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A perturbation solution for head fluctuations in a coastal leaky aquifer system considering water table over-height

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Abstract This paper develops a new analytical solution for the aquifer system, which comprises an unconfined aquifer on the top, a semi-confined aquifer at the bottom and an aquitard between them. This new solution is derived from the Boussinesq equation for the unconfined aquifer and one-dimensional leaky confined flow equation for the lower aquifer using the perturbation method, considering the water table over-height at the remote boundary. The head fluctuation predicted from this solution is generally greater than the one solved from the linearized Boussinesq equation when the ratio of the tidal amplitude to the thickness of unconfined aquifer is large. It is found that both submarine groundwater discharges from upper and lower aquifers increase with tidal amplitude–aquifer thickness ratio and may be underestimated if the discharge is calculated based on the average head fluctuation. The effects of the aquifer parameters and linearization of the Boussinesq equation on the normalized head fluctuation are also investigated.

Key words perturbation analysis; Boussinesq equation; tidal fluctuation; submarine groundwater discharge

Une solution par la méthode des perturbations pour les fluctuations de charge dans un système aquifère côtier semi-captif tenant compte de la surélévation de la surface libre

Résumé Cet article développe une nouvelle solution analytique pour un système aquifère comprenant un aquifère libre supérieur, un aquifère semi-captif inférieur, séparés par un aquitard. Cette nouvelle solution est dérivée de l'équation de Boussinesq pour l'aquifère libre et de l'équation unidimensionnelle d'écoulement pour l'aquifère semi-captif, en utilisant la méthode des perturbations et en tenant compte de la surélévation de la nappe phréatique à la frontière la plus éloignée. La fluctuation de la charge obtenue à partir de cette solution est généralement plus importante que celle obtenue par l'équation de Boussinesq linéarisée lorsque le rapport entre l'amplitude de la marée et l'épaisseur de l'aquifère libre est grande. On constate que les débits des eaux souterraines sous-marines pour les deux aquifères supérieur et inférieur augmentent avec le rapport de l'amplitude des marées à l'épaisseur de l'aquifère et peuvent être sous-estimés si le débit est calculé sur la base des fluctuations de charge moyennes. Les effets des paramètres de l'aquifère et de la linéarisation de l'équation de Boussinesq sur la fluctuation de la charge normalisée sont également étudiés.

Mots clefs analyse de perturbations; équation de Boussinesq; fluctuation de la marée; débit des eaux souterraines sous-marines

1 INTRODUCTION

The study of groundwater in coastal areas is of practical interest for many researchers for economic and environmental reasons. Related investigations are concentrated on several different topics such as parameter estimation (e.g. Pandit *et al.* 1991, Li *et al.* 2006) and groundwater head fluctuations (e.g. Jiao and Tang 1999, Jeng *et al.* 2002, Li *et al.* 2007). Among them, tide-induced groundwater head fluctuation is an important issue, which is useful in studying other topics. The approaches commonly

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used to investigate the groundwater fluctuation in a tidal aquifer system include analytical methods (e.g. Jiao and Tang 1999, Chuang and Yeh 2008), numerical methods (e.g. Ataie-Ashtiani *et al.* 1999, Li and Jiao 2002, Chen and Hsu 2004, Liu *et al.* 2008), and experimental studies (e.g. Nielsen 1990, Turner 1993, Ataie-Ashtiani *et al.* 1999, Uchiyama *et al.* 2000, Cartwright *et al.* 2004, Robinson *et al.* 2006). In these studies, two types of problem are considered for the coastal aquifer system; one involves single layer aquifers (e.g. Parlange *et al.* 1984, Song *et al.* 2007), while the other is concerned with multiple layered aquifer systems (e.g. Li and Jiao 2001a, 2001b, Jeng *et al*. 2002).

Many coastal aquifer systems comprise an upper unconfined aquifer, a bottom semi-confined aquifer and an aquitard in between (e.g. White and Roberts 1994, Chen and Jiao 1999). The analytical solutions developed for the problems commonly rely on some simplifications and/or assumptions. For example, the water table fluctuation in the unconfined aquifer is assumed constant or the Boussinesq equation is linearized. Jiao and Tang (1999) investigated the influence of leakage on tidal response in a leaky aquifer system analytically without considering the water table fluctuation in the unconfined aquifer. Li *et al.* (2001) used the linearized Boussinesq equation for the unconfined aquifer and presented an approximate analytical solution to investigate the dynamic effect of unconfined aquifer on groundwater fluctuation in confined aquifers. Li and Jiao (2001a) derived an analytical solution to investigate the influence of leakage and storativity of the aquitard on tidal response in a coastal leaky aquifer system. They assumed that the water table of the upper unconfined aquifer remains constant. Also assuming a constant head for the unconfined aquifer, Li and Jiao (2001b) developed an analytical solution for a coastal leaky aquifer system in which the aquitard extends a finite distance under the sea. Jeng *et al.* (2002) developed an analytical solution for a coastal aquifer system, which allows water table fluctuations in response to the tide. However, their solution was based on the linearized Boussinesq equation for the unconfined aquifer. Chuang and Yeh (2007) developed a mathematical model, also based on the linearized Boussinesq equation, to investigate the effects of tidal fluctuations and aquitard leakage on the groundwater head of the confined aquifer extending an infinite distance under the sea.

The problem of water table over-height, arisen from the nonlinear effect, is an interesting issue in the coastal aquifers. Based on the Boussinesq equation, many researchers (e.g. Knight 1981, Parlange *et al.* 1984, Li and Jiao 2003) studied the over-height problem. Parlange *et al.* (1984) derived an approximate analytical solution for groundwater level fluctuation subject to a tidal boundary condition in an unconfined aquifer. They developed a second-order solution for the Boussinesq equation using the perturbation approach. Li and Jiao (2003) developed an exact asymptotic solution and approximate perturbation solution for multi-sinusoidal components of the sea tide in a coastal aquifer system. Their results showed that the mean groundwater level of the aquifer system stands considerably above the mean sea level, even in the absence of net inland recharge of groundwater and rainfall. Song *et al.* (2007) presented a perturbation solution derived from the Boussinesq equation for one-dimensional tidal groundwater flow in an unconfined aquifer. Their solution adopted a perturbation parameter that was by definition less than one, and thus was applicable to a wide range of physical conditions within the constraint of the Boussinesq approximation. Their approach avoids a secular term in the third-order perturbation equation of Parlange *et al.* (1984).

