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Design of distributed amplify-and-forward relay networks for multi-input multi-output transmission

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Abstract: Relays can potentially enhance the transmission performance of multi-input multi-output (MIMO) systems. A parallel single-antenna relay network has additional advantages in flexibility, diversity and cost, but also poses significant design problems because the absence of inter-antenna connections over different relays makes the underlying mathematical problems much more difficult to solve. In this study, the authors consider the design of parallel amplify-and-forward relay networks. More specifically, the authors consider the design of relay gains to maximise the system capacity. As no closed-form analytic solution can be found, the authors first develop an iterative algorithm to find a locally optimal solution. Since algorithmic optimisation provides little insight into the analytical properties of the solution, they also attempt analytical solutions for several asymptotic noise conditions. It turns out that the solutions involve some methods to select the optimal subsets of relays for signal forwarding. The authors analyse the resulting capacity outage diversity orders and confirm the analysis with simulation results.

1 Introduction

Relays have been considered a useful means for coverage extension and capacity enhancement of wireless systems [1]. Among all conceivable relaying strategies, two have received the most attention: amplify-and-forward (AF) [2] and decode-and-forward (DF) [3]. In AF systems the relays amplify or beamform the received signals without further processing, whereas in DF systems they decode (or demodulate if there is no channel coding) the received signals and transmit the re-encoded (or remodulated) signals to the destination. Besides the forwarding strategies, an important subject in relay system design is the overall wireless system architecture. In this, because of the capacity advantage of multi-input multi-output (MIMO) transmission over single-input single-output (SISO) transmission, many have sought to incorporate some MIMO concepts one way or another. The present work is concerned with AF-based distributed relay networks, whose architecture will be described further later.

The simplest relay-aided transmission system consists of three nodes: source, relay (cooperator) and destination [4]. To facilitate MIMO transmission, an intuitive approach is to install multiple antennas on one or more of the nodes. For simplicity, consider the situation where the source and the destination have an equal number of antennas. A case with a single-antenna source (SAS) and a single relay (SR) equipped with multiple antennas (MAR, multiple-antenna relay) is considered in [5]. A natural extension to have a multiple-antenna source (MAS) and an SR–MAR to enable spatial multiplexing [6, 7]. In studies of MAS–SR–MAR systems, the multiple antennas on each terminal are usually assumed to be fully connected and may have arbitrary interconnection weights. In this case, known matrix theory can be used to decompose an MIMO transmission channel into parallel SISO links (via e.g. the singular value decomposition or the QR decomposition). Each spatially multiplexed signal stream can then be transported over one parallel link, and the matrix decomposition can be viewed as simultaneous beamforming for these streams. Typical performance measures, such as the signal-to-interference-plus-noise ratio (SINR) or the mean-square error in received signal values, can be expressed in terms of the parameters of the decomposed channel. System optimisation may then become essentially a problem of power allocation among the individual streams [6, 7].

On the other hand, use of multiple, parallel relays (PR) has also been considered by many researchers and shown to be potentially beneficial in various aspects [8–22]. For example, it is found that an increased number of relays can benefit the system capacity [8]. In fact, the PR can function as virtual transmitter antennas and effect transmitter diversity either in the form of distributed space–time coding [9, 10] or in the form of distributed beamforming [11–13]. The corresponding diversity order has been examined in [11, 14], respectively. Moreover, parallel relaying has also been studied in the contexts of sensor networks [15], two-way relaying [16] and secrecy communication [17]. However, despite the potential benefits, the fact that the relays are not connected but stand in parallel raises a cooperation problem which, if not dealt with, could severely limit the realisable benefit.

To see why, let L be the number of PR and N_i $(1 \le i \le L)$ the number of antennas on relay i. Let M denote the number of antennas on the source terminal as well as that on the destination terminal. Consider first the simplest case where each terminal has only one antenna, that is SAS–PR–SAR where SAR stands for single-antenna relay, and $M = N_i = 1\ \forall i$ [12, 13]. In this case, the relays effectively constitute a distributed beamformer for the single signal stream. Applying the same design philosophy to an MAS–PR–MAR system with $M > 1$ and $N_i > 1\ \forall i$, there can be $M_S = \min\{M,\ N_i\ \ \forall i\}$ concurrent signal streams. The beamforming techniques used in MAS–SR–MAR systems can be extended to this scenario with a twist [18, 19]. That is, the available antennas on the relays can be used to provide M_S parallel subchannels between the source and the destination. Systems operating in the above ways have been considered in some works [5–7, 12, 13, 18, 19]. In terms of capacity, however, such systems suffer from two consequences. First, the number of supported subchannels (i.e. the number of concurrent spatially multiplexed streams) does not grow with the relay number L, but is upper-bounded by M_S. Secondly, to increase the number of streams we need to ensure that all relays are equipped with sufficient antennas. Designs that can obviate the above limits are of interest and importance.

In this work we consider the design of MAS–PR–SAR systems (where $N_i = 1$ and $\Sigma N_i = L$) to support multiple signal streams. More specifically, we consider the design of AF relay forwarding gains for maximisation of system capacity. Previously, Jin *et al.* [20] considered the case where the relays had equal gain and analysed the statistics of the resulting ergodic capacity. Chen *et al.* [21] considered the minimisation of transmission power subject to per-stream SINR targets. The problem is related to system capacity, but somewhat indirectly. Bae and Lee [22] proposed algorithms for capacity optimisation under the condition that the product of the source-to-relay and the relay-to-destination channel matrices was asymptotically diagonal in the limit of a large number of relays. However in sum, there is as yet no extensive work on the design of distributed parallel relay networks for capacity maximisation. Actually, the relationship between number of relay terminals and system capacity also needs to be further clarified. The present work is motivated by these observations.

We consider two approaches to maximising the capacity of a distributed relay network with presence of perfect channel state information. The first is algorithmic, as so far no closed-form solution to the problem exists. However, although algorithmic optimisation can yield good results, it provides little insight into the analytical properties of the solutions. We thus also attempt an analytical approach. As no closed-form solution can be obtained for the general situation, we consider two asymptotic situations that are more amenable to analysis. In one of them the relay noise dominates the overall noise in the received signal at the destination and in the other the destination terminal noise dominates. Alternatively, these two situations can also be viewed as providing two upper bounds to the system capacity.

The rest of this paper is organised as follows. Section 2 formulates the problem. Section 3 derives the algorithmic solution. Section 4 derives suboptimal analytical solutions for the two asymptotic noise conditions mentioned above. Section 5 presents some simulation results. Finally, Section 6 gives some concluding remarks.

2 Problem formulation

We consider two-slot, two-hop transmission from a source terminal to a destination terminal through the help of a distributed relay network; there is no direct link between the source and the destination. Both the source and the destination terminals are equipped with M antennas. The distributed relay network is composed of L single-antenna relay terminals. To preserve the degree of freedom provided by the source and the destination antennas, we assume $L \geq M$. Each relay performs AF relaying. Fig. 1 illustrates the system model.

Let $\boldsymbol{x} \in \mathbb{C}^M$ denote the signals transmitted from the source and $\boldsymbol{y} \in \mathbb{C}^M$ that received at the destination, where $\mathbb{C}^M$ denotes the set of $M \times 1$ vectors of complex numbers. Let $\boldsymbol{G}^{\mathrm{H}} \in \mathbb{C}^{L\times M}$ be the matrix of MIMO channel coefficients between the source terminal antennas and the relays, where $\mathbb{C}^{L\times M}$ denotes the set of $L \times M$ matrices of complex numbers and superscript H denotes Hermitian transpose. Similarly, let $\boldsymbol{F} \in \mathbb{C}^{M\times L}$ be the channel matrix between the relays and the destination terminal antennas. The received signals at the relays are assumed to be subject to additive complex white Gaussian noise (AWGN) $\boldsymbol{n}_{\mathrm{R}} \sim \mathcal{CN}(0, \sigma_{\mathrm{R}}^2\boldsymbol{I}_{\mathrm{L}})$, where $\boldsymbol{I}_{\mathrm{L}}$ denotes the $L \times L$ identity matrix. Likewise, the received signal at the destination is subject to AWGN $\boldsymbol{n}_{\mathrm{D}} \sim \mathcal{CN}(0, \sigma_{\mathrm{D}}^2\boldsymbol{I}_M)$. Let the ith relay have the complex gain $r(i)$. Then the end-to-end transmission behaviour can be written as

$$\boldsymbol{y} = \boldsymbol{F}\boldsymbol{R}\boldsymbol{G}^{\mathrm{H}}\boldsymbol{x} + \boldsymbol{F}\boldsymbol{R}\boldsymbol{n}_{\mathrm{R}} + \boldsymbol{n}_{\mathrm{D}} \tag{1}$$

where $\boldsymbol{R}$ is a diagonal matrix with $r(i)$ as its ith diagonal element.

Assume that the source transmits independent streams over its M antennas. For all practical designs, the transmission powers of the source and the relays are limited. Therefore let the source antennas have equal finite transmission power σ_x^2. Further assume that the relays are subject to a total power limit P_{R}. Hence we have

$$\begin{aligned} P_{\mathrm{R}} &\geq \mathrm{tr}(E\{(\boldsymbol{R}\boldsymbol{G}^{\mathrm{H}}\boldsymbol{x} + \boldsymbol{R}\boldsymbol{n}_{\mathrm{R}})(\boldsymbol{R}\boldsymbol{G}^{\mathrm{H}}\boldsymbol{x} + \boldsymbol{R}\boldsymbol{n}_{\mathrm{R}})^{\mathrm{H}}\}) \\ &= \sum_{i=1}^{L}(\sigma_{\mathrm{R}}^2 + \sigma_x^2\|\boldsymbol{g}_i\|^2)|r(i)|^2 \triangleq \sum_{i=1}^{L} p(i)|r(i)|^2 \end{aligned} \tag{2}$$

where $E\{\cdot\}$ denotes expectation, $\|\cdot\|$ denotes the two-norm of a vector, $\mathrm{tr}(\cdot)$ means matrix trace, $\boldsymbol{g}_i$ is the ith column of $\boldsymbol{G}$ and $p(i)$ denotes the sum of input signal and noise powers at relay i.

