Ramanujan J (2012) 28:25–43
DOI 10.1007/s11139-011-9341-y

A basis for $S_k(\Gamma_0(4))$ and representations of integers as sums of squares

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Received: 11 February 2011 / Accepted: 18 August 2011 / Published online: 11 February 2012

Abstract In this paper, we find a basis for the space $S_k(\Gamma_0(4))$ of cusp forms of even weight k for the congruence subgroup $\Gamma_0(4)$ in terms of Eisenstein series. As an application, we obtain formulas for $r_{4s}(n)$, the number of ways to represent a nonnegative integer n as a sum of $4s$ integral squares.

Keywords Modular forms (one variable) · Period polynomials · Fourier coefficients of modular forms · Sum of squares

Mathematics Subject Classification (2000) Primary 11F11 · Secondary 11F30 · 11F67

1 Introduction and statements of results

Throughout the paper, we assume that k is an even positive integer. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and let $M_k(\Gamma)$ and $S_k(\Gamma)$ be the space of modular forms and the space of cusp forms of weight k on Γ, respectively.

The first author was partially supported by Grant-in-Aid for Scientific Research (No. 22540096), Japan Society for the Promotion of Science. The second author was partially supported by Grant 99-2115-M-009-011-MY3 of the National Science Council, Taiwan (ROC). Part of the work was done while the second author visited the first author at Tsuda College. He would like to thank Tsuda College for the great hospitality.

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For $S_k(\Gamma_0(2))$, we have given in [4] a basis $\{E_{2j}^{i\infty} E_{k-2j}^0 \mid j = 2, \dots, d+1\}$, where $E_n^{i\infty}$ and E_n^0 are normalized Eisenstein series of weight n for the cusps $i\infty$ and 0, respectively, and $d = \dim S_k(\Gamma_0(2)) = \lfloor k/4 \rfloor - 1$. The existence of such a basis was suggested in [1, 6]. In this paper, we will find a basis for $S_k(\Gamma_0(4))$. The main motivation is to obtain formulas for $r_s(n)$, the number of ways to represent a non-negative integer n as a sum of s integral squares.

To state our main results, let us first recall that the group $\Gamma_0(4)$ has 3 cusps, represented by $i\infty$, 0, and $1/2$. To each cusp α and each even integer $k \ge 4$, we may associate an Eisenstein series

$$\mathcal{E}_k^\alpha(\tau) := \sum_{d/c \sim \alpha} \frac{1}{(c\tau - d)^k},$$

where the sum runs over all cusps d/c equivalent to α under $\Gamma_0(4)$. More explicitly, we have

$$\begin{aligned} \mathcal{E}_k^{i\infty}(\tau) &= \frac{1}{2} \sum_{(c,d)=1, 4|c} \frac{1}{(c\tau - d)^k} = \frac{1}{2^k - 1}\big(2^k E_k(4\tau) - E_k(2\tau)\big), \\ \mathcal{E}_k^0(\tau) &= \frac{1}{2} \sum_{(c,d)=(c,4)=1} \frac{1}{(c\tau - d)^k} = \frac{2^k}{2^k - 1}\big(E_k(\tau) - E_k(2\tau)\big), \\ \mathcal{E}_k^{1/2}(\tau) &= \frac{1}{2} \sum_{(c,d)=1, 2|c, 4\nmid c} \frac{1}{(c\tau - d)^k} \\ &= \frac{1}{2^k - 1}\big(-E_k(\tau) + \big(2^k + 1\big)E_k(2\tau) - 2^k E_k(4\tau)\big), \end{aligned} \tag{1.1}$$

where

$$E_k(\tau) = 1 + \frac{(2\pi i)^k}{\Gamma(k)\zeta(k)} \sum_{n=1}^\infty \sigma_{k-1}(n) q^n = 1 - \frac{2k}{B_k} \sum_{n=1}^\infty \sigma_{k-1}(n) q^n, \qquad q = e^{2\pi i \tau},$$

is the Eisenstein series of weight k on $\mathrm{SL}_2(\mathbb{Z})$ and B_k is the kth Bernoulli number; see Lemma 3.2 of [10] for calculation of Fourier expansions of the Eisenstein series. In fact, the series $S_k(0,1)$, $S_k(1,0)$, and $S_k(1,1)$ in Lemma 3.2 of [10] are essentially our $E_k^{i\infty}$, E_k^0, and $E_k^{1/2}$ here, respectively. This is because $\Gamma_0(4)$ is conjugate to $\Gamma(2)$ by $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. These Eisenstein series have the property that

$$\lim_{\tau \to i\infty} \big(\mathcal{E}_k^\alpha\big|_k \gamma\big)(\tau) = \begin{cases} 1 & \text{if } a/c \sim \alpha, \\ 0 & \text{if } a/c \not\sim \alpha, \end{cases}$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. In particular, if $\alpha \not\sim \beta$, then $E_{k_1}^\alpha E_{k_2}^\beta$ is a cusp form of weight $k_1 + k_2$ on $\Gamma_0(4)$.

In order to simplify the expressions in the statements of our theorems, we rescale the Eisenstein series and define

$$E_k^{i\infty}(\tau) = \mathcal{E}_k^{i\infty}(\tau), \qquad E_k^0(\tau) = \big(E_k^{i\infty}\big|_k W_4\big)(\tau) = 2^{-k}\mathcal{E}_k^0(\tau), \tag{1.2}$$

where W_4 denotes the Atkin–Lehner involution on $M_k(\Gamma_0(4))$. In addition, for $k=2$, we also define the Eisenstein series $E_2^{\alpha}(\tau)$ using the Fourier expansions given in (1.1). These Eisenstein series $E_2^{\alpha}(\tau)$ are not modular forms, but using the transformation property of $E_2(\tau)$, it can be easily verified that for any modular form f of weight k on $\Gamma_0(4)$, the functions

$$E_2^{i\infty}(\tau)f(\tau)-\frac{1}{\pi i k}f'(\tau),\qquad E_2^{0}(\tau)f(\tau)-\frac{1}{\pi i k}f'(\tau)$$

are modular forms of weight $k+2$ on $\Gamma_0(4)$. (See (3.6) and (3.7) below.) Now we can give our basis for $S_k(\Gamma_0(4))$.

Theorem 1.1 *Let $k\geq 6$ be an even integer. Then the sets*

$$\left\{E_2^{i\infty}E_{k-2}^{0}-\frac{1}{\pi i(k-2)}E_{k-2}^{0}{}'\right\}\cup\left\{E_n^{i\infty}E_{k-n}^{0}\mid n=4,6,\ldots,k-4\right\}$$

and

$$\left\{E_2^{0}E_{k-2}^{i\infty}-\frac{1}{\pi i(k-2)}E_{k-2}^{i\infty}{}'\right\}\cup\left\{E_n^{0}E_{k-n}^{i\infty}\mid n=4,6,\ldots,k-4\right\}$$

are both bases for $S_k(\Gamma_0(4))$.

