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GUIDING CENTER DRIFT INDUCED BY HOMOGENIZATION

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The guiding center drift induced by the homogenization of the Lorentz forces is studied. It generates memory effects. The memory (or nonlocal) kernel is described by the Volterra integral equation. The memory kernel can be characterized explicitly in terms of a Radon measure. It describes the extra velocity drift. By way of velocity drift, we view the Gauss's law with polarization charges.

Keywords: Guiding center; homogenization; memory effect; two-scale convergence; Radon measure; Volterra integral equation.

AMS Subject Classification: 35B27, 35B35

1. Introduction

In many cases of practical interest, the motion in a magnetic field of an electrically charged particle (such as an electron or ion in a plasma) can be treated as the superposition of a relatively fast circular motion around a point called the guiding center and a relatively slow drift of this point. The drift speeds may differ from various species depending on their charge states, masses, or temperatures, possibly resulting in electric currents or chemical separation. The rare plasma theory of a single

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charged particle tells us about when the particle moving in a perturbation magnetic field can be separated into a fast, oscillatory component — the gyromotion — and a slow component obtained by averaging out the gyromotion (see Refs. 8, 18 and 24 for physical background and references therein). Therefore it is interesting to study the guiding center motion from the point of view of homogenization.

Let Ω be a bounded open set in $\mathbb{R}^3$ and δ denote the small parameter, $0 < \delta \ll 1$. Let $U^\delta(x,t) = (u_1^\delta(x,t), u_2^\delta(x,t), u_3^\delta(x,t))^t \in \mathbb{R}^3$ be the velocity of the particles and m the mass; then the equation of motion for a particle with charged particle q in electromagnetic fields $E(x,t) \in L^\infty(0,T;L^2(\Omega))$ and $B^\delta(x,t)$ is modeled by

$$m\frac{d}{dt}U^\delta(x(t),t) = q\left(E(x,t) + U^\delta(x,t) \times \left(B^\delta(x,t) + \frac{M}{\delta}\right)\right), \tag{1.1}$$

where $(x,t) \in \Omega \times (0,T)$ and $M(x) = (M_1(x), M_2(x), M_3(x))^t \in L^\infty(\Omega)$ is a magnetic field. The initial data is complemented by

$$U^\delta(x,0) = U_0(x) = (u_1(x,0), u_2(x,0), u_3(x,0))^t \in L^2(\Omega). \tag{1.2}$$

For simplicity we will assume the magnetic field B^δ propagates in the fixed direction $\mathbf{b}$ with magnitude w^δ, i.e.

$$B^\delta(x,t) = w^\delta(x,t)\mathbf{b} = w^\delta(x,t)(b_1,b_2,b_3)^t, \quad |\mathbf{b}| = 1, \tag{1.3}$$

where the sequence of scalar measurable functions $\{w^\delta\}_\delta$ satisfies the bounds

$$a_- \le w^\delta(x,t) \le a_+, \quad \text{a.e. in } \Omega \times (0,T). \tag{1.4}$$

The function w^δ is used to describe the microscopic nonhomogeneous media and varying local characteristics. It follows from (1.4) that the sequence of measurable function $\{w^\delta\}_\delta$ is uniformly bounded in $L^\infty(\Omega \times (0,T))$, so that according to the Banach–Alaoglu–Bourbaki theorem, we may extract a subsequence still denoted by $\{w^\delta\}_\delta$ with

$$w^\delta \stackrel{w}{\rightharpoonup} w \quad \text{weakly}^* \text{ in } L^\infty(\Omega \times (0,T)).$$

Thus the homogenization theory studies the behavior of the associated solution sequence $\{U^\delta\}_\delta$ as $\delta \to 0$ and asks whether average behavior can be discerned from (1.1). To obtain a more accurate description of the limiting behavior of (1.1), it is more efficient to apply the two-scale convergence method introduced by Nguetseng[22,23] and Allaire.[3] The basic idea is to consider the behavior of the homogenization process not only from the macroscopic point of view, but also from the microscopic one, introducing an additional microscopic variable. To this end, we will look for a formal asymptotic expansion of U^δ. The aim is serving as a function of δ for $\delta \to 0$ and the heuristic device is to consider that U^δ and w^δ in Eq. (1.1) having two-scale expansions given respectively by (see Refs. 9 and 25 and references therein)

$$U^\delta(x,t) = U_0(x,t,\tau) + \delta U_1(x,t,\tau) + \delta^2 U_2(x,t,\tau) + \cdots \tag{1.5}$$

and

$$w^{\delta}(x,t) = w_0(x,t,\tau) + \delta w_1(x,t,\tau) + \delta^2 w_2(x,t,\tau) + \cdots, \tag{1.6}$$

where U_i and $w_i, i = 0, 1, 2, \ldots$, are $\mathcal{T}$-periodic functions of the fast variable $\tau = \frac{t}{\delta}$. Furthermore, we also assume the zero mean conditions

$$\langle U_i \rangle = \langle w_i \rangle = 0, \quad i = 1, 2, 3, \ldots,$$

where $\langle f \rangle$ denotes the average value of f over one period $\mathcal{T}$:

$$\langle f \rangle \equiv \frac{1}{\mathcal{T}} \int_{\mathcal{T}} f(\tau) d\tau.$$

We then substitute the expansion in (1.5)–(1.6) into (1.1) and employ the corresponding chain rule

$$\frac{d}{dt} \to \frac{d}{dt} + \frac{1}{\delta}\frac{\partial}{\partial \tau} \tag{1.7}$$

to obtain

$$\begin{aligned} m\left(\frac{d}{dt} + \frac{1}{\delta}\frac{\partial}{\partial \tau}\right)&(U_0(x,t,\tau) + \delta U_1(x,t,\tau) + \delta^2 U_2(x,t,\tau) + \cdots) \\ &= q\Bigg[E + (U_0(x,t,\tau) + \delta U_1(x,t,\tau) + \delta^2 U_2(x,t,\tau) + \cdots) \\ &\quad \cdot (w_0(x,t,\tau) + \delta w_1(x,t,\tau) + \delta^2 w_2(x,t,\tau) + \cdots)\mathbf{b} + \frac{M}{\delta}\Bigg], \end{aligned} \tag{1.8}$$

then equate power of $O(\delta^{-1})$ and $O(1)$ respectively in (1.8) to derive the constraint microscopic equation

$$m\frac{\partial}{\partial \tau} U_0(x,t,\tau) = U_0(x,t,\tau) \times M, \tag{1.9}$$

and

$$\begin{aligned} m\frac{d}{dt}U_0(x,t,\tau) &+ m\frac{\partial}{\partial \tau}U_1(x,t,\tau) \\ &= q(E(x,t) + U_0(x,t,\tau) \times w_0(x,t,\tau)\mathbf{b} + U_1(x,t,\tau) \times M). \end{aligned} \tag{1.10}$$

The macroscopic equation is derived by averaging Eq. (1.10) over the fast variable τ

$$m\frac{d}{dt}\langle U_0(x,t,\tau)\rangle = q(E(x,t) + \langle U_0(x,t,\tau) \times w_0(x,t,\tau)\rangle \mathbf{b}), \tag{1.11}$$

where the third term on the right-hand side of (1.10) vanishes because of the zero mean condition.