The total noise vector $\boldsymbol{F}\boldsymbol{R}\boldsymbol{n}_{\mathrm{R}} + \boldsymbol{n}_{\mathrm{D}}$ in (1) at the destination is, in general, spatially correlated. To find the system capacity,

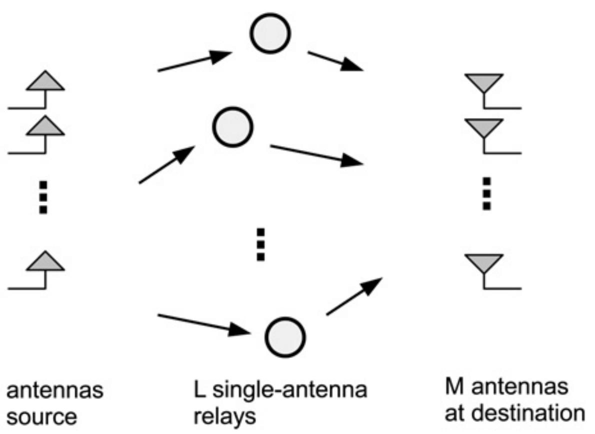

Fig. 1 *MIMO system with distributed relays*

consider whitening the noise by premultiplying $\boldsymbol{y}$ with $\boldsymbol{W}^{-1/2}$ at the destination, where $\boldsymbol{W}$ is the autocorrelation matrix of the noise given by

$$\begin{aligned}\boldsymbol{W} &= E\{(\boldsymbol{FRn}_{\mathrm{R}} + \boldsymbol{n}_{\mathrm{D}})(\boldsymbol{FRn}_{\mathrm{R}} + \boldsymbol{n}_{\mathrm{D}})^{\mathrm{H}}\} \\ &= \sigma_{\mathrm{D}}^2[\boldsymbol{I}_M + \sigma^2(\boldsymbol{FR})(\boldsymbol{FR})^{\mathrm{H}}]\end{aligned} \quad (3)$$

with $\sigma \triangleq \sigma_{\mathrm{R}}/\sigma_{\mathrm{D}}$. Let $\boldsymbol{H} \triangleq \boldsymbol{FRG}^{\mathrm{H}}$ represent the noise-free equivalent end-to-end channel matrix. The system capacity is then a function of $\boldsymbol{R}$ as [23]

$$\begin{aligned}\log_2 C(\boldsymbol{R}) &\triangleq \log_2 \det(\boldsymbol{I}_M + \sigma_x^2 \boldsymbol{H}^{\mathrm{H}}\boldsymbol{W}^{-1}\boldsymbol{H}) \\ &= \log_2 \det(\boldsymbol{W} + \sigma_x^2 \boldsymbol{HH}^{\mathrm{H}}) - \log_2 \det(\boldsymbol{W})\end{aligned} \quad (4)$$

where $\det(\cdot)$ denotes matrix determinant. The optimisation problem can be stated as

$$\boldsymbol{R}_{\mathrm{opt}} = \arg\max_{\boldsymbol{R}} C(\boldsymbol{R}) \quad (5)$$

subject to

$$\sum_{i=1}^{L} p(i)|r(i)|^2 = \sum_{i=1}^{L} (\sigma_{\mathrm{R}}^2 + \sigma_x^2 \|\boldsymbol{g}_i\|^2)|r(i)|^2 \le P_{\mathrm{R}} \quad (6)$$

For convenience, term $C(\boldsymbol{R})$ as the 'capacity measure'.

Unfortunately, the above optimisation problem is not convex. We know of no efficient solution for the problem. As mentioned, we will present two approaches to a solution. Before that, we first tie up some theoretical loose ends in the next subsection.

2.1 Capacity scaling with relay power

The inequality power constraint (6) naturally prompts one to think: is it possible to simplify the constraint by considering only the equality therein without impacting the optimality of the solution? Or, alternatively, given a set of $r(i)$, $1 \le i \le L$, that satisfies (6) with inequality, will the system capacity be improved by scaling the $r(i)$ values to reach equality in (6)? Intuitively, the answer may seem to be a no-brainer, as increasing the transmission power should be beneficial to the signal-to-noise ratio (SNR) and thus the capacity. However mathematically, owing to the presence of some matrices in (4) and (1), a solid proof nevertheless requires some work. We give the proof in Appendix 1. (Some intermediate results in the proof will also be useful later in system design.) For convenience, we state the result as a theorem.

Theorem 1 (Capacity scaling): If the complex relay gains are scaled by a common factor $s \in \mathbb{C}$ with $|s| > 1$, then $C(s\boldsymbol{R}) > C(\boldsymbol{R})$.

Therefore it is confirmed that scaling up of the relay gains can increase system capacity, and the optimisation constraint (6) can be simplified to

$$\sum_{i=1}^{L} p(i)|r(i)|^2 = \sum_{i=1}^{L} (\sigma_{\mathrm{R}}^2 + \sigma_x^2 \|\boldsymbol{g}_i\|^2)|r(i)|^2 = P_{\mathrm{R}} \quad (7)$$

That is, the relay network should always transmit at the maximum allowed total power.

Next, one may wonder if the capacity could increase without bound if the total relay transmission power tends to infinity. Intuitively, the answer may appear to be another no-brainer because, from (1), the quality of the source-to-relay links should effect a cap on the attainable data rate however high the relay power can be. However again, a solid mathematical proof requires a few lines of reasoning. We likewise state the result as a theorem and prove it in Appendix 2.

Theorem 2 (Asymptotic capacity at high relay power): As $|s| \to \infty$, $C(s\boldsymbol{R})$ is upper-bounded by $\det[\boldsymbol{I}_M + (\sigma_x^2/\sigma_{\mathrm{R}}^2)\boldsymbol{GG}^{\mathrm{H}}]$ and it approaches the upper bound if and only if $\boldsymbol{G}$ and $\boldsymbol{FR}$ span the same row space.

With Theorem 2, it is verified that $C(\boldsymbol{R})$ is upper-bounded even if the relays may transmit at infinite power.

3 System design via algorithmic optimisation

As noted previously, no closed-form solution is known for the capacity optimisation problem at hand. Thus we consider an algorithmic solution in this section, wherein the relay gains are optimised iteratively. In doing so, however, we need to update the value of $C(\boldsymbol{R})$ repeatedly with each change of the relay gains. Although, in principle, this can done by using (4), the computational load appears formidable. To solve this problem, we limit each adjustment of the relay gains to a form that corresponds to some low-rank updates of the matrices entering (4). Then the associated matrix computations can be carried out less arduously. Specifically, in each iteration, we replace one relay gain by some value $\alpha \in \mathbb{C}$. The other relay gains are multiplied by a factor $\beta \in \mathbb{R}_+$ (where $\mathbb{R}_+$ stands for the set of positive real numbers) such that the power constraint (7) is satisfied. The factors α and β are chosen to maximise $C(\boldsymbol{R})$.

In more detail, let $\boldsymbol{r} = [r(1), \ldots, r(L)]^{\mathrm{T}}$ denote the vector of relay gains where superscript T denotes transpose. Let $\boldsymbol{r}_0$ be the same as $\boldsymbol{r}$ except that its ith element is replaced by zero, and let $\boldsymbol{r}_{\mathrm{u}}$ denote the gain vector after the above-described update. Then

$$\boldsymbol{r}_0 = (\boldsymbol{I}_L - \boldsymbol{S}_i)\boldsymbol{r}, \quad \boldsymbol{r}_{\mathrm{u}} = \beta\boldsymbol{r}_0 + \alpha\boldsymbol{S}_i\boldsymbol{1} \quad (8)$$

where $\boldsymbol{S}_i$ denotes the 'selection matrix' whose elements are all zero except for a 1 at the ith diagonal position and $\boldsymbol{1}$ represents an all-ones vector. Clearly, α is subject to the constraint

$$0 \le |\alpha| < \sqrt{P_{\mathrm{R}}/p(i)} \quad (9)$$

and for given α, we have (letting $r_0(i)$ be the ith element in $\boldsymbol{r}_0$)

$$\beta = \sqrt{\frac{P_{\mathrm{R}} - |\alpha|^2 p(i)}{\sum_i p(i)|r_0(i)|^2}} \quad (10)$$

Let $\boldsymbol{R}_0 = \mathrm{diag}(\boldsymbol{r}_0)$ and $\boldsymbol{R}_{\mathrm{u}} = \mathrm{diag}(\boldsymbol{r}_{\mathrm{u}})$, where $\mathrm{diag}(\boldsymbol{v})$ stands for a diagonal matrix that has the ith element of the (column or row) vector $\boldsymbol{v}$ as its ith diagonal element. Then the noise-free equivalent end-to-end channel matrix after gain updating is given by

$$\boldsymbol{H}_{\mathrm{u}} = \beta\boldsymbol{H}_0 + \alpha\boldsymbol{f}_i\boldsymbol{g}_i^{\mathrm{H}} \quad (11)$$

where $\boldsymbol{H}_0 = \boldsymbol{FR}_0\boldsymbol{G}^{\mathrm{H}}$ and $\boldsymbol{H}_{\mathrm{u}} = \boldsymbol{FR}_{\mathrm{u}}\boldsymbol{G}^{\mathrm{H}}$. The autocorrelation

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matrix of the received noise vector at the destination becomes

$$\begin{aligned}\boldsymbol{W}_{\mathrm{u}} &= \sigma_{\mathrm{D}}^2[\boldsymbol{I}_M + (\sigma\beta)^2(\boldsymbol{F}\boldsymbol{R}_0)(\boldsymbol{F}\boldsymbol{R}_0)^{\mathrm{H}}] + |\sigma_{\mathrm{R}}\alpha|^2\boldsymbol{f}_i\boldsymbol{f}_i^{\mathrm{H}} \\ &\triangleq \sigma_{\mathrm{D}}^2\boldsymbol{W}_0 + |\sigma_{\mathrm{R}}\alpha|^2\boldsymbol{f}_i\boldsymbol{f}_i^{\mathrm{H}}\end{aligned} \tag{12}$$

Note that $\boldsymbol{H}_{\mathrm{u}}$ is different from $\beta\boldsymbol{H}_0$ by a rank-1 matrix and that $\boldsymbol{W}_{\mathrm{u}}$ is different from $\sigma_{\mathrm{D}}^2\boldsymbol{W}_0$ also by a rank-1 matrix.

To proceed, we note that the design problem can now be decomposed into three subproblems: (i) how to express $C(\boldsymbol{R}_{\mathrm{u}})$ in terms of α and β; (ii) how to optimise the values of α and β; and (iii) how to iterate. We address these subproblems in the order below.