As mentioned earlier, our motivation to study $S_k(\Gamma_0(4))$ is to obtain exact formulas for $r_s(n)$, the number of ways to represent a nonnegative integer n as a sum of s integral squares. (See [2, 5] for surveys of the long and rich history of this problem.) To see the connection between $S_k(\Gamma_0(4))$ and $r_s(n)$, let us recall that the generating function for $r_s(n)$ is

$$\Theta(\tau)^s=\left(\sum_{n\in\mathbb{Z}}q^{n^2}\right)^s,\qquad q=e^{2\pi i\tau}.$$

When s is even, we have

$$\Theta(\gamma\tau)^s=\left(\frac{-1}{d}\right)^{s/2}(c\tau+d)^{s/2}\Theta(\tau)^s$$

for all $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(4)$, where $(\frac{-1}{d})$ is the Legendre symbol. Thus, for a positive integer s, the function $\Theta(\tau)^{4s}$ is a linear combination of Eisenstein series $E_{2s}^{i\infty}(\tau)$, $E_{2s}^{0}(\tau)$, $E_{2s}^{1/2}(\tau)$, and the functions in Theorem 1.1. In fact, we can do a little better.

The theta function $\Theta(\tau)$ satisfies

$$\Theta\left(-\frac{1}{4\tau}\right)=\sqrt{\frac{2\tau}{i}}\,\Theta(\tau).$$

It follows that

$$\Theta^{4s}|_{2s}W_4=(-1)^s\Theta^{4s}.$$

In other words, $\Theta^{4s} \in M_{2s}(\Gamma_0(4), (-1)^s)$, the $(-1)^s$-Atkin–Lehner eigensubspace of $M_{2s}(\Gamma_0(4))$. Moreover, $\Theta(\tau)$ vanishes at the cusp $1/2$. (This is because $\Theta(\tau)$ has the infinite product representation $\eta(2\tau)^5/\eta(\tau)^2\eta(4\tau)^2$. Thus, the zeros of $\Theta(\tau)$ must be at cusps. From the above transformation, we conclude that $\Theta(\tau)$ must vanish at $1/2$.) Therefore, we have

$$\Theta(\tau)^{4s} \in \mathbb{C}\big(E_{2s}^{i\infty}(\tau) + (-1)^s E_{2s}^0(\tau)\big) \oplus S_{2s}\big(\Gamma_0(4), (-1)^s\big).$$

Now we have

$$\dim S_k\big(\Gamma_0(4), +\big) = \left\lfloor \frac{k}{4} \right\rfloor - 1, \qquad \dim S_k\big(\Gamma_0(4), -\big) = \frac{k}{2} - \left\lfloor \frac{k}{4} \right\rfloor - 1.$$

From the dimension formulas and Theorem 1.1, we easily obtain bases for $S_k(\Gamma_0(4), \pm 1)$.

Corollary 1.2 *If $k \geq 8$ is an even integer, then*

$$\big\{E_n^{i\infty} E_{k-n}^0 + E_n^0 E_{k-n}^{i\infty} \mid n = 4, 6, \ldots, 2\lfloor k/4 \rfloor\big\}$$

is a basis for $S_k(\Gamma_0(4), +)$. In particular, if $k \equiv 0 \bmod 4$, then $\Theta(\tau)^{2k}$ is a linear combination of $E_k^{i\infty}(\tau) + E_k^0(\tau)$ and the functions above.

Corollary 1.3 *If $k \geq 6$ is an even integer, then*

$$\left\{E_2^{i\infty} E_{k-2}^0 - E_2^0 E_{k-2}^{i\infty} - \frac{1}{\pi i(k-2)}\big({E_{k-2}^0}' - {E_{k-2}^{i\infty}}'\big)\right\}$$
$$\cup \big\{E_n^{i\infty} E_{k-n}^0 - E_n^0 E_{k-n}^{i\infty} \mid n = 4, 6, \ldots, k - 2\lfloor k/4 \rfloor - 2\big\}$$

is a basis for $S_k(\Gamma_0(4), -)$. In particular, if $k \equiv 2 \bmod 4$, then $\Theta(\tau)^{2k}$ is a linear combination of $E_k^{i\infty}(\tau) - E_k^0(\tau)$ and the functions above.

We remark that since $\Gamma_0^+(4)$ is conjugate to $\Gamma_0(2)$ by $\left(\begin{smallmatrix} 1 & 1/2 \\ 0 & 1 \end{smallmatrix}\right)$, we can first obtain a basis for $S_k(\Gamma_0(2))$ and apply $\tau \mapsto \tau + 1/2$ to the basis to get a basis for $S_k(\Gamma_0(4), +)$, and consequently exact formulas for $r_s(n)$. This is the approach adopted in [6]. However, this method only work for the cases $8|s$. Also, the basis for $S_k(\Gamma_0(4), +)$ obtained in this way is different from the basis in Corollary 1.2.

Another result of similar nature is given by K. Kilger. In his Ph.D. thesis [7], K. Kilger obtained bases for $S_k(\Gamma_0(N))$, $N = 1, \ldots, 4$, using modular symbols. His bases in the case $N = 4$ are similar, but different from ours.

Example 1.4 Here we give some formulas for $r_{4s}(n)$. In the following, we let

$$f_{s,0} := E_{2s}^{i\infty} + (-1)^s E_{2s}^0,$$

$$f_{s,2} := E_2^{i\infty} E_{2s-2}^0 + (-1)^s E_2^0 E_{2s-2}^{i\infty} - \frac{1}{\pi i(2s-2)}\big({E_{2s-2}^0}' + (-1)^s {E_{2s-2}^{i\infty}}'\big),$$

$$f_{s,n} := E_n^{i\infty} E_{2s-n}^0 + (-1)^s E_n^0 E_{2s-n}^{i\infty} \quad (n \geq 4,\ n \text{ even}).$$

By comparing suitably many Fourier coefficients, we find

$$\begin{aligned}
\Theta^8 &= f_{2,0},\\
\Theta^{12} &= f_{3,0} + f_{3,2},\\
\Theta^{16} &= f_{4,0} + \frac{17}{16} f_{4,4},\\
\Theta^{20} &= f_{5,0} + \frac{17}{31} f_{5,2} - \frac{134}{93} f_{5,4},\\
\Theta^{24} &= f_{6,0} + \frac{43928}{18657} f_{6,4} - \frac{6848}{18657} f_{6,6},\\
\Theta^{28} &= f_{7,0} + \frac{2073}{5461} f_{7,2} - \frac{1561873}{737235} f_{7,4} + \frac{460309}{245745} f_{7,6},\\
\Theta^{32} &= f_{8,0} + \frac{11379631232}{4392213525} f_{8,4} - \frac{13142016}{6506983} f_{8,6} + \frac{967923424}{627459075} f_{8,8},\\
\Theta^{36} &= f_{9,0} + \frac{929569}{3202291} f_{9,2} - \frac{2997123429668}{1165073523075} f_{9,4} + \frac{817033178804}{317747324475} f_{9,6}\\
&\quad - \frac{130045826398}{35305258275} f_{9,8}.
\end{aligned}$$

We now indicate how Theorem 1.1 is proved. We shall see that Theorem 1.1 is, in fact, a consequence of linear independence of certain period polynomials of cusp forms on $S_k(\Gamma_0(4))$.