Let us remark that the constraint microscopic equation (1.9) shows the evolution of velocity in microscopic variable τ under the fast gyromotion along the external magnetic field M, through the average on a period $\mathcal{T}$, and affects the average behavior of the velocity. Equation (1.11) describes the effects after averaging the gyromotion. The second term on the right-hand side of (1.10) shows the contribution of the fast gyromotion.

To give a rigorous mathematical analysis of the above asymptotic expansion, it is interesting to study the motion of the guiding center (gyromotion center) connecting with Eq. (1.1) by homogenization. Indeed, as we will show in this paper, the memory effect induced by homogenization occurs because of the nonuniform magnetic field. The memory term describing by an integral operator shows the extra drift velocity which is perpendicular to the magnetic field relative to the guiding center.

When the external field $M = 0$, the homogenization of the Lorentz equation was studied by Amirat, Hamdache and Ziani in Ref. 4 for time-independent case (see Ref. 16 for the time-dependent case). The guiding center motion is formulated by the kinetic equation, in particular the Vlasov equation for collisionless case. This makes the research of the homogenization of Vlasov and Vlasov–Poisson equations interesting and challenge. The homogenization problem about Vlasov–Poisson system with strong magnetic field is studied by Frénod and Sonnendrücker in Ref. 12. They introduce a new rapid time scale induced by the strong magnetic field and perform the method of time-periodic homogenization, to deduce the local problem satisfied by the profile associated to the solution; then, by taking the time mean value of that problem to deduce the effective equation. The homogenization of the Vlasov (or Vlasov–Poisson) equation they obtained gives a good mathematical proof of the guiding center approximation in plasma physics and it shows that in a cloud of particles the mutual influence of particles can be expressed in term of their apparent motion without any additional term. When a strong electric field, orthogonal to the strong magnetic field, is added in the Vlasov equation, they showed that a drift velocity will be induced by homogenization. However, the memory effect does not appear in this situation. The various asymptotic limits of solutions to the Vlasov–Poisson equation in the presence of a strong external magnetic field is discussed in Ref. 13 by Golse and Saint-Raymont (see Ref. 1, 11 and 21).

The organization of the paper is as follows. In Sec. 2, we prove the main result concerning the homogenization of the Lorentz equation with oscillating magnetic field. We show that the limiting equation (homogenized equation) is an integro-differential equation. The memory kernel is described by the Volterra equation. Section 3 is devoted to the characterization of the memory kernel. We consider the special structures of $\{w^\delta\}_\delta$ and represent the weak limits and the memory kernels explicitly in terms of a Radon measure. In Sec. 4, we consider the guiding center motion which can be separated into the component parallel to the magnetic field and the component perpendicular to magnetic field. The motion of the component perpendicular to magnetic field represents a drift motion concerned with respect to the guiding center (gyromotion center). From the polarization drift, we deduce Gauss's law with the polarization charges.

2. Homogenization and Memory Effect

As mentioned in Sec. 1, the two-scale convergence was introduced by Nguetseng[22,23] and Allaire[3] as an efficient tool to study the homogenization problem. It is an alternative approach of the energy method introduced by Tartar to treat the homogenization problem. Allaire also developed the theory further by studying some general properties of two-scale convergence. Moreover he used two-scale convergence to analyze several homogenization problems, both linear and nonlinear. The two-scale convergence is intermediary between strong and weak convergences. It means that in practical applications there are homogenization problems where the solutions do not have classical limit and the weak limit is not a satisfactory approximation of the solution, the asymptotic behavior of the solution can be characterized by so-called *two-scale limit* (see Refs. 10, 13, 20, 22, 23 and 29 for the detail and applications).

We denote by $C^\infty_\#(Y)$ the space of infinitely differentiable functions defined on $Y = [0,1]^3$ and extended to $\mathbb{R}^3$ by Y-periodicity. For $p > 1$ and an open subset $\Omega \subset \mathbb{R}^3$, $L^p(\Omega; C^\infty_\#(Y))$ is the space of functions of $L^p(\Omega)$ with value in $C^\infty_\#(Y)$. A bounded sequence $\{u^\delta\}_\delta$ of $L^p(\Omega)$ is said (weakly) two-scale converge to $u(x,y) \in L^p(\Omega \times Y)$ if and only if

$$\lim_{\delta\to 0}\int_\Omega u^\delta(x)\psi\left(x,\frac{x}{\delta}\right)dx = \int_\Omega\int_Y u(x,y)\psi(x,y)dydx, \tag{2.1}$$

for any function $\psi(x,y) \in \mathcal{D}(\Omega; C^\infty_\#(Y))$ that is Y-periodic with respect to the second argument. This definition is justified by the following compactness theorem which was first proved by Nguetseng[22] and then further developed by Allaire in Ref. 3.

Theorem 2.1. *Let $\psi(x,x/\delta)$ be measurable in Ω and $\psi(x,y) \in L^p(\Omega; C^\infty_\#(Y))$, $1 < p < \infty$. Then for $\delta > 0$ we have*

$$\left\|\psi\left(x,\frac{x}{\delta}\right)\right\|_{L^p(\Omega)} \le \|\psi(x,y)\|_{L^p(\Omega;C^\infty_\#(Y))} \equiv \left[\int_\Omega \sup_{y\in Y}|\psi(x,y)|^p dx\right]^{\frac{1}{p}}. \tag{2.2}$$

Moreover, if $\psi(x,y) \in L^p(\Omega; C^\infty_\#(Y))$ then

$$\lim_{\delta\to 0}\int_\Omega \psi^p\left(x,\frac{x}{\delta}\right)dx = \int_\Omega\int_Y \psi^p(x,y)dydx \tag{2.3}$$

and $\psi(x,\frac{x}{\delta})$ two-scale converges to $\psi(x,y)$.

Theorem 2.2. *Let $\{u^\delta\}_\delta$ be a bounded sequence in $L^p(\Omega)$, $1 < p \le \infty$. Then, there exist a subsequence still denoted by $\{u^\delta\}_\delta$ and a function $u(x,y) \in L^p(\Omega \times Y)$ such that u^δ two-scale converges to $u(x,y)$.*

The proof is similar to the L^2 case as given by Allaire in Ref. 3 with modification (see also Refs. 19 and 29). Therefore, the proof is omitted. We now focus our attention to derive the prior estimates that are available for the Lorentz equation. First of all, we notice that its solution $U^\delta(x,t)$ satisfies the following estimate.

Lemma 2.1. *Let $E \in L^\infty(0,T;L^2(\Omega))$. Under assumptions* (1.1)–(1.4), *there exists a constant C independent of δ such that the solution U^δ of the Lorentz equation* (1.1) *satisfies*

$$\|U^\delta\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{2.4}$$

and therefore $\{U^\delta\}_\delta$ is bounded in $L^2(\Omega \times (0,T))$. Thus, by Theorem 2.2, *we have the two-scale limit* (*after passing to subsequence*)

$$U^\delta(x,t) \to \bar{U}(x,t,\tau) \tag{2.5}$$

in $L^2(\Omega \times (0,T) \times \mathcal{T})$, where $\mathcal{T} = (0,1)$.

