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Effective Control Charts for Monitoring Multivariate Process Dispersion

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When monitoring process dispersion, it is common to pay more attention to dispersion increases than to decreases for practical reasons. Nonetheless, it is also important to detect dispersion decreases for two reasons: (i) it deserves further investigations as to why the process has improved; and (ii) if the process has changed, the settings of the control chart would need to be adjusted for effective future monitoring. In this paper, we first propose an effective control chart for detecting multivariate dispersion decreases in phase II process monitoring, which is constructed using the same approach as that of the one-sided likelihood-ratio-test-based multivariate chart proposed recently in the literature for detecting dispersion increases. We then discuss a combined charting scheme by combining these two one-sided charts for detecting either dispersion increases or decreases. Comparative simulation studies show that the proposed combined control charting scheme outperforms several existing two-sided control charts in terms of the average run length when the process dispersion indeed increases or decreases. Two real-life examples are presented to demonstrate the applicability of the proposed charts.

Keywords: ARL-biased; combined chart; multivariate process dispersion; one-sided likelihood ratio test; phase II monitoring

1. Introduction

1.1. Literature review and motivation

The control chart is a popular and effective online statistical process control tool for process improvement. Usually, the process mean is monitored using location charts such as the $\bar{X}$-chart, and the process dispersion is monitored using dispersion charts such as the *R*-chart. Although a lot more research has been devoted to developing control charts for monitoring process mean in the literature, it is just as important to develop control charts for monitoring process dispersion.

Increases in dispersion most likely would cause some level of deterioration in the quality of the process output and lead to excessive defective units. Conversely, decreases in dispersion would eventually result in improved process/product performance, fewer defects, and lower cost. Although detecting increases in process dispersion is necessary for preventing more defective products from being produced, detecting dispersion decreases has its own merits. First, the process of searching for the causes of the decrease may lead to substantial process improvements. Second, if the process dispersion indeed has decreased, the control limits of the control chart would need to be adjusted accordingly for effective future monitoring. Control charts designed for detecting general dispersion changes (either increase or decrease) may be able to serve this purpose, whereas an effective control chart specifically designed for detecting dispersion decreases faster than general-purposed charts would prove to be beneficial in practice.

Several control charts have been proposed for detecting increases in process dispersion; see, for example, Crowder and Hamilton,[1] Chang and Gan,[2] and Shu and Jiang[3] for the univariate case, and Sakata[4] and Calvin[5] for the multivariate case. Recently, based on the likelihood ratio test (LRT) statistic of a one-sided test, Yen and Shiau[6] proposed an effective control chart specifically designed for detecting dispersion increases for multivariate processes. They also demonstrated via a simulation study that this one-sided chart indeed outperforms several popular existing two-sided charts in terms of the average run length (ARL) when the process dispersion increases. On the other hand, monitoring decreases in dispersion has received less attention, and only a few control schemes for the univariate case have been proposed. See, for example, Nelson,[7] Acosta-Mejia,[8] Huwang *et al*.,[9] and Yeh *et al*.[10] As modern processes are getting more complicated, quite often, the quality of a manufacturing process or product is characterized by multiple quality characteristics. However, to the best of our knowledge, no multivariate control chart has been proposed in the literature specifically for detecting dispersion decreases.

The first objective of this study is to propose a control chart based on the one-sided LRT statistic, which can detect multivariate dispersion decreases more effectively than most existing multivariate control charts. The proposed chart extends the notion of the

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chart proposed by Yen and Shiau[6] for detecting dispersion increases. Note that these one-sided charts designed for detecting just increases or just decreases are suitable only when the practitioner knows exactly in which direction the process dispersion would go when it is out of control.

When both directions are possible, it is conventional to use a two-sided chart that can detect both out-of-control scenarios. However, Pignatiello *et al.*[11] pointed out that an *R*-chart with equal-tail-probability limits is ARL-biased, for which they referred to a chart having its out-of-control ARL possibly larger than its in-control ARL (similar to a biased test for testing a two-sided alternative such as that considered in Pachares).[12] Recently, Huwang *et al.*[9] also showed that a two-sided equal-tail-probability exponentially weighted moving average (EWMA) chart for monitoring the variance (MacGregor and Harris[13]) is ARL-biased. Similar results for the case of individual observations (i.e., when the subgroup size $n=1$) were also reported in Yeh *et al.*[10] In addition, Lowry *et al.*[14] reported that, when using the *R* or *S* chart, it is much more difficult to detect decreases than increases in variance. Similar phenomena for the EWMA chart were also discussed in Huwang *et al.*[9] and Yeh *et al.*[10]

Acosta-Mejia[8] proposed combining two one-sided sample-range-based cumulative sum (CUSUM) charts for monitoring both increases and decreases in variance. Acosta-Mejia *et al.*[15] also proposed a combined chart for monitoring overall changes in variance and showed that their proposed chart outperforms existing two-sided charts. Similar results for the univariate case under $n=1$ were also reported in Yeh *et al.*[10] Inspired by these works, the second objective of this paper is to develop an effective monitoring scheme for simultaneous monitoring of increases and decreases in multivariate process dispersion by combing two effective one-sided control charts.

There has been some research in the last two decades devoted to multivariate process dispersion monitoring. The control charting approaches to monitoring changes in the covariance matrix include multivariate Shewhart, multivariate exponentially weighted moving average (MEWMA), multivariate cumulative sum (MCUSUM), and nonparametric control charts. See, for example, Alt,[16] Alt and Smith,[17] Tang and Barnett,[18,19] Levinson *et al.*,[20] Yeh *et al.*,[21] Runger and Testik,[22] Yeh *et al.*,[23] Djauhari,[24] Reynolds and Stoumbos,[25,26] Huwang *et al.*,[27] Hawkins and Maboudou-Tchao[28], Chenouri *et al.*[29], and Costa and Machado[30]. Excellent reviews of these developments can be found in, for example, Yeh *et al.*[31] and Bersimis *et al.*[32]

1.2. Preliminaries

Suppose that the $p \times 1$ quality characteristic vector $\boldsymbol{X}$ of a multivariate process of interest is distributed as a multivariate normal distribution, denoted by $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with unknown mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Most of the existing charting techniques for monitoring $\boldsymbol{\Sigma}$ are centered on two-sided tests of the hypotheses

$$H_0\text{: } \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0 \text{ versus } H_1\text{: } \boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}_0, \tag{1}$$

where $\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}_0$ are the current and the in-control process covariance matrices of the quality characteristic vector $\boldsymbol{X}$, respectively. These techniques are typically designed to detect general changes in $\boldsymbol{\Sigma}$.

In general, more attention would be paid to the case of dispersion increases than decreases. Presumably, the detecting power of a one-sided test would be larger than that of the corresponding two-sided test if the process dispersion indeed changes in the direction as stipulated in the alternative hypothesis. Yen and Shiau[6] proposed an effective one-sided control chart for monitoring increases in dispersion based on the LRT statistic for testing:

$$H_0\text{: } \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0 \text{ versus } H_1\text{: } \boldsymbol{\Sigma} \geqslant \boldsymbol{\Sigma}_0 \quad \text{and} \quad \boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}_0, \tag{2}$$

where $\boldsymbol{\Sigma} \geqslant \boldsymbol{\Sigma}_0$ means that $\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_0$ is positive semidefinite (p.s.d.). They reported that their one-sided chart outperforms several existing two-sided control charts in terms of the ARL.

As discussed earlier, it is also important to detect decreases in dispersion. An analogous one-sided control chart based on the LRT statistic for testing the following hypotheses will be proposed and studied in this paper:

$$H_0\text{: } \boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0 \text{ versus } H_1\text{: } \boldsymbol{\Sigma} \leqslant \boldsymbol{\Sigma}_0 \quad \text{and} \quad \boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}_0, \tag{3}$$

where $\boldsymbol{\Sigma} \leqslant \boldsymbol{\Sigma}_0$ means that $\boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}$ is p.s.d..

Mathematically, the alternative hypothesis in (3) means that $\boldsymbol{a}'\boldsymbol{\Sigma}\boldsymbol{a} \leqslant \boldsymbol{a}'\boldsymbol{\Sigma}_0\boldsymbol{a}$ for all $p \times 1$ vectors $\boldsymbol{a}$ and that the inequality holds for some $\boldsymbol{a} \neq 0$. Thus, the H_1 in (3) in a sense indicates that the process dispersion has decreased because the variance of every possible linear combination of $\boldsymbol{X}$, var($\boldsymbol{a}'\boldsymbol{X}$) is less than or equal to that when the process is in control.

The rest of the paper is organized as follows. Section 2 describes the proposed one-sided LRT-based control chart for detecting dispersion decreases. Section 3 compares via simulations the ARL performance of the proposed one-sided control chart with that of some existing two-sided control charts based on various tests of H_0: $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$ versus H_1: $\boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}_0$. Section 4 discusses in detail the proposed LRT-based combined control chart. It also compares the proposed combined chart with some existing two-sided charts. Section 5 applies the two proposed charts to two real-life data sets to demonstrate their applicability and effectiveness. Section 6 concludes the paper with a brief summary and discussion.

2. One-sided likelihood-ratio-test-based control chart for monitoring dispersion decreases

Assume that at time t, a subgroup of n $p \times 1$ random vectors $\boldsymbol{X}_{t1}, \cdots, \boldsymbol{X}_{tn}$ is sampled from the process. Each $\boldsymbol{X}_{tj}$ follows the p-dimensional normal distribution $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with both $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ unknown. When the process is in control, $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$. Because in this study, we only focus on phase II monitoring of $\boldsymbol{\Sigma}$, we assume that $\boldsymbol{\Sigma}_0$ is known (or can be estimated quite accurately from in-control phase I data) to avoid the complication of the parameter estimation effects as most of the research do in the literature. Note that the in-control location parameter $\boldsymbol{\mu}_0$ needs not to be assumed known. To test if the process dispersion has decreased, that is, $\boldsymbol{\Sigma} \leqslant \boldsymbol{\Sigma}_0$, we consider the LRT of the hypotheses (3).

Let

$$\bar{\boldsymbol{X}}_t = \frac{1}{n}\sum_{j=1}^{n}\boldsymbol{X}_{tj} \quad \text{and} \quad \mathbf{S}_t = \frac{1}{n}\sum_{j=1}^{n}\left(\boldsymbol{X}_{tj} - \bar{\boldsymbol{X}}_t\right)\left(\boldsymbol{X}_{tj} - \bar{\boldsymbol{X}}_t\right)' \tag{4}$$

denote, respectively, the sample mean and sample covariance matrix of the n observations obtained at time t. Then $\mathbf{B}_t \equiv n\mathbf{S}_t$ follows a Wishart distribution with $n-1$ degrees of freedom and covariance matrix $\boldsymbol{\Sigma}$, denoted by $W_p(n-1, \boldsymbol{\Sigma})$. Note that $\mathbf{S}_t$ is positive definite with probability 1, if and only if $n > p$ (Dykstra[33]).