The objective of this paper is to develop an analytical solution using perturbation analysis for a coastal leaky aquifer system, comprising an upper unconfined aquifer, a lower semi-confined aquifer, and an aquitard between them. The one-dimensional leaky confined flow equation for the lower aquifer coupled with the Boussinesq equation for the upper aquifer is used to describe the groundwater fluctuations in the coastal leaky aquifer system. The second-order perturbation method is then applied to develop the solution for the coupled equations. The mean water table is assumed above the mean sea level based on the work of Li and Jiao (2003). The solution from the Boussinesq equation for the unconfined aquifer accounts for the water table over-height, which was generally considered for cases of thin unconfined aquifer and/or large tidal amplitude. The solution of Jeng *et al.* (2002) obtained from the linearized Boussinesq equation may therefore be considered as a special case of this newly derived solution. The effects of aquifer parameters and linearization of the Boussinesq equation on the behaviour of the groundwater level fluctuations in the coastal leaky aquifer system are examined. Finally, the issues of tide-induced submarine groundwater discharges in the upper and lower aquifers and leakage flux through the aquitard are addressed.

2 PROBLEM SET-UP AND BOUNDARY CONDITIONS

Figure 1 shows a coastal leaky aquifer system with an upper unconfined aquifer, a semi-confined aquifer at the bottom with an impermeable base, and an aquitard between them. The formations of both upper and lower aquifers are isotropic and homogenous. The effects of tidal fluctuation on the groundwater flow in both aquifers are considered. These two aquifers interact with leakage through aquitard, which is linearly proportional to the head difference between two aquifers. Consider that the storage of the aquitard is very small and negligible. The origin of the x-axis is located at the intersection of the mean sea surface and beach face. The x-axis is horizontal, positive landward and perpendicular to the coastal line and both aquifers are assumed semi-infinite. The flow velocity in the aquifer system is essentially horizontal and the thickness of the unconfined aquifer below the mean sea level is considered equal to or larger than the tidal amplitude.

Under those assumptions and using the Boussinesq equation for the upper aquifer and leaky confined flow equation for the lower aquifer, the governing equations of the head fluctuations in both aquifers are, respectively (Bear 1972, Jiao and Tang 1999), expressed as:

$$S_y \frac{\partial W}{\partial t} = K_1 \frac{\partial}{\partial x}\left(W \frac{\partial W}{\partial x}\right) + L(H - W) \tag{1a}$$

$$S_2 \frac{\partial H}{\partial t} = T_2 \frac{\partial^2 H}{\partial x^2} + L(W - H) \tag{1b}$$

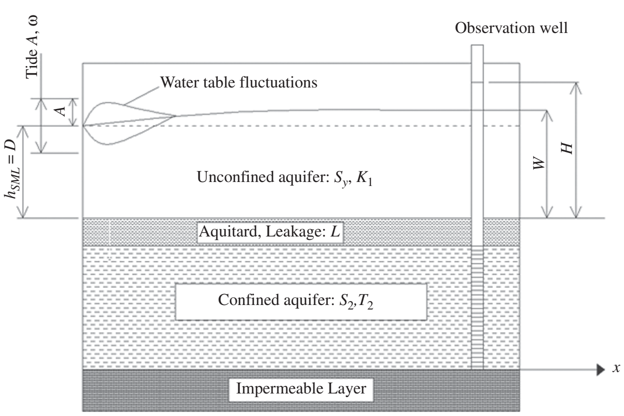

Fig. 1 Schematic diagram of a coastal leaky aquifer system containing an upper unconfined aquifer, a lower semi-confined aquifer, and an aquitard in between.

where W and H are the hydraulic heads in the upper and lower aquifers respectively, S_y and K_1 are the specific yield and hydraulic conductivity of the upper aquifer, respectively, S_2 and T_2 are the storativity and transmissivity of the lower aquifer, respectively, and L is the specific leakage of the aquitard. The tidal boundary conditions for both aquifers are written as:

$$W(0,t) = H(0,t) = D + Ae^{i\omega t} \tag{2a}$$

where D is the thickness of the unconfined aquifer below mean sea level and A and ω are the tidal amplitude and frequency, respectively. The boundary conditions for equations (1) and (2) at the remote inland side may be written as:

$$\left.\frac{\partial W}{\partial x}\right|_{x\to\infty} = \left.\frac{\partial H}{\partial x}\right|_{x\to\infty} = 0 \tag{2b}$$

Li and Jiao (2003) showed that the mean groundwater level of the aquifer system stands considerably above the mean sea level due to the water table over-height. They proved that in both unconfined and semi-confined aquifers, when the specific leakage of the aquitard is larger than zero and $x \to \infty$, the water table over-height is equal to:

$$\begin{aligned} W(\infty,t) - D &= H(\infty,t) - D \\ &= \left(D + \frac{T_2}{K_1}\right)\left(\sqrt{1 + \frac{\alpha_L}{2}} - 1\right) \end{aligned} \tag{2c}$$

where $\alpha_L = A^2 K_1^2 \big/ (DK_1 + T_2)^2$.