For convenience, let us set $w = k - 2$. Assume that N is an integer with $N > 1$. For a cusp form $f \in S_{w+2}(\Gamma_0(N))$ and an integer n with $0 \leq n \leq w$, we let

$$r_n(f) := \int_0^{i\infty} f(z) z^n \, dz \tag{1.3}$$

be the nth *period* of f. Since $r_n : S_{w+2}(\Gamma_0(N)) \to \mathbb{C}$ is a linear functional, there exists a unique cusp form $R_{\Gamma_0(N),w,n}(z) \in S_{w+2}(\Gamma_0(N))$ such that

$$r_n(f) = c_w(f, R_{\Gamma_0(N),w,n}), \quad c_w := 2^{-1}(2i)^{w+1} \tag{1.4}$$

for all cusp forms f of the same weight on $\Gamma_0(N)$. Here

$$(f, g) := \iint_{\Gamma_0(N)\backslash\mathbb{H}} f(z)\overline{g(z)} y^w \, dx \, dy, \quad z = x + iy, \tag{1.5}$$

denotes the Petersson inner product of f and g. We now explain the relation between $R_{\Gamma_0(4),w,n}$ and $E_n^{i\infty} E_{k-n}^0$.

Using Rankin's method [12] and following the argument in the proof of Proposition 2 of [6], we can show that if f is a newform of weight k on $\Gamma_0(4)$, then for even integers $n > k/2$, we have

$$\left(f, E_n^{i\infty} E_{k-n}^0\right) = c_{k,n} L(f, k-1) L(f, n),$$

where $L(f, s)$ denotes the L-function associated to f and $c_{k,n}$ is a constant depending on k and n. (See Proposition 3.1 below.) For oldforms from $S_k(\mathrm{SL}_2(\mathbb{Z}))$ and $S_k(\Gamma_0(2))$, there are also similar formulas. On the other hand, from the definitions (1.3) and (1.4) of r_n and $R_{\Gamma_0(4),w,n}$, it is easy to see that

$$(f, R_{\Gamma_0(4),w,n}) = c'_{k,n} L(f, n+1)$$

for some constant $c'_{k,n}$ independent of f. Therefore, even though $E_n^{i\infty} E_{k-n}^0$ is not precisely a multiple of $R_{\Gamma_0(4),w,n-1}$, we can still deduce linear independence among $E_n^{i\infty} E_{k-n}^0$ from that among $R_{\Gamma_0(4),w,n}$.

To obtain linear independence among $R_{\Gamma_0(4),w,n}$, we consider *period polynomials* $r(f)$ which for cusp forms $f \in S_k(\Gamma_0(N))$ for general N are defined by

$$r(f)(X) := \int_0^{i\infty} f(z)(X-z)^w \, dz.$$

Furthermore, even and odd period polynomials $r^+(f)$ and $r^-(f)$ are defined by

$$r^{\pm}(f)(X) := \frac{1}{2}\left\{r(f)(X) \pm r(f)(-X)\right\}.$$

The period polynomials for $R_{\Gamma_0(N),w,n}$ are computed in [4] and will be crucial in our proof of Theorem 1.1. To state the formula, we let $B_m(x)$ (resp., B_m) denote the mth Bernoulli polynomial (resp., number). By $B_m^0(x)$, we denote the mth Bernoulli polynomial without its B_1-term (see [9, page 208]):

$$B_m^0(x) := \sum_{\substack{0 \le i \le m \\ i \ne 1}} \binom{m}{i} B_i x^{m-i} = \sum_{\substack{0 \le i \le m \\ i \text{ even}}} \binom{m}{i} B_i x^{m-i}.$$

For an integer n with $0 < n < w$, let

$$\tilde{n} = w - n$$

and define a polynomial $S_{N,w,n}$ in X by

$$S_{N,w,n}(X) := \frac{N^{\tilde{n}} X^w}{\tilde{n}+1} B_{\tilde{n}+1}^0\left(\frac{1}{NX}\right) - \frac{1}{n+1} B_{n+1}^0(X).$$

Then the period polynomials $r^{\pm}(R_{\Gamma_0(N),w,n})$ are given as follows [4].

Theorem 1.5 [4, Theorem 1.1] *Let N be an integer greater than* 1. *For an even integer n with $0 < n < w$, we have*

$$r^-(R_{\Gamma_0(N),w,n})(X) = S_{N,w,n}(X).$$

Also, for an odd integer n with $0 < n < w$, *we have*

$$\begin{aligned} r^+(R_{\Gamma_0(N),w,n})(X) \\ &= S_{N,w,n}(X) \\ &\quad - \frac{(w+2)B_{n+1}B_{\tilde{n}+1}}{(n+1)(\tilde{n}+1)B_{w+2}}\left(\frac{X^w}{N}\prod_{p|N}\frac{1-p^{-(n+1)}}{1-p^{-(w+2)}} - \frac{1}{N^{n+1}}\prod_{p|N}\frac{1-p^{-(\tilde{n}+1)}}{1-p^{-(w+2)}}\right), \end{aligned}$$

where p *runs over all prime divisors of* N.

In the sequel, we focus on the case $N = 4$. Furthermore, we consider only vector spaces over $\mathbb{C}$, and linear independence means that of over $\mathbb{C}$. First, we will prove the following theorem:

Theorem 1.6 *The polynomials*

$$S_{4,w,n}(X) \quad (n = 2, 4, \ldots, w-2)$$

are linearly independent.

Note that an analogous result for $\Gamma_0(2)$ was obtained in [4], where explicit evaluation of Hankel determinants formed by Bernoulli numbers is the main ingredient. Here the key to our proof of Theorem 1.6 is the 2-adic ordinal of the coefficients of $S_{4,w,n}(X)$. The method used here is not applicable to the case $\Gamma_0(2)$. (This is due to the fact that $\mathrm{ord}_2(4) = 2$, but $\mathrm{ord}_2(2) = 1$.)

By the similar argument as for proving Theorem 1.6, we can derive the following result.

Theorem 1.7

(1)

$$\{R_{\Gamma_0(4),w,n} \mid n = 1, 3, \ldots, w-3\}$$

form a basis for $S_{w+2}(\Gamma_0(4))$.

(2)

$$\{R_{\Gamma_0(4),w,n} \mid n = 2, 4, \ldots, w-2\}$$

form a basis for $S_{w+2}(\Gamma_0(4))$.

Remark 1.8 We now recall the formula

$$R_{\Gamma_0(N),w,n}|_{w+2}W_N = (-1)^{n+1}N^{w/2-n}R_{\Gamma_0(N),w,\tilde{n}}$$

in [4, page 330] for the Atkin–Lehner involution W_N. In Theorem 1.7(1), the basis can be replaced by

$$\{R_{\Gamma_0(4),w,n} \mid n = 3, 5, \ldots, w-1\},$$

deleting $n = 1$ and adding $n = w - 1$. These correspond to each other by the Atkin–Lehner involution.

Now, by Theorem 1.7, we know that $f = 0$ if $(f, R_{\Gamma_0(4),w,n}) = 0$ for all $n = 1, 3, \ldots, w-3$ (or $n = 2, 4, \ldots, w-2$, respectively). This leads us to the following $\Gamma_0(4)$-version of the Eichler–Shimura–Manin theorem (see [3, 9, 11, 13]).

Corollary 1.9 *Let* $f \in S_{w+2}(\Gamma_0(4))$.