For monitoring increases in dispersion, Yen and Shiau[6] derived the LRT statistic for testing the hypotheses (2) and proposed to construct a one-sided control chart with the following monitoring statistic

$$T_I = \begin{cases} n\sum_{i=1}^{p_I^*}[(d_i - 1) - \log d_i] & , \text{for } p_I^* > 0 \\ 0 & , \text{for } p_I^* = 0 \end{cases}, \tag{5}$$

where $d_1 \geqslant \cdots \geqslant d_p > 0$ are the roots of $|\mathbf{S}_t - d\boldsymbol{\Sigma}_0| = 0$ and p_I^* is the number of $d_i > 1$, that is, $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_{p_I^*} > 1 \geqslant d_{p_I^*+1} \geqslant \cdots \geqslant d_p > 0$. Similarly, for monitoring decreases in dispersion, we derive and propose, in this paper, the following monitoring statistic

$$T_D = \begin{cases} n\sum_{i=1}^{p_D^*}[(d_i - 1) - \log d_i] & , \text{for } p_D^* > 0 \\ 0 & , \text{for } p_D^* = 0 \end{cases}, \tag{6}$$

where p_D^* is the number of $0 < d_i < 1$, that is, $0 < d_1 \leqslant d_2 \leqslant \cdots \leqslant d_{p_D^*} < 1 \leqslant d_{p_D^*+1} \leqslant d_{p_D^*+2} \leqslant \cdots \leqslant d_p$. The derivation of the LRT statistic in (6) is similar to that for (5) as given in Yen and Shiau[6] and hence is omitted.

The rejection region for the LRT of the hypotheses (3) ((2)) are $\{T_D > T_D(\alpha)\}$ ($\{T_I > T_I(\alpha)\}$), where the critical value $T_D(\alpha)$ ($T_I(\alpha)$) is chosen such that the significance level equals α. In other words, $T_D(\alpha)$ ($T_I(\alpha)$) is the $(1-\alpha)$th quantile of the distribution of T_D (T_I). Consequently, the proposed control chart for monitoring dispersion decreases (increases) plots T_D (T_I) against sampling sequence with the upper control limit (UCL) set at $T_D(\alpha)$ ($T_I(\alpha)$). The chart signals whenever T_D (T_I) exceeds $T_D(\alpha)$ ($T_I(\alpha)$).

Yen and Shiau[6] proved that the statistic T_I is invariant to the distribution parameters, which implies that, without loss of generality, it can be assumed that the in-control parameters $\boldsymbol{\mu}_0 = 0$ and $\boldsymbol{\Sigma}_0 = \mathbf{I}_p$ when studying the distribution of T_I under H_0. This property also holds for T_D.

The exact distribution of T_D is rather difficult to obtain analytically. We used Monte Carlo simulations to estimate the critical value of T_D in this paper. We simulated 1,000,000 values of T_D and obtained the $(1-\alpha)$th quantile of the empirical distribution of T_D. Such a procedure was then repeated 100 times, resulting in 100 $T_D(\alpha)$ estimates. The upper control limit, $\text{UCL}_{p,n,\alpha}$, which depends on p, n, and α, was then estimated by taking the average of the 100 $T_D(\alpha)$ estimates.

Listed in Table I are the $\text{UCL}_{p,n,\alpha}$'s and the corresponding standard errors (in parentheses) obtained by simulation for the T_D-based control chart for $p = 2, 3, 4$, $n = 5, 10, 15, 20, 25, 30, 35, 40$, and $\alpha = 0.05, 0.01, 0.0027$. The UCL seems to decrease as n becomes larger, for given p and α. On the other hand, the UCL increases as p increases, given n and α. Note that the standard error increases as α becomes smaller, which is typical when estimating quantiles.

3. Performance comparison

In this section, we compare the proposed control chart with three existing control charts in terms of the ARL performance. The three existing techniques based on various two-sided tests of (2), which were also considered in Yen and Shiau,[6] include the "two-sided LRT" and "two-sided modified-LRT" control charts and the decomposition-based control chart by Tang and Barnett.[18,19] We did not include the well-known $|S|$-chart, the so-called sample generalized variance chart, in the comparative study because Tang and Barnett[18,19] had shown that their decomposition-based chart outperforms the $|S|$ chart. The superiority was also confirmed by Yeh *et al.*[31] We also did not include the widely used Hotelling T^2 chart (Hotelling[34]) in the comparison because it was mainly designed for detecting mean changes (but notorious for the confounding of location and dispersion changes).

In the following, we briefly discuss the three two-sided control charts. First, the two-sided LRT control chart is based on the statistic (Anderson,[35] p. 439)

$$\lambda^* = \left|\mathbf{S}_t\boldsymbol{\Sigma}_0^{-1}\right|^{n/2} exp\left\{-\frac{n}{2}tr\mathbf{S}_t\boldsymbol{\Sigma}_0^{-1} + \frac{pn}{2}\right\}. \tag{7}$$

Unfortunately, the two-sided LRT based on λ^* is biased. However, by replacing n by $n-1$ in (7), one can obtain an unbiased two-sided LRT based on the following modified likelihood ratio statistic (Anderson[35], p. 440):

Table I. The control limits and their standard errors (in parentheses) of the T_D-based control chart for various p, n, and α

		$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.0027$
$p=2$	$n=5$	12.0739 (0.0016)	17.7394 (0.0037)	22.2362 (0.0065)
	$n=10$	8.8973 (0.0013)	13.3262 (0.0030)	16.8419 (0.0050)
	$n=15$	8.0055 (0.0012)	12.1225 (0.0025)	15.3951 (0.0050)
	$n=20$	7.5507 (0.0010)	11.5204 (0.0023)	14.6795 (0.0045)
	$n=25$	7.2723 (0.0012)	11.1521 (0.0025)	14.2381 (0.0043)
	$n=30$	7.0786 (0.0011)	10.8968 (0.0023)	13.9410 (0.0044)
	$n=35$	6.9346 (0.0010)	10.7061 (0.0023)	13.7181 (0.0041)
	$n=40$	6.8198 (0.0011)	10.5551 (0.0025)	13.5414 (0.0044)
$p=3$	$n=5$	22.9058 (0.0023)	31.4373 (0.0053)	38.1781 (0.0097)
	$n=10$	14.7134 (0.0016)	20.2710 (0.0031)	24.5485 (0.0057)
	$n=15$	12.9198 (0.0014)	17.9484 (0.0030)	21.8259 (0.0052)
	$n=20$	12.0682 (0.0015)	16.8711 (0.0030)	20.5722 (0.0049)
	$n=25$	11.5527 (0.0013)	16.2167 (0.0026)	19.8181 (0.0059)
	$n=30$	11.2030 (0.0011)	15.7773 (0.0029)	19.3201 (0.0052)
	$n=35$	10.9458 (0.0011)	15.4473 (0.0025)	18.9358 (0.0050)
	$n=40$	10.7451 (0.0013)	15.1975 (0.0025)	18.6467 (0.0054)
$p=4$	$n=5$	46.3231 (0.0039)	62.6317 (0.0100)	75.7670 (0.0177)
	$n=10$	22.3340 (0.0020)	29.1848 (0.0042)	34.3739 (0.0078)
	$n=15$	19.0444 (0.0018)	25.0528 (0.0038)	29.5973 (0.0060)
	$n=20$	17.5777 (0.0016)	23.2429 (0.0032)	27.5159 (0.0058)
	$n=25$	16.7219 (0.0017)	22.1899 (0.0033)	26.3228 (0.0061)
	$n=30$	16.1467 (0.0014)	21.4832 (0.0029)	25.5341 (0.0059)
	$n=35$	15.7282 (0.0015)	20.9815 (0.0028)	24.9543 (0.0060)
	$n=40$	15.4085 (0.0013)	20.5919 (0.0027)	24.5220 (0.0062)

$$\lambda^{*(mod)} = \left(\frac{e}{n-1}\right)^{\frac{p(n-1)}{2}} \left|\mathbf{B}_t\boldsymbol{\Sigma}_0^{-1}\right|^{(n-1)/2} exp\left\{-\frac{1}{2}tr\mathbf{B}_t\boldsymbol{\Sigma}_0^{-1}\right\}; \tag{8}$$

see Sugiura and Nagao.[36] The control chart based on (8) will be referred to as the two-sided modified-LRT control chart.

Finally, assuming that Σ_0 is known, Tang and Barnett[18,19] proposed a multivariate Shewhart chart based on decomposing $\mathbf{B}_t/(n-1)$ into a sum of a series of independent χ^2 statistics. For more details, see Tang and Barnett.[18,19] This control chart will be referred to as the TB-decomposed control chart.

3.1. Comparisons

Denote the ARL of the in-control process and out-of-control process by ARL_0 and ARL_1, respectively. Let T be the charting statistic of a control chart. To estimate the ARL, we first generate N statistics, $T_1, \ldots, T_N$, for a very large number N, and compute the proportion of the T_i's that exceed the control limit set for achieving a preset false-alarm rate α. The aforementioned steps are repeated b times, thus resulting in b proportions. The b proportions are then averaged, and the reciprocal of the average is taken as the ARL estimate, denoted by $\widehat{\mathrm{ARL}}$. The standard error of $\widehat{\mathrm{ARL}}$ can be obtained as follows. Note that, multiplying each proportion by N, the b statistics thus obtained are independent and identically distributed (i.i.d.) as Binomial(N, θ), where θ is the probability that the statistic T of a randomly selected sample exceeds the control limit. When the process is in control, $\theta=\alpha$, the false-alarm rate. Because $\widehat{\mathrm{ARL}}$ is the reciprocal of the maximum likelihood estimator (MLE) of θ, then, by the asymptotic efficiency property of MLE, it can easily be shown that $\widehat{\mathrm{ARL}}$ follows a limiting normal distribution with mean $1/\theta$ and standard deviation $\sqrt{\frac{1-\theta}{Nb\theta^3}}$. Therefore, the standard error of this ARL estimator can be calculated by

$$\left[\widehat{\mathrm{ARL}}^2(\widehat{\mathrm{ARL}}-1)/(Nb)\right]^{\frac{1}{2}}. \tag{9}$$

Alternatively, because the reciprocal of each proportion is an estimate of ARL, it is very common to take the average of these b i.i.d. ARL estimates as $\widehat{\mathrm{ARL}}$ and the sample standard deviation of the b reciprocals divided by $\sqrt{b}$ as its standard error. We remark that the difference between the two approaches is negligible when N is large, such as the N used in our simulation studies. However, when N is not large enough, our simulation study indicates that the first approach provides a more accurate estimate of ARL than the second approach, in the sense that its $\widehat{\mathrm{ARL}}$ is closer to the true ARL with a smaller standard error.