3 PERTURBATION SOLUTION

For the convenience of expression, dimensionless parameters $\overline{W} = W\big/D$ and $\overline{H} = H\big/D$ are adopted herein. The tidal amplitude-aquifer thickness ratio δ, denoted as the ratio of the tidal amplitude A to the mean thickness of the unconfined aquifer D, is chosen as the perturbation parameter. Detailed derivation of the perturbation solutions for the governing equations, equations (1a) and (1b), with appropriate boundary conditions, equations (2a)–(2c), is given in Appendix A and the solutions of $W(x, t)$ and $H(x, t)$ for the upper and lower aquifers are, respectively:

$$\begin{aligned} W(x,t) &= D\overline{W} \\ &= D[1 + \delta \overline{W}_1 + \delta^2 \overline{W}_2 + O(\delta^3)] \end{aligned} \tag{3a}$$

and

$$\begin{aligned} H(x,t) &= D\overline{H} \\ &= D[1+\delta\overline{H}_1+\delta^2\overline{H}_2+O(\delta^3)] \end{aligned} \tag{3b}$$

where the solutions of $\overline{W}_1$ and $\overline{H}_1$ are similar to the solutions of Jeng *et al*. (2002) and can be expressed, respectively, as:

$$\overline{W}_1 = \mathrm{Re}[(a_1 e^{-\lambda_1 x} + a_2 e^{-\lambda_2 x})e^{i\omega t}] \tag{4a}$$

and

$$\overline{H}_1 = \mathrm{Re}[(b_1 e^{-\lambda_1 x} + b_2 e^{-\lambda_2 x})e^{i\omega t}] \tag{4b}$$

and the solutions of $\overline{W}_2$ and $\overline{H}_2$ are, respectively, written as:

$$\begin{aligned} \overline{W}_2 = \mathrm{Re}[(&c_1 e^{-2\lambda_1 x} + c_2 e^{-2\lambda_2 x} + c_3 e^{-\lambda_3 x} \\ &+ c_4 e^{\tau_1 x} + c_5 e^{\tau_2 x})e^{2i\omega t} + c_6 e^{-\alpha x} \\ &+ c_7 e^{-\beta x} + B] \end{aligned} \tag{4c}$$

and

$$\begin{aligned} \overline{H}_2 = \mathrm{Re}[(&d_1 e^{-2\lambda_1 x} + d_2 e^{-2\lambda_2 x} + d_3 e^{-\lambda_3 x} \\ &+ d_4 e^{\tau_1 x} + d_5 e^{\tau_2 x})e^{2i\omega t} + d_6 e^{-\alpha x} \\ &+ d_7 e^{-\beta x} + B] \end{aligned} \tag{4d}$$

where Re denotes the real part of the complex expression, $i=\sqrt{-1}$, and the coefficients a_1, a_2, b_1, b_2, c_1 to c_7, d_1 to d_7, λ_1, λ_2, λ_3, τ_1, τ_2, α, β and B are, respectively, defined as:

$$a_1 = -\frac{-S_y i\omega + K_1 D\lambda_2^2}{K_1 D(\lambda_1^2-\lambda_2^2)} \tag{5a}$$

$$a_2 = \frac{-S_y i\omega + K_1 D\lambda_1^2}{K_1 D(\lambda_1^2-\lambda_2^2)} \tag{5b}$$

$$b_1 = -\frac{T_2\lambda_2^2 - S_2 i\omega}{T_2(\lambda_1^2-\lambda_2^2)} \tag{5c}$$

$$b_2 = \frac{T_2\lambda_1^2 - S_2 i\omega}{T_2(\lambda_1^2-\lambda_2^2)} \tag{5d}$$

$$c_1 = \frac{-2a_1^2\lambda_1^2\left(2iS_2\omega - 4T_2\lambda_1^2 + L\right)}{E_1} \tag{5e}$$

$$c_2 = \frac{-2a_2^2\lambda_2^2\left(2iS_2\omega - 4T_2\lambda_2^2 + L\right)}{E_2} \tag{5f}$$

$$c_3 = \frac{-a_1 a_2\lambda_3^2\left(2iS_2\omega - T_2\lambda_3^2 + L\right)}{E_3} \tag{5g}$$

$$c_4 = -\frac{\{4c_1\lambda_1^2 + 4c_2\lambda_2^2 + c_3\lambda_3^2 - \tau_2^2 (c_1+c_2+c_3) + 2a_1^2\lambda_1^2 + 2a_2^2\lambda_2^2 + a_1 a_2\lambda_3^2\}}{\tau_1^2-\tau_2^2} \tag{5h}$$

$$c_5 = \frac{\{4c_1\lambda_1^2 + 4c_2\lambda_2^2 + c_3\lambda_3^2 - \tau_1^2 (c_1+c_2+c_3) + 2a_1^2\lambda_1^2 + 2a_2^2\lambda_2^2 + a_1 a_2\lambda_3^2\}}{\tau_1^2-\tau_2^2} \tag{5i}$$

$$c_6 = \frac{E_4}{\alpha^2-\beta^2}\left(T_2 - \frac{L}{\alpha^2}\right) \tag{5j}$$

$$c_7 = \left(1-\frac{T_2\beta^2}{L}\right)\left[E_4 - B + \frac{E_4 L}{\alpha^2\left(\alpha^2-\beta^2\right)}\right] \tag{5k}$$

$$d_1 = \frac{2a_1^2\lambda_1^2 L}{E_5} \tag{5l}$$

$$d_2 = \frac{2a_2^2\lambda_2^2 L}{E_6} \tag{5m}$$

$$d_3 = \frac{a_1 a_2\lambda_3^2 L}{E_7} \tag{5n}$$

$$d_4 = \frac{\tau_2^2 (d_1+d_2+d_3) - 4d_1\lambda_1^2 - 4d_2\lambda_2^2 - d_3\lambda_3^2}{\tau_1^2-\tau_2^2} \tag{5o}$$

$$d_5 = -\frac{\tau_1^2 (d_1+d_2+d_3) - 4d_1\lambda_1^2 - 4d_2\lambda_2^2 - d_3\lambda_3^2}{\tau_1^2-\tau_2^2} \tag{5p}$$

$$d_6 = E_4 - B + \frac{E_4 L}{\alpha^2\left(\alpha^2-\beta^2\right)} \tag{5q}$$

$$d_7 = -E_4 - \frac{E_4 L}{\alpha^2\left(\alpha^2-\beta^2\right)} \tag{5r}$$

$$\lambda_1 = \sqrt{\frac{1}{2}\left(-b+\sqrt{b^2-4c}\right)} \tag{5s}$$

$$\lambda_2 = \sqrt{\frac{1}{2}\left(-b - \sqrt{b^2 - 4c}\right)} \tag{5t}$$

$$\lambda_3 = \lambda_1 + \lambda_2 \tag{5u}$$

$$\tau_1 = -\sqrt{bb + \sqrt{bb^2 - 4cc}} \tag{5v}$$

$$\tau_2 = -\sqrt{bb - \sqrt{bb^2 - 4cc}} \tag{5w}$$

$$\alpha = \sqrt{\frac{L}{T_2}} \tag{5x}$$

$$\beta = \sqrt{\frac{K_1 D + T_2}{K_1 D T_2 L}} \tag{5y}$$

$$B = \frac{D^2}{A^2}\left(\frac{K_1 D + T_2}{K_1 D}\right)\left(\sqrt{1 + \frac{\alpha_L}{2}} - 1\right) \tag{5z}$$