Proof. We will apply the standard energy method to prove that $\{U^\delta\}_\delta$ is a bounded sequence in $L^\infty(0,T;L^2(\Omega))$. We multiply the Lorentz equation (1.1) by U^δ and apply the Cauchy–Schwarz inequality to obtain

$$\frac{d}{dt}\|U^\delta(\cdot,t)\|_{L^2}^2 \leq \frac{2q}{m}\|E(\cdot,t)\|_{L^2}\|U^\delta(\cdot,t)\|_{L^2}. \tag{2.6}$$

Then integrating over the time variable t and using initial condition, we obtain the Gronwall-type inequality

$$\|U^\delta(\cdot,t)\|_{L^2} \leq \|U_0\|_{L^2} + \frac{2q}{m}\int_0^t \|E(\cdot,s)\|_{L^2} ds. \tag{2.7}$$

Thus $\{U^\delta\}_\delta$ is bounded in $L^\infty(0,T;L^2(\Omega))$, hence bounded in $L^2(\Omega \times (0,T))$ for bounded measure Ω. Therefore the two-scale limit (2.5) follows immediately from Theorem 2.2. □

The following lemma follows immediately from the fact that $U^\delta(x,t)$ converges to $\bar{U}(x,t,\tau)$ in the sense of two-scale limit. In other words, the two-scale method justifies mathematically the formal asymptotic expansion as mentioned in Sec. 1.

Lemma 2.2. *Assume the hypothesis of Lemma* 2.1. *Let $\bar{U}_0(x,\tau) \in L^2(\Omega \times \mathcal{T})$. Then, there exist subsequences still denoted by $\{U^\delta(x,t)\}_\delta$, $\{B^\delta(x,t)\}_\delta$ and functions $\bar{U} \in L^2(\Omega \times (0,T) \times \mathcal{T})$ and $\bar{B} \in L^\infty(\Omega \times (0,T) \times \mathcal{T})$ such that $U^\delta(x,t)$ and $B^\delta(x,t)$ two-scale converge to $\bar{U}(x,t,\tau)$ and $\bar{B}(x,t,\tau)$, respectively. Furthermore, $\bar{U}(x,t,\tau)$ and $\bar{B}(x,t,\tau)$ satisfy the two-scale limit system*

$$m\frac{\partial}{\partial\tau}\bar{U}(x,t,\tau) = q\bar{U}(x,t,\tau) \times M, \tag{2.8}$$

$$m\frac{d}{dt}\bar{U}(x,t,\tau) = q(E(x,t) + \bar{U}(x,t,\tau) \times \bar{B}(x,t,\tau)).$$

$$\bar{U}(x,0,\tau) = \bar{U}_0(x,\tau). \tag{2.9}$$

Proof. The two-scale convergence of $B^\delta = \omega^\delta(x,t)\mathbf{b}$ to $\bar{B} = \bar{\omega}(x,t,\tau)\mathbf{b}$ follows from the two-scale of convergence $\omega^\delta(x,t) \to \bar{\omega}(x,t,\tau)$. The weak formulation of the Lorentz equation (1.1) is obtained by multiplying the admissible test function $\phi(x,t,\frac{t}{\delta}) = \phi(x,t,\tau) \in \mathcal{D}(\Omega \times (0,T); C^\infty_\sharp(\mathcal{T}))$ and then integrating over the space-time domain $\mathcal{O} = \Omega \times (0,T)$:

$$\begin{aligned}&\int_{\mathcal{O}} m\frac{d}{dt}U^\delta(x,t)\phi\left(x,t,\frac{t}{\delta}\right)dxdt\\ &\quad = \int_{\mathcal{O}} qE(x,t)\phi\left(x,t,\frac{t}{\delta}\right)dxdt + \int_{\mathcal{O}} qU^\delta(x,t)\\ &\qquad \cdot\left(B^\delta(x,t) + \frac{M}{\delta}\right)\phi\left(x,t,\frac{t}{\delta}\right)dxdt. \end{aligned} \tag{2.10}$$

Then integrating by parts, Eq. (2.10) becomes

$$\begin{aligned}&-\int_{\mathcal{O}} mU^\delta(x,t)\left[\frac{d}{dt}\phi\left(x,t,\frac{t}{\delta}\right) + \frac{1}{\delta}\frac{\partial}{\partial\tau}\phi\left(x,t,\frac{t}{\delta}\right)\right]dxdt\\ &\quad = \int_{\mathcal{O}} qE(x,t)\phi\left(x,t,\frac{t}{\delta}\right)dxdt\\ &\qquad + \int_{\mathcal{O}} qU^\delta(x,t)\times\left(B^\delta(x,t) + \frac{M}{\delta}\right)\phi\left(x,t,\frac{t}{\delta}\right)dxdt. \end{aligned} \tag{2.11}$$

To discuss the two-scale limit, we rewrite Eq. (2.11) as

$$\begin{aligned}&-\int_{\mathcal{O}} \delta mU^\delta(x,t)\frac{d}{dt}\phi\left(x,t,\frac{t}{\delta}\right)dxdt - \int_{\mathcal{O}} mU^\delta(x,t)\frac{\partial}{\partial\tau}\phi\left(x,t,\frac{t}{\delta}\right)dxdt\\ &\quad = \int_{\mathcal{O}} \delta qE(x,t)\phi\left(x,t,\frac{t}{\delta}\right)dxdt + \int_{\mathcal{O}} \delta qU^\delta(x,t)\times B^\delta(x,t)\phi\left(x,t,\frac{t}{\delta}\right)dxdt\\ &\qquad + \int_{\mathcal{O}} qU^\delta(x,t)\times M\phi\left(x,t,\frac{t}{\delta}\right)dxdt. \end{aligned} \tag{2.12}$$

Since U^δ is bounded in $L^\infty(0,T;L^2(\Omega))$ and B^δ is bounded in $L^\infty(\Omega\times(0,T))$, we see that the first, third and fourth terms of (2.12) converge to 0. Therefore employing Lemma 2.1 and passing to the two-scale limit of Eq. (2.12) yields

$$-\int_{\mathcal{S}} m\bar{U}(x,t,\tau)\frac{\partial}{\partial\tau}\phi(x,t,\tau)dxdtd\tau = \int_{\mathcal{S}} q\bar{U}(x,t,\tau)\times M\phi(x,t,\tau)dxdtd\tau, \tag{2.13}$$

then integrating by parts we obtain

$$\int_{\mathcal{S}} m\frac{\partial}{\partial\tau}\bar{U}(x,t,\tau)\phi(x,t,\tau)dxdtd\tau = \int_{\mathcal{S}} q\bar{U}(x,t,\tau)\times M\phi(x,t,\tau)dxdtd\tau, \quad (2.14)$$

where $\mathcal{S} = \mathcal{O}\times\mathcal{T}$. This shows that $\bar{U}$ is a weak solution of the microscopic constraint in Eq. (2.8).