Assume that $\boldsymbol{\Sigma}_0$ has decreased to $\boldsymbol{\Sigma}$, that is, $\boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}$ is p.s.d. and $\boldsymbol{\Sigma} \neq \boldsymbol{\Sigma}_0$. When simulating the distribution of T_D under H_0, without loss of generality, we can assume that $\boldsymbol{\Sigma}_0 = \mathbf{I}_p$ because the distribution of T_D is invariant in $\boldsymbol{\Sigma}_0$ as discussed earlier. For simplicity, we consider $p=2$. Express $\mathbf{\Sigma}$ as $\begin{bmatrix} \Delta_1 & \rho\sqrt{\Delta_1\Delta_2} \\ \rho\sqrt{\Delta_1\Delta_2} & \Delta_2 \end{bmatrix}$, where ρ is the correlation coefficient. Note that when studying dispersion decreases, we consider $\Delta_i \leqslant 1$, $i=1,2$. Also note that $\mathbf{\Sigma}_0 - \mathbf{\Sigma}$ being p.s.d. restricts ρ to

$$|\rho| \leqslant \left[\frac{(1-\Delta_1)(1-\Delta_2)}{\Delta_1\Delta_2}\right]^{\frac{1}{2}}. \tag{10}$$

Thus, the case when only the correlation changes, that is, $\Delta_1=\Delta_2=1$ and $\rho \neq 0$, is not an out-of-control scenario as set in the alternative hypothesis H_1 of (3).

In our simulation study, setting $\alpha=0.0027$ (which results in $\text{ARL}_0=370$), we consider the cases of $n=5, 10$. The in-control and out-of-control covariance matrices are $\mathbf{\Sigma}_0 = \mathbf{I}$ and $\mathbf{\Sigma}$, respectively. The following scenarios of $\mathbf{\Sigma}$ are considered:

(i) $\Delta_1=\Delta_2=c$ and $\rho=0$ (that is, $\mathbf{\Sigma} = c\mathbf{\Sigma}_0$) for $c=0.9$, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1.
(ii) $\Delta_1 \neq \Delta_2$ and $\rho=0$ for the following eight combinations: $(\Delta_1, \Delta_2)=(0.8,1)$, (0.6,1), (0.4,1), (0.2,1), (0.8,0.6), (0.6,0.4), (0.4,0.2), (0.2,0.8).
(iii) For $\rho \neq 0$, under the condition (10), we choose $|\rho|=0.2$ and 0.4 for the following four combinations: $(\Delta_1, \Delta_2)=(0.6,0.6)$, (0.6,0.4), (0.4,0.4), (0.4,0.2). Note that these four combinations are selected from scenarios (i) and (ii) so that we can study the effect of ρ on the ARL performance.

In the simulation study, for each scenario, we take $b\,(=100)$ replications of $N\,(=1{,}000{,}000)$ simulated values of the four competing control charting statistics for each replicate to obtain $\widehat{\text{ARL}}$ along with its standard error. From Table I, for $\alpha=0.0027$, the control limits of the proposed T_D-based chart are 22.2362 and 16.8419 for $n=5$ and 10, respectively. As for the two-sided LRT and two-sided modified-LRT charts, the control limits are, respectively, 22.6815 and 17.6769 for $n=5$; and 17.536 and 15.4539 for $n=10$. It was confirmed by simulation that, with the aforementioned control limits, $\text{ARL}_0 \approx 370$, and the corresponding standard error is around 0.7118. The control limit of the TB-decomposed control chart is $\chi^2_3(0.9973) = 14.1563$ for both $n=5$, 10. Table II ($n=5$) and Table III ($n=10$) list the estimates of ARL_1 and their standard errors (in parentheses) of the four control charts under comparison for the scenarios described earlier. The following observations can be made:

- When dispersion decreases, the proposed T_D-based one-sided control chart outperforms the three competing control charts in all the cases tested.
- It is interesting to note that, opposite to the increase case, the two-sided LRT chart has a better ARL_1 performance than the two-sided modified-LRT chart. The worst performer is the TB-decomposed control chart and it is ARL-biased.
- For the effect of ρ, because the eigenvalues of $\mathbf{\Sigma}_0 - \mathbf{\Sigma}$ depend on ρ through ρ^2, the sign of ρ does not play any role in the ARL_1 performance. Also, a larger subgroup size n leads to a smaller ARL_1 value. For fixed n and ρ, the ARL_1 decreases when both Δ_1 and Δ_2 decrease, or when one decreases and the other is fixed. The ARL_1 also decreases when $|\rho|$ increases from 0 to 0.4, an interesting phenomenon that was also found in Yen and Shiau[6] for the increase case.

3.2. Comparing the chart performance for increases versus decreases in dispersion

Lowry *et al.*[14] reported that detecting decreases in variance is much harder than detecting increases in the univariate case. In this subsection, we study this same issue for the multivariate case with the control charts based on T_D (for decreases) and T_I (for increases).

Although the magnitude of a dispersion decrease can only range in (0,1), the range for a dispersion increase is $(1,\infty)$. Thus, for a fair comparison, it is more reasonable to use the logarithm scale for the size of changes, that is, log(*c*), so that the two ranges become $(0,\infty)$ and $(-\infty,0)$. For the case of $\mathbf{\Sigma} = c\mathbf{\Sigma}_0$ with $c=1.25$, 1.5, 1.75, 2, 2.25, 2.5, 2.75, 3 for dispersion increases and $c = \frac{1}{1.25}, \frac{1}{1.5}, \frac{1}{1.75}, \frac{1}{2}, \frac{1}{2.25}, \frac{1}{2.5}, \frac{1}{2.75}, \frac{1}{3}$ for decreases, we compare the ARL_1 values of the one-sided chart with that of the three competing charts discussed earlier. By comparing the T_I and T_D-based control charts, the following are observed.

- From the statistics T_I and T_D, it can be seen that one separates the eigenvalues d_is of $\mathbf{S}_t\mathbf{\Sigma}_0^{-1}$ into two disjoint sets, one for T_I and the other for T_D. Hence, the rejection regions of testing (2) and (3) are disjoint.
- Figure 1 depicts the ARL_1 curves of all four control charts under comparison for both cases of monitoring increases and decreases for $n=5$, 10. The scale of the shift size c is in natural logarithm. Observed from the ARL_1's, we find that the T_I-based chart has a better performance than the T_D-based chart. The comparison is based on the same absolute logarithm scale of values c (for example, $\log(1.25)=|\log(0.8)|$, $\log(2)=|\log(0.5)|$, $\log(2.5)=|\log(0.4)|$). Take $\log(1.25)=\log(0.8)$ as an example, the ARL_1 value of the T_I and T_D-based charts when $n=5$ are 69.211 and 233.475, respectively. The results shown in Figure 1 confirm that detecting decreases in dispersion is also more difficult than detecting increases in the multivariate case, similar to the univariate case as reported in Lowry *et al.*[14]

As for the three competing control charts under comparison in this paper, the two-sided LRT control chart is ARL-biased for monitoring increases in dispersion, whereas the TB-decomposed control chart is ARL-biased for detecting dispersion decreases. The two-sided modified-LRT chart is not only ARL-unbiased but also performs better for detecting dispersion increases than decreases, just like the T_I chart versus T_D chart as described earlier.

Table II. The ARL_1 and their standard errors (in parentheses) of the T_D-based and two-sided control charts for $p=2$ and $n=5$

$p=2$		$n=5$			
			Two-sided		
Δ_1	Δ_2	One-sided	LRT	Modified-LRT	TB-decomposed
[$\rho=0$]					
0.9	0.9	298.983 (0.5161)	314.880 (0.5579)	353.000 (0.6623)	496.845 (1.106)
0.8	0.8	233.475 (0.3560)	254.553 (0.4053)	310.482 (0.5462)	537.819 (1.246)
0.7	0.7	175.510 (0.2319)	195.367 (0.2724)	252.949 (0.4015)	479.370 (1.049)
0.6	0.6	124.864 (0.1390)	140.358 (0.1657)	189.433 (0.2600)	364.898 (0.6961)
0.5	0.5	82.6634 (0.0747)	93.2962 (0.0896)	129.793 (0.1473)	244.197 (0.3808)
0.4	0.4	49.4864 (0.0345)	55.8158 (0.0413)	79.3682 (0.0703)	142.781 (0.1700)
0.3	0.3	25.5262 (0.0126)	28.6359 (0.0151)	41.0587 (0.0260)	69.1756 (0.0571)
0.2	0.2	10.3866 (0.0033)	11.5322 (0.0037)	16.3339 (0.0064)	25.0818 (0.0123)
0.1	0.1	2.8130 (0.0004)	3.0347 (0.0004)	3.9901 (0.0007)	5.2730 (0.0011)
0.8	1	292.969 (0.5006)	305.901 (0.5342)	338.103 (0.6208)	399.009 (0.7960)
0.6	1	211.343 (0.3065)	225.849 (0.3387)	263.982 (0.4281)	313.678 (0.5547)
0.4	1	128.709 (0.1455)	138.392 (0.1622)	166.308 (0.2138)	184.734 (0.2504)
0.2	1	52.6129 (0.0378)	56.6234 (0.0422)	68.9434 (0.0568)	67.5800 (0.0551)
0.8	0.6	170.073 (0.2211)	188.601 (0.2583)	242.327 (0.3765)	472.288 (1.025)
0.6	0.4	77.8419 (0.0682)	87.8123 (0.0818)	121.782 (0.1338)	236.375 (0.3627)
0.4	0.2	21.8535 (0.0100)	24.4871 (0.0119)	34.8693 (0.0203)	60.2424 (0.0464)
0.2	0.8	43.6235 (0.0285)	48.4802 (0.0334)	64.2977 (0.0512)	81.0882 (0.0726)
[$\rho=0.2$]					
0.6	0.6	117.658 (0.1271)	131.958 (0.1510)	176.398 (0.2336)	353.679 (0.6642)
0.6	0.4	73.3948 (0.0625)	82.6247 (0.0747)	114.051 (0.1213)	229.145 (0.3461)
0.4	0.4	46.7310 (0.0316)	52.6636 (0.0379)	74.6288 (0.0640)	137.985 (0.1615)
0.4	0.2	20.7560 (0.0092)	23.2491 (0.0110)	33.0219 (0.0187)	58.3553 (0.0442)
[$\rho=0.4$]					
0.6	0.6	96.2540 (0.0939)	106.761 (0.1098)	139.138 (0.1635)	315.542 (0.5596)
0.6	0.4	60.5702 (0.0468)	67.8713 (0.0555)	92.2621 (0.0881)	206.056 (0.2951)
0.4	0.4	38.8405 (0.0239)	43.6855 (0.0285)	61.3143 (0.0476)	123.844 (0.1373)
0.4	0.2	17.4736 (0.0071)	19.5435 (0.0084)	27.5368 (0.0142)	52.5819 (0.0378)