3.1 $C(\boldsymbol{R}_u)$ as function of α and β

To start, note from (4) that we have

$$\log_2 C(\boldsymbol{R}_{\mathrm{u}}) = \log_2 \det(\boldsymbol{W}_{\mathrm{u}} + \sigma_x^2\boldsymbol{H}_{\mathrm{u}}\boldsymbol{H}_{\mathrm{u}}^{\mathrm{H}}) - \log_2 \det(\boldsymbol{W}_{\mathrm{u}}) \tag{13}$$

To express it in terms of α and β, we invoke some low-rank updating formulas for matrix inverses and determinants as follows.

From matrix theory [24], if $\boldsymbol{B} = \boldsymbol{A} + \boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{H}}$ where $\boldsymbol{A}$ is a full-rank square matrix and $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$ are vectors, then

$$\begin{aligned}\boldsymbol{B}^{-1} &= \boldsymbol{A}^{-1} - \frac{\boldsymbol{A}^{-1}\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{A}^{-1}}{1 + \boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{A}^{-1}\boldsymbol{u}_1}, \\ \det(\boldsymbol{B}) &= \det(\boldsymbol{A})(1 + \boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{A}^{-1}\boldsymbol{u}_1)\end{aligned} \tag{14}$$

However, the above equations only deal with rank-1 updates. In our work, we also need a formula for rank-2 update of the determinant, which can be obtained using the rank-1 update formulas: Consider $\boldsymbol{C} = \boldsymbol{B} + \boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{H}}$. Then

$$\begin{aligned}\det(\boldsymbol{C}) &= \det(\boldsymbol{B})(1 + \boldsymbol{v}_2^{\mathrm{H}}\boldsymbol{B}^{-1}\boldsymbol{u}_2) \\ &= \det(\boldsymbol{A})(1 + \boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{A}^{-1}\boldsymbol{u}_1) \\ &\quad\times\left[1 + \boldsymbol{v}_2^{\mathrm{H}}\left(\boldsymbol{A}^{-1} - \frac{\boldsymbol{A}^{-1}\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{A}^{-1}}{1 + \boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{A}^{-1}\boldsymbol{u}_1}\right)\boldsymbol{u}_2\right] \\ &= \det(\boldsymbol{A})[(1 + \boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{A}^{-1}\boldsymbol{u}_1)(1 + \boldsymbol{v}_2^{\mathrm{H}}\boldsymbol{A}^{-1}\boldsymbol{u}_2) \\ &\quad - \boldsymbol{v}_2^{\mathrm{H}}\boldsymbol{A}^{-1}\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{A}^{-1}\boldsymbol{u}_2]\end{aligned} \tag{15}$$

Now we apply these update formulae to $\boldsymbol{C}(\boldsymbol{R})_{\mathrm{u}}$. First, consider the term $\det(\boldsymbol{W}_{\mathrm{u}})$. From (12) it can readily be seen to be a polynomial in $|\alpha|^2$ and β^2. Applying the rank-1 determinant update formula to it results in

$$\det(\boldsymbol{W}_{\mathrm{u}}) = \sigma_{\mathrm{D}}^{2M}\det(\boldsymbol{W}_0)(1 + |\sigma\alpha|^2\boldsymbol{f}_i^{\mathrm{H}}\boldsymbol{W}_0^{-1}\boldsymbol{f}_i) \tag{16}$$

Its dependence on $|\alpha|^2$ and β^2 can be expressed more concretely in terms of an eigenvalue decomposition of $(\boldsymbol{F}\boldsymbol{R}_0)(\boldsymbol{F}\boldsymbol{R}_0)^{\mathrm{H}}$

$$(\boldsymbol{F}\boldsymbol{R}_0)(\boldsymbol{F}\boldsymbol{R}_0)^{\mathrm{H}} = \boldsymbol{V}_1\boldsymbol{\Sigma}_1\boldsymbol{V}_1^{\mathrm{H}} \tag{17}$$

Then, letting $e_1(i)$ denote the ith eigenvalue (i.e. the ith diagonal element of Σ_1), we have

$$\begin{aligned}\det(\boldsymbol{W}_{\mathrm{u}}) &= \sigma_{\mathrm{D}}^{2M}\prod_{i=1}^{M}[1 + (\sigma\beta)^2 e_1(i)] \\ &\quad\times\{1 + |\sigma\alpha|^2\boldsymbol{f}_i^{\mathrm{H}}\boldsymbol{V}_1[\boldsymbol{I}_M + (\sigma\beta)^2\boldsymbol{\Sigma}_1]^{-1}\boldsymbol{V}^{\mathrm{H}}\boldsymbol{f}_i\}\end{aligned} \tag{18}$$

where the leading product also appears in the common denominator of the braced quantity.

Next, consider the term $\det(\boldsymbol{W}_{\mathrm{u}} + \sigma_x^2\boldsymbol{H}_{\mathrm{u}}\boldsymbol{H}_{\mathrm{u}}^{\mathrm{H}})$ for which we have

$$\begin{aligned}\boldsymbol{W}_{\mathrm{u}} + \sigma_x^2\boldsymbol{H}_{\mathrm{u}}\boldsymbol{H}_{\mathrm{u}}^{\mathrm{H}} &= (\sigma_{\mathrm{D}}^2\boldsymbol{W}_0 + \beta^2\sigma_x^2\boldsymbol{H}_0\boldsymbol{H}_0^{\mathrm{H}}) + (\alpha^{\mathrm{H}}\beta\sigma_x^2)(\boldsymbol{H}_0\boldsymbol{g}_i)\boldsymbol{f}_i^{\mathrm{H}} \\ &\quad + \boldsymbol{f}_i[(\alpha\beta\sigma_x^2)(\boldsymbol{H}_0\boldsymbol{g}_i)^{\mathrm{H}} + (|\alpha\boldsymbol{g}_i|^2\sigma_x^2 \\ &\quad + |\sigma_{\mathrm{R}}\alpha|^2)\boldsymbol{f}_i^{\mathrm{H}}]\end{aligned} \tag{19}$$

Using the rank-2 update formula (15) for matrix determinants with the following identifications of variables

$$\begin{aligned}&\boldsymbol{C} \leftrightarrow \boldsymbol{W}_{\mathrm{u}} + \sigma_x^2\boldsymbol{H}_{\mathrm{u}}\boldsymbol{H}_{\mathrm{u}}^{\mathrm{H}} \\ &\boldsymbol{A} \leftrightarrow \sigma_{\mathrm{D}}^2[\boldsymbol{I}_M + (\sigma\beta)^2\boldsymbol{F}\boldsymbol{R}_0(\boldsymbol{F}\boldsymbol{R}_0)^{\mathrm{H}}] + \beta^2\sigma_x^2\boldsymbol{H}_0\boldsymbol{H}_0^{\mathrm{H}}, \quad \boldsymbol{u}_1 \leftrightarrow \boldsymbol{f}_i \\ &\boldsymbol{v}_1 \leftrightarrow (\alpha^{\mathrm{H}}\beta\sigma_x^2)(\boldsymbol{H}_0\boldsymbol{g}_i) + (|\alpha\boldsymbol{g}_i|^2\sigma_x^2 + |\sigma_{\mathrm{R}}\alpha|^2)\boldsymbol{f}_i \\ &\boldsymbol{u}_2 \leftrightarrow \boldsymbol{H}_0\boldsymbol{g}_i \quad \text{and} \quad \boldsymbol{v}_2 \leftrightarrow (\alpha\beta\sigma_x^2)\boldsymbol{f}_i\end{aligned}$$

we obtain

$$\begin{aligned}\det(\boldsymbol{W}_{\mathrm{u}} + \sigma_x^2\boldsymbol{H}_{\mathrm{u}}\boldsymbol{H}_{\mathrm{u}}^{\mathrm{H}}) &= \det(\boldsymbol{A})[(1 + \boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{A}^{-1}\boldsymbol{u}_1)(1 + \boldsymbol{v}_2^{\mathrm{H}}\boldsymbol{A}^{-1}\boldsymbol{u}_2) \\ &\quad - \boldsymbol{v}_2^{\mathrm{H}}\boldsymbol{A}^{-1}\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{A}^{-1}\boldsymbol{u}_2]\end{aligned} \tag{20}$$

As in the case of $\det(\boldsymbol{W}_{\mathrm{u}})$, its polynomial functional dependence on α and β can be brought out more concretely with an eigenvalue decomposition of a constituent factor of $\boldsymbol{A}$

$$\boldsymbol{F}\boldsymbol{R}_0(\boldsymbol{F}\boldsymbol{R}_0)^{\mathrm{H}} + \sigma_{\mathrm{R}}^{-2}\sigma_x^2\boldsymbol{H}_0\boldsymbol{H}_0^{\mathrm{H}} = \boldsymbol{V}_2\boldsymbol{\Sigma}_2\boldsymbol{V}_2^{\mathrm{H}} \tag{21}$$