Since $\bar{U}$ solves Eq. (2.8), the two-scale limit of (2.11) is given by

$$\begin{aligned} -m&\int_{\mathcal{S}} \bar{U}(x,t,\tau)\frac{d}{dt}\phi(x,t,\tau)dxdtd\tau \\ &= \int_{\mathcal{S}} qE(x,t)\phi(x,t,\tau)dxdtd\tau + \int_{\mathcal{S}} q\bar{U}(x,t,\tau)\times\bar{B}(x,t,\tau)\phi(x,t,\tau)dxdtd\tau, \end{aligned} \quad (2.15)$$

or (after integrating by parts)

$$\begin{aligned} m&\int_{\mathcal{S}} \frac{d}{dt}\bar{U}(x,t,\tau)\phi(x,t,\tau)dxdtd\tau \\ &= \int_{\mathcal{S}} qE(x,t)\phi(x,t,\tau)dxdtd\tau + \int_{\mathcal{S}} q\bar{U}(x,t,\tau)\times\bar{B}(x,t,\tau)\phi(x,t,\tau)dxdtd\tau. \end{aligned} \quad (2.16)$$

Thus $\bar{U}$ solves (2.9) and this ends the proof of Lemma 2.2. □

Remark. Since $E\in L^\infty(0,T;L^2(\Omega))$, the local existence and uniqueness for $\bar{U}\in L^2(\Omega\times(0,T)\times\mathcal{T})$ of the two-scale limit system (2.8)–(2.9) follows from the standard energy estimate, Gronwall inequality and the fixed-point argument. The proof is standard and therefore is omitted.

Equation (2.9) means that the two-scale limit of the Lorentz equation has the same form as the original one, which satisfies the constraint equation (2.8). We note that Eq. (2.8) has a solution that can be represented by

$$\bar{U}(x,t,\tau) = e^{-\frac{q}{m}N\tau}\bar{V}(x,t), \quad (2.17)$$

where $\bar{V}(x,t) = \bar{U}(x,t,0)$ and

$$N = \begin{pmatrix} 0 & M_3(x) & -M_2(x) \\ -M_3(x) & 0 & M_1(x) \\ M_2(x) & -M_1(x) & 0 \end{pmatrix}.$$

In order to realize Eq. (2.17), it needs to analyze $e^{-\frac{q}{m}N\tau}$. By straightforward computation we have the recursive relation

$$N^3 = -\|M\|^2N, \quad N^4 = -\|M\|^2N^2, \quad N^5 = \|M\|^4N, \quad N^6 = \|M\|^4N^2, \ldots$$

and so on where $\|M\|$ is the norm of the magnetic field $M = (M_1, M_2, M_3)$. Therefore we have obtained the Rodrigues' rotation formula ($n = 1, 2, \ldots$)

$$\begin{aligned} e^N &= I + \left[I - \frac{\|M\|^2}{3!} + \frac{\|M\|^4}{5!} - \cdots + (-1)^{n-1}\frac{\|M\|^{2n-2}}{(2n-1)!} + \cdots\right] N \\ &\quad + \left[\frac{1}{2!} - \frac{\|M\|^2}{4!} + \frac{\|M\|^4}{6!} - \cdots + (-1)^{n-1}\frac{\|M\|^{2n-2}}{(2n)!} + \cdots\right] N^2 \\ &= I + \frac{\sin\|M\|}{\|M\|}N + \frac{1-\cos\|M\|}{\|M\|^2}N^2. \end{aligned}$$

This formula provides an algorithm to compute the exponential map without computing the full matrix exponent. Then simple change of variable yields

$$e^{-\frac{q}{m}N\tau} = I - \frac{\sin(\|M\|\frac{q}{m}\tau)}{\|M\|}N + \frac{1-\cos(\|M\|\frac{q}{m}\tau)}{\|M\|^2}N^2,$$

and the expression in terms of the trigonometric functions also shows that $e^{-\frac{q}{m}N\tau}$ is a periodic function of τ which is essential for Lemma 2.2. In our setting the functions defined are always of period one in τ-variable. Thus without loss of generality, we may assume M is a unit vector $\|M\| = 1$ and $\frac{q}{m} = 2\pi$ by proper rescaling. Thus we have the right periodicity for $e^{-\frac{q}{m}N\tau}$. For the remainder here as well as for the rest of this paper, we always assume $e^{-\frac{q}{m}N\tau}$ is one-periodic in τ. However, without further confusion we prefer to keep the term $\frac{q}{m}$ and write $e^{-\frac{q}{m}N\tau}$ as

$$e^{-\frac{q}{m}N\tau} = I - \sin(q\tau/m)N + (1-\cos(q\tau/m))N^2. \tag{2.18}$$

As N is skew-symmetric, the operator $e^{-\frac{q}{m}N\tau}$ is unitary for all τ. By way of (2.17), we can rewrite Eq. (2.9) as

$$m\frac{d}{dt}\bar{V}(x,t) = qe^{\frac{q}{m}N\tau}E(x,t) + qe^{\frac{q}{m}N\tau}[e^{-\frac{q}{m}N\tau}\bar{V}(x,t) \times \bar{B}(x,t,\tau)]. \tag{2.19}$$

To obtain the homogenized equation, we rewrite $(e^{-\frac{q}{m}N\tau}\bar{V}) \times \bar{B}$ as

$$e^{-\frac{q}{m}N\tau}\bar{V}(x,t) \times \bar{B}(x,t,\tau) = -\bar{w}(x,t,\tau)J(x,t)e^{-\frac{q}{m}N\tau}\bar{V}(x,t), \tag{2.20}$$

where the skew-symmetric matrix function $J(x,t)$ is given by

$$J(x,t) = \begin{pmatrix} 0 & b_3(x,t) & -b_2(x,t) \\ -b_3(x,t) & 0 & b_1(x,t) \\ b_2(x,t) & -b_1(x,t) & 0 \end{pmatrix}.$$

Therefore, the two-scale limit equation (2.19) becomes

$$\frac{\partial}{\partial t}\bar{V}(x,t) - \bar{A}(x,t,\tau)\bar{V}(x,t) = \frac{q}{m}e^{\frac{q}{m}N\tau}E(x,t) \tag{2.21}$$

for $(x,t,\tau) \in \Omega \times (0,T) \times \mathcal{T}$ and the matrix function $\bar{A}$ is given by

$$\bar{A}(x,t,\tau) = -\frac{q}{m}\bar{w}(x,t,\tau)e^{\frac{q}{m}N\tau}J(x,t)e^{-\frac{q}{m}N\tau}. \tag{2.22}$$

Without loss of generality, we assume the trivial initial condition, then by Duhamel principle the solution $\bar{V}(x,t)$ of (2.21) is represented as

$$\bar{V}(x,t) = (\bar{L})^{-1}g \equiv \int_0^t \bar{G}(x,s,t,\tau)e^{\frac{q}{m}N\tau}\frac{q}{m}E(x,s)ds, \tag{2.23}$$

where

$$\bar{G}(x,s,t,\tau) \equiv \Phi(x(t),t,\tau)\Phi(x(s),s,\tau)^{-1} \tag{2.24}$$

is the Green's function associated with the initial value problem of the first-order linear differential operator $\bar{L}$ defined by (2.21), i.e.

$$\bar{L}\bar{V}(x,t) := \frac{\partial}{\partial t}\bar{V}(x,t) - \bar{A}(x,t,\tau)\bar{V}(x,t).$$

Here Φ is the unique solution of the matrix differential equation

$$\frac{d}{dt}\Phi(x,t,\tau) - \bar{A}(x,t,\tau)\Phi(x,t,\tau) = 0, \quad \Phi(x,0,\tau) = I, \tag{2.25}$$

where I is the 3×3 identity matrix. The matrix function $\Phi(x,t,\tau)$ so defined is called the *matrizant* of the system (2.25).