4. A combined chart based on the two one-sided likelihood-ratio-test-based control charts

4.1. A combined likelihood-ratio-test-based control chart

For monitoring the variance in the univariate case, as pointed out by Acosta-Mejia,[8] Acosta-Mejia *et al.*,[15] and Yeh *et al.*,[10] combining two one-sided charts leads to better performance in detecting overall variance changes than does a two-sided chart. As stated earlier, our second objective in this paper is to study whether combining the two effective one-sided charts for detecting multivariate dispersion will do better than the two-sided charts. The answer is, it depends on how we split the overall false-alarm rate to the two individual charts. In our case, the combined chart signals an out-of-control alarm if

$$T_I > T_I(\alpha_I) \quad \text{or} \quad T_D > T_D(\alpha_D),$$

where the critical values $T_I(\alpha_I)$ and $T_D(\alpha_D)$ are taken as the control limits, which are obtained by controlling the type I error probability. Although mathematically, the two rejection regions may not be disjoint, our simulation study indicated that they are disjoint in practice. This could be because each test takes care of one side of the alternative, and that the two test statistics, T_I and T_D are calculated with two disjoint sets of eigenvalues, $\{d_i|d_i>1\}$ and $\{d_i|0<d_i<1\}$. Hence, the type I error probability for the combined control chart is practically $\alpha_I+\alpha_D$. This property makes our search for appropriate values of (α_I, α_D) much easier.

Assuming, without loss of generality, that $\boldsymbol{\mu}_0=\mathbf{0}$ and $\boldsymbol{\Sigma}_0=\mathbf{I}_p$, we generate $N=1,000,000$ independent samples of size n, each from $N_p(\mathbf{0}, \mathbf{I}_p)$. For each sample, we compute the eigenvalues of the sample covariance matrix S_t and the statistics T_I and T_D. Then, for a given α_I (α_D), the control limit is the $100(1-\alpha_I)$ ($100(1-\alpha_D)$) percentile of the N simulated values of T_I (T_D). To make the combined chart perform well in both directions of dispersion changes, α_I and α_D (satisfying $\alpha=\alpha_I+\alpha_D$) are chosen by a search algorithm to obtain a potentially ARL-unbiased combined chart.

4.2. Unequal-tail-probability control limits

As discussed earlier, in the univariate case, a chart with equal-tail-probability limits for detecting changes in dispersion is ARL-biased. Extending to the multivariate case, consider $p=2$ and use $\alpha_I=\alpha_D=\alpha/2$ for the proposed combined chart. For the case of $\boldsymbol{\Sigma}=c\boldsymbol{\Sigma}_0$,

Table III. The ARL_1 and their standard errors (in parentheses) of the T_D-based and two-sided control charts for $p=2$ and $n=10$

$p=2$			$n=10$		
			Two-sided		
Δ_1	Δ_2	One-sided	LRT	Modified-LRT	TB-decomposed
[$\rho=0$]					
0.9	0.9	231.379 (0.3512)	264.566 (0.4295)	320.872 (0.5739)	447.483 (0.9455)
0.8	0.8	135.346 (0.1569)	165.032 (0.2114)	220.625 (0.3270)	361.246 (0.6857)
0.7	0.7	73.8704 (0.0631)	91.9113 (0.0876)	128.586 (0.1452)	220.068 (0.3257)
0.6	0.6	37.0430 (0.0222)	45.9560 (0.0308)	65.3752 (0.0525)	111.343 (0.1170)
0.5	0.5	16.9510 (0.0068)	20.6787 (0.0092)	29.1816 (0.0155)	48.0918 (0.0330)
0.4	0.4	7.1178 (0.0018)	8.4318 (0.0023)	11.5158 (0.0037)	17.8846 (0.0074)
0.3	0.3	2.8616 (0.0004)	3.2408 (0.0005)	4.1384 (0.0007)	5.8390 (0.0013)
0.2	0.2	1.3075 (0.0001)	1.3822 (0.0001)	1.5626 (0.0001)	1.8825 (0.0002)
0.1	0.1	1.0011 ($<10^{-5}$)	1.0020 ($<10^{-5}$)	1.0050 ($<10^{-5}$)	1.0132 ($<10^{-5}$)
0.8	1	218.002 (0.3211)	241.937 (0.3755)	282.737 (0.4746)	330.960 (0.6012)
0.6	1	102.165 (0.1028)	116.061 (0.1245)	142.131 (0.1689)	168.282 (0.2177)
0.4	1	33.7151 (0.0193)	38.1904 (0.0233)	46.9748 (0.0319)	53.6292 (0.0389)
0.2	1	5.9238 (0.0013)	6.5425 (0.0015)	7.7469 (0.0020)	8.3353 (0.0023)
0.8	0.6	67.9924 (0.0557)	83.8212 (0.0763)	115.561 (0.1237)	204.400 (0.2915)
0.6	0.4	14.9257 (0.0056)	18.1365 (0.0075)	25.3001 (0.0125)	42.7605 (0.0276)
0.4	0.2	2.3444 (0.0003)	2.6242 (0.0003)	3.2625 (0.0005)	4.6052 (0.0009)
0.2	0.8	4.9007 (0.0010)	5.6664 (0.0012)	7.0973 (0.0018)	8.5422 (0.0024)
[$\rho=0.2$]					
0.6	0.6	31.9588 (0.0178)	39.3761 (0.0244)	55.0788 (0.0405)	102.047 (0.1026)
0.6	0.4	13.1879 (0.0046)	15.9629 (0.0062)	21.9999 (0.0101)	39.8826 (0.0249)
0.4	0.4	6.4225 (0.0015)	7.5859 (0.0020)	10.2652 (0.0031)	16.8478 (0.0067)
0.4	0.2	2.1987 (0.0002)	2.4509 (0.0003)	3.0184 (0.0004)	4.4419 (0.0008)
[$\rho=0.4$]					
0.6	0.6	20.1843 (0.0088)	24.1900 (0.0117)	32.0725 (0.0179)	72.8507 (0.0618)
0.6	0.4	8.9502 (0.0025)	10.6660 (0.0033)	14.1725 (0.0051)	30.7313 (0.0168)
0.4	0.4	4.6744 (0.0009)	5.4539 (0.0012)	7.1485 (0.0018)	13.5552 (0.0048)
0.4	0.2	1.8174 (0.0002)	1.9974 (0.0002)	2.3868 (0.0003)	3.8726 (0.0007)

Figure 2 depicts the ARL_1 for various values of c. Because some of these values are greater than 370, this demonstrates that using equal-tail probabilities for the proposed combined chart also leads to an ARL-biased chart. Hence, we suggest using unequal tail probabilities to construct the control limits of the proposed combined control chart. We demonstrated in Section 3 that the power of the one-sided chart based on T_D for monitoring decreases in dispersion is worse than that of the one-sided chart based on T_I for monitoring increases in dispersion. Therefore, it is necessary to set

$$\alpha_I < \alpha/2 \quad \text{and} \quad \alpha_D = \alpha - \alpha_I. \tag{11}$$

Through computer search, we have found many combinations of (α_I,α_D) satisfying (11), and at the same time, the corresponding combined charts are most likely to be ARL-unbiased. In this paper, for $p=2$ and $\alpha=0.0027$, we present ten such combinations: (0.000515, 0.002185), (0.000415, 0.002285), (0.000395, 0.002305), (0.000375, 0.002325), (0.000275, 0.002425) for $n=5$ and (0.000715, 0.001985), (0.000635, 0.002065), (0.000615, 0.002085), (0.000595, 0.002105), (0.000515, 0.002185) for $n=10$. For each control chart, we generate N (=200,000) simulated values of T_I (T_D) to get the $(1-\alpha_I)$th ($(1-\alpha_D)$th) quantile as an estimate of the upper control limit. To get more precision, we repeat the procedure b (=100) times and take the average of these b estimates as the estimate of the upper control limit. The standard error of this estimate is obtained as before. Table IV lists for practitioners the upper control limits and the corresponding standard errors (in parentheses) for the ten (α_I,α_D)'s considered. These combinations were chosen because their ARL_1 values are smaller than that of the two-sided modified-LRT control chart for all the cases under study.

4.3. Comparisons and discussions

Similar to (10) for the decrease case, the restricted range of ρ for the case of dispersion increases is

$$|\rho| \leqslant \left[\frac{(\Delta_1-1)(\Delta_2-1)}{\Delta_1\Delta_2}\right]^{\frac{1}{2}} \tag{12}$$

under the condition that $\mathbf{\Sigma}-\mathbf{\Sigma}_0$ is p.s.d.. Thus, we also do not consider the case when $\Delta_1=\Delta_2=1$ and $\rho\neq 0$ as an out-of-control scenario.