Note that the coefficients E_1, E_2 and E_3 in equations (5e)–(5g) are, respectively, defined as:

$$E_1 = 4T_2 K_1 D \lambda_1^2 \left(\frac{2i\omega S_y + L}{K_1 D} + \frac{2i\omega S_2 + L}{T_2} - 4\lambda_1^2\right) + 2S_y S_2 \omega \left(2\omega - \frac{iL}{S_2} - \frac{iL}{S_y}\right) \tag{6a}$$

$$E_2 = 4T_2 K_1 D \lambda_2^2 \left(\frac{2i\omega S_y + L}{K_1 D} + \frac{2i\omega S_2 + L}{T_2} - 4\lambda_2^2\right) + 2S_y S_2 \omega \left(2\omega - \frac{iL}{S_2} - \frac{iL}{S_y}\right) \tag{6b}$$

$$E_3 = T_2 K_1 D \lambda_3^2 \left(\frac{2i\omega S_2 + L}{T_2} + \frac{2i\omega S_y + L}{K_1 D} - \lambda_3^2\right) + 2S_y S_2 \omega \left(2\omega - \frac{iL}{S_2} - \frac{iL}{S_y}\right) \tag{6c}$$

The coefficient E_4 in equations (5j), (5k), (5q) and (5r) is defined as:

$$E_4 = \frac{BL\beta^2}{L\beta^2 - T_2 L - \frac{L^2}{T_2}} \tag{6d}$$

The coefficients E_5, E_6, and E_7 in equations (5l)–(5n) are, respectively, defined as:

$$E_5 = -4T_2 \lambda_1^2 \left(\frac{2i\omega S_y + L}{K_1 D} + \frac{2i\omega S_2 + L}{T_2} - 4\lambda_1^2\right) + \frac{2S_y S_2 \omega}{K_1 D}\left(2\omega + \frac{iL}{S_y} + \frac{iL}{S_2}\right) \tag{6e}$$

$$E_6 = -4T_2 \lambda_2^2 \left(\frac{2i\omega S_y + L}{K_1 D} + \frac{2i\omega S_2 + L}{T_2} - 4\lambda_2^2\right) + \frac{2S_y S_2 \omega}{K_1 D}\left(2\omega + \frac{iL}{S_y} + \frac{iL}{S_2}\right) \tag{6f}$$

$$E_7 = -T_2 \lambda_3^2 \left(\frac{2i\omega S_y + L}{K_1 D} + \frac{2i\omega S_2 + L}{T_2} - \lambda_3^2\right) + \frac{2S_y S_2 \omega}{K_1 D}\left(2\omega + \frac{iL}{S_y} + \frac{iL}{S_2}\right) \tag{6g}$$

The coefficients b and c in both equations (5s) and (5t) are, respectively, defined as:

$$b = -i\omega\left(\frac{S_y}{T_1} + \frac{S_2}{T_2}\right) - L\left(\frac{1}{T_1} + \frac{1}{T_2}\right) \tag{6h}$$

$$c = -\frac{S_y S_2 \omega^2}{T_1 T_2} + \frac{Li\omega\left(S_y + S_2\right)}{T_1 T_2} \tag{6i}$$

Finally, the coefficients bb and cc in both equations (5v) and (5w) are, respectively, defined as:

$$\mathrm{bb} = 2i\omega\left(\frac{S_y}{T_1} + \frac{S_2}{T_2}\right) + \frac{T_2 + K_1 D}{L} \tag{6j}$$

$$\mathrm{cc} = -\frac{4S_y S_2 \omega^2}{T_1 T_2} \tag{6k}$$

3.1 Leakage flux and submarine groundwater discharges

The mean water levels in the aquifer system are defined as (Li and Jiao 2003):

$$\hat{W} = \frac{1}{P}\int_t^{t+P} W(x,\tau)\,\mathrm{d}\tau \tag{7a}$$

$$\hat{H} = \frac{1}{P}\int_t^{t+P} H(x,\tau)\,\mathrm{d}\tau \tag{7b}$$

where $\hat{W}$ and $\hat{H}$ are mean water levels in the upper and lower aquifers and P is the tidal period defined as $2\pi/\omega$. Substituting equations (3a) and (3b) into equations (7a) and (7b), respectively, results in:

$$\hat{W} = A\delta\left(c_6 \mathrm{e}^{-\alpha x} + c_7 \mathrm{e}^{-\beta x} + B\right) \tag{8a}$$

$$\hat{H} = A\delta\left(d_6 \mathrm{e}^{-\alpha x} + d_7 \mathrm{e}^{-\beta x} + B\right) \tag{8b}$$

Based on Darcy's law, the time interval leakage flux, $F_L(x)$, through the aquitard is defined as (Li and Jiao 2003):

$$F_L(x) = L\left(\hat{W} - \hat{H}\right) = LA\delta\left[(c_6 - d_7)\mathrm{e}^{-\alpha x} + (c_7 - d_7)\mathrm{e}^{-\beta x}\right] \tag{9}$$

The submarine groundwater discharges from the upper and lower aquifers, Q_U and Q_L, may, respectively, be calculated using following equations:

$$Q_U = K_1 \int_t^{t+P} W(0,t) \max\left[0, \left.\frac{\partial W}{\partial x}\right|_{x=0}\right] \mathrm{d}t \tag{10a}$$

$$Q_L = T_2 \int_t^{t+P} \max\left[0, \left.\frac{\partial H}{\partial x}\right|_{x=0}\right] \mathrm{d}t \tag{10b}$$

Note that these definitions are different from those defined by Li and Jiao (2003).