We can rewrite Eq. (2.21) as the differential integral equation:

$$\frac{d}{dt}\bar{V}(x,t) = \frac{q}{m}\int_0^t \bar{A}(x,t,\tau)\bar{G}(x,s,t,\tau)e^{\frac{q}{m}N\tau}E(x,s)ds + \frac{q}{m}e^{\frac{q}{m}N\tau}E(x,t). \tag{2.26}$$

For simplicity we define the two-scale correction matrix function $\bar{C}$ by

$$\bar{C}(x,s,t,\tau) = \bar{A}(x,t,\tau)\bar{G}(x,s,t,\tau)e^{\frac{q}{m}N\tau} - \widetilde{A}(x,t,\tau)G(x,s,t), \tag{2.27}$$

where

$$G(x,s,t) = \int_{\mathcal{T}} \bar{G}(x,s,t,\tau)e^{\frac{q}{m}N\tau}d\tau \tag{2.28}$$

and

$$\widetilde{A}(x,t,\tau) = -\frac{q}{m}w(x,t)e^{\frac{q}{m}N\tau}J(x,t)\gamma, \quad \gamma = \int_{\mathcal{T}} e^{-\frac{q}{m}N\tau}d\tau. \tag{2.29}$$

Here γ denotes the average correction function. Substituting (2.27)–(2.29) into (2.26), we obtain

$$\frac{d}{dt}\bar{V}(x,t) = \widetilde{A}(x,t)\bar{V}(x,t) + \frac{q}{m}e^{\frac{q}{m}N\tau}E(x,t) + \frac{q}{m}\int_0^t \bar{C}(x,s,t,\tau)E(x,s)ds. \tag{2.30}$$

The averaging equation can be obtained by multiplying $e^{-\frac{q}{m}N\tau}$ and integrating the variable τ on Eq. (2.30), we have the homogenized equation for $U(x,t)$

$$m\frac{d}{dt}U(x,t) = -qw(x,t)J(x,t)U(x,t) + qE(x,t) + \int_0^t C(x,s,t)qE(x,s)ds, \tag{2.31}$$

where

$$C(x,s,t) = \int_{\mathcal{T}} e^{-\frac{q}{m}N\tau}\bar{C}(x,s,t,\tau)d\tau.$$

Next, we introduce the kernel $D(x,s,t)$, the solution of the resolvent (or Volterra–Green) equation, given by

$$D(x,s,t) = C(x,s,t) - \int_s^t C(x,s,\sigma)D(x,\sigma,t)d\sigma. \tag{2.32}$$

Then integrating by parts and using the condition $D(x,s,s) = 0$, we deduce from (2.32) that

$$m\frac{d}{dt}U(x,t) = -qw(x,t)J(x,t)U(x,t) + qE(x,t) - \int_0^t K(x,s,t)U(x,s)ds \tag{2.33}$$

or

$$m\frac{d}{dt}U(x,t) = q(E(x,t) + U(x,t)\times B) - \int_0^t K(x,s,t)U(x,s)ds, \tag{2.34}$$

where the kernel K is given by

$$K(x,s,t) = m\frac{d}{ds}D(x,s,t) - qw(x,s)D(x,s,t)J(x,t), \tag{2.35}$$

with $(x,s,t) \in \Omega\times(0,T)\times(0,T)$. We thus have proved the following theorem.

Theorem 2.3. *Under the hypotheses* (1.1)–(1.3), *there exist a subsequence of* $\{w^\delta\}_\delta$ *and a kernel* K *defined on* $\Omega\times(0,T)\times(0,T)$, *measurable in* x *and* t, *such that* U^δ *converges weakly in* $L^2(\Omega\times(0,T))$ *to* U *solution of* (2.34) *with resolvent* D *defined in* $\Omega\times(0,T)\times(0,T)$ *solving* (2.32) *and the kernel* K *is given by* (2.35).

This theorem also answers the typical question of the homogenization theory. If the solutions U^δ of the problems $\mathcal{L}^\delta U^\delta = g$ converge weakly to U, here

$$\mathcal{L}^\delta U^\delta = m\frac{d}{dt}U^\delta(x,t) + qw^\delta(x,t)J(x,t)U^\delta(x,t), \tag{2.36}$$

can an operator $\mathcal{L}$ be found such that U is a solution of the problem $\mathcal{L}U = g$, and is $\mathcal{L}$ of the same type as $\mathcal{L}^\delta$? The answer is negative. Indeed, it is given by

$$\mathcal{L}U \equiv m\frac{d}{dt}U(x,t) + qw(x,t)J(x,t)U(x,t) + \int_0^t K(x,s,t)U(x,s)ds \tag{2.37}$$

which is an integro-differential operator, i.e. the homogenization process generates memory or nonlocal effects described by integro-differential equations (see Refs. 2, 4–7, 14, 15, 17, 26–28).

Remark 1. The above argument is still held for the nontrivial initial data $U_0(x) \neq 0$. In this situation the homogenization equation (2.34) becomes

$$m\frac{d}{dt}U(x,t) = q(E(x,t) + U(x,t) \times B) - \int_0^t K(x,s,t)U(x,s)ds + \Psi(x,t)U_0(x), \tag{2.38}$$

where

$$\Psi(x,t) = \int_0^t D(x,s,t)\chi(x,s)ds + \chi(x,t), \tag{2.39}$$

and

$$\chi(x,t) = \int_T me^{-\frac{q}{m}N\tau}\bar{A}(x,t,\tau)\bar{A}_1(x,\tau)\gamma^{-1}d\tau, \quad \bar{A}_1(x,\tau) = \int_0^t \bar{A}(x,s,\tau)ds.$$

Remark 2. Taking the average of Eq. (2.17) in τ (over $\mathcal{T}$), we have

$$U(x,t) = \left\langle \frac{q}{m}e^{-\frac{q}{m}N\tau} \right\rangle \bar{V}(x,t) \equiv \left(\int_{\mathcal{T}} e^{-\frac{q}{m}N\tau}d\tau\right) \bar{V}(x,t).$$

Then applying Rodrigues' rotation formula (2.18) yields

$$\int_{\mathcal{T}} e^{-\frac{q}{m}N\tau}d\tau = I - C_1N + C_2N^2$$

for some constants C_1 and C_2. Thus we derive the representation for U:

$$U(x,t) = (I - C_1N + C_2N^2)\bar{V}(x,t). \tag{2.40}$$

It is therefore enough to find the equation satisfied by $\bar{V}(x,t)$. We note that $\bar{V}(x,t)$ does not oscillate, and we can average directly on (2.21) to get

$$\frac{\partial}{\partial t}\bar{V}(x,t) - \langle\bar{A}\rangle(x,t)\bar{V}(x,t) = \left\langle \frac{q}{m}e^{\frac{q}{m}N\tau} \right\rangle E(x,t). \tag{2.41}$$

However, (2.41) is not the equation looking for since the coefficient $\langle\bar{A}\rangle(x,t)$ is not the weak limit. To proceed, we have to rewrite (2.41) as

$$\begin{aligned}\frac{\partial}{\partial t}\bar{V}(x,t) - A(x,t)\bar{V}(x,t) - (\langle\bar{A}\rangle(x,t) - A(x,t))\bar{V}(x,t)\\ = \frac{q}{m}\langle e^{\frac{q}{m}N\tau}\rangle E(x,t),\end{aligned} \tag{2.42}$$

where $A(x,t)$ is the average of $\widetilde{A}(x,t,\tau)$ on $\mathcal{T}$. Analogous to the defect measure, the third term of (2.42) is to characterize failure of strong convergence. Indeed, following the same procedure as (2.21)–(2.36) we can represent the correction term by an integral. Thus the memory effect does occur in this situation. The memory term can be explained the interactions or resonance effects between electron and the fields, and part of energy is absorbed and given back later. For this reason, the effective equation, the homogenized equation, must be like an equation with added integral term used for memory effects.