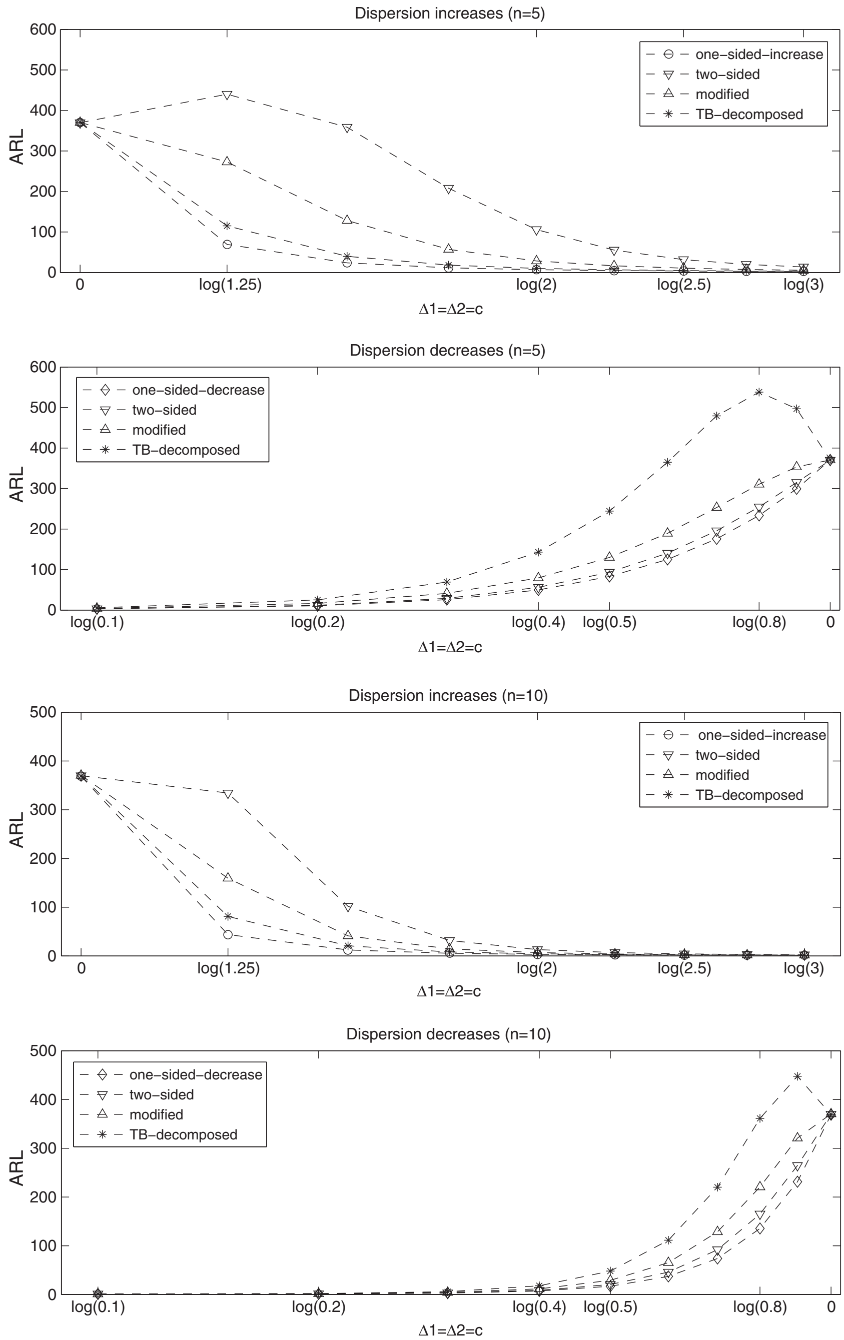

Figure 1. The average run length curves of the four control charts under comparison for both dispersion increases and decreases ($n=5$, 10)

Setting $\alpha=0.0027$ and considering $p=2$ and $n=5$, 10, the following three out-of-control scenarios for $\mathbf{\Sigma}$ are considered:

(i) $\Delta_1=\Delta_2=c$ and $\rho=0$ (that is, $\mathbf{\Sigma}=c\mathbf{\Sigma}_0$) for $c=1.25$, 1.35, 1.5, 1.75, 2, 2.25, 2.5, 2.75, 3 (for increases) and 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1 (for decreases).

(ii) $\Delta_1\neq\Delta_2$ and $\rho=0$ for the following eight combinations: $(\Delta_1,\Delta_2)=(1.25,1)$, (1.75,1), (2.25,1), (2.75,1), (1.25,1.75), (1.75,2.25), (2.75,1.25), (2.25,2.75) (for increases) and (0.8,1), (0.6,1), (0.4,1), (0.2,1), (0.8,0.6), (0.6,0.4), (0.4,0.2), (0.2,0.8) (for decreases).

(iii) To study the effect of ρ on ARL performance, for $\rho\neq0$, under conditions (10) and (12), we choose $|\rho|=0.2$ and 0.4 for the following eight combinations: $(\Delta_1,\Delta_2)=(1.75,1.75)$, (1.75,2.25), (2.25,2.25), (2.25,2.75) (for increases) and (0.6,0.6), (0.6,0.4), (0.4,0.4), (0.4,0.2) (for decreases).

Tables V (for $n=5$) and Table VI (for $n=10$) give the simulated ARL$_1$'s and their standard errors (in parentheses) of the proposed combined chart, two-sided LRT, two-sided modified-LRT, and TB-decomposed control charts for the scenarios (i)–(iii) described earlier. From Tables V and VI, we summarize the comparisons as follows.

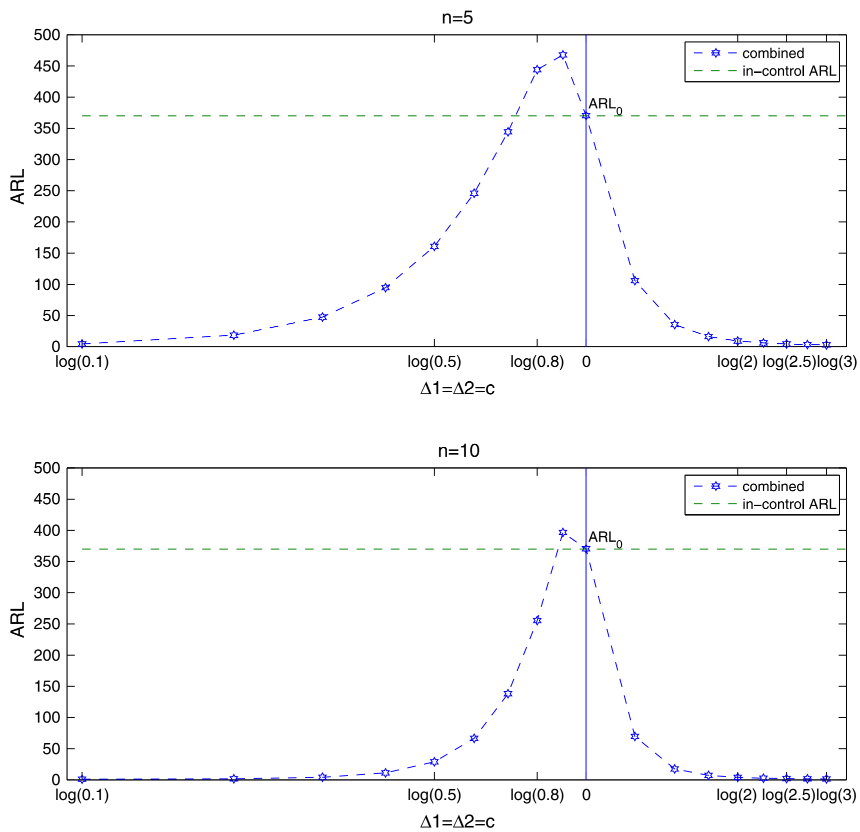

Figure 2. The average run length curves of the proposed combined chart with equal tail probabilities ($\alpha_I = \alpha_D = 0.00135$) for $n = 5$, 10, and $p = 2$

Table IV. The control limits and their standard errors (in parentheses) of the combined chart for five combinations of (α_I,α_D) when $p = 2$

α_I α_D	0.000515 0.002185	0.000415 0.002285	0.000395 0.002305	0.000375 0.002325	0.000275 0.002425
UCL	11.0242 22.9694	11.4214 22.8159	11.5120 22.7870	11.6090 22.7574	12.1762 22.6138
($n = 5$)	(0.0083) (0.0076)	(0.0089) (0.0074)	(0.0090) (0.0072)	(0.0091) (0.0072)	(0.0115) (0.0071)
α_I α_D	0.000715 0.001985	0.000635 0.002065	0.000615 0.002085	0.000595 0.002105	0.000515 0.002185
UCL	11.3660 17.6482	11.5879 17.5438	11.6478 17.5187	11.7108 17.493	11.9770 17.3956
($n = 10$)	(0.00679) (0.00546)	(0.0070) (0.0054)	(0.0071) (0.0054)	(0.0072) (0.0054)	(0.0080) (0.0050)

- For the combined control chart, considering the five combinations of (α_I, α_D), increasing the α_I (especially for $\alpha_I = 0.000515$ when $n = 5$ and 0.000715 when $n = 10$) will result in smaller ARL_1'*s* for detecting dispersion increases, whereas producing larger ARL_1'*s* for detecting dispersion decreases. Similarly, the larger the α_D is (especially for $\alpha_D = 0.002425$ when $n = 5$ and 0.002185 when $n = 10$), the smaller the ARL_1 value is for detecting dispersion decreases.
- For all the combinations of Δ_1 and Δ_2 in the scenarios (i)–(iii), the ARL_1 for $n = 10$ is smaller than that for $n = 5$. On the other hand, for fixed n, ARL_1 gets smaller when c is farther away from the in-control value $c = 1$ on either side. When n, ρ, and one of Δ_1 and Δ_2, say Δ_2, are fixed, the ARL_1 decreases when Δ_1 is farther away from 1. Similar observations regarding the effect of ρ on the chart performance as those discussed earlier in Section 3.1 can also be made here.
- Regardless of whether the dispersion increases or decreases, the ARL values of the proposed combined control chart (for the presented combinations of (α_I,α_D)) are smaller than 370 for all the cases tested. It outperforms the two-sided modified-LRT control chart, which is also ARL-unbiased. As discussed earlier, the TB-decomposed chart is ARL-biased when detecting dispersion decreases, whereas the two-sided LRT chart is ARL-biased when detecting dispersion increases.

It is noted that when α_I increases, the proposed combined chart would gain the power for detecting dispersion increases, but lose that for detecting decreases; and the situation is opposite for α_D. To see this more clearly, Figure 3 displays the ARL curves of three combined charts corresponding to $\alpha_I = 0.000515$, 0.000395, and 0.000275 (for $n = 5$) along with that of the three existing charts for the scenario (i). The plot (not shown) for the case of $n = 10$ is similar. We remark that there is only a small range of α_I for the combined chart to perform better than the two-sided modified-LRT chart in both directions; also, when α_I gets too large or too small, the combined chart will become ARL-biased.