4 RESULTS AND DISCUSSION

In this section, some examples are used to illustrate the use of the present solution and investigate the effects of aquifer and perturbation parameters on the head fluctuation, leakage flux and submarine groundwater discharges in a coastal leaky aquifer system. To describe the problem clearly, two new variables are defined, i.e. $w = (W - D)/A$ and $h = (H - D)/A$ in which w and h represent normalized head fluctuations in the upper and lower aquifers, respectively. Jeng *et al.*'s (2002) solution obtained from the linearized Boussinesq equation can be considered as a special case of the present solution by neglecting the second-order perturbation, δ^2. The results by the present solution are compared with those of Jeng *et al.*'s solution to explore the second-order effect. The difference between these two solutions is defined as $D_w = (w_p - w_J)$ for the upper aquifer and $D_h = (h_p - h_J)$ for the lower aquifer where the subscripts p and J denote the present solution and Jeng *et al.*'s solution, respectively. For comparison, the values of aquifer parameters are chosen the same as those in Jeng *et al.*'s study, i.e. $S_y = 0.3$, $S_2 = 0.001$, $D = 10$ m, $K_1 = 200$ m/d, $T_2 = 2000$ m^2/d, $L = 1$/d, $A = 2.5$ m and $\omega = 2\pi$ rad/d.

4.1 The effect of tidal amplitude-aquifer thickness ratio on head fluctuation

Figure 2 shows the spatial distributions of w and h at $t = 24$ h for $L = 0.2$ d^{-1} and $A = 0.1$, 1.5 and 2.5 m (i.e. $\delta = 0.01$, 0.15 and 0.25). The temporal fluctuations of w and h at $x = 100$ m are presented in Fig. 3 for $L = 0.2$ d^{-1} and $\delta = 0.01$, 0.15 and 0.25. Figures 2 and 3 indicate that the normalized groundwater heads predicted by Jeng *et al.*'s (2002) solution are close to those of the present solution when δ is equal to 0.01. As δ increases, the difference between the present solution and Jeng *et al.*'s (2002) solution increase. The difference between the present solution and Jeng *et al.*'s (2002) solution comes from the term $\delta^2\overline{W}_2$ for unconfined aquifers shown in equation (3a) and $\delta^2\overline{H}_2$ for semi-confined aquifers in equation (3b). For large values of δ, δ^2 becomes very large and the difference between these two solutions is very significant. Therefore, the error from the solution by linearized

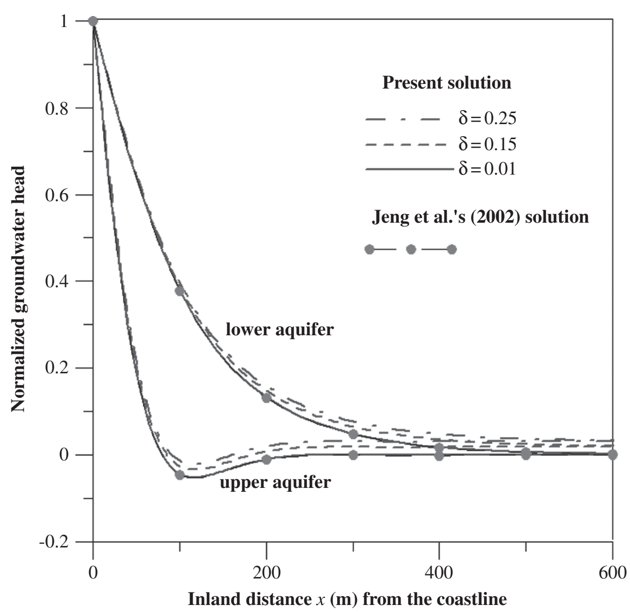

Fig. 2 Spatial distributions of normalized groundwater heads, $w = (W - D)/A$ and $h = (H - D)/A$ and at $t = 24$ h with $L = 0.2$ d^{-1} and different values of δ.

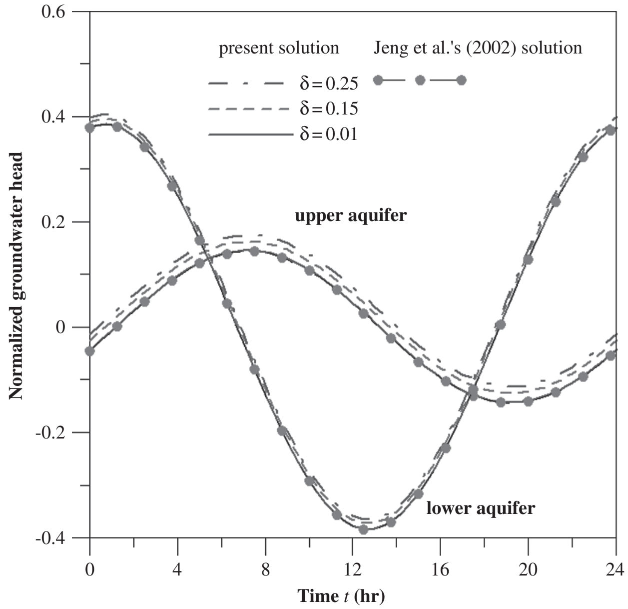

Fig. 3 Temporal fluctuations of normalized groundwater heads, $w = (W - D)/A$ and $h = (H - D)/A$ at $x = 100$ m with $L = 0.2$ d^{-1} and different values of δ.

