\epsilon \le \tilde{\epsilon}_*$, the solution of (3.4) *satisfies* (3.6).

By the energy method, partition of unity, and Lemmas 3.2, 4.2 and 4.3, we also see

For any $\tau \in (0, 1]$, $\delta > 0$, there are $\mu, \tilde{\epsilon}_* \in (0, 1)$ such that, for any $\epsilon \in [\tilde{\epsilon}_*, 1]$, the solution of (3.4) *satisfies* (3.6).

Combining above two results, we know that (3.6) of Lemma 3.3 holds.

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