Table V. The ARL_1 and their standard errors (in parentheses) of the combined and the two-sided control charts for $p=2$ and $n=5$

$n=5$		Combined chart					Two-sided charts		
	α_I	0.000515	0.000415	0.000395	0.000375	0.000275	Two-sided	Modified-	TB-decomp
Δ_1	Δ_2 α_D	0.002185	0.002285	0.002305	0.002325	0.002425	LRT chart	LRT chart	chart
[$\rho=0$]									
1.25	1.25	182.427	203.422	208.390	213.792	245.670	444.612	273.774	115.093
		(0.5495)	(0.6472)	(0.6711)	(0.6974)	(0.8593)	(2.0940)	(1.0111)	(0.2749)
1.35	1.35	116.011	131.017	134.632	138.466	163.251	426.376	207.989	73.0108
		(0.2782)	(0.3341)	(0.3480)	(0.3630)	(0.4650)	(1.9664)	(0.6691)	(0.1385)
1.5	1.5	60.6654	68.4594	70.3712	72.4871	85.7688	361.089	128.347	39.8668
		(0.1048)	(0.1257)	(0.1311)	(0.1370)	(0.1766)	(1.5322)	(0.3239)	(0.0556)
1.75	1.75	25.4883	28.3380	29.0164	29.7647	34.5769	208.336	57.1628	18.1600
		(0.0282)	(0.0331)	(0.0343)	(0.0357)	(0.0448)	(0.6708)	(0.0958)	(0.0168)
2	2	13.3973	14.6453	14.9445	15.2717	17.3485	105.350	28.5795	10.1459
		(0.0106)	(0.0121)	(0.0125)	(0.0129)	(0.0157)	(0.2406)	(0.0336)	(0.0069)
2.25	2.75	8.3002	8.5371	9.0294	9.2767	10.6090	55.5172	16.4173	6.5760
		(0.0050)	(0.0052)	(0.0057)	(0.0060)	(0.0074)	(0.0917)	(0.0144)	(0.0035)
2.5	2.5	5.7406	5.8798	6.1680	6.3151	7.0826	31.7536	10.5236	4.7092
		(0.0028)	(0.0029)	(0.0031)	(0.0033)	(0.0039)	(0.0394)	(0.0073)	(0.0020)
2.75	2.75	4.3120	4.5586	4.6173	4.6812	5.0737	19.9077	7.3812	3.6347
		(0.0018)	(0.0019)	(0.0020)	(0.0020)	(0.0023)	(0.0194)	(0.0042)	(0.0013)
3	3	3.4316	3.6023	3.6424	3.6865	3.9551	13.4558	5.5271	2.9596
		(0.0012)	(0.0013)	(0.0013)	(0.0014)	(0.0015)	(0.0106)	(0.0026)	(0.0009)
0.9	0.9	347.554	337.405	335.616	333.511	323.881	313.745	353.070	495.872
		(1.4468)	(1.3838)	(1.3728)	(1.3599)	(1.3014)	(1.2407)	(1.4814)	(2.4666)
0.8	0.8	286.393	274.959	272.769	270.519	259.642	255.060	310.039	535.705
		(1.0819)	(1.0177)	(1.0055)	(0.9931)	(0.9337)	(0.9091)	(1.2187)	(2.7699)
0.7	0.7	217.002	207.723	206.130	204.459	195.982	195.731	252.963	473.350
		(0.7132)	(0.6678)	(0.6602)	(0.6521)	(0.6119)	(0.6108)	(0.8979)	(2.3004)
0.6	0.6	154.820	147.997	146.793	145.605	139.847	141.401	191.037	365.430
		(0.4294)	(0.4012)	(0.3963)	(0.3915)	(0.3685)	(0.3747)	(0.5889)	(1.5599)
0.5	0.5	101.368	97.1628	96.3721	95.5804	91.8113	93.2975	130.250	245.405
		(0.2271)	(0.2131)	(0.2105)	(0.2079)	(0.1956)	(0.2004)	(0.3311)	(0.8579)
0.4	0.4	60.1249	57.6616	57.2145	56.7611	54.5995	55.5833	79.0986	142.800
		(0.1034)	(0.0971)	(0.0959)	(0.0948)	(0.0894)	(0.0918)	(0.1563)	(0.3802)
0.3	0.3	30.9166	29.7078	29.4825	29.2566	28.1810	28.6862	41.1490	69.3633
		(0.0378)	(0.0356)	(0.0352)	(0.0348)	(0.0329)	(0.0338)	(0.0583)	(0.1282)
0.2	0.2	12.3451	11.9033	11.8211	11.7371	11.3442	11.5274	16.3257	25.1002
		(0.0093)	(0.0088)	(0.0087)	(0.0086)	(0.0082)	(0.0084)	(0.0143)	(0.0276)
0.1	0.1	3.1940	3.1086	3.0930	3.0771	3.0017	3.0369	3.9937	5.2814
		(0.0011)	(0.0010)	(0.0010)	(0.0010)	(0.0010)	(0.0010)	(0.0016)	(0.0024)
1	1	370.117	370.233	370.727	370.700	370.837	369.898	370.501	370.693
		(1.5901)	(1.5908)	(1.5939)	(1.5938)	(1.5947)	(1.5886)	(1.5925)	(1.5938)
[$\rho=0$]									
1.25	1	269.847	288.679	292.864	297.486	323.478	407.598	317.541	206.849
		(0.9894)	(1.0949)	(1.1188)	(1.1454)	(1.2989)	(1.8378)	(1.2633)	(0.6636)
1.75	1	66.3940	74.0113	75.8098	77.7820	90.5563	270.128	112.686	48.8971
		(0.1201)	(0.1414)	(0.1466)	(0.1524)	(0.1916)	(0.9909)	(0.2663)	(0.0757)
2.25	1	22.7755	24.9236	25.4409	25.9992	29.5393	109.515	39.5475	18.1628
		(0.0238)	(0.0273)	(0.0281)	(0.0291)	(0.0353)	(0.2551)	(0.0549)	(0.0168)
2.75	1	11.2772	12.1208	12.3222	12.5417	13.8990	46.4183	18.3754	9.4983
		(0.0081)	(0.0090)	(0.0093)	(0.0095)	(0.0112)	(0.0700)	(0.0171)	(0.0062)
1.25	1.75	52.2713	58.5448	60.0815	61.7412	72.4315	293.798	104.098	32.2161
		(0.0837)	(0.0993)	(0.1033)	(0.1076)	(0.1369)	(1.1241)	(0.2364)	(0.0403)
1.75	2.25	12.9404	14.1077	14.3900	14.6988	16.6359	93.7726	26.8216	9.5835
		(0.01000)	(0.0114)	(0.0118)	(0.0122)	(0.0147)	(0.2020)	(0.0305)	(0.0063)
2.25	2.75	5.6949	6.0701	6.1591	6.2560	6.8567	30.4234	10.3371	4.6230
		(0.0028)	(0.0031)	(0.0031)	(0.0032)	(0.0037)	(0.0369)	(0.0071)	(0.0020)

(*Continues*)

Table V. *Continued.*

$n=5$			Combined chart					Two-sided charts		
Δ_1	Δ_2	α_I α_D	0.000515 0.002185	0.000415 0.002285	0.000395 0.002305	0.000375 0.002325	0.000275 0.002425	Two-sided LRT chart	Modified-LRT chart	TB-decomp chart
2.75	1.25		10.7817	11.0454	11.1395	12.2696	12.5781	48.2621	17.8314	8.6187
			(0.0075)	(0.0078)	(0.0079)	(0.0092)	(0.0096)	(0.0742)	(0.0164)	(0.0053)
[$\rho=0.2$]										
1.75	1.75		22.1800	24.4308	24.9756	25.5726	29.3464	151.037	45.5483	17.0671
			(0.0228)	(0.0264)	(0.0274)	(0.0284)	(0.0349)	(0.4137)	(0.0680)	(0.0153)
1.75	2.25		11.9014	12.9049	13.1473	13.4113	15.0608	72.9049	23.1441	9.2679
			(0.0088)	(0.0100)	(0.0103)	(0.0106)	(0.0126)	(0.1382)	(0.0244)	(0.0060)
2.25	2.25		7.8399	8.4199	8.5583	8.7095	9.6444	44.4463	14.7104	6.4399
			(0.0046)	(0.0051)	(0.0053)	(0.0054)	(0.0063)	(0.0655)	(0.0122)	(0.0034)
2.25	2.75		5.4940	5.8391	5.9211	6.0097	6.5551	25.8864	9.58968	4.5767
			(0.0026)	(0.0029)	(0.0029)	(0.0030)	(0.0035)	(0.0289)	(0.0063)	(0.0019)
[$\rho=0.4$]										
1.75	1.75		16.2703	17.6750	18.0106	18.3719	20.6587	77.2699	28.1743	14.3516
			(0.0142)	(0.0161)	(0.0166)	(0.0171)	(0.0205)	(0.1509)	(0.0328)	(0.0117)
1.75	2.25		9.6935	10.3984	10.5669	10.7508	11.8847	42.4239	16.4006	8.3856
			(0.0064)	(0.0071)	(0.0073)	(0.0075)	(0.0088)	(0.0611)	(0.0144)	(0.0051)
2.25	2.25		6.7557	7.1870	7.2890	7.4001	8.08238	27.6544	11.2275	6.0366
			(0.0036)	(0.0040)	(0.0041)	(0.0042)	(0.0048)	(0.0319)	(0.0080)	(0.0030)
2.25	2.75		5.1848	5.2782	5.3113	5.7057	5.8114	17.9707	7.9092	4.4190
			(0.0024)	(0.0024)	(0.0025)	(0.0028)	(0.0029)	(0.0166)	(0.0047)	(0.0018)
[$\rho=0$]										
0.8	1		339.173	330.087	328.526	326.947	318.497	308.718	341.507	400.986
			(1.3949)	(1.3390)	(1.3295)	(1.3199)	(1.2690)	(1.2109)	(1.4091)	(1.7932)
0.6	1		252.691	243.739	242.286	240.662	232.504	226.116	263.286	313.716
			(0.8964)	(0.8491)	(0.8416)	(0.8331)	(0.7910)	(0.7586)	(0.9535)	(1.2405)
0.4	1		155.491	149.553	148.423	147.243	141.941	138.060	165.710	185.099
			(0.4322)	(0.4076)	(0.4030)	(0.3982)	(0.3768)	(0.3614)	(0.4756)	(0.5616)
0.2	1		64.1978	61.6052	61.1174	60.6404	58.3473	56.6815	68.9822	67.5459
			(0.1141)	(0.1072)	(0.1060)	(0.1047)	(0.0988)	(0.0946)	(0.1272)	(0.1232)
0.8	0.6		211.506	202.298	200.706	199.017	190.934	190.309	244.445	478.538
			(0.6862)	(0.6418)	(0.6342)	(0.6262)	(0.5884)	(0.5855)	(0.8528)	(2.3383)
0.6	0.4		95.8447	91.7705	91.0365	90.2845	86.6889	88.0406	122.245	236.041
			(0.2087)	(0.1955)	(0.1932)	(0.1908)	(0.1794)	(0.1837)	(0.3010)	(0.8092)
0.4	0.2		26.4296	25.3954	25.2060	25.0166	24.1059	24.5286	34.9046	60.3489
			(0.0298)	(0.0281)	(0.0277)c	(0.0274)	(0.0259)	(0.0266)	(0.0455)	(0.1040)
0.2	0.8		51.5425	50.9160	50.7174	48.7413	48.3682	48.5092	64.3588	81.0461
			(0.0819)	(0.0804)	(0.0780)	(0.0753)	(0.0744)	(0.0748)	(0.1146)	(0.1621)
[$\rho=0.2$]										
0.6	0.6		145.378	139.029	137.895	136.797	131.286	132.330	177.552	353.989
			(0.3906)	(0.3652)	(0.3608)	(0.3565)	(0.3351)	(0.3391)	(0.5275)	(1.4872)
0.6	0.4		89.9337	86.1954	85.5180	84.8162	81.4684	82.6078	114.030	228.686
			(0.1896)	(0.1779)	(0.1758)	(0.1736)	(0.1634)	(0.1669)	(0.2711)	(0.7716)
0.4	0.4		56.7609	54.4676	54.0523	53.6075	51.5534	52.4787	74.3677	138.276
			(0.0948)	(0.0891)	(0.0880)	(0.0869)	(0.0820)	(0.0842)	(0.1424)	(0.3623)
0.4	0.2		25.0526	24.0846	23.9109	23.7274	22.8649	23.2634	33.0781	58.4811
			(0.0275)	(0.0259)	(0.0256)	(0.0253)	(0.0239)	(0.0245)	(0.0419)	(0.0991)
[$\rho=0.4$]										
0.6	0.6		118.665	113.662	112.769	111.829	107.365	107.054	139.789	316.416
			(0.2878)	(0.2698)	(0.2666)	(0.2633)	(0.2476)	(0.2465)	(0.3682)	(1.2566)
0.6	0.4		74.1372	71.1023	70.5440	69.9567	67.2527	67.8900	92.4313	206.264
			(0.1418)	(0.1331)	(0.1316)	(0.1299)	(0.1224)	(0.1242)	(0.1976)	(0.6608)
0.4	0.4		47.3060	45.3854	45.0403	44.6784	43.0012	43.7086	61.3450	123.548
			(0.0720)	(0.0676)	(0.0668)	(0.0660)	(0.0623)	(0.0639)	(0.1066)	(0.3058)