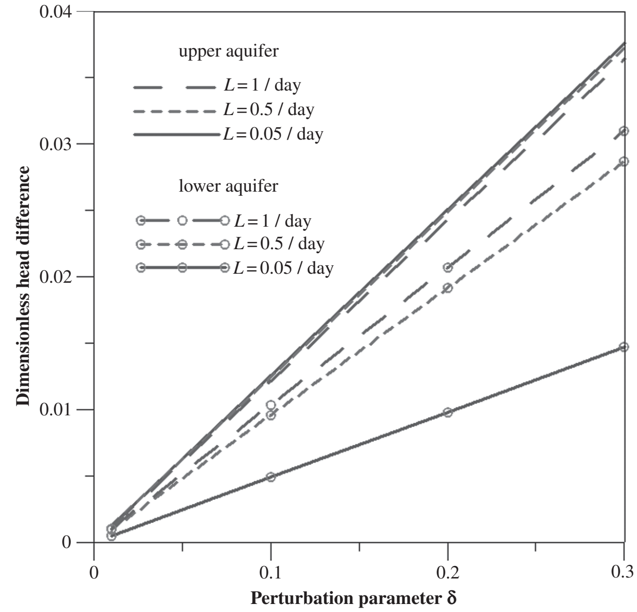

Fig. 4 Dimensionless head differences in the upper and lower aquifers versus perturbation parameter at $x = 100$ m and $t = 24$ h for different values of leakage.

Boussinesq equation increases with δ. Figure 2 also demonstrates that the difference increases with the landward distance from coastline and approaches a constant when the landward distance is beyond the threshold value. Moreover, Fig. 3 shows that the time lag is independent of δ in this case.

4.2 The effects of leakage and transmissivity on head fluctuation

Figure 4 shows the lines of dimensionless head difference (D_w and D_h) *versus* δ at $x = 100$ m and $t = 24$ h when $L = 1$, 0.5 and 0.05 d^{-1}. This figure indicates that both D_w and D_h increase linearly with δ at different leakages: D_w increases with decreasing leakage in the upper aquifer, while D_h increases with leakage in the lower aquifer. In addition, the influence of leakage on D_h is larger than that on D_w. Figure 4 also displays the slopes of the lines in the lower aquifer are smaller than those in the upper aquifer when the leakage ranges from 0.05 to 1.0 /day. Based on equation (2c), the value of water table over-height is independent of the aquifer storativities S_y and S_2. Therefore, only the effect of transmissivity on the head fluctuations in both upper and lower aquifers is discussed herein. The curves of D_w and D_h *versus* x for different transmissivity ratios ($T_r = K_1 D/T_2$) are shown in Fig. 5 for $L = 0.5$ d^{-1}, $t = 24$ h and $\delta = 0.25$. Figure 5 indicates that T_r has a significant effect on D_w and

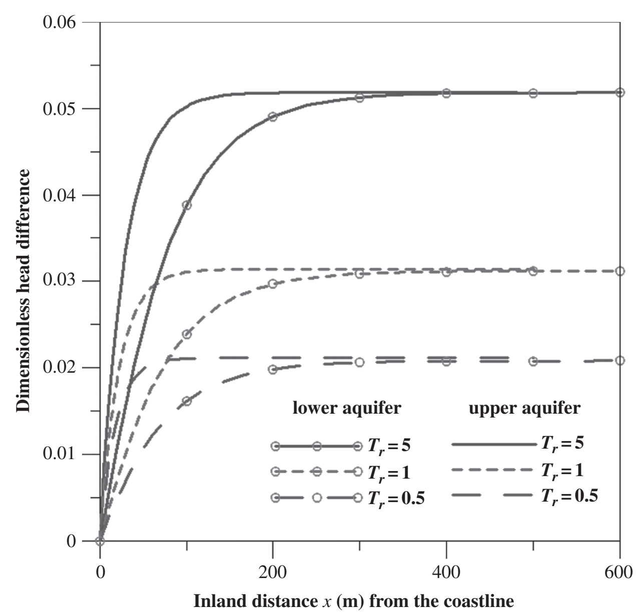

Fig. 5 Dimensionless head differences in the upper and lower aquifers versus inland distance at $t = 24$ h with $L = 0.2$ d^{-1}, $\delta = 0.25$, and different transmissivity ratios (T_r).

D_h. This figure demonstrates that D_h becomes constant when $x > 500$ m for T_r ranging from 0.5 to 5; however, D_w becomes constant when $x > 100$ m for $T_r = 0.5$, $x > 150$ m for $T_r = 1$ and $x > 200$ m for $T_r = 5$. Figure 5 also shows that both D_w and D_h increase with T_r, but the threshold distance is

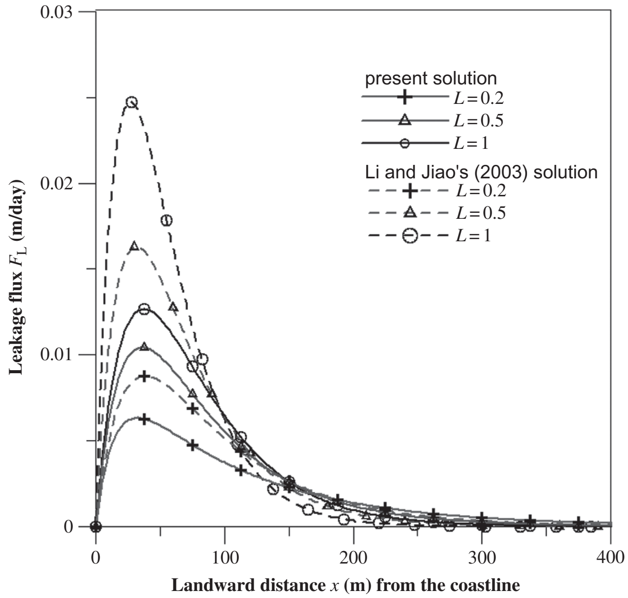

Fig. 6 Leakage flux F_L *versus* landward distance from the coastline.