3. Characterization of the Memory Kernel

In this section we will use the Radon measure to characterize the memory kernel K. We assume that $\{w^\delta\}_\delta$ is a sequence of measurable functions that satisfies the bounds

$$a_- \le w^\epsilon\left(\frac{t}{\delta}\right) \le a_+, \quad \text{a.e. in } \Omega \tag{3.1}$$

and

$$B^\delta = w^\delta\left(\frac{t}{\delta}\right)\mathbf{b}, \quad \mathbf{b} = (b_1, b_2, b_3)^t. \tag{3.2}$$

Here we also assume $\mathbf{b} = (M_1, M_2, M_3)^t$ is a constant vector, and therefore

$$J = N = \begin{pmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{pmatrix}.$$

One should notice that in this section the spatial domain Ω need not be an open set of $\mathbb{R}^3$ and may be any measure space endowed with measure having no atoms. It follows from (3.1) that there exists a two-scale limit $\bar{w}(\tau)$ of w^δ such that, after extracting a subsequence,

$$w^\delta \xrightarrow{2} \bar{w}(\tau) \quad \text{in } L^\infty(\mathcal{T}) \tag{3.3}$$

and the weak limit w of $\{w^\delta\}_\delta$ given by

$$w = \int_{\mathcal{T}} \bar{w}(\tau)d\tau. \tag{3.4}$$

Note that from (2.22) we also have

$$\bar{A} = -\frac{q}{m}\bar{w}(\tau)J. \tag{3.5}$$

For convenience, we will assume $\mathbf{b} = (b_1, b_2, b_3)^t$ is a unit vector $|\mathbf{b}| = 1$ then direct calculation shows

$$J^2 = \mathbf{b} \otimes \mathbf{b} - I, \quad J^3 + J = 0. \tag{3.6}$$

The minimal polynomial of J has three simple roots $0, i, -i$ and by Lagrange interpolation formula for e^{tJ} it will take the form $1 + \sin tx + (1 - \cos t)x^2$. Therefore the matrizant of the system (2.25) can be represented as

$$\begin{aligned} \Phi(t,\tau) &= \exp\Big(-t\frac{q}{m}\bar{w}(\tau)J\Big) \\ &= I - \sin t\frac{q}{m}\bar{w}(\tau)J + \Big(1 - \cos t\frac{q}{m}\bar{w}(\tau)\Big)J^2, \end{aligned} \tag{3.7}$$

hence

$$\begin{aligned}\bar{G}(s,t,\tau) &= \Phi(t,\tau)\Phi(s,\tau)^{-1}\\ &= \Phi(t-s,\tau) = \exp\Big(-(t-s)\frac{q}{m}\bar{w}(\tau)J\Big)\end{aligned} \tag{3.8}$$

and

$$\bar{C}(s,t,\tau) = \bar{A}(\tau)\bar{G}(s,t,\tau)e^{\frac{q}{m}J\tau} - \widetilde{A}(\tau)G(s,t). \tag{3.9}$$

The fluctuation part is therefore given by

$$\begin{aligned}C &= \int_{\mathcal{T}} e^{-\frac{q}{m}N\tau}\bar{C}(s,t,\tau)d\tau\\ &= \int_{\mathcal{T}} -\frac{q}{m}(\bar{w}(\tau)e^{-\frac{q}{m}J\tau} - w\gamma)Je^{-(t-s)\frac{q}{m}\bar{w}(\tau)J}e^{\frac{q}{m}J\tau}d\tau.\end{aligned} \tag{3.10}$$

To derive the explicit form of the memory kernel K we have to obtain the resolvent kernel D of (2.32) first. As mentioned in Sec. 2, the key step in deriving the kernel K is of the same type as the function C with respect to a Radon measure due to the resolvent equation. This representation is very important because it tells how the memory effect, produced in the macroscopic equation, depends on the way the sequence $\{w^\delta\}$ oscillates. We now prove the representation lemma directly related to the resolvent equation (2.32).

Lemma 3.1. *There exists a Radon measure μ defined on $\mathcal{T}$ such that the solution $D(s,t)$ of the resolvent equation* (2.32) *is given explicitly by*

$$D(s,t) = -\int_{\mathcal{T}} \frac{q}{m}(\bar{w}(\tau)e^{-\frac{q}{m}J\tau} - w\gamma)Je^{-(t-s)\frac{q}{m}\bar{w}(\tau)J}e^{\frac{q}{m}J\tau}d\mu(\tau). \tag{3.11}$$

Proof. For fixed $s,t \in [0,T]$, we denote by $\mathcal{H}$ as the set

$$\mathcal{H} \equiv \{\langle\phi_{\bar{s},\bar{t}}u,v\rangle : \Lambda \to \mathbb{R} | \bar{s},\bar{t} \in [0,T]; u,v \in \mathbb{R}^3\} \equiv \{\phi_{\bar{s},\bar{t}}\},$$

where $\phi \in C([0,T]\times[0,T]\times\mathcal{T};\mathbf{M}_{3\times 3})$. Let V be the vector space generated by $\mathcal{H}$; then it is obvious that V is the subspace of the space $C(\mathcal{T})$. We define a linear operator $T: V \to \mathbb{R}$ by

$$\begin{aligned}\langle T, \langle\phi_{\bar{s},\bar{t}}u,v\rangle\rangle &= \int_{\mathrm{cal}\ T}\langle\phi_{\bar{s},\bar{t}}(\tau)u,v\rangle d\tau\\ &\quad + \int_0^T \chi_{[\bar{s},\bar{t}]}(\bar{\sigma})\left\langle\int_{\mathcal{T}}\phi_{\bar{\sigma},\bar{t}}(\tau)d\tau D(\bar{s},\bar{\sigma})u,v\right\rangle d\bar{\sigma}.\end{aligned} \tag{3.12}$$

From the definition (3.12), it is easy to see that

$$|\langle T,\phi_{\bar{s},\bar{t}}u,v\rangle| \le C\|\langle\phi_{\bar{s},\bar{t}}u,v\rangle\|_{C(\mathcal{T})},$$

where C is a constant. This shows that $\{T\}$ are bounded functionals on V. It follows from Hahn–Banach theorem, there exists a bounded functional $\{\mathcal{L}\}$ on $C(\mathcal{T})$ such that $\mathcal{L}|_V = T$; therefore applying the Riesz representation theorem we deduce that there exists a Radon measure $\{\mu\}$ on $\mathcal{T}$ such that

$$\langle \mathcal{L}, \psi \rangle = \int_{\mathcal{T}} \psi(\tau) d\mu(\tau) \quad \forall\, \psi \in C(\mathcal{T}). \tag{3.13}$$