(1) *If* $r_1(f) = r_3(f) = \cdots = r_{w-3}(f) = 0$, *then* $f = 0$.
(2) *If* $r_2(f) = r_4(f) = \cdots = r_{w-2}(f) = 0$, *then* $f = 0$.

The proof of Theorems 1.6 and 1.7 will be given in Sect. 2. Then in Sect. 3, we will deduce Theorem 1.1 from Theorem 1.7.

2 Proofs of Theorems 1.6 and 1.7

In this section, we give proofs for Theorems 1.6 and 1.7. First, we recall 2-adic ordinal of a rational number.

Definition 2.1 For a rational number x, let us express x as

$$x = 2^a \frac{q}{p},$$

where a, p, q are integers such that $(p, q) = 1$ and p, q are odd. Then the 2-adic ordinal $\mathrm{ord}_2(x)$ of x is defined by

$$\mathrm{ord}_2(x) := a.$$

We need the following elementary properties of 2-adic ordinal.

Lemma 2.2 *For* $x, y \in \mathbb{Q}$, *it holds that*

$$\mathrm{ord}_2(xy) = \mathrm{ord}_2(x) + \mathrm{ord}_2(y), \tag{2.1}$$

$$\mathrm{ord}_2(x+y) = \mathrm{ord}_2(x), \qquad \textit{if } \mathrm{ord}_2(x) < \mathrm{ord}_2(y), \tag{2.2}$$

$$\mathrm{ord}_2(x+y) \geq \mathrm{ord}_2(x) + 1, \qquad \textit{if } \mathrm{ord}_2(x) = \mathrm{ord}_2(y), \tag{2.3}$$

$$\mathrm{ord}_2(B_{2n}) = -1, \qquad \textit{if } n \geq 1. \tag{2.4}$$

Proof Proofs of (2.1), (2.2) and (2.3) are straightforward and we omit them. We note that (2.4) follows from the well-known Clausen–von Staudt Theorem on the Bernoulli numbers (see, e.g., [8]). □

Here we recall the polynomial $S_{4,w,n}(X)$ for an integer n with $0 < n < w$:

$$S_{4,w,n}(X) = \frac{4^{\tilde{n}} X^w}{\tilde{n}+1} B^0_{\tilde{n}+1}\left(\frac{1}{4X}\right) - \frac{1}{n+1} B^0_{n+1}(X).$$

We set

$$a_{ij} := \text{the coefficient of } X^{2j-1} \text{ in } S_{4,w,2i}(X) \quad (i, j = 1, 2, \dots, w/2-1).$$

We will show in Lemma 2.4 that

$$\det_{\substack{1\le i\le w/2-1\\ 1\le j\le w/2-1}} [a_{ij}] \neq 0. \tag{2.5}$$

To do so, we need the following lemma:

Lemma 2.3 *The* 2*-adic ordinal* $\mathrm{ord}_2(a_{ij})$ *of* a_{ij} *satisfies the following*:

$$\begin{aligned}
&\mathrm{ord}_2(a_{i,i}) = -2, && \textit{for } i = 1, 2, \dots, w/2-1,\\
&\mathrm{ord}_2(a_{i,i+1}) = 0, && \textit{for } i = 1, 2, \dots, w/2-2,\\
&\mathrm{ord}_2(a_{i,i+k}) \ge 4(k-1)+1, && \textit{for } i = 1, 2, \dots, w/2-1;\ k = 2, 3, \dots, w/2-1-i,\\
&\mathrm{ord}_2(a_{i,j}) \ge -1, && \textit{for } j < i.
\end{aligned}$$

Proof We expand $S_{4,w,2i}(X)$ as

$$\begin{aligned}
S_{4,w,2i}(X) &= \frac{4^{w-2i}X^w}{w-2i+1} B^0_{w-2i+1}\left(\frac{1}{4X}\right) - \frac{1}{2i+1} B^0_{2i+1}(X)\\
&= \frac{1}{w-2i+1} \sum_{\ell=0,\ \ell \text{ even}}^{w-2i+1} 4^{\ell-1} \binom{w-2i+1}{\ell} B_\ell X^{2i-1+\ell}\\
&\quad - \frac{1}{2i+1} \sum_{\ell=0,\ \ell \text{ even}}^{2i+1} \binom{2i+1}{\ell} B_\ell X^{2i+1-\ell}\\
&= \frac{1}{w-2i+1} \sum_{j=i}^{w/2} 4^{2j-2i-1} \binom{w-2i+1}{2j-2i} B_{2j-2i} X^{2j-1}\\
&\quad - \frac{1}{2i+1} \sum_{j=1}^{i+1} \binom{2i+1}{2i-2j+2} B_{2i-2j+2} X^{2j-1}.
\end{aligned}$$

Then we know

$$a_{ii} = \frac{1}{w-2i+1} 4^{-1} \binom{w-2i+1}{0} B_0 - \frac{1}{2i+1} \binom{2i+1}{2} B_2,$$

and we have $\mathrm{ord}_2(a_{ii}) = \mathrm{ord}_2(4^{-1}) = -2$. We also know

$$a_{ii+1} = \frac{1}{w-2i+1} 4^1 \binom{w-2i+1}{2} B_2 - \frac{1}{2i+1} \binom{2i+1}{0} B_0,$$

and we have $\mathrm{ord}_2(a_{ii+1}) = \mathrm{ord}_2(-1/(2i+1)) = 0$.

Now, for a_{ii+k} and a_{ij} $(j < i)$, we see

$$a_{ii+k} = \frac{1}{w-2i+1} 4^{2k-1} \binom{w-2i+1}{2k} B_{2k},$$
$$a_{ij} = -\frac{1}{2i+1} \binom{2i+1}{2i-2j+2} B_{2i-2j+2}.$$

Hence we have $\mathrm{ord}_2(a_{ii+k}) \geq \mathrm{ord}_2(4^{2k-1}/2) = 4k-3$ for $k = 2, 3, \ldots, w/2-1-i$, and $\mathrm{ord}_2(a_{ij}) \geq \mathrm{ord}_2(B_{2i-2j+2}) = -1$ for $j < i$.

This completes the proof. □

The following lemma is crucial in our proofs of Theorems 1.6 and 1.7.

Lemma 2.4 *Set*

$$D = \det_{\substack{1 \leq i \leq w/2-1 \\ 1 \leq j \leq w/2-1}} [a_{ij}].$$

Then

$$\mathrm{ord}_2(D) = -w + 2.$$

In particular, we have

$$\det_{\substack{1 \leq i \leq w/2-1 \\ 1 \leq j \leq w/2-1}} [a_{ij}] \neq 0.$$

Proof Let us set $d = w/2 - 1$, and let id denote the identity element of the symmetric group S_d of degree d.

From Lemma 2.3, we know that

$$\mathrm{ord}_2(a_{ii}) = -2 \quad \text{and} \quad \mathrm{ord}_2(a_{ij}) \geq -1 \quad \text{if } i \neq j.$$

Therefore, for an element σ in S_d, we have

$$\mathrm{ord}_2(a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{d\sigma(d)}) = -2d = -w + 2 \quad \text{if } \sigma = \mathrm{id},$$
$$\mathrm{ord}_2(a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{d\sigma(d)}) \geq -w + 1 \quad \text{if } \sigma \neq \mathrm{id}.$$

Noting that the determinant D is given by

$$D = \sum_{\sigma} \varepsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{d\sigma(d)},$$

where the sum runs over all elements in the permutation group S_d, and $\varepsilon(\sigma)$ denotes $+1$ or -1 according to whether the permutation σ is even or odd, we have

$$\mathrm{ord}_2(D) = \mathrm{ord}_2(a_{11} a_{22} \cdots a_{dd}) = -w + 2.$$

This proves the lemma. □

Now we are ready to give proofs of Theorems 1.6 and 1.7.