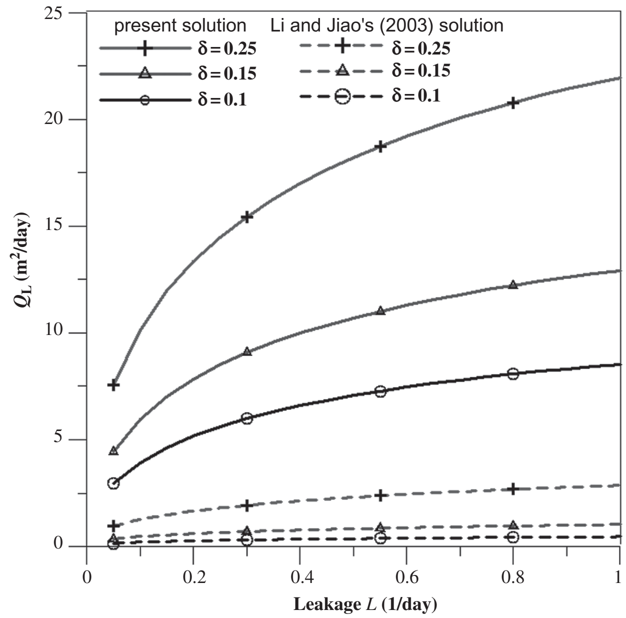

Fig. 7 Submarine groundwater discharge from the lower aquifer, Q_L *versus* leakage L with different values of δ for the present solution and Li and Jiao's (2003) solution.

independent of T_r for the lower aquifer; however, it increases with T_r for the upper aquifer.

4.3 Investigation on the changes of the leakage flux and submarine groundwater discharge

Figure 6 shows the leakage flux, F_L *versus* landward distance, x, with different values of the leakage L for the present solution and Li and Jiao's (2003) solution. This figure depicts that the leakage flux is zero at both aquifers and increases gradually with the landward distance, reaches a maximum value at some distance, and then decreases predicted by both solutions. The maximum leakage flux occurs almost at the same landward distance for all curves. In addition, the present solution predicts lower leakage fluxes than those of Li and Jiao's (2003) solution when $x < 100$ m, implying that the leakage flux may be overestimated if based on the average head fluctuations in the upper and lower aquifers. It is interesting to note that the leakage flux increases with L near the coastal line and decreases with increasing L as the distance increases.

Figure 7 shows the submarine groundwater discharge from the lower aquifer, i.e. Q_L *versus* L with different δ estimated from the present method and Li and Jiao's (2003) method. Those results shown in the figure indicate that the present method gives higher estimations than those of Li and Jiao's (2003) method. This figure also shows that the difference of Q_L values predicted by these two solutions increases with L. In addition, Q_L estimated from both solutions increases with δ and the rate of increase in Q_L also increases with δ.

Figure 8 shows the discharge from the upper aquifer, i.e. Q_U *versus* L with different δ for the present method and Li and Jiao's (2003) method. This figure demonstrates that Q_U decreases with increasing

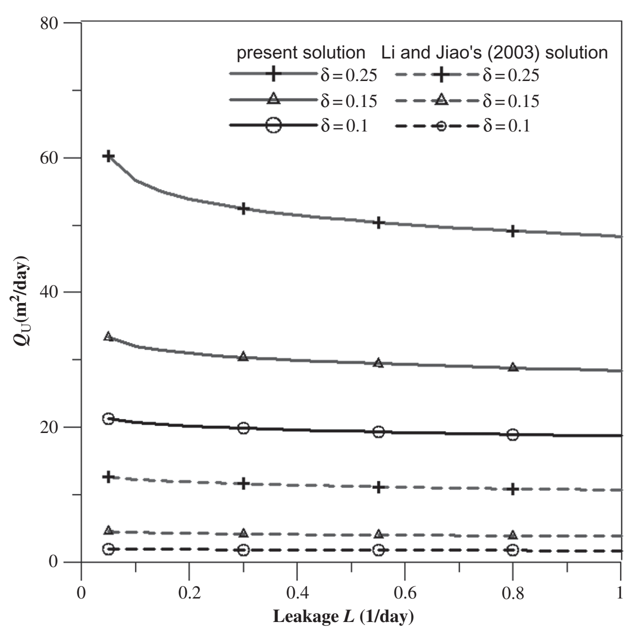

Fig. 8 Submarine groundwater discharge from the upper aquifer, Q_U *versus* leakage L with different values of δ.

L for all δ and increases with δ for all leakage. However, the rate of decrease in Q_U is higher for the present solution than that for Li and Jiao's (2003) solution. At all δ and L, the present solution gives higher predictions for Q_U than Li and Jiao's (2003) solution.

5 CONCLUDING REMARKS

This paper presents a nonlinear mathematical model, with the Boussinesq equation for the upper unconfined aquifer and leaky confined flow equation for the lower aquifer, to simulate head fluctuation for a coastal leaky aquifer system, which contains an upper unconfined aquifer, an aquitard, and a lower semi-confined aquifer. The solution of the model can describe the tide-induced head fluctuation in a coastal leaky aquifer system for the case that the tidal amplitude–aquifer thickness ratio is large. This solution can be considered as a generalization of Jeng *et al.*'s (2002) solution, which was derived from the linearized Boussinesq equation. In both upper and lower aquifers, the head fluctuation predicted by the present method is generally greater than that of Jeng *et al.*'s (2002) solution. The difference between these two solutions increases with both the inland distance and the magnitude of tidal amplitude–aquifer thickness ratio when the distance is smaller than a threshold. However, the difference increases as the leakage decreases in the upper aquifer while the difference increases with leakage in the lower aquifer. The effect of the transmissivity ratio (T_r) on the difference is significant and the differences increase with T_r in both upper and lower aquifers. Both the maximum leakage flux and the submarine groundwater discharge from the lower aquifer may be overestimated if the discharge is calculated based on the difference of average head fluctuation between the upper and lower aquifers. In addition, both submarine groundwater discharges from upper and lower aquifers increase with the tidal amplitude–aquifer thickness ratio. However, the discharge from the upper aquifer decreases with increasing leakage while the one from the lower aquifer increases with leakage.