Choosing $\psi(\tau) = \langle \phi_{s,t}(\tau)u, v\rangle$, then from (3.12)–(3.13) we obtain

$$\begin{aligned}\int_{\mathcal{T}} \langle \phi_{s,t}(\tau)u, v\rangle d\mu(\tau) &= \langle \mathcal{L}, \langle \phi_{s,t}(\tau)u, v\rangle\rangle \\ &= \int_{\mathcal{T}} \langle \phi_{s,t}(\tau)u, v\rangle d\tau + \int_0^T \left\langle \int_{\mathcal{T}} \phi_{\sigma,t}(\tau) dy d\tau D(s,\sigma)u, v \right\rangle d\sigma\end{aligned}$$

for any $u, v \in \mathbb{R}^3$, thus we have

$$\begin{aligned}\int_{\mathcal{T}} \phi_{s,t}(\tau) d\mu(\tau) &= \langle \mathcal{L}, \phi_{s,t}(\tau)\rangle \\ &= \int_{\mathcal{T}} \phi_{s,t}(\tau) dy d\tau + \int_0^T \left[\int_{\mathcal{T}} \phi_{\sigma,t}(\tau) d\tau\right] D(s,\sigma) d\sigma.\end{aligned} \tag{3.14}$$

In particular, let

$$\phi_{s,t}(\tau) = -\frac{q}{m}(\bar{w}(\tau)e^{-\frac{q}{m}J\tau} - w\gamma)Je^{-(t-s)\frac{q}{m}\bar{w}(\tau)J}e^{\frac{q}{m}J\tau}$$

and use Eqs. (2.32) and (3.14), we derive the relation

$$\begin{aligned}D(s,t) &= \int_{\mathcal{T}} \phi_{s,t}(\tau) d\mu(\tau) \\ &= \int_{\mathcal{T}} \phi_{s,t}(\tau) dy d\tau + \int_0^T \left[\int_{\mathcal{T}} \phi_{\sigma,t}(\tau) d\tau\right] D(s,\sigma) d\sigma\end{aligned}$$

or equivalently

$$D(s,t) = -\int_{\mathcal{T}} \frac{q}{m}(\bar{w}(\tau)e^{-\frac{q}{m}J\tau} - w\gamma)Je^{-(t-s)\frac{q}{m}\bar{w}(\tau)J}e^{\frac{q}{m}J\tau} d\mu(\tau). \tag{3.15}$$

This completes the proof of Lemma 3.1. □

By way of the Lemma 3.1, we can derive the memory kernel $K(s,t)$ directly

$$\begin{aligned}K(s,t) = K(t-s) &= m\frac{dD}{ds} - q\omega DJ \\ &= -\int_{\mathcal{T}} \frac{q^2}{m}(\bar{w}(\tau)e^{-\frac{q}{m}J\tau} - w\gamma)(\bar{w}(\tau) - w) \\ &\quad \cdot J^2 e^{-\frac{q}{m}(t-s)\bar{w}(\tau)J}e^{\frac{q}{m}J\tau} d\mu(\tau).\end{aligned} \tag{3.16}$$

Theorem 3.1. *Let the sequence of scalar functions* $\{w^\delta\}_\delta$ *satisfy* (3.1). *Then, up to a subsequence of* $\delta \to 0$, *there exists a kernel* K *associated with* $\{w^\delta\}_\delta$ *and* *defined on* $\Omega \times (0,T)$ *such that the sequence* $\{U^\delta\}$, *solutions to* (1.1), *converges in* $L^2(\Omega \times (0,T))$ *weakly to* U, *a solution of integral differential equation*

$$\begin{aligned} m\frac{d}{dt}U(x,t) &= q(E(x,t) + U(x,t) \times B) \\ &\quad - \int_0^t K(t-s)U(x,s)\,ds + \Psi(x,t)U_0(x), \\ U(x,0) &= U_0(x), \end{aligned} \tag{3.17}$$

where w *is the weak* limit of* w^δ *and the memory kernel* K *is measurable in* $x \in \Omega$ *and admits the integral representation* (3.16) *with a family of positive parametrized measures* $\{\mu\}$ *defined on* $\mathcal{T}$ *and* $\Psi(x,t)$ *is defined by Eq.* (2.39).

Remark. As mentioned in Remark 2 after Theorem 2.3, the memory effect does also occur in this case and the integral is of convolution type because of time translation invariant.

4. Drifts and Polarization Induced by Homogenization

In this section, we analyze the motion of charged particles from the point of view of homogenization. In order to meaningfully describe some physical quantities, we will assume the trivial initial condition $U^\delta(x,0) = U_0(x) = 0$. First we rewrite Eq. (3.17) as

$$m\frac{d}{dt}U(x,t) = q(E(x,t) + U(x,t) \times B) - \omega^2 J^2 \int_0^t \mathcal{K}(t-s)U(x,s)ds$$

or, using the property of the matrix J,

$$m\frac{d}{dt}U(x,t) = q(E(x,t) + U(x,t) \times B) - \int_0^t \mathcal{K}(t-s)U(x,s)ds \times B \times B, \tag{4.1}$$

where

$$\mathcal{K}(t-s) \equiv \int_{\mathcal{T}} \frac{q^2(\bar{w}(\tau)e^{-\frac{q}{m}J\tau} - w\gamma)(\bar{w}(\tau) - w)}{m\omega^2} e^{-\frac{q}{m}(t-s)\bar{w}(\tau)J} e^{\frac{q}{m}J\tau} d\mu(\tau). \tag{4.2}$$

Therefore the homogenized equation can be written as

$$m\frac{d}{dt}U(x,t) = q(E(x,t) + U(x,t) \times B) - (W_\perp(x,t) \times B) \times B, \tag{4.3}$$

where

$$W_\perp(x,t) = \int_0^t \mathcal{K}(t-s)U(x,s)ds. \tag{4.4}$$

To proceed, we decompose U as follows:

$$U(x,t) = U_\parallel(x,t) + \tilde{U}_\perp(x,t), \tag{4.5}$$

where $U_\parallel$ is parallel to the magnetic field B and $\tilde{U}_\perp$ is perpendicular to B. Obviously $U_\parallel$ satisfied the first-order differential equation

$$m\frac{d}{dt}U_\parallel(x,t) = qE_\parallel(x,t), \tag{4.6}$$

by taking the parallel part of (4.3). Let $\Omega_c = \frac{q|B|}{m}$ denote the Larmor frequency (see Ref. 8 for the physical explanation); then the perpendicular part $\tilde{U}_\perp$ can be further decomposed into

$$\tilde{U}_\perp(x,t) = U_\perp(x,t) + U_E(x,t) + U_P(x,t), \tag{4.7}$$

where $U_E(x,t)$ and $U_P(x,t)$ are given, respectively, by

$$U_E(x,t) = \frac{E_\perp \times B}{|B|^2} + \frac{W_\perp \times B}{q} \tag{4.8}$$

and

$$U_P(x,t) = \frac{m}{q|B|^2}\frac{\partial E_\perp}{\partial t} + \frac{m}{q^2}\frac{\partial W_\perp}{\partial t} = \frac{1}{\Omega_c|B|}\frac{\partial E_\perp}{\partial t} + \frac{|B|}{q\Omega_c}\frac{\partial W_\perp}{\partial t}. \tag{4.9}$$