Then, letting $e_2(i)$ denote the ith eigenvalue, we have $\det(\boldsymbol{A}) = \sigma_{\mathrm{D}}^{2M}\prod_{i=1}^{M}[1 + (\sigma\beta)^2 e_2(i)]$ and

$$\begin{aligned}&\det(\boldsymbol{W}_{\mathrm{u}} + \sigma_x^2\boldsymbol{H}_{\mathrm{u}}\boldsymbol{H}_{\mathrm{u}}^{\mathrm{H}}) \\ &= \det(\boldsymbol{A})[(1 + l_1\boldsymbol{p}_1^{\mathrm{H}}\boldsymbol{p}_1 + l_2\boldsymbol{p}_2^{\mathrm{H}}\boldsymbol{p}_1)(1 + l_2^{\mathrm{H}}\boldsymbol{p}_1^{\mathrm{H}}\boldsymbol{p}_2) \\ &\quad - (l_2^{\mathrm{H}}\boldsymbol{p}_1^{\mathrm{H}}\boldsymbol{p}_1)(l_1\boldsymbol{p}_1^{\mathrm{H}}\boldsymbol{p}_2 + l_2\boldsymbol{p}_2^{\mathrm{H}}\boldsymbol{p}_2)] \\ &= \sigma_{\mathrm{D}}^{2M}\prod_{i=1}^{M}[1 + (\sigma\beta)^2 e_2(i)][1 + l_1|\boldsymbol{p}_1|^2 + 2\Re(l_2\boldsymbol{p}_2^{\mathrm{H}}\boldsymbol{p}_1) \\ &\quad + |l_2|^2(|\boldsymbol{p}_1^{\mathrm{H}}\boldsymbol{p}_2|^2 - |\boldsymbol{p}_1|^2|\boldsymbol{p}_2|^2)]\end{aligned} \tag{22}$$

where $\Re(\cdot)$ denotes the real part of a quantity and we have made the following definitions to simplify the notation

$$l_1 \triangleq (|\alpha\boldsymbol{g}_i|^2\sigma_x^2 + |\sigma_{\mathrm{R}}\alpha|^2)\sigma_{\mathrm{D}}^{-2}, \quad l_2 \triangleq \alpha\beta\sigma_x^2\sigma_{\mathrm{D}}^{-2} \tag{23}$$

$$\begin{aligned}\boldsymbol{p}_1 &\triangleq [\boldsymbol{I}_M + (\sigma\beta)^2\boldsymbol{\Sigma}_2]^{-(1/2)}\boldsymbol{V}_2^{\mathrm{H}}\boldsymbol{f}_i, \\ \boldsymbol{p}_2 &\triangleq [\boldsymbol{I}_M + (\sigma\beta)^2\boldsymbol{\Sigma}_2]^{-(1/2)}\boldsymbol{V}_2^{\mathrm{H}}(\boldsymbol{H}_0\boldsymbol{g}_i)\end{aligned} \tag{24}$$

In summary, combining (13), (18) and (22), we can express $C(\boldsymbol{R}_u)$ as a function of α and β. Thus we now turn to the problem of finding α and β that maximise $C(\boldsymbol{R}_u)$.

3.2 Optimisation of α and β

To start, note that none of the terms constituting $\det(\boldsymbol{W}_u)$ and $\det(\boldsymbol{W}_u + \sigma_x^2 \boldsymbol{H}_u \boldsymbol{H}_u^H)$ depend on the phase of α except $\Re(l_2 \boldsymbol{p}_2^H \boldsymbol{p}_1)$ that appears in (22). As a result, for any given $|\alpha|$ and β, $C(\boldsymbol{R}_u)$ can be maximised by choosing the phase of α such that $\Re(l_2 \boldsymbol{p}_2^H \boldsymbol{p}_1)$ is maximised. This can be achieved by letting $\angle\alpha = \angle \boldsymbol{p}_1^H \boldsymbol{p}_2$, so that $\Re(l_2 \boldsymbol{p}_2^H \boldsymbol{p}_1) = |l_2 \boldsymbol{p}_2^H \boldsymbol{p}_1|$. The problem thus reduces to one of finding the best $|\alpha|$ and β. However since there is a one-to-one relation between $|\alpha|$ and β [see (10)], we only need to solve for β. After some straightforward algebra based on (13), (18), (22), we can show that the optimal β is one that maximises the following function

$$q(\beta) \triangleq \frac{1 + l_1|\boldsymbol{p}_1|^2 + 2|l_2 \boldsymbol{p}_2^H \boldsymbol{p}_1| + |l_2|^2(|\boldsymbol{p}_1^H \boldsymbol{p}_2|^2 - |\boldsymbol{p}_1|^2|\boldsymbol{p}_2|^2)}{1 + |\sigma\alpha|^2 \boldsymbol{f}_i^H \boldsymbol{V}_1[\boldsymbol{I}_M + (\sigma\beta)^2 \boldsymbol{\Sigma}_1]^{-1} \boldsymbol{V}^H \boldsymbol{f}_i} \times \prod_{i=1}^{M} \frac{1 + (\sigma\beta)^2 e_2(i)}{1 + (\sigma\beta)^2 e_1(i)} \tag{25}$$

where $|\alpha|$, l_1, l_2, $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are all functions of β.

Owing to the complicated nature of (25), there is, in general, no closed-form solution for the optimal β. We need to resort to a search technique, and such techniques are innumerable. A simplest one is non-iterative line search, in which one examines a sufficiently dense subset of all admissible values of β to find the one maximising $q(\beta)$. From (9) and (10), the set of admissible values of β are given by

$$0 < \beta \leq \sqrt{\frac{P_R}{\Sigma_{i=1}^{L} p(i)|r_0(i)|^2}} \tag{26}$$

A second method is to iteratively update a trial solution to β by solving a low-order polynomial approximation to $q(\beta)$ in each iteration. For example, one may, in each iteration, use a quadratic approximation obtained by taking the second-order Taylor series expansions of $q(\beta)$ around some β value and take the β value that maximises the quadratic approximation as the updated trial solution. If this value should fall outside the admissible range given in (26), we may replace it by the nearest boundary value of the range. In addition, if the resulting $q(\beta)$ value should decrease in some iteration, then we may stop the iteration and revert to an earlier solution.

In fact, to find the optimal β one need not work with $q(\beta)$ directly. Any monotone increasing function of $q(\beta)$ can be used in its stead. For example, since, from (25), $q(\beta)$ is a product of multiple factors, it may be easier to consider maximising a logarithm of $q(\beta)$ than $q(\beta)$ itself, for then products become sums. This approach is taken in our implementation of the quadratic approximation method. Moreover, in implementing the quadratic approximation method we have chosen to take the Taylor series expansion at $\beta = 1$. The reason is that, since we adjust one relay gain at a time, the overall optimisation process belongs to the category of alternating optimisation which is guaranteed to converge to a local optimum [25]. Upon convergence, the values in $\boldsymbol{r}_u$ will change little from one iteration to the next. In other words, the optimal β values will approach unity upon convergence of the overall algorithm. Hence a series expansion around $\beta = 1$ should provide a good approximation to the performance surface in the later stages of algorithm progression and benefit its convergence behaviour there. In summary, in our implementation of the quadractic approximation method we seek to maximise $q_l(\beta) \triangleq \log q(\beta)$. For it we define

$$q_a(\beta) \triangleq q_l(1) + q_l'(1)(\beta - 1) + \frac{q_l''(1)}{2}(\beta - 1)^2 \tag{27}$$

where $q_l'(1)$ and $q_l''(1)$ are the first and second derivatives of $q_l(\beta)$ evaluated at $\beta = 1$. The solution to the equation $q_a'(\beta) = 0$ is then taken to be the current trial solution for β.

3.3 Overall algorithm

We summarise the proposed successive optimisation algorithm as follows. It finds a locally optimal solution.

1. Select some relay i for gain adjustment, where i can be chosen in round-robin fashion, for example.
2. Perform eigenvalue decomposition of $(\boldsymbol{F}\boldsymbol{R}_0)(\boldsymbol{F}\boldsymbol{R}_0)^H$ and $\boldsymbol{F}\boldsymbol{R}_0(\boldsymbol{F}\boldsymbol{R}_0)^H + \sigma_R^{-2}\sigma_x^2 \boldsymbol{H}_0 \boldsymbol{H}_0^H$ as in (17) and (21) to find $\boldsymbol{\Sigma}_i$ and $\boldsymbol{V}_i$, $i = 1, 2$.
3. Solve for the β that maximises $q(\beta)$ as given in (25) by a search method, such as the line search or the quadratic approximation method described in the last subsection. Obtain the corresponding $|\alpha|$ using (10) and let $\angle\alpha = \angle \boldsymbol{p}_1^H \boldsymbol{p}_2$.
4. Update the relay gains by setting the gain of the ith relay to α and multiplying the gains of all other relays by β.
5. Exit if some stopping criteria are satisfied; otherwise go to step 1.

4 System design via suboptimal analytical solutions

Although algorithmic optimisation can yield good results, it provides little insight into the analytical properties of the solutions. We thus consider an analytical approach in this section. Since no closed-form solution can be obtained for the general situation, we consider several simplified situations which are more amenable to analysis. In particular, note that in AF systems the receiver noise arises from two sources: the relay noise $\boldsymbol{n}_R$ and the destination terminal noise $\boldsymbol{n}_D$. The design problem becomes mathematically more tractable when one of the two dominates in the overall receiver noise so that the other may be ignored. The results obtained from ignoring one noise source may be viewed as upper bounds on system capacity or as asymptotic performance of the system. For convenience, we term the two simplified conditions the relay noise-dominant condition and the destination noise-dominant condition, respectively.

Interestingly, closed-form analytical solutions are not available for arbitrary L even in these simplified conditions. However such solutions can be found if L is restricted to some specific values depending on M. It thus prompts a (suboptimal) relay selection approach wherein a judiciously selected subset of the relays is used to participate in signal transmission and the subset size is such that an analytical solution exists. This approach also helps us to study the resulting capacity outage diversity and compare it to that of single-hop MIMO systems with or without antenna

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selection [26, 27]. For convenience of reference, Appendix 3 contains a brief overview of some concepts related to the capacity outage diversity of single-hop MIMO systems.