Proofs of Theorems 1.6 and 1.7 In Lemma 2.4, we proved that

$$\det_{\substack{1\le i\le w/2-1\\ 1\le j\le w/2-1}}[a_{ij}]\neq 0. \tag{2.6}$$

Since

$$a_{ij} = \text{the coefficient of } X^{2j-1} \text{ in } S_{4,w,2i}(X) \quad (i,j=1,2,\ldots,w/2-1),$$

the inequality (2.6) shows that $S_{4,w,2i}$ $(i=1,2,\ldots,w/2-1)$ are linearly independent. This implies Theorem 1.6.

Next we note that

$$\begin{aligned} a_{ij} &= \text{the coefficient of } X^{2j-1} \text{ in } S_{4,w,2i}(X)\\ &= \text{the coefficient of } X^{2j-1} \text{ in } r^-(R_{\Gamma_0(4),w,2i})(X)\\ &= -\binom{w}{2j-1} r_{w-2j+1}(R_{\Gamma_0(4),w,2i})\\ &= -\binom{w}{2j-1} c_w(R_{\Gamma_0(4),w,2i}, R_{\Gamma_0(4),w,w-2j+1}). \end{aligned}$$

Then, from (2.6), we have

$$\prod_{j=1}^{w/2-1}\left(-\binom{w}{2j-1}c_w\right)\det_{\substack{1\le i\le w/2-1\\ 1\le j\le w/2-1}}\left[(R_{\Gamma_0(4),w,2i}, R_{\Gamma_0(4),w,w-2j+1})\right]\neq 0.$$

From this, it follows that

$$\det_{\substack{1\le i\le w/2-1\\ 1\le j\le w/2-1}}\left[(R_{\Gamma_0(4),w,2i}, R_{\Gamma_0(4),w,w-2j+1})\right]\neq 0. \tag{2.7}$$

This implies that $R_{\Gamma_0(4),w,2i}$, $i=1,2,\ldots,w/2-1$, are linearly independent, and so are $R_{\Gamma_0(4),w,w-2j+1}$, $j=1,2,\ldots,w/2-1$. Now taking into account the dimension of $S_{w+2}(\Gamma_0(4))$, we conclude that both $\{R_{\Gamma_0(4),w,n} \mid n=2,4,\ldots,w-2\}$ and $\{R_{\Gamma_0(4),w,n} \mid n=3,5,\ldots,w-1\}$ are bases of $S_{w+2}(\Gamma_0(4))$. By applying the Atkin–Lehner involution, we know that $\{R_{\Gamma_0(4),w,n} \mid n=1,3,\ldots,w-3\}$ also form a basis for $S_{w+2}(\Gamma_0(4))$. This completes the proof of Theorem 1.7. □

3 Proof of Theorem 1.1 and Corollaries 1.2 and 1.3

In the following proposition, a *newform* in $S_k(\Gamma_0(N))$ means a normalized Hecke eigenform in the newform subspace of $S_k(\Gamma_0(N))$. Also, the Petersson inner product of two cusp forms f and g in $S_k(\Gamma_0(4))$ is defined as (1.5).

Proposition 3.1 Analogue of [6, Proposition 2] *Let $k \geq 6$ be an even integer. For an integer ℓ with $2 \leq \ell \leq k/2-2$, let $E_{2\ell}^{0}$ and $E_{k-2\ell}^{i\infty}$ be the Eisenstein series defined in* (1.2), *and set*

$$c_{k,\ell} = \frac{(k-2)!}{(4\pi)^{k-1}} \cdot \frac{4\ell}{B_{2\ell}} \cdot \frac{1}{1-2^{2\ell}} \cdot \frac{1}{\big(1-2^{2\ell-k}\big)\zeta(k-2\ell)}.$$

(1) *If f is a newform in $S_k(\Gamma_0(4))$, then we have*

$$\big(f, E_{2\ell}^{0} E_{k-2\ell}^{i\infty}\big) = c_{k,\ell} L(f, k-1) L(f, k-2\ell).$$

(2) *If f is a newform in $S_k(\Gamma_0(2))$ with $f|_k W_2 = \epsilon_f f$, then for $g(\tau) = f(\tau)$ or $f(2\tau)$, we have*

$$\big(g, E_{2\ell}^{0} E_{k-2\ell}^{i\infty}\big) = c_{k,\ell}\big(1+\epsilon_f 2^{-k/2}\big) L(f, k-1) L(g, k-2\ell).$$

(3) *If f is a Hecke eigenform in $S_k(\mathrm{SL}_2(\mathbb{Z}))$ with $T_2 f = \lambda_f f$, then for $g(\tau) = f(\tau)$, $f(2\tau)$, or $f(4\tau)$, we have*

$$\big(g, E_{2\ell}^{0} E_{k-2\ell}^{i\infty}\big) = c_{k,\ell}\big(1+2^{-k+1}(1-\lambda_f)\big) L(f, k-1) L(g, k-2\ell).$$

Moreover, the same formulas hold for $\ell = 1$ or $k/2-1$ if $E_{2\ell}^{0} E_{k-2\ell}^{i\infty}$ is replaced by

$$E_2^{0}(\tau) E_{k-2}^{i\infty}(\tau) - \frac{1}{\pi i(k-2)} \frac{d}{d\tau} E_{k-2}^{i\infty}(\tau),$$
$$E_2^{i\infty}(\tau) E_{k-2}^{0}(\tau) - \frac{1}{\pi i(k-2)} \frac{d}{d\tau} E_{k-2}^{0}(\tau),$$

respectively.

Proof The proof follows the argument in [6, Proposition 2], so parts of the proof will be sketchy.

We first consider the case $2 \leq \ell < (k-1)/4$. Let $f(\tau) = \sum a_n q^n \in S_k(\Gamma_0(4))$. According to (1.1),

$$E_{2\ell}^{0}(\tau) = \frac{4\ell}{B_{2\ell}(1-2^{2\ell})} \sum_{n=1}^{\infty} \big(\sigma_{2\ell-1}(n) - \sigma_{2\ell-1}(n/2)\big) q^n = \sum_{n=1}^{\infty} e_{2\ell}(n) q^n.$$

By Rankin's method, we have

$$\big(f, E_{2\ell}^{0} E_{k-2\ell}^{i\infty}\big) = \frac{(k-2)!}{(4\pi)^{k-1}} \mathcal{L}_{f,\ell}(k-1), \tag{3.1}$$

where

$$\mathcal{L}_{f,\ell}(s) = \sum_{n=1}^{\infty} e_{2\ell}(n) a(n) n^{-s}. \tag{3.2}$$

(See [12] and [14, pages 144–146] for more details.)