It is straightforward to show that $U_\perp$ satisfies

$$m\frac{d}{dt}U_\perp(x,t) + m\frac{d}{dt}U_P(x,t) = qU_\perp(x,t) \times B. \tag{4.10}$$

Assuming $W_\perp$ and $E_\perp$ have a harmonic time dependence, that is $W_\perp$ and $E_\perp \sim e^{-i\omega t}$, with a characteristic angular frequency ω, then we have

$$\begin{aligned}\frac{|m\frac{dU_P}{dt}|}{|qU_\perp \times B|} &= \frac{|\frac{m^2}{q|B|^2}\frac{\partial^2 E_\perp}{\partial t^2} + \frac{m^2}{q^2}\frac{\partial^2 W_\perp}{\partial t^2}|}{q|U_\perp||B|} \\ &\le \frac{|U_E|}{|B||U_\perp|}\left(\frac{\omega}{\Omega_c}\right)^2 + \frac{|B||W_\perp|}{q|U_\perp|}\left(\frac{\omega}{\Omega_c}\right)^2. \end{aligned} \tag{4.11}$$

If the characteristic angular frequency is much smaller than the cyclotron frequency, $\omega \ll \Omega_c$, with $\frac{|W_\perp|}{|U_\perp|}$ and $\frac{|U_E|}{|U_\perp|}$ are also small, then we have

$$\left|\frac{d}{dt}U_P\right| \ll |qU_\perp(x,t) \times B|. \tag{4.12}$$

Thus it is sufficient to approximate (4.10) by

$$m\frac{d}{dt}U_\perp(x,t) = qU_\perp(x,t) \times B. \tag{4.13}$$

Therefore, $U_\perp$ is concerned with the usual circular motion of the charged particle about the magnetic field, and is independent of the variations of the electric field. Superposed upon this circular motion velocity are the drift velocities, the equations (4.8) and (4.9). We note that Eq. (4.8) describes the electric drift velocity, and Eq. (4.9) shows the effect of polarization drift velocity.

In the following we deduce the polarization for many particles (electrons or ions) by way of the above single particle homogenized equation. We note that for many physical situations in rare plasma, without loss of the physical meanings, we can omit the interactions of the electrons or ions with each other. And therefore the behavior of the many particles can be served as the sum of the individual particles. To do this, for small volume δV, the polarization current density J_p is the rate of flow positive and negative charges across unit area, and is given by

$$\begin{aligned} J_p = \frac{1}{\delta V}\sum_i q_i U_{Pi} &= \frac{1}{\delta V}\sum_i \left(\frac{m_i}{|B|^2}\frac{\partial E_\perp}{\partial t} + \frac{m_i}{q_i}\frac{\partial W_{i\perp}}{\partial t}\right) \\ &\equiv \rho_m\left(\frac{1}{|B|^2}\frac{\partial E_\perp}{\partial t} + \frac{\partial \mathcal{W}_\perp}{\partial t}\right), \end{aligned} \tag{4.14}$$

where $\frac{\partial \mathcal{W}_\perp}{\partial t} = \frac{1}{\delta V}\sum_i \frac{m_i}{\rho_m q_i}\frac{\partial W_{i\perp}}{\partial t}$ and the summation is taking over all positive and negative charges contained in δV, and ρ_m is the mass density of the plasma. Equation (4.14) means that the polarization effect in a plasma is due to the time variation of the electric field. The contribution of a steady electric field does not result in a polarization field, since the ions and electrons can move around to preserve quasineutrality. Because the plasma behaves like a dielectric, the polarization current density J_p can be introduced by way of the dielectric coefficients of the plasma. For this purpose, we separate the total current density J_T into the polarization current density J_p and the current density J_0, the effect of other sources,

$$J_T = J_p + J_0. \tag{4.15}$$

Combining the polarization current density J_p and $\epsilon_0\frac{\partial E_\perp}{\partial t}$, we obtain

$$\begin{aligned} \epsilon_0\frac{\partial E_\perp}{\partial t} + \rho_m\left(\frac{1}{|B|^2}\frac{\partial E_\perp}{\partial t} + \frac{\partial \mathcal{W}_\perp}{\partial t}\right) &= \epsilon_0\left(1 + \frac{\rho_m}{\epsilon_0|B|^2}\right)\frac{\partial E_\perp}{\partial t} + \rho_m\frac{\partial \mathcal{W}_\perp}{\partial t} \\ &\equiv \epsilon\frac{\partial E_\perp}{\partial t} + \rho_m\frac{\partial \mathcal{W}_\perp}{\partial t}, \end{aligned} \tag{4.16}$$

where

$$\epsilon = \epsilon_0\left(1 + \frac{\rho_m}{\epsilon_0|B|^2}\right) \tag{4.17}$$

is the effective electric permittivity perpendicular to the magnetic, and the term $\rho_m\frac{\partial \mathcal{W}_\perp}{\partial t}$ is the extra homogenized polarization electric. From another viewpoint, the resulting polarization charge density ρ_p and polarization current density J_p satisfy the charge continuity equation. The polarization charge density ρ_p can be viewed as

$$\rho_p = -\frac{\rho_m}{|B|^2}(\nabla\cdot E_\perp) - \rho_m(\nabla\cdot\mathcal{W}_\perp). \tag{4.18}$$

From Eq. (4.14), the flux of polarization current density J_p is given by

$$\nabla\cdot J_p = \frac{\rho_m}{|B|^2}\frac{\partial}{\partial t}(\nabla\cdot E_\perp) + \rho_m\frac{\partial}{\partial t}(\nabla\cdot\mathcal{W}_\perp), \tag{4.19}$$

and therefore Eqs. (4.18) and (4.19) make clear that the charge continuity equation is satisfied

$$\frac{\partial \rho_p}{\partial t} + \nabla \cdot J_p = 0. \tag{4.20}$$

The total charge can be separated as

$$\rho = \rho_0 + \rho_p, \tag{4.21}$$

where the charge ρ_0 is associated with the current J_0 through the charge continuity equation

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot J_0 = 0. \tag{4.22}$$

When the parallel component of the electric field vanishes, the electric flux satisfies

$$\begin{aligned} \epsilon_0 \nabla \cdot E = \rho &= \rho_0 + \rho_p \\ &= \rho_0 - \frac{\rho_m}{|B|^2}(\nabla \cdot E_\perp) - \rho_m(\nabla \cdot \mathcal{W}_\perp) \\ &= \rho_0 - \epsilon_0 \left[\frac{\rho_m}{\epsilon_0 |B|^2}(\nabla \cdot E_\perp) + \frac{\rho_m}{\epsilon_0}(\nabla \cdot \mathcal{W}_\perp) \right] \\ &\equiv \rho_0 - \epsilon_0 \nabla \cdot \mathcal{P}, \end{aligned} \tag{4.23}$$

where

$$\nabla \cdot \mathcal{P} \equiv \left(\frac{\rho_m}{\epsilon_0 |B|^2} + \frac{\rho_m}{\epsilon_0} \right) \nabla \cdot (E_\perp + \mathcal{W}_\perp)$$

is the polarization field. Let $D = E + \mathcal{P}$ then the Gauss's law becomes

$$\nabla \cdot D = \frac{\rho_0}{\epsilon_0}. \tag{4.24}$$

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