4.1 Destination noise-dominant condition

From (3), when the destination noise dominates in the overall noise we have $\boldsymbol{W} \gtrsim \sigma_{\rm D}^2 \boldsymbol{I}$. Then from (4) we obtain

$$C(\boldsymbol{R}) \lesssim \det\left[\boldsymbol{I}_M + \frac{\sigma_x^2}{\sigma_{\rm D}^2}\boldsymbol{H}^{\rm H}\boldsymbol{H}\right] \tag{28}$$

Hence the capacity is approximately that of an $M \times M$ single-hop point-to-point MIMO system with channel matrix $\boldsymbol{H}$ and transmitted signal-to-received noise power ratio (transmit-to-receive SNR) $\sigma_x^2/\sigma_{\rm D}^2$. However, even in this rather simplified condition, no general solution is available to the optimisation problem (5) for arbitrary $L > M$. However an analytical solution can be obtained for $L = M$. Thus we consider a relay selection approach wherein M relays are selected to perform the relaying. Let the total of $\begin{pmatrix} L \\ M \end{pmatrix}$ selections be indexed from 1 to $\begin{pmatrix} L \\ M \end{pmatrix}$. For the kth selection define the corresponding optimisation target based on (28) as

$$C_{\rm D}^{(k)}(\boldsymbol{R}_{\rm D}) \triangleq \det\left[\boldsymbol{I}_M + \frac{\sigma_x^2}{\sigma_{\rm D}^2}(\boldsymbol{F}_k\boldsymbol{R}_{\rm D}\boldsymbol{G}_k^{\rm H})(\boldsymbol{F}_k\boldsymbol{R}_{\rm D}\boldsymbol{G}_k^{\rm H})^{\rm H}\right] \tag{29}$$

where $\boldsymbol{F}_k \in \mathbb{C}^{M\times M}$ and $\boldsymbol{G}_k \in \mathbb{C}^{M\times M}$, respectively, denote the submatrices of $\boldsymbol{F}$ and $\boldsymbol{G}$ constructed by collecting the columns corresponding to the active relays in the kth selection, and $\boldsymbol{R}_{\rm D}$ denotes the diagonal matrix of relay gains of the active relays. Let $\boldsymbol{r}_{\rm D}(i)$ be the ith diagonal term in $\boldsymbol{R}_{\rm D}$. In high SNR

$$\begin{aligned} C_{\rm D}^{(k)}(\boldsymbol{R}_{\rm D}) &\lesssim \det\left[\frac{\sigma_x^2}{\sigma_{\rm D}^2}(\boldsymbol{F}_k\boldsymbol{R}_{\rm D}\boldsymbol{G}_k^{\rm H})(\boldsymbol{F}_k\boldsymbol{R}_{\rm D}\boldsymbol{G}_k^{\rm H})^{\rm H}\right] \\ &= \left(\frac{\sigma_x}{\sigma_{\rm D}}\right)^{2M}\det(\boldsymbol{F}_k\boldsymbol{F}_k^{\rm H})\det(\boldsymbol{G}_k\boldsymbol{G}_k^{\rm H})\prod_i |r_{\rm D}(i)|^2 \end{aligned} \tag{30}$$

To maximise $C_{\rm D}^{(k)}(\boldsymbol{R}_{\rm D})$ subject to the power constraint (7) given $\boldsymbol{F}_k$ and $\boldsymbol{G}_k$, we equivalently find the optimum $\boldsymbol{r}_{\rm D}$ such that

$$\boldsymbol{r}_{\rm D} = \arg\max_{\boldsymbol{r}_{\rm D}} \prod_i |r_{\rm D}(i)| \tag{31}$$

subject to

$$\sum_{i=1}^{M}(\sigma_{\rm R}^2 + \sigma_x^2\|\boldsymbol{g}_i^{(k)}\|^2)|r_{\rm D}(i)|^2 = P_{\rm R} \tag{32}$$

where $\boldsymbol{r}_{\rm D} = [r_{\rm D}(1), \ldots, r_{\rm D}(M)]^{\rm T}$ and $\boldsymbol{g}_i^{(k)}$ denotes the ith column of $\boldsymbol{G}_k$. Employing the Lagrange multiplier technique leads to the optimum relay power allocation as

$$|r_{\rm D}(i)| = \sqrt{\frac{P_{\rm R}}{M(\sigma_{\rm R}^2 + \sigma_x^2\|\boldsymbol{g}_i^{(k)}\|^2)}} \tag{33}$$

Denote the resulting $C_{\rm D}^{(k)}(\boldsymbol{R}_{\rm D})$ by $C_{\rm DO}^{(k)}(\boldsymbol{R}_{\rm D})$. The final solution is then given by the optimal selection

$$\bar{k} \triangleq \arg\max_k C_{\rm DO}^{(k)}(\boldsymbol{R}_{\rm D}) \tag{34}$$

together with its corresponding optimum relay power allocation.

To see its performance, substitute (33) into (30) and assume $\sigma_x^2\|\boldsymbol{g}_i^{(k)}\|^2 \gg \sigma_{\rm R}^2$ (i.e. consider the high SNR limit). Then we obtain an upper bound for any $C_{\rm DO}^{(k)}(\boldsymbol{R}_{\rm D})$ as

$$C_{\rm DO}^{(k)}(\boldsymbol{R}_{\rm D}) \lesssim \left(\frac{\sigma_x}{\sigma_{\rm D}}\right)^{2M}\det(\boldsymbol{F}_k\boldsymbol{F}_k^{\rm H})\frac{\det(\boldsymbol{G}_k\boldsymbol{G}_k^{\rm H})}{\prod_i\|\boldsymbol{g}_i^{(k)}\|^2}\left(\frac{P_{\rm R}}{\sigma_x^2 M}\right)^M \tag{35}$$

A simpler upper bound can be obtained by considering a QR decomposition of $\boldsymbol{G}_k^{\rm H}$ as $\boldsymbol{G}_k^{\rm H} = \boldsymbol{QT}$, where $\boldsymbol{Q}$ is a unitary matrix and $\boldsymbol{T}$ is an upper triangular matrix. Denote the ith column of $\boldsymbol{T}$ by $\boldsymbol{t}_i$ and the ith diagonal term of $\boldsymbol{T}$ by $\boldsymbol{T}(i, i)$. Then $\|\boldsymbol{t}_i\|^2 = \|\boldsymbol{g}_i^{(k)}\|^2$ because $\boldsymbol{t}_i$ and $\boldsymbol{g}_i^{(k)}$ are related by a unitary transform $\boldsymbol{Q}$ and $|T(i, i)|^2\|\boldsymbol{t}_i\|^2$. Consequently,

$$\begin{aligned} \frac{\det(\boldsymbol{G}_k\boldsymbol{G}_k^{\rm H})}{\prod_i\|\boldsymbol{g}_i^{(k)}\|^2} &= \frac{|\det(\boldsymbol{Q})|^2|\det(\boldsymbol{T})|^2}{\prod_i\|\boldsymbol{g}_i^{(k)}\|^2} \\ &= \prod_{i=1}^{M}\frac{|\boldsymbol{T}(i, i)|^2}{\|\boldsymbol{g}_i^{(k)}\|^2} \le \prod_{i=1}^{M}\frac{\|\boldsymbol{t}_i\|^2}{\|\boldsymbol{g}_i^{(k)}\|^2} = 1 \end{aligned} \tag{36}$$

where equality holds only when $\boldsymbol{G}_k$ has orthogonal columns. Substituting into (35) yields the desired upper bound

$$C_{\rm DO}^{(k)}(\boldsymbol{R}_{\rm D}) < \det(\boldsymbol{F}_k\boldsymbol{F}_k^{\rm H})\left(\frac{P_{\rm R}}{\sigma_{\rm D}^2 M}\right)^M \tag{37}$$

Thus we obtain an upper bound $C_{\rm DU}$ on the capacity measure for the suboptimal solution as

$$\begin{aligned} C_{\rm DO}^{(\bar{k})}(\boldsymbol{R}_{\rm D}) &< \det(\boldsymbol{F}_{\bar{k}}\boldsymbol{F}_{\bar{k}}^{\rm H})\left(\frac{P_{\rm R}}{\sigma_{\rm D}^2 M}\right)^M \\ &\le \max_k \det(\boldsymbol{F}_k\boldsymbol{F}_k^{\rm H})\left(\frac{P_{\rm R}}{\sigma_{\rm D}^2 M}\right)^M \triangleq C_{\rm DU} \end{aligned} \tag{38}$$

$\log_2 C_{\rm DU}$ is actually the asymptotic capacity of an $(M, L; M)_S$ system at transmit-to-receive SNR $P_{\rm R}^M/(\sigma_{\rm D}^2 M)^M$, where $(X, Y; Z)_S$ denotes an $X \times Y$ single-hop point-to-point MIMO system wherein the receiver selects, out of the total Y received antenna signals, the Z that yields the maximum capacity for receiver processing. (See the review in Appendix 3.)

By (35) we may also obtain a lower bound for $C_{\rm DO}^{(\bar{k})}(\boldsymbol{R}_{\rm D})$. Letting

$$\underline{k} = \arg\max_k \frac{\det(\boldsymbol{G}_k\boldsymbol{G}_k^{\rm H})}{\Pi_i\|\boldsymbol{g}_i^{(k)}\|^2} \tag{39}$$

we have the lower bound $C_{\rm DL}$ as

$$C_{\rm DL} \triangleq C_{\rm DO}^{(\underline{k})}(\boldsymbol{R}_{\rm D}) \le \max_k C_{\rm DO}^{(k)}(\boldsymbol{R}_{\rm D}) \tag{40}$$

When $L \gg M$, it becomes more likely to find a set of M nearly orthogonal columns in $\boldsymbol{G}$. In this case, we will have

$\det(\boldsymbol{G}_{\underline{k}}\boldsymbol{G}_{\underline{k}}^{\mathrm{H}})/\prod_i \|\boldsymbol{g}_i^{(k)}\|^2 \lesssim 1$ and thus

$$C_{\mathrm{DL}} = C_{\mathrm{DO}}^{(k)}(\boldsymbol{R}_{\mathrm{D}}) \lesssim \det(\boldsymbol{F}_{\underline{k}}\boldsymbol{F}_{\underline{k}}^{\mathrm{H}})\left(\frac{P_{\mathrm{R}}}{\sigma_{\mathrm{D}}^2 M}\right)^M \tag{41}$$

Now since $\underline{k}$ is a selection based on $\boldsymbol{G}$ without taking $\boldsymbol{F}$ into consideration and since from (41) $\log_2 C_{\mathrm{DL}}$ resembles the form of the capacity of an $M \times M$ single-hop point-to-point MIMO system with channel matrix $\boldsymbol{F}_{\underline{k}}$ at transmit-to-receive SNR $P_{\mathrm{R}}^M/(\sigma_{\mathrm{D}}^2 M)^M$, we can view $\log_2 C_{\mathrm{DL}}$ as the capacity of an $(M, M; M)_S$ system.

Therefore from (38) and (40) we conclude that in the destination-noise dominant condition, the performance of relay selection with optimal power allocation is asymptotically upper-bounded by that of the $(M, L; M)_S$ MIMO antenna selection system and lower-bounded by that of $(M, M; M)_S$. The capacity outage diversity order is thus similarly bounded by that of these two systems. Simulation results in the next section will show that, although the above derivation has been carried out mostly assuming asymptotic conditions, relay selection systems operating in practical conditions exhibit some similar performance characteristics.

4.2 Relay noise-dominant condition

We now turn to the relay noise-dominant situation. Again, no closed-form general solution can be found for arbitrary values of L and M, but a solution can be found if they are related in a specific way. We thus again propose a relay selection scheme.