Now assume $f(\tau)$ is a newform in $S_k(\Gamma_0(4))$. Then

$$\mathcal{L}_{f,\ell}(s) = \frac{4\ell}{B_{2\ell}(1-2^{2\ell})}\left(\sum_{n=1}^{\infty}\sigma_{2\ell-1}(n)a(n)n^{-s} - \sum_{n=1}^{\infty}\sigma_{2\ell-1}(n/2)a(n)n^{-s}\right).$$

Following the computation in [6, page 822], we find that the first sum above is equal to

$$\frac{L(f,s)L(f,s-2\ell+1)}{\zeta^{(2)}(2s-2\ell-k+2)},$$

where $\zeta^{(2)}(s) := (1-2^{-s})\zeta(s)$. Also, because f is assumed to be a newform on $\Gamma_0(4)$, we have $a(2n)=0$ for all n and the second sum above is simply 0. Upon setting $s=k-1$, we get the formula in Part (1) for the case $2 \le \ell < (k-1)/4$.

We next assume that f is a newform in $S_k(\Gamma_0(2))$. For the case $g=f$, aside from a difference in the scalars, the proof is exactly the same as the proof of (i) of Proposition 2 in [6] and we find

$$\mathcal{L}_{f,\ell} = \frac{4\ell}{B_{2\ell}(1-2^{2\ell})}\frac{L(f,s)L(f,s-2\ell+1)}{\zeta^{(2)}(2s-2\ell-k+2)},$$

from which we obtain the formula in the case $g=f$. We now consider $g(\tau)=f(2\tau)$. Letting $b_\ell = 4\ell/B_{2\ell}(1-2^{2\ell})$, by (3.2), we have

$$\begin{aligned}\mathcal{L}_{g,\ell}(s) &= b_\ell\sum_{n=1}^{\infty}\sigma_{2\ell-1}(n)a(n/2)n^{-s} - b_\ell\sum_{n=1}^{\infty}\sigma_{2\ell-1}(n/2)a(n/2)n^{-s}\\ &= 2^{-s}b_\ell\sum_{n=1}^{\infty}\sigma_{2\ell-1}(2n)a(n)n^{-s} - 2^{-s}b_\ell\sum_{n=1}^{\infty}\sigma_{2\ell-1}(n)a(n)n^{-s}. \qquad (3.3)\end{aligned}$$

Inserting the identity

$$\sigma_{2\ell-1}(2n) = \left(1+2^{2\ell-1}\right)\sigma_{2\ell-1}(n) - 2^{2\ell-1}\sigma_{2\ell-1}(n/2)$$

into the equation, we obtain

$$\begin{aligned}\mathcal{L}_{g,\ell}(s) &= 2^{-s+2\ell-1}b_\ell\left(\sum_{n=1}^{\infty}\sigma_{2\ell-1}(n)a(n)n^{-s} - \sum_{n=1}^{\infty}\sigma_{2\ell-1}(n/2)a(n)n^{-s}\right)\\ &= 2^{-s+2\ell-1}\mathcal{L}_{f,\ell}(s) = 2^{-s+2\ell-1}b_\ell\frac{L(f,s)L(f,s-2\ell+1)}{\zeta^{(2)}(2s-2\ell-k+2)}\\ &= b_\ell\frac{L(f,s)L(g,s-2\ell+1)}{\zeta^{(2)}(2s-2\ell-k+2)}. \qquad (3.4)\end{aligned}$$

Setting $s=k-1$, we get the formula in Part (2) for the case $2 \le \ell < (k-1)/4$.

We now consider the case when f is a normalized Hecke eigenform in $S_k(\mathrm{SL}_2(\mathbb{Z}))$. Again, when $g=f$, the proof of the formula is almost the same as the proof of (ii) of

Proposition 2 in [6]. Then when $g(\tau)=f(2\tau)$, a computation analogous to (3.3) and (3.4) gives us the claimed formula. The proof of the case $g(\tau)=f(4\tau)$ is similar. This completes the proof of the case $2\le\ell<(k-1)/4$.

We next consider the case $(k+1)/4<\ell\le k/2-2$. Using the fact that the Atkin–Lehner involution W_4 is a Hermitian operator with respect to the Petersson inner product, we have

$$\bigl(f, E_{2\ell}^0 E_{k-2\ell}^{i\infty}\bigr)=\bigl(f|W_4, E_{k-2\ell}^0 E_{2\ell}^{i\infty}\bigr).$$

When f is a newform in $S_k(\Gamma_0(4))$ with $f|_k W_4=\epsilon_f f$, by the formula in Part (1) with ℓ replaced by $k/2-\ell$, this is equal to

$$\bigl(f, E_{2\ell}^0 E_{k-2\ell}^{i\infty}\bigr)=\epsilon_f\bigl(f, E_{k-2\ell}^0 E_{2\ell}^{i\infty}\bigr)=\epsilon_f c_{k,k/2-\ell}L(f,k-1)L(f,2\ell).$$

Then from the functional equation

$$\left(\frac{2\pi}{\sqrt{4}}\right)^{-s}\Gamma(s)L(f,s)=\epsilon_f(-1)^{k/2}\left(\frac{2\pi}{\sqrt{4}}\right)^{-(k-s)}\Gamma(k-s)L(f,k-s)$$

and the identity

$$\zeta(2n)=-\frac{(2\pi i)^{2n}}{\Gamma(2n)}\frac{B_{2n}}{4n} \tag{3.5}$$

for integers $n\ge 1$, we get

$$\bigl(f, E_{2\ell}^0 E_{k-2\ell}^{i\infty}\bigr)=c_{k,k/2-\ell}L(f,k-1)L(f,2\ell)=c_{k,\ell}L(f,k-1)L(f,k-2\ell).$$

Now assume that f is a newform in $S_k(\Gamma_0(2))$ with $f|_k W_2=\epsilon_f f$. Then

$$(f|_k W_4)(\tau)=(2\tau)^{-k}f(-1/4\tau)=\epsilon_f(2\tau)^{-k}\bigl(2\sqrt{2}\tau\bigr)^k f(2\tau)=\epsilon_f 2^{k/2}f(2\tau),$$

and consequently, for $g(\tau)=f(\tau)$,

$$\bigl(g, E_{2\ell}^0 E_{k-2\ell}^{i\infty}\bigr)=\epsilon_f 2^{k/2}\bigl(h, E_{k-2\ell}^0 E_{2\ell}^{i\infty}\bigr)$$

with $h(\tau)=f(2\tau)$. Applying the formula in Part (2) with ℓ replaced by $k/2-\ell$, we get

$$\begin{aligned}\bigl(g, E_{2\ell}^0 E_{k-2\ell}^{i\infty}\bigr)&=\epsilon_f 2^{k/2}c_{k,k/2-\ell}\bigl(1+\epsilon_f 2^{-k/2}\bigr)L(f,k-1)L(h,2\ell)\\&=2^{-2\ell}c_{k,k/2-\ell}\bigl(1+\epsilon_f 2^{k/2}\bigr)L(f,k-1)L(g,2\ell).\end{aligned}$$

Then from the functional equation for $L(f,s)$ and (3.5), we establish the formula in Part (2) for the case $g(\tau)=f(\tau)$. The proof of the case $g(\tau)=f(2\tau)$ is similar.