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APPENDIX A

Derivation of the solutions

Define the dimensionless parameters $\overline{W} = W/D$, $\overline{H} = H/D$, and $\delta = A/D$. Substituting these parameters into equations (1) and (2), the governing equations and boundary conditions can then be obtained as:

$$S_y \frac{\partial \overline{W}}{\partial t} = K_1 D \frac{\partial}{\partial x}\left(\overline{W}\frac{\partial \overline{W}}{\partial x}\right) + L\left(\overline{H} - \overline{W}\right) \quad \text{(A1)}$$

$$S_2 \frac{\partial \overline{H}}{\partial t} = T_2 \frac{\partial^2 \overline{H}}{\partial x^2} + L\left(\overline{W} - \overline{H}\right) \quad \text{(A2)}$$

$$\overline{W}(0,t) = \overline{H}(0,t) = 1 + \delta e^{i\omega t} \quad \text{(A3)}$$

$$\overline{W}(\infty,t) = \overline{H}(\infty,t) = 1 + \frac{1}{D}\left(D + \frac{T_2}{K_1}\right)\left(\sqrt{1+\frac{\alpha_L}{2}} - 1\right) \quad \text{(A4)}$$

The perturbation method is adopted herein to develop the solution of equations (A1) and (A2). The following two assumptions are used to obtain $\overline{W}$ and $\overline{H}$:

$$\overline{W} = \overline{W}_0 + \delta \overline{W}_1 + \delta^2 \overline{W}_2 + O\left(\delta^3\right) \quad \text{(A5)}$$

$$\overline{H} = \overline{H}_0 + \delta \overline{H}_1 + \delta^2 \overline{H}_2 + O\left(\delta^3\right) \quad \text{(A6)}$$

where $\delta = A/D$ is the perturbation parameter. Substituting both equations (A5) and (A6) into equations (A1) and (A2) results in following equations for $\left(\delta^0\right)$:

$$S_y \frac{\partial \overline{W}_0}{\partial t} = K_1 D \overline{W}_0 \frac{\partial^2 \overline{W}_0}{\partial x^2} + K_1 D \left(\frac{\partial \overline{W}_0}{\partial x}\right)^2 + L\left(\overline{H}_0 - \overline{W}_0\right) \quad \text{(A7)}$$

$$S_2 \frac{\partial \overline{H}_0}{\partial t} = T_2 \frac{\partial^2 \overline{H}_0}{\partial x^2} + L\left(\overline{W}_0 - \overline{H}_0\right) \quad \text{(A8)}$$

$$\overline{W}_0(0,t) = \overline{H}_0(0,t) = \overline{W}_0(\infty,t) = \overline{H}_0(\infty,t) = 1 \quad \text{(A9)}$$

Accordingly, we get the results of $\overline{W}_0 = \overline{H}_0 = 1$. For $\left(\delta^1\right)$ the governing equation and associated boundary conditions are:

$$S_y \frac{\partial \overline{W}_1}{\partial t} = K_1 D \frac{\partial^2 \overline{W}_1}{\partial x^2} + L\left(\overline{H}_1 - \overline{W}_1\right) \quad \text{(A10)}$$

$$S_2 \frac{\partial \overline{H}_1}{\partial t} = T_2 \frac{\partial^2 \overline{H}_1}{\partial x^2} + L\left(\overline{W}_1 - \overline{H}_1\right) \quad \text{(A11)}$$

$$\overline{W}_1(0,t) = \overline{H}_1(0,t) = \delta\, e^{i\omega t} \quad \text{(A12)}$$

$$\overline{W}_1(\infty,t) = \overline{H}_1(\infty,t) = 0 \quad \text{(A13)}$$

The solution for equations (A10)–(A13) is obtained using the method of separation of variables (Jeng *et al.* 2002); the results are expressed in equations (4a) and (4b). For $\left(\delta^2\right)$ the related governing equation and its associated boundary conditions are:

$$S_y \frac{\partial \overline{W}_2}{\partial t} = K_1 D \frac{\partial^2 \overline{W}_2}{\partial x^2} + K_1 D \left[\overline{W}_1 \frac{\partial^2 \overline{W}_1}{\partial x^2} + \left(\frac{\partial \overline{W}_1}{\partial x} \right)^2 \right] + L\left(\overline{H}_2 - \overline{W}_2\right) \quad \text{(A14)}$$

$$S_2 \frac{\partial \overline{H}_2}{\partial t} = T_2 \frac{\partial^2 \overline{H}_2}{\partial x^2} + L\left(\overline{W}_2 - \overline{H}_2\right) \quad \text{(A15)}$$

$$\overline{W}_2(0,t) = \overline{H}_2(0,t) = 0 \quad \text{(A16)}$$

$$\overline{W}_2(\infty,t) = \overline{H}_2(\infty,t) = B \quad \text{(A17)}$$

Based on the method of separation of variables, the variables $\overline{W}_2$ and $\overline{H}_2$ are assumed as:

$$\overline{W}_2 = F_1(x)\, e^{2i\omega t} + F_2(x) \quad \text{(A18)}$$

$$\overline{H}_2 = G_1(x)\, e^{2i\omega t} + G_2(x) \quad \text{(A19)}$$

Substituting equations (A18)–(A23) into equations (A14) and (A15) and using boundary condition, equations (A16) and (A17), leads to the following results:

$$F_1(x) = c_1 e^{-2\lambda_1 x} + c_2 e^{-2\lambda_2 x} + c_3 e^{-\lambda_3 x} + c_4 e^{\tau_1 x} + c_5 e^{\tau_2 x} \quad \text{(A20)}$$

$$F_2(x) = c_6 e^{-\alpha x} + c_7 e^{-\beta x} + B \quad \text{(A21)}$$

$$G_1 = d_1 e^{-2\lambda_1 x} + d_2 e^{-2\lambda_2 x} + d_3 e^{-\lambda_3 x} + d_4 e^{\tau_1 x} + d_5 e^{\tau_2 x} \quad \text{(A22)}$$

$$G_2(x) = d_6 e^{-\alpha x} + d_7 e^{-\beta x} + B \quad \text{(A23)}$$

where coefficients c_1 to c_7, d_1 to d_7, λ_1, λ_2, λ_3, τ_1, τ_2, α, β and B are defined as equations (5e)–(5z), respectively, in the text.