To start, let N out of the L relays be selected to participate in the relaying, where $N \geq M$ but is otherwise undetermined for the moment. Altogether there are $\binom{L}{N}$ selections. For the jth selection we let $\boldsymbol{F}_j \in \mathbb{C}^{M\times N}$ and $\boldsymbol{G}_j \in \mathbb{C}^{M\times N}$ denote the corresponding channel submatrices of $\boldsymbol{F}$ and $\boldsymbol{G}$, respectively. From the derivation up to (52) in Appendix 1 we may infer that, in the relay noise-dominant situation

$$C_{\mathrm{R}}^{(j)}(\boldsymbol{R}_{\mathrm{R}}) \lesssim \det\left(\boldsymbol{I}_M + \frac{\sigma_x^2}{\sigma_{\mathrm{R}}^2}\boldsymbol{G}_j\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{H}}\boldsymbol{G}_j^{\mathrm{H}}\right) \tag{42}$$

where $\boldsymbol{R}_{\mathrm{R}}$ is the diagonal matrix of relay gains, $\boldsymbol{V}_j \in \mathbb{C}^{N\times M}$ is the matrix of right singular vectors of $\boldsymbol{F}_j\boldsymbol{R}_{\mathrm{R}}$ with its jth column corresponding to the jth largest singular value of $\boldsymbol{F}_j\boldsymbol{R}_{\mathrm{R}}$. Comparing with the situation addressed in Appendix 2 [in particular, see (62)] we find that relay noise-dominant systems behave similarly to systems with very high relay transmission power. Hence by Theorem 2, $C_{\mathrm{R}}^{(j)}(\boldsymbol{R}_{\mathrm{R}})$ is upper-bounded as

$$C_{\mathrm{R}}^{(j)}(\boldsymbol{R}_{\mathrm{R}}) \leq \det\left(\boldsymbol{I}_M + \frac{\sigma_x^2}{\sigma_{\mathrm{R}}^2}\boldsymbol{G}_j\boldsymbol{G}_j^{\mathrm{H}}\right) \tag{43}$$

where equality holds and $C_{\mathrm{R}}^{(j)}(\boldsymbol{R}_{\mathrm{R}})$ is maximised if the rows of $\boldsymbol{F}_j\boldsymbol{R}_{\mathrm{R}}$ span the same space as that of $\boldsymbol{G}_j$. Note that, contrary to the destination noise-dominant case, in the present case the total relay transmission power does not affect the performance at all, only the row space of $\boldsymbol{F}_j\boldsymbol{R}_{\mathrm{R}}$ matters. The relay network should try to align the row space of $\boldsymbol{F}_j\boldsymbol{R}_{\mathrm{R}}$ with that of $\boldsymbol{G}_j$, which is a beamforming problem.

To proceed, let $\boldsymbol{f}_i^{(j)\mathrm{H}}$ denote the ith row of $\boldsymbol{F}_j$. Let $\boldsymbol{O}_j \in \mathbb{C}^{N\times(N-M)}$ be a matrix of basis vectors for the orthogonal complement of the row space of $\boldsymbol{G}_j$; that is, $\boldsymbol{O}_j$ is such that $\boldsymbol{G}_j\boldsymbol{O}_j = \boldsymbol{0}$ where $\boldsymbol{0}$ denotes a zero matrix. Also, let $\boldsymbol{\Phi}_i^{(j)} = \mathrm{diag}(\boldsymbol{f}_i^{(j)\mathrm{H}})\boldsymbol{O}_j$. Immediately we have

$$\boldsymbol{r}_{\mathrm{R}}^{\mathrm{T}}\boldsymbol{\Phi}_i^{(j)} = \boldsymbol{r}_{\mathrm{R}}^{\mathrm{T}}\mathrm{diag}(\boldsymbol{f}_i^{(j)\mathrm{H}})\boldsymbol{O}_j = \boldsymbol{f}_i^{(j)\mathrm{H}}\boldsymbol{R}_{\mathrm{R}}\boldsymbol{O}_j \tag{44}$$

where $\boldsymbol{r}_{\mathrm{R}} \in \mathbb{C}^N$ is the vector formed by the diagonal elements of $\boldsymbol{R}_{\mathrm{R}}$. To make the row space of $\boldsymbol{F}_j\boldsymbol{R}_{\mathrm{R}}$ equal to that of $\boldsymbol{G}_j$, therefore we may equivalently find $\boldsymbol{r}_{\mathrm{R}}$ such that $\boldsymbol{r}_{\mathrm{R}}^{\mathrm{T}}\boldsymbol{\Phi}_i^{(j)} = \boldsymbol{0}$ $\forall_i$. For this, define

$$\boldsymbol{\Phi}_j \triangleq [\boldsymbol{\Phi}_1^{(j)}\ \boldsymbol{\Phi}_2^{(j)}\ \cdots\ \boldsymbol{\Phi}_M^{(j)}] \in \mathbb{C}^{N\times N(N-M)} \tag{45}$$

Then the optimal solution or beamformer $\boldsymbol{r}_{\mathrm{R}}$ should be such that $\boldsymbol{r}_{\mathrm{R}}^{\mathrm{T}}\boldsymbol{\Phi}_j = \boldsymbol{0}$. The existence of such a solution would require $\boldsymbol{\Phi}_j$ to have a non-empty null column space. Therefore let $M(N-M) < N$. Combined with the earlier assumption that $N \geq M$, the only choice is $N = M + 1$ for any $M \geq 2$.

In conclusion, the final solution is given by the selection

$$\bar{j} = \arg \max_{1\leq j\leq \binom{L}{M+1}} C_{\mathrm{R}}^{(j)}(\boldsymbol{R}_{\mathrm{R}}) \tag{46}$$

where for each j, $\boldsymbol{R}_{\mathrm{R}}$ is given by $\mathrm{diag}(\boldsymbol{r}_{\mathrm{R}})$ with $\boldsymbol{r}_{\mathrm{R}}$ being the solution to the equation $\boldsymbol{r}_{\mathrm{R}}^{\mathrm{T}}\boldsymbol{\Phi}_j = \boldsymbol{0}$ and 'normalised' such that $\Sigma_{i=1}^{M+1}(\sigma_{\mathrm{R}}^2 + \sigma_x^2\|\boldsymbol{g}_i^{(j)}\|^2)|r_{\mathrm{R}}(i)|^2 = P_{\mathrm{R}}$ [where $\boldsymbol{g}_i^{(j)}$ is the ith column of $\boldsymbol{G}_j$ and $r_{\mathrm{R}}(i)$ is the ith element of $\boldsymbol{r}_{\mathrm{R}}$].

Regarding its performance, from (43) we see that the resulting capacity measure is approximately given by C_{RO} as follows

$$\max_j C_{\mathrm{R}}^{(j)}(\boldsymbol{R}_{\mathrm{R}}) \simeq \max_j \det\left(\boldsymbol{I}_M + \frac{\sigma_x^2}{\sigma_{\mathrm{R}}^2}\boldsymbol{G}_j\boldsymbol{G}_j^{\mathrm{H}}\right) \triangleq C_{\mathrm{RO}} \tag{47}$$

where the middle expression indicates that $\log_2 C_{\mathrm{RO}}$ should behave similarly to an $(M, L; M+1)_S$ MIMO antenna selection system with transmit-to-receive SNR $\sigma_x^2/\sigma_{\mathrm{R}}^2$. We will not develop upper and lower bounds to the capacity performance as in the destination noise-dominant case because (47) is already a good approximation.

5 Simulation results

In presenting the simulation results, we first consider the performance of the iterative algorithm. We arbitrarily let $M = 4$, $\sigma_x^2 = 1$, $P_{\mathrm{R}} = 10$ and $\sigma_{\mathrm{R}}^2 = \sigma_{\mathrm{D}}^2 = 0.1$. We consider two relay network sizes: $L = 6$ and 12. The channel matrices $\boldsymbol{F}$ and $\boldsymbol{G}$ are generated by letting all their elements be independent and identically distributed (i.i.d.) complex Gaussian random variables. The relays are initialised to an identical gain that satisfies the power constraint (7). Thus their initial performance also serves as a benchmark to compare algorithm results with.

Fig. 2 illustrates the progression of average capacity with number of iterations under two methods of solving for the relay gains adjustment factor β: line search and quadratic approximation, where the former has a much higher computational complexity than the latter. The results show that line search performs better than quadratic approximation, but both show a qualitatively similar convergence behaviour and the final results after convergence are quite close.

Next, we consider the methods derived under the two dominant noise assumptions. This serves two main purposes. First, their performance is compared with two

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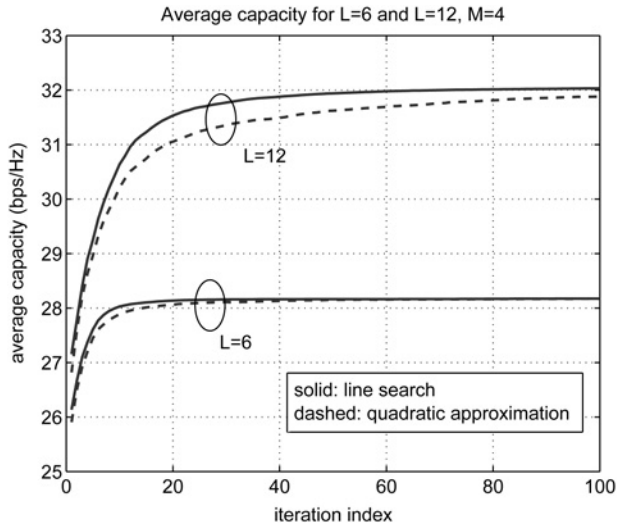

Fig. 2 *Progression of average capacity with number of iterations in algorithmic optimisation*

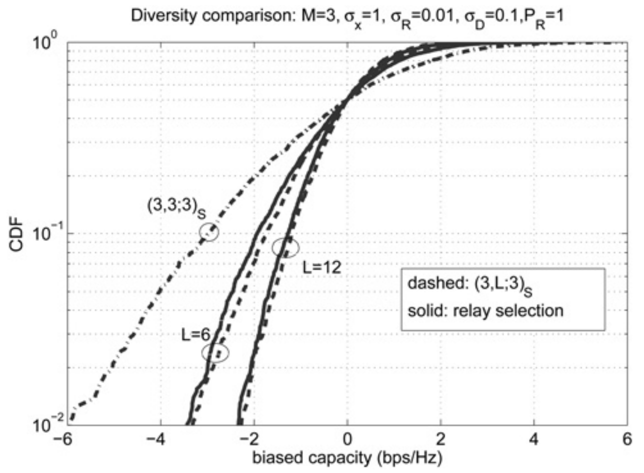

Fig. 4 *Horizontally 'biased' capacity CDF curves of distributed relay systems for diversity comparison*

benchmarks, namely, that of equal-gain allocation and that obtained with our iterative algorithm. Secondly, their capacity outage behaviour is observed. In this we also look at how close the bounds C_{DU} and C_{DL} and the approximation C_{RO} are to the actual results.