(*Continues*)

Table V. *Continued.*

$n=5$			Combined chart					Two-sided charts		
Δ_1	Δ_2	α_I α_D	0.000515 0.002185	0.000415 0.002285	0.000395 0.002305	0.000375 0.002325	0.000275 0.002425	Two-sided LRT chart	Modified-LRT chart	TB-decomp chart
0.4	0.2		21.0468 (0.0211)	20.2362 (0.0199)	20.0880 (0.0196)	19.9395 (0.0194)	19.2246 (0.0184)	19.5465 (0.0188)	27.5379 (0.0317)	52.7095 (0.0846)

Table VI. The ARL_1 and their standard errors (in parentheses) of the combined and the two-sided control charts for $p=2$ and $n=10$

$n=10$			Combined chart					Two-sided charts		
Δ_1	Δ_2	α_I α_D	0.000715 0.001985	0.000635 0.002065	0.000615 0.002085	0.000595 0.002105	0.000515 0.002185	Two-sided LRT chart	Modified-LRT chart	TB-decomp chart
[$\rho=0$]										
1.25	1.25		105.209 (0.2402)	113.404 (0.2689)	115.722 (0.2772)	118.205 (0.2862)	129.266 (0.3274)	335.402 (1.3715)	160.786 (0.4545)	81.2546 (0.1628)
1.35	1.35		55.0231 (0.0904)	59.1732 (0.1009)	60.3196 (0.1039)	61.5572 (0.1071)	67.1666 (0.1222)	219.135 (0.7237)	91.1760 (0.1936)	44.4824 (0.0656)
1.5	1.5		24.5234 (0.0266)	26.1133 (0.0293)	26.5648 (0.0300)	27.0406 (0.0309)	29.1209 (0.0345)	101.643 (0.2280)	41.2840 (0.0586)	20.8205 (0.0207)
1.75	1.75		9.2092 (0.0059)	9.6659 (0.0064)	9.8017 (0.0065)	9.9343 (0.0066)	10.5325 (0.0073)	31.6083 (0.0391)	14.4905 (0.0119)	8.2378 (0.0050)
2	2		4.7955 (0.0021)	4.9764 (0.0022)	5.0265 (0.0023)	5.0803 (0.0023)	5.3084 (0.0025)	13.0719 (0.0102)	6.9090 (0.0038)	4.4011 (0.0018)
2.25	2.25		3.0662 (0.0010)	3.1553 (0.0010)	3.1801 (0.0011)	3.2062 (0.0011)	3.3182 (0.0011)	6.8626 (0.0037)	4.0989 (0.0016)	2.8684 (0.0009)
2.5	2.5		2.2466 (0.0006)	2.2970 (0.0006)	2.3109 (0.0006)	2.3257 (0.0006)	2.3885 (0.0006)	4.2783 (0.0017)	2.8248 (0.0009)	2.1310 (0.0005)
2.75	2.75		1.8030 (0.0004)	1.8342 (0.0004)	1.8429 (0.0004)	1.8520 (0.0004)	1.8914 (0.0004)	3.0192 (0.0010)	2.1623 (0.0005)	1.7310 (0.0003)
3	3		1.5419 (0.0003)	1.5629 (0.0003)	1.5686 (0.0003)	1.5746 (0.0003)	1.6006 (0.0003)	2.3302 (0.0006)	1.7794 (0.0004)	1.4927 (0.0002)
0.9	0.9		294.014 (1.1254)	285.343 (1.0759)	283.274 (1.0642)	281.314 (1.0532)	274.190 (1.0134)	265.305 (0.9645)	321.807 (1.2889)	449.549 (2.1290)
0.8	0.8		178.674 (0.5326)	172.282 (0.5042)	171.724 (0.5017)	170.227 (0.4952)	165.160 (0.4732)	165.899 (0.4764)	221.080 (0.7334)	362.207 (1.5393)
0.7	0.7		97.1147 (0.2129)	93.7383 (0.2019)	92.9303 (0.1992)	92.1540 (0.1967)	89.0777 (0.1869)	92.1222 (0.1966)	129.160 (0.3270)	219.474 (0.7254)
0.6	0.6		47.8917 (0.0733)	46.3034 (0.0697)	45.9410 (0.0689)	45.5804 (0.0681)	44.1263 (0.0648)	46.0766 (0.0692)	65.4834 (0.1176)	111.350 (0.2616)
0.5	0.5		21.4046 (0.0216)	20.7646 (0.0206)	20.6149 (0.0204)	20.4664 (0.0202)	19.8653 (0.0193)	20.6785 (0.0205)	29.1854 (0.0347)	48.0834 (0.0738)
0.4	0.4		8.6665 (0.0054)	8.4461 (0.0052)	8.3941 (0.0051)	8.3423 (0.0051)	8.1451 (0.0049)	8.4377 (0.0052)	11.5247 (0.0084)	17.8969 (0.0165)
0.3	0.3		3.3075 (0.0011)	3.2444 (0.0011)	3.2294 (0.0011)	3.2147 (0.0011)	3.1581 (0.0010)	3.2413 (0.0011)	4.1385 (0.0016)	5.8373 (0.0029)
0.2	0.2		1.3956 (0.0002)	1.3832 (0.0002)	1.3802 (0.0002)	1.3773 (0.0002)	1.3660 (0.0002)	1.3822 (0.0002)	1.5626 (0.0003)	1.8820 (0.0004)
0.1	0.1		1.0021 ($<10^{-5}$)	1.0020 ($<10^{-5}$)	1.0019 ($<10^{-5}$)	1.0019 ($<10^{-5}$)	1.0017 ($<10^{-5}$)	1.0020 ($<10^{-5}$)	1.0050 ($<10^{-5}$)	1.0132 ($<10^{-5}$)
1	1		370.076 (1.5898)	370.055 (1.5896)	370.343 (1.5915)	369.365 (1.5852)	370.028 (1.5895)	369.174 (1.5840)	369.365 (1.5852)	370.892 (1.5950)

(*Continues*)

Table VI. *Continued.*

$n=10$			Combined chart					Two-sided charts		
Δ_1	Δ_2	α_I / α_D	0.000715 / 0.001985	0.000635 / 0.002065	0.000615 / 0.002085	0.000595 / 0.002105	0.000515 / 0.002185	Two-sided LRT chart	Modified-LRT chart	TB-decomp chart
[$\rho=0$]										
1.25	1		198.961	211.195	214.484	218.012	233.825	351.130	230.862	160.799
			(0.6260)	(0.6847)	(0.7008)	(0.7181)	(0.7978)	(1.4692)	(0.7827)	(0.4545)
1.75	1		27.0185	28.6382	29.0923	29.5761	31.7195	71.0162	36.3381	23.4215
			(0.0308)	(0.0337)	(0.0345)	(0.0354)	(0.0393)	(0.1329)	(0.0483)	(0.0248)
2.25	1		8.2382	8.5835	8.6799	8.7820	9.2217	17.9978	10.6023	7.6064
			(0.0050)	(0.0053)	(0.0054)	(0.0055)	(0.0059)	(0.0166)	(0.0074)	(0.0044)
2.75	1		4.1460	4.2717	4.3061	4.3426	4.5006	7.5755	5.0624	3.9431
			(0.0016)	(0.0017)	(0.0018)	(0.0018)	(0.0019)	(0.0043)	(0.0023)	(0.0015)
1.25	1.75		20.4036	21.6194	21.9563	22.3140	23.9191	69.5681	31.7445	16.4998
			(0.0201)	(0.0220)	(0.0225)	(0.0230)	(0.0256)	(0.1288)	(0.0394)	(0.0145)
1.75	2.25		4.6491	4.8189	4.8659	4.9160	5.1305	12.0845	6.5916	4.2181
			(0.0020)	(0.0021)	(0.0021)	(0.0022)	(0.0023)	(0.0090)	(0.0035)	(0.0017)
2.25	2.75		2.2356	2.2852	2.2988	2.3132	2.3757	4.2110	2.8032	2.1134
			(0.0006)	(0.0006)	(0.0006)	(0.0006)	(0.0006)	(0.0017)	(0.0008)	(0.0005)
2.75	1.25		3.7739	3.8861	3.9168	3.9497	4.0922	7.5436	4.8540	3.6460
			(0.0014)	(0.0016)	(0.0015)	(0.0015)	(0.0016)	(0.0043)	(0.0021)	(0.0013)
[$\rho=0.2$]										
1.75	1.75		8.0219	8.3802	8.4799	8.5875	9.0515	23.0817	11.8257	7.6239
			(0.0048)	(0.0051)	(0.0052)	(0.0053)	(0.0057)	(0.0243)	(0.0087)	(0.0044)
1.75	2.25		4.3273	4.4742	4.5148	4.5579	4.7428	10.1944	5.9340	4.0714
			(0.0018)	(0.0019)	(0.0019)	(0.0019)	(0.0021)	(0.0069)	(0.0030)	(0.0016)
2.25	2.25		2.9426	3.0228	3.0449	3.0683	3.1705	6.1390	3.8491	2.8234
			(0.0010)	(0.0010)	(0.0010)	(0.0010)	(0.0010)	(0.0031)	(0.0015)	(0.0009)
2.25	2.75		2.2000	2.2349	2.2477	2.2612	2.3185	3.9475	2.7087	2.1008
			(0.0006)	(0.0005)	(0.0006)	(0.0006)	(0.0006)	(0.0015)	(0.0008)	(0.0005)
[$\rho=0.4$]										
1.75	1.75		5.8719	6.0873	6.1468	6.2105	6.4824	12.2542	7.5341	6.0346
			(0.0031)	(0.0029)	(0.0031)	(0.0032)	(0.0034)	(0.0092)	(0.0043)	(0.0030)
1.75	2.25		3.6065	3.7097	3.7379	3.7678	3.8979	6.8412	4.5342	3.6158
			(0.0014)	(0.0013)	(0.0014)	(0.0014)	(0.0015)	(0.00370)	(0.00191)	(0.00131)
2.25	2.25		2.6280	2.6886	2.7052	2.7228	2.7996	4.6598	3.2436	2.6541
			(0.0008)	(0.0008)	(0.0008)	(0.0008)	(0.0008)	(0.0020)	(0.0011)	(0.0008)
2.25	2.75		2.0533	2.0911	2.1014	2.1123	2.1598	3.3162	2.4488	2.0458
			(0.0005)	(0.0005)	(0.0005)	(0.0005)	(0.0005)	(0.0011)	(0.0007)	(0.0005)
[$\rho=0$]										
0.8	1		273.538	265.827	264.016	262.412	254.130	242.757	284.139	333.934
			(1.0098)	(0.9673)	(0.9574)	(0.9487)	(0.9041)	(0.8440)	(1.0691)	(1.3625)
0.6	1		132.423	128.083	127.139	126.135	122.724	116.120	142.207	168.913
			(0.3395)	(0.3229)	(0.3193)	(0.3155)	(0.3028)	(0.2786)	(0.3779)	(0.4894)
0.40.4	1		43.5446	42.1234	41.7933	41.4533	40.2210	38.2751	47.0963	53.7468
			(0.0635)	(0.0604)	(0.0597)	(0.0590)	(0.0563)	(0.0523)	(0.0715)	(0.0873)
0.2	1		7.2723	7.0777	7.0322	6.9867	6.8101	6.5445	7.7498	8.3410
			(0.0041)	(0.0039)	(0.0039)	(0.0038)	(0.0037)	(0.0035)	(0.0045)	(0.0051)
0.8	0.6		89.2351	86.1638	85.4672	84.7537	81.6257	83.8086	115.485	203.845
			(0.1874)	(0.1778)	(0.1756)	(0.1734)	(0.1639)	(0.1705)	(0.2763)	(0.6492)
0.6	0.4		18.7702	18.2198	18.0888	17.9601	17.4446	18.1194	25.2685	42.6998
			(0.0177)	(0.0169)	(0.0167)	(0.0165)	(0.0158)	(0.0168)	(0.0278)	(0.0617)
0.4	0.2		2.6724	2.6258	2.6148	2.6039	2.5641	2.6246	3.2629	4.6038
			(0.0008)	(0.0008)	(0.0007)	(0.0007)	(0.0007)	(0.0008)	(0.0011)	(0.0020)
0.2	0.8		5.9349	5.7875	5.7528	5.7176	5.5836	5.6684	7.1022	8.5415
			(0.0030)	(0.0028)	(0.0028)	(0.0028)	(0.0027)	(0.0027)	(0.0039)	(0.0053)