Now assume that f is a Hecke eigenform in $S_k(\mathrm{SL}_2(\mathbb{Z}))$ with $T_2 f=\lambda_f f$. We have

$$(f|_k W_4)(\tau)=(2\tau)^{-k}f(-1/4\tau)=2^k f(4\tau),$$

and thus, for $g(\tau)=f(\tau)$,

$$\bigl(g, E_{2\ell}^0 E_{k-2\ell}^{i\infty}\bigr)=2^k\bigl(h, E_{k-2\ell}^0 E_{2\ell}^{i\infty}\bigr)$$

with $h(\tau) = f(4\tau)$. Using the formula in Part (3), we derive that

$$\begin{aligned}\left(g, E_{2\ell}^{0} E_{k-2\ell}^{i\infty}\right) &= 2^k c_{k,k/2-\ell}\left(1 + 2^{-k+1}(1-\lambda_f)\right) L(f, k-1) L(h, 2\ell) \\ &= 2^{k-4\ell} c_{k,k/2-\ell}\left(1 + 2^{-k+1}(1-\lambda_f)\right) L(f, k-1) L(f, 2\ell).\end{aligned}$$

Then, by the functional equation for $L(f,s)$ and (3.5) again, we see that the formula in Part (3) holds for $g(\tau) = f(\tau)$. The proof of the cases $g(\tau) = f(2\tau)$ and $g(\tau) = f(4\tau)$ is similar. This completes the proof of the formulas for $2 \le \ell \le k/2-2$.

Finally, let us consider the cases $\ell = 1$ and $\ell = k/2-1$. Assume that $\ell = 1$. We first recall the well-known transformation formula

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{6}{\pi i} c(c\tau+d) + (c\tau+d)^2 E_2(\tau),$$

which can be proved easily by considering the logarithmic derivative of the two sides of $\eta((a\tau+b)/(c\tau+d))^{24} = (c\tau+d)^{12}\eta(\tau)^{24}$, where $\eta(\tau)$ is the Dedekind eta function. It follows that the Eisenstein series $E_2^0(\tau) = (E_2(\tau) - E_2(2\tau))/3$ satisfies

$$E_2^0\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{1}{\pi i} c(c\tau+d) + (c\tau+d)^2 E_2^0(\tau) \tag{3.6}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$. Also, since $E_{k-2}^{i\infty}(\tau)$ is a modular form of weight $k-2$, we have

$$E_{k-2}^{i\infty\,\prime}\left(\frac{a\tau+b}{c\tau+d}\right) = (k-2)c(c\tau+d)^{k-1} E_{k-2}^{i\infty}(\tau) + (c\tau+d)^k E_{k-2}^{i\infty}(\tau). \tag{3.7}$$

Thus,

$$h(\tau) = E_2^0 E_{k-2}^{i\infty}(\tau) - \frac{1}{\pi i(k-2)} E_{k-2}^{i\infty\,\prime}(\tau)$$

is a cusp form of weight k on $\Gamma_0(4)$. Now we have

$$E_{k-2}^{i\infty}(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} \frac{1}{(c\tau+d)^{k-2}},$$

where Γ_∞ is the subgroup generated by $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and for $\gamma\in\Gamma_\infty\backslash\Gamma_0(4)$, we write $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$. It follows that, for $f\in S_k(\Gamma_0(4))$,

$$\begin{aligned}\overline{(f,h)} &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)}\iint_{\Gamma_0(4)\backslash\mathbb{H}}\overline{f(\tau)}\left(\frac{E_2^0(\tau)}{(c\tau+d)^{k-2}}+\frac{1}{\pi i}\frac{c}{(c\tau+d)^{k-1}}\right)y^k\,\frac{dx\,dy}{y^2}\\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)}\iint_{\gamma(\Gamma_0(4)\backslash\mathbb{H})}\overline{f\left(\gamma^{-1}\tau\right)}\\ &\quad\times\left(\frac{E_2^0(\gamma^{-1}\tau)}{(c\gamma^{-1}\tau+d)^{k-2}}+\frac{1}{\pi i}\frac{c}{(c\gamma^{-1}\tau+d)^{k-1}}\right)\\ &\quad\times\left(\operatorname{Im}\gamma^{-1}\tau\right)^k\frac{dx\,dy}{y^2},\end{aligned}$$

where we write $\tau=x+iy$. From the transformation formula (3.6), we get

$$\frac{E_2^0(\gamma^{-1}\tau)}{(c\gamma^{-1}\tau+d)^{k-2}}+\frac{1}{\pi i}\frac{c}{(c\gamma^{-1}\tau+d)^{k-1}}=(c\tau-a)^kE_2^0(\tau).$$

Consequently, if $f(\tau)=\sum a(n)q^n$ and $E_2^0(\tau)=\sum e_2(n)q^n$, we have

$$\begin{aligned}\overline{(f,h)} &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)}\iint_{\gamma(\Gamma_0(4)\backslash\mathbb{H})}\overline{f(\tau)}E_2^0(\tau)y^k\,\frac{dx\,dy}{y^2}\\ &= \int_0^\infty\int_0^1\sum_{m,n=1}^\infty\overline{a(m)}e_2(n)e^{2\pi i(n-m)x}e^{-2\pi(m+n)y}y^{k-2}\,dx\,dy\\ &= \frac{\Gamma(k-1)}{(4\pi)^{k-1}}\sum_{n=1}^\infty\overline{a(n)}e_2(n)n^{-(k-1)}=\frac{(k-2)!}{(4\pi)^{k-1}}\overline{\mathcal{L}_{f,1}(k-1)},\end{aligned}$$

and we are back to (3.1). Therefore, the formulas in the statement of the proposition hold if we replace $E_2^0E_{k-2}^{i\infty}$ by $h=E_2^0E_{k-2}^{i\infty}-{E_{k-2}^{i\infty}}'/\pi i(k-2)$. Finally, the case $E_2^{i\infty}E_{k-2}^0-{E_{k-2}^0}'/\pi i(k-2)$ can be proved by applying the Atkin–Lehner involution, as what we did for the case $(k+1)/4<\ell\le k/2-2$. This completes the proof of the proposition. □

We now prove Theorem 1.1 and Corollaries 1.2 and 1.3.

Proof of Theorem 1.1 Let $k\ge 6$ be an even integer and let

$$d=\dim S_k\left(\Gamma_0(4)\right)=\frac{k}{2}-2.$$

Let $h_1=E_2^0E_{k-2}^{i\infty}-{E_{k-2}^{i\infty}}'/\pi i(k-2)$ and $h_j=E_{2j}^0E_{k-2j}^{i\infty}$ for $j=2,\dots,d$. As in Proposition 3.1, by a *newform* in $S_k(\Gamma_0(N))$, we mean a normalized Hecke eigenform in the newform subspace of $S_k(\Gamma_0(N))$. We first choose a basis for $S_k(\Gamma_0(4))$ to be

$$\begin{aligned}&\big\{f(\tau), f(2\tau), f(4\tau): \ f \text{ a Hecke eigenform in } S_k\big(\mathrm{SL}_2(\mathbb{Z})\big)\big\}\\ &\quad\cup\big\{f(\tau), f(2\tau): \ f \text{ a newform in } S_k\big(\Gamma_0(2)\big)\big\}\\ &\quad\cup\big\{f(\tau): \ f \text{ a newform in } S_k\big(\Gamma_0(4)\big)\big\}\end{aligned}$$