For the destination noise-dominant case, Fig. 3 shows some cumulative distribution function (CDF) curves of the obtained capacity at $M = 3$, $L = 6$, $\sigma_x = 1$, $\sigma_D = 0.1$ and $P_R = 1$. Not surprisingly, the iterative algorithm performs better than the suboptimal relay-selection solutions, and the equal-gain allocation performs worse. The C_{DU} curve is rather close to the iterative algorithm results at the same σ_D/σ_R ratio. As to the relay-selection solutions, we see that the capacity performance drops as σ_R increases (which worsens the SNR). However even though the destination noise becomes less dominant with increasing σ_R, the capacity outage diversity order (indicated by the slope of the curve) remains similar and similar to that of C_{DU}.

Next, we consider how the diversity order varies with number of relays (L). Fig. 4 shows some results with all system parameters the same as above except for a fixed $\sigma_R = 10^{-2}$ and a variable L. As the purpose is to examine the diversity order behaviour but not the actual capacity, we 'bias' the CDF curves horizontally to make their 50% points co-located at zero capacity. The curves verify that the proposed relay selection method indeed yields a similar diversity order to C_{DU}, and the diversity order increases (i.e. the CDF curve steepens) with number of relays. To compare with the diversity behaviour of C_{DL}, we also show a curve for a $(3, 3; 3)_S$ system.

Now consider the design based on the relay noise-dominant assumption. Fig. 5 shows some results. Again, the iterative algorithm performs better and the equal-gain worse. We verify that the diversity order behaviour at $\sigma_R/\sigma_D = 10$ is similar to that of a $(3, 6; 4)_S$ system (the behaviour of C_{RO}). As expected, capacity drops as σ_D increases (which lowers the SNR and also makes the relay noise less dominant). However the great difference with Fig. 3 is the reduction in diversity order (i.e. reduction in steepness of CDF curve) with reduced relay noise dominance.

In Fig. 6, we compare the capacity CDFs of distributed relay networks of different sizes, all designed with the relay selection method for the relay noise-dominant condition, with the $(M, L; M+1)_S$ MIMO antenna selection systems. We see that the performance of the latter tightly upper bounds the corresponding distributed relay systems.

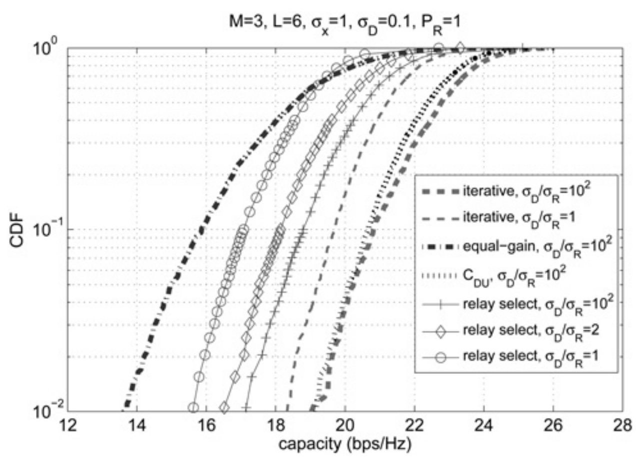

Fig. 3 *Capacity CDF of distributed relay system designed under destination noise-dominant assumption*

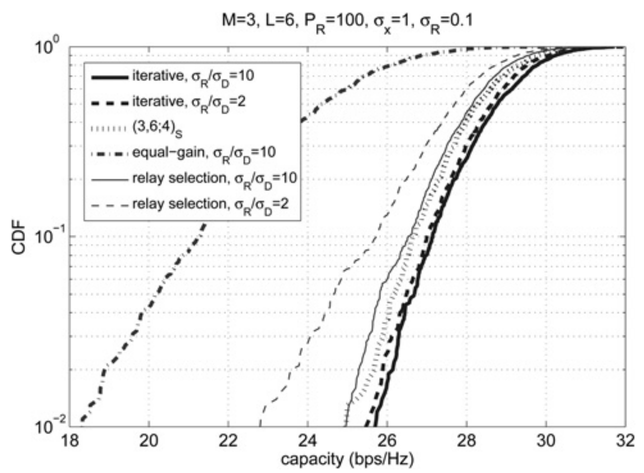

Fig. 5 *Capacity CDF of distributed relay system designed under relay noise-dominant assumption*

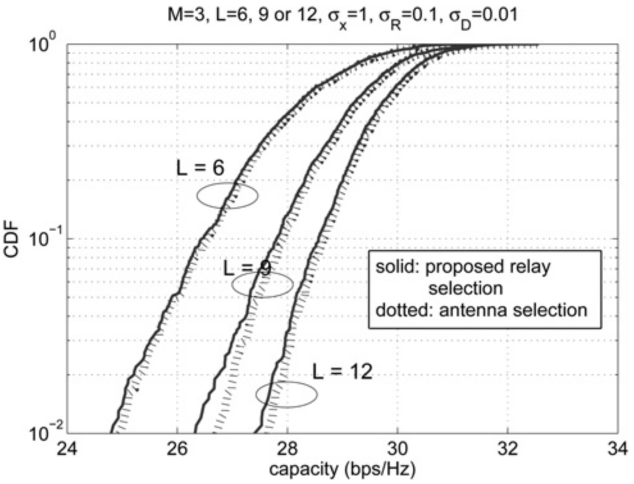

Fig. 6 *Comparison of diversity orders of capacity CDFs of distributed relay systems of different sizes designed under relay noise-dominant assumption. Solid lines show the results for relay systems designed using the proposed method, whereas dotted lines show the behaviour of* $(M, L; M+1)_S$ *antenna selection systems*

6 Conclusion

We considered the design of distributed AF relay networks for two-hop MIMO transmission. More specifically, we considered the determination of relay gains for maximisation of system capacity. As no closed-form analytical solution could be found for the problem, we considered two alternative approaches. One approach was algorithmic, for which we derived an efficient iterative algorithm. Since the algorithmic solution gave little insight into the analytical properties of the solution, we also took an analytical approach, assuming some asymptotic noise conditions. The analytical approach resulted in several relay selection-type of solutions and facilitated an analysis of the diversity behaviour of the solutions. It turned out that their capacity diversity performance behaved similarly to some single-hop point-to-point MIMO antenna selection systems previously analysed by other researchers.

Some simulation results were presented. The results showed that, not surprisingly, the iterative algorithm did yield better designs than the relay selection methods, but at the cost of a substantially higher computational complexity. More significantly, they also confirmed our outage diversity analysis and verified that increasing the number of relays could enhance the outage diversity performance.

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9 Appendix 1: Proof of Theorem 1

First, it is easy to verify for (4) that $C(s\boldsymbol{R}) = C(|s|\boldsymbol{R})$. Thus, without loss of generality, we assume $s \in \mathbb{R}_+$ (the set of positive real numbers) hereunder.

Consider a singular value decomposition of $\boldsymbol{FR}$ as

$$\boldsymbol{FR} = \boldsymbol{U\Lambda V}^{\mathrm{H}} \tag{48}$$

where for convenience we let $\boldsymbol{\Lambda}$ be $M \times M$. Thus $\boldsymbol{U} \in \mathbb{C}^{M\times M}$ is the matrix of left singular vectors as usual, but the matrix of right singular vectors $\boldsymbol{V}$ becomes $L \times M$, that is, $\boldsymbol{V} \in \mathbb{C}^{L\times M}$. Further, let the singular values on the diagonal of $\boldsymbol{\Lambda}$ be arranged in descending numerical order, and let λ_i denote the ith value therein. Substituting the above into (3) and (4), we obtain

$$\boldsymbol{W} = \boldsymbol{U\Sigma U}^{\mathrm{H}} \tag{49}$$

$$\begin{aligned} C(\boldsymbol{R}) &= \det\{\boldsymbol{I}_M + \sigma_x^2 \boldsymbol{G}[(\boldsymbol{FR})^{\mathrm{H}}\boldsymbol{W}^{-1}(\boldsymbol{FR})]\boldsymbol{G}^{\mathrm{H}}\} \\ &= \det[\boldsymbol{I}_M + \sigma_x^2 \boldsymbol{G}(\boldsymbol{V}\overline{\boldsymbol{\Sigma}}\boldsymbol{V}^{\mathrm{H}})\boldsymbol{G}^{\mathrm{H}}] \end{aligned} \tag{50}$$

where $\boldsymbol{\Sigma}$ and $\overline{\boldsymbol{\Sigma}}$ are diagonal matrices with their ith diagonal terms given by

$$\boldsymbol{\Sigma}(i, i) = \sigma_{\mathrm{D}}^2 + (\sigma_{\mathrm{R}}|\lambda_i|)^2 \tag{51}$$

$$\overline{\boldsymbol{\Sigma}}(i, i) = \frac{|\lambda_i|^2}{\sigma_{\mathrm{D}}^2 + (\sigma_{\mathrm{R}}|\lambda_i|)^2} = \frac{|\lambda_i|^2}{\boldsymbol{\Sigma}(i, i)} \tag{52}$$