(*Continues*)

Table VI. *Continued.*

$n=10$			Combined chart					Two-sided charts		
		α_I	0.000715	0.000635	0.000615	0.000595	0.000515	Two-sided	Modified-	TB-decomp
Δ_1	Δ_2	α_D	0.001985	0.002065	0.002085	0.002105	0.002185	LRT chart	LRT chart	chart
[$\rho=0.2$]										
0.6	0.6		41.1430	39.8038	39.4928	39.1850	38.0150	39.4933	55.1601	102.277
			(0.0583)	(0.0554)	(0.0548)	(0.0541)	(0.0517)	(0.0548)	(0.0908)	(0.2302)
0.6	0.4		16.5463	16.0612	15.9483	15.8380	15.4047	15.9709	22.0159	39.9115
			(0.0146)	(0.0139)	(0.0138)	(0.0136)	(0.0131)	(0.0138)	(0.0226)	(0.0557)
0.4	0.4		7.7960	7.5994	7.5531	7.5074	7.3383	7.5922	10.2691	16.8506
			(0.0045)	(0.0044)	(0.0043)	(0.0043)	(0.0041)	(0.0044)	(0.0070)	(0.0150)
0.4	0.2		2.4976	2.4555	2.4455	2.4356	2.3966	2.4519	3.0197	4.4430
			(0.0007)	(0.0007)	(0.0007)	(0.0007)	(0.0007)	(0.0007)	(0.0010)	(0.0018)
[$\rho=0.4$]										
0.6	0.6		25.8062	25.0026	24.8133	24.6240	23.8554	24.1704	32.0291	72.9145
			(0.0287)	(0.0274)	(0.0271)	(0.0268)	(0.0255)	(0.0260)	(0.0399)	(0.1383)
0.6	0.4		11.0973	10.7890	10.7167	10.6444	10.3733	10.6836	14.1897	30.7822
			(0.0079)	(0.0076)	(0.0075)	(0.0074)	(0.0071)	(0.0074)	(0.0115)	(0.0376)
0.4	0.4		5.5969	5.4646	5.4333	5.4023	5.2896	5.4547	7.1518	13.5648
			(0.0027)	(0.0026)	(0.0026)	(0.0025)	(0.0025)	(0.0026)	(0.0040)	(0.0108)
0.4	0.2		2.0300	2.0001	1.9930	1.9858	1.9589	1.9977	2.3869	3.8746
			(0.0005)	(0.0005)	(0.0004)	(0.0004)	(0.0004)	(0.0005)	(0.0006)	(0.0015)

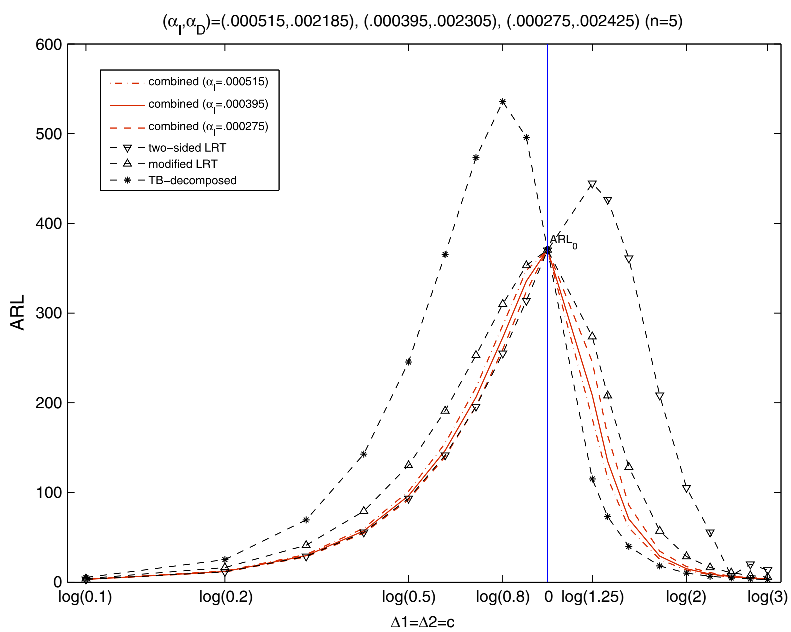

Figure 3. The average run length curves of the control charts under study for c for $p=2$, $n=5$ and x-axis in log scale for c

The combined chart is based on the two one-sided hypotheses (2) and (3), whereas the existing two-sided control charts studied in this paper are all based on the two-sided hypotheses (1). Because there are many $\mathbf{\Sigma}-\mathbf{\Sigma}_0$ that are neither positive semidefinite nor negative semidefinite, it is clear that the union of the sets under the alternative hypotheses of (2) and (3) is smaller than the set of the alternative hypothesis of (1), that is,

$$\{H_1\colon \mathbf{\Sigma} \geqslant \mathbf{\Sigma}_0 \text{ and } \mathbf{\Sigma} \neq \mathbf{\Sigma}_0\} \cup \{H_1\colon \mathbf{\Sigma} \geqslant \mathbf{\Sigma}_0 \text{ and } \mathbf{\Sigma} \neq \mathbf{\Sigma}_0\} \subsetneq \{H_1\colon \mathbf{\Sigma} \neq \mathbf{\Sigma}_0\}. \quad (13)$$

Thus, if the out-of-control scenario considered is outside of the alternative hypotheses of (2) and (3), the proposed combined chart may not result in better performance than others. Nevertheless, when detecting dispersion increases or decreases and the change range of ρ satisfies (10) and (12), our proposed combined chart gives a more satisfactory performance than the existing two-sided charts considered in the current paper.

5. Examples

In this section, we first illustrate the application of using the proposed T_D-based control chart for monitoring dispersion decreases with a real-life example. Then, the same data set and another real-life data set as given in Yen and Shiau[6] are used to demonstrate the proposed combined control chart.

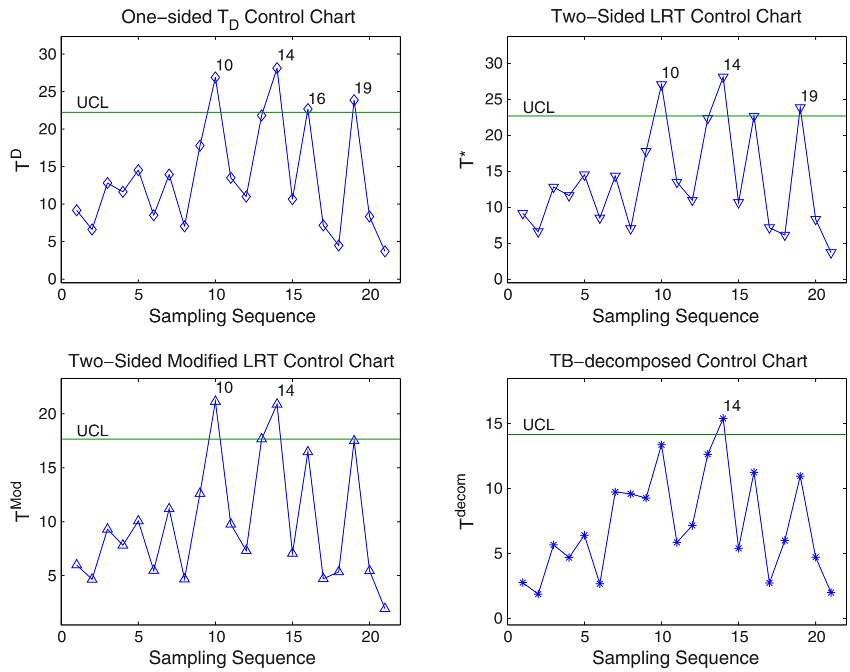

Figure 4. The one-sided T_D, two-sided likelihood ratio test, two-sided modified likelihood ratio test, and TB-decomposed control charts for 21 new samples of the integrated circuit component example