and label the functions by $g_1, \ldots, g_d$. We also let f_i denote the corresponding newform from which g_i originates. Consider the $d \times d$ matrix

$$A = \big[(g_i, h_j)\big]_{i,j=1,\ldots,d}$$

formed by the Petersson inner product of g_i and h_j. Since $\{g_i\}$ is a basis for $S_k(\Gamma_0(4))$, $\{h_j\}$ is a basis if and only if $\det A \neq 0$. Now by the formulas in Proposition 3.1, we have

$$\det A = \left(\prod_{j=1}^{d} c_{k,j}\right)\left(\prod_{i=1}^{d} b_i L(f_i, k-1)\right)\det\big[L(g_i, k-2j)\big]_{i,j=1,\ldots,d},$$

where

$$b_i = \begin{cases} 1 + 2^{-k+1}(1+\lambda_{f_i}) & \text{if } f_i \text{ is a Hecke eigenform in } S_k(\Gamma(1)) \\ & \text{with } T_2 f_i = \lambda_{f_i} f_i, \\ 1 + \epsilon_{f_i} 2^{-k/2} & \text{if } f_i \text{ is a newform in } S_k(\Gamma_0(2)) \text{ with } f_i|_k W_2 = \epsilon_{f_i} f_i, \\ 1 & \text{if } f_i \text{ is a newform in } S_k(\Gamma_0(4)). \end{cases}$$

The numbers $c_{k,j}$ are clearly nonzero. Also, since f_i are assumed to be normalized Hecke eigenforms, we know that $b_i L(f_i, k-1) \neq 0$. Therefore, to show that $\det A \neq 0$, it suffices to show that $\det[L(g_i, k-2j)] \neq 0$.

Now by (1.3), we have

$$\begin{aligned} L(g_i, k-2j) &= \frac{(-2\pi i)^{k-2j}}{\Gamma(k-2j)} \int_0^{i\infty} g_i(\tau)\tau^{k-2j-1}\, d\tau = \frac{(-2\pi i)^{k-2j}}{\Gamma(k-2j)} r_{k-2j-1}(g_i) \\ &= \frac{(-2\pi i)^{k-2j}}{2\Gamma(k-2j)}(2i)^{k-1}(g_i, R_{k-2j-1}), \end{aligned}$$

where $R_n = R_{\Gamma_0(4),k-2,n}$ is the cusp form in $S_k(\Gamma_0(4))$ characterized by the property (1.4). Thus, $\det[L(g_i, k-2j)] \neq 0$ if and only if $\det[(g_i, R_{k-2j-1})] \neq 0$. However, $\{R_{k-2j-1}\}_{j=1}^d$ is a basis of $S_k(\Gamma_0(4))$ by Theorem 1.7 and Remark 1.8, and so is $\{g_i\}_{i=1}^d$ by the assumption. Hence we know that $\det[(g_i, R_{k-2j-1})] \neq 0$, and we can conclude that the set

$$\left\{E_2^0 E_{k-2}^{i\infty} - \frac{1}{\pi i(k-2)} E_{k-2}^{i\infty\,\prime}\right\} \cup \left\{E_n^0 E_{k-n}^{i\infty} \mid n = 4, 6, \ldots, k-4\right\}$$

is a basis for $S_k(\Gamma_0(4))$. Applying the Atkin–Lehner involution to this basis, we see that the other set in the statement of theorem is also a basis. □

Proofs of Corollaries 1.2 and 1.3 Let $W : S_k(\Gamma_0(4)) \to S_k(\Gamma_0(4))$ be defined by $W(f) = f|_k W_4$ for any f in $S_k(\Gamma_0(4))$. Let I denote the identity automorphism of $S_k(\Gamma_0(4))$. Since $W^2 = I$, we have

$$S_k\big(\Gamma_0(4), +\big) = \mathrm{Ker}(I - W) = \mathrm{Im}(I + W),$$
$$S_k\big(\Gamma_0(4), -\big) = \mathrm{Ker}(I + W) = \mathrm{Im}(I - W).$$

Now, from Theorem 1.1, we know

$$\left\{E_2^{i\infty} E_{k-2}^0 - \frac{1}{\pi i(k-2)} {E_{k-2}^0}'\right\} \cup \left\{E_n^{i\infty} E_{k-n}^0 \mid n = 4, 6, \ldots, k-4\right\}$$

is a basis for $S_k(\Gamma_0(4))$. Then the set

$$\left\{E_2^{i\infty} E_{k-2}^0 - \frac{1}{\pi i(k-2)} {E_{k-2}^0}'\right\} \cup \left\{E_n^{i\infty} E_{k-n}^0 + E_n^0 E_{k-n}^{i\infty} \mid n = 4, 6, \ldots, 2\lfloor k/4\rfloor\right\}$$
$$\cup \left\{E_n^{i\infty} E_{k-n}^0 \mid n = 2\lfloor k/4\rfloor + 2, 2\lfloor k/4\rfloor + 4, \ldots, k-4\right\}$$

is also a basis for $S_k(\Gamma_0(4))$. In particular,

$$\left\{E_n^{i\infty} E_{k-n}^0 + E_n^0 E_{k-n}^{i\infty} \mid n = 4, 6, \ldots, 2\lfloor k/4\rfloor\right\}$$

is linearly independent. Furthermore, since $E_n^{i\infty} E_{k-n}^0 + E_n^0 E_{k-n}^{i\infty} \in S_k(\Gamma_0(4), +)$ and $\dim S_k(\Gamma_0(4), +) = \lfloor \frac{k}{4}\rfloor - 1$, we know

$$\left\{E_n^{i\infty} E_{k-n}^0 + E_n^0 E_{k-n}^{i\infty} \mid n = 4, 6, \ldots, 2\lfloor k/4\rfloor\right\}$$

is a basis for $S_k(\Gamma_0(4), +)$.

Next, from Theorem 1.1, we know that $S_k(\Gamma_0(4), -) = \mathrm{Im}(I - W)$ is spanned by

$$\left\{E_2^{i\infty} E_{k-2}^0 - E_2^0 E_{k-2}^{i\infty} - \frac{1}{\pi i(k-2)}\left({E_{k-2}^0}' - {E_{k-2}^{i\infty}}'\right)\right\}$$
$$\cup \left\{E_n^{i\infty} E_{k-n}^0 - E_n^0 E_{k-n}^{i\infty} \mid n = 4, 6, \ldots, k-4\right\}.$$

Then $S_k(\Gamma_0(4), -)$ is also spanned by the set

$$\left\{E_2^{i\infty} E_{k-2}^0 - E_2^0 E_{k-2}^{i\infty} - \frac{1}{\pi i(k-2)}\left({E_{k-2}^0}' - {E_{k-2}^{i\infty}}'\right)\right\}$$
$$\cup \left\{E_n^{i\infty} E_{k-n}^0 - E_n^0 E_{k-n}^{i\infty} \mid n = 4, 6, \ldots, k - 2\lfloor k/4\rfloor - 2\right\}.$$

Now, noting that $\dim S_k(\Gamma_0(4), -) = k/2 - \lfloor k/4\rfloor - 1$, we conclude the set above is a basis of $S_k(\Gamma_0(4), -)$. □

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