By scaling $\boldsymbol{R}$ to $s\boldsymbol{R}$ and using $\boldsymbol{W}_s$ to denote the resulting noise correlation matrix at the destination (in place of $\boldsymbol{W}$), we find that

$$\boldsymbol{W}_s = \sigma_{\mathrm{D}}^2\boldsymbol{I} + (s\,\sigma_{\mathrm{R}})^2(\boldsymbol{FR})(\boldsymbol{FR})^{\mathrm{H}} = \boldsymbol{U\Sigma}_s\boldsymbol{U}^{\mathrm{H}} \tag{53}$$

$$\begin{aligned} C(s\boldsymbol{R}) &= \det\{\boldsymbol{I}_M + \sigma_x^2 \boldsymbol{G}[s^2(\boldsymbol{FR})^{\mathrm{H}}\boldsymbol{W}_s^{-1}(\boldsymbol{FR})]\boldsymbol{G}^{\mathrm{H}}\} \\ &= \det[\boldsymbol{I}_M + \sigma_x^2 \boldsymbol{G}(\boldsymbol{V}\overline{\boldsymbol{\Sigma}}_s\boldsymbol{V}^{\mathrm{H}})\boldsymbol{G}^{\mathrm{H}}] \end{aligned} \tag{54}$$

where $\boldsymbol{\Sigma}_s$ and $\overline{\boldsymbol{\Sigma}}_s$ are diagonal matrices with their ith diagonal terms given by

$$\boldsymbol{\Sigma}_s(i, i) = \sigma_{\mathrm{D}}^2 + (s\,\sigma_{\mathrm{R}}|\lambda_i|)^2 \tag{55}$$

$$\overline{\boldsymbol{\Sigma}}_s(i, i) = \frac{s^2|\lambda_i|^2}{\sigma_{\mathrm{D}}^2 + (s\,\sigma_{\mathrm{R}}|\lambda_i|)^2} = a(i)\overline{\boldsymbol{\Sigma}}(i, i) \tag{56}$$

with $a(i)$ defined as

$$a(i) \triangleq \frac{s^2[\sigma_{\mathrm{D}}^2 + (\sigma_{\mathrm{R}}|\lambda_i|)^2]}{\sigma_{\mathrm{D}}^2 + s^2(\sigma_{\mathrm{R}}|\lambda_i|)^2} \tag{57}$$

For $s^2 > 1$, we have $a(i) > 1\ \forall i \in [1,\ M]$. Then since $|\lambda_i| \geq |\lambda_j|$ for all $i, j \in [1, M]$, we obtain $a(i) \leq a(j)\ \forall i \leq j$.

Let $\boldsymbol{a} = [a(1), \ldots, a(M)]^{\mathrm{T}}$ and $\boldsymbol{a}_\Delta = [a_\Delta(1), \ldots, a_\Delta(M)]^{\mathrm{T}}$ where $a_\Delta(i) = a(i) - a(1),\ 1 \leq i \leq M$. The elements of $\boldsymbol{a}_\Delta$ are all non-negative. Further, let $\mathrm{diag}(\boldsymbol{v})$ denote the diagonal matrix having the ith element of vector $\boldsymbol{v}$ as its ith diagonal element. Then from (52) and (56), $\overline{\boldsymbol{\Sigma}}_s$ can be expressed as the sum of two diagonal matrices as

$$\begin{aligned} \overline{\boldsymbol{\Sigma}}_s &= \mathrm{diag}(\boldsymbol{a})\overline{\boldsymbol{\Sigma}} = [a(1)\boldsymbol{I}_M + \mathrm{diag}(\boldsymbol{a}_\Delta)]\overline{\boldsymbol{\Sigma}} \\ &= a(1)\overline{\boldsymbol{\Sigma}} + \underbrace{\mathrm{diag}(\boldsymbol{a}_\Delta)\overline{\boldsymbol{\Sigma}}}_{\triangleq \overline{\boldsymbol{\Sigma}}_\Delta} \end{aligned} \tag{58}$$

Therefore $\overline{\boldsymbol{\Sigma}}_s$ is the sum of two non-negative diagonal matrices. As a result, from (50), (54) and (58) we have

$$C(s\boldsymbol{R}) = \det[\boldsymbol{I}_M + a(1)\boldsymbol{\Omega} + \boldsymbol{\Omega}_\Delta] \tag{59}$$

where

$$\boldsymbol{\Omega} \triangleq \sigma_x^2\boldsymbol{G}(\boldsymbol{V}\overline{\boldsymbol{\Sigma}}\boldsymbol{V}^{\mathrm{H}})\boldsymbol{G}^{\mathrm{H}}, \quad \boldsymbol{\Omega}_\Delta \triangleq \sigma_x^2\boldsymbol{G}(\boldsymbol{V}\overline{\boldsymbol{\Sigma}}_\Delta\boldsymbol{V}^{\mathrm{H}})\boldsymbol{G}^{\mathrm{H}} \tag{60}$$

Note that $\boldsymbol{\Omega}_\Delta$ is positive-semidefinite.

We now invoke an inequality concerning the eigenvalues of sum matrices [28, Section 6.4]. Specifically, let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Hermitian matrices and let $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$. Let the respective ith largest eigenvalues be denoted $\rho_i(\boldsymbol{A})$, $\rho_i(\boldsymbol{B})$ and $\rho_i(\boldsymbol{C})$. Then $\rho_i(\boldsymbol{C}) \geq \rho_i(\boldsymbol{A})$ and $\rho_i(\boldsymbol{C}) \geq \rho_i(\boldsymbol{B})$.

Applying the above inequality to (59), we obtain

$$C(s\boldsymbol{R}) \geq \det[\boldsymbol{I}_M + a(1)\Omega] > \det(\boldsymbol{I}_M + \Omega) = C(\boldsymbol{R}) \tag{61}$$

where the second inequality is due to $a(1) > 1$.

10 Appendix 2: Proof of Theorem 2

From (56), as $|s| \to \infty$ the significance of σ_{D} vanishes. Then $\overline{\boldsymbol{\Sigma}}_s(i, i) \lesssim \sigma_{\mathrm{R}}^{-2}$ and

$$\begin{aligned} C(s\boldsymbol{R}) &\lesssim \det[\boldsymbol{I}_M + (\sigma_x/\sigma_R)^2\boldsymbol{GVV}^{\mathrm{H}}\boldsymbol{G}^{\mathrm{H}}] \\ &= \det[\boldsymbol{I}_L + (\sigma_x/\sigma_R)^2\boldsymbol{G}^{\mathrm{H}}\boldsymbol{GVV}^{\mathrm{H}}] \end{aligned} \tag{62}$$

where $\boldsymbol{V}$ is the matrix of right singular vectors of $\boldsymbol{FR}$ as given in (48). For convenience, let $\rho_i(\boldsymbol{M})$ denote the ith largest eigenvalue of a matrix $\boldsymbol{M}$ that has real eigenvalues. We have

$$\rho_1(\boldsymbol{G}^{\mathrm{H}}\boldsymbol{G}) \geq \rho_2(\boldsymbol{G}^{\mathrm{H}}\boldsymbol{G}) \geq \cdots \geq \rho_M(\boldsymbol{G}^{\mathrm{H}}\boldsymbol{G}) > 0 \tag{63}$$

$$\rho_i(\boldsymbol{VV}^{\mathrm{H}}) = 1, \quad 1 \leq i \leq M \tag{64}$$

$$\rho_i(\boldsymbol{G}^{\mathrm{H}}\boldsymbol{G}) = \rho_i(\boldsymbol{VV}^{\mathrm{H}}) = 0, \quad M+1 \leq i \leq L \tag{65}$$

Now, it is known that for any two non-negative-definite Hermitian matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, we have the eigenvalue relation $\rho_i(\boldsymbol{AB}) \leq \rho_i(\boldsymbol{A})\rho_1(\boldsymbol{B})$ [28, Section 6.6]. Hence

$$\rho_i(\boldsymbol{G}^{\mathrm{H}}\boldsymbol{GVV}^{\mathrm{H}}) \leq \rho_i(\boldsymbol{G}^{\mathrm{H}}\boldsymbol{G})\rho_1(\boldsymbol{VV}^{\mathrm{H}}) = \rho_i(\boldsymbol{G}^{\mathrm{H}}\boldsymbol{G}) \tag{66}$$

and finally

$$C(s\boldsymbol{R}) \leq \det[\boldsymbol{I}_M + (\sigma_x/\sigma_{\mathrm{R}})^2\boldsymbol{GG}^{\mathrm{H}}]$$

11 Appendix 3: capacity outage diversity of MIMO antenna selection systems

Since in Section 4 we use concepts of MIMO antenna selection systems for diversity analysis, we briefly review some known results relevant to this topic.

Consider a point-to-point MIMO system with M transmitter antennas and N receiver antennas, where $N \geq M$. Let $\boldsymbol{H} \in \mathbb{C}^{N\times M}$ be the channel matrix and let the transmitted signal-to-received noise power ratio (transmit-to-receive SNR) ρ^2. Then the system capacity is given by

$$\log_2 C(\boldsymbol{H}) = \log_2 \det(\boldsymbol{I}_M + \rho^2 \boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}) \tag{67}$$

For a flat-fading $\boldsymbol{H}$, a statistical lower bound is [27]

$$\log_2 C(\boldsymbol{H}) \geq \sum_{i=1}^{M} \log_2 (1 + \rho^2 \gamma^2_{N-i+1}) \tag{68}$$

where γ^2_{N-i+1} denotes a gamma-distributed random variable with $N-i+1$ degrees of freedom. This lower bound indicates that the capacity of an $M \times N$ MIMO system is statistically equivalent to or better than that of a system composed of M parallel independent single-input multi-output (SIMO) subsystems wherein the ith subsystem performs maximal-ratio combining (MRC) on $N-M+i$ receiver antennas. In other words, the overall capacity outage diversity of an $M \times N$ MIMO system is bounded between $N-M+1$ and N.

Consider a system where the receiver selects M out of its N antennas for use in signal detection. Let $\boldsymbol{H}_S$ be the $M \times M$ channel matrix of the resulting MIMO channel. This matrix contains the M rows in $\boldsymbol{H}$ that correspond to the selected receiver antennas. There are $\binom{N}{M}$ possible antenna choices. Let $(M, N; M)_S$ denote a system wherein the antennas are chosen to maximise the capacity. Then the system capacity can be described as

$$\log_2 C_S = \max_{\boldsymbol{H}_S} \log_2 \det(\boldsymbol{I} + \rho^2 \boldsymbol{H}_S^{\mathrm{H}}\boldsymbol{H}_S) \tag{69}$$

It is shown in [26] that the capacity of such a system is again statistically equivalent to or better than a MIMO system composed of M parallel independent SIMO subsystems wherein the ith subsystem performs antenna selection by choosing one out of $N-i+1$ receiver antennas.

In a nutshell, both the full system $(M, N; N)_s$ and the receiver antenna selection system $(M, N; M)_s$ can be statistically modelled as a set of parallel SIMO transmissions and thus share a similar capacity outage diversity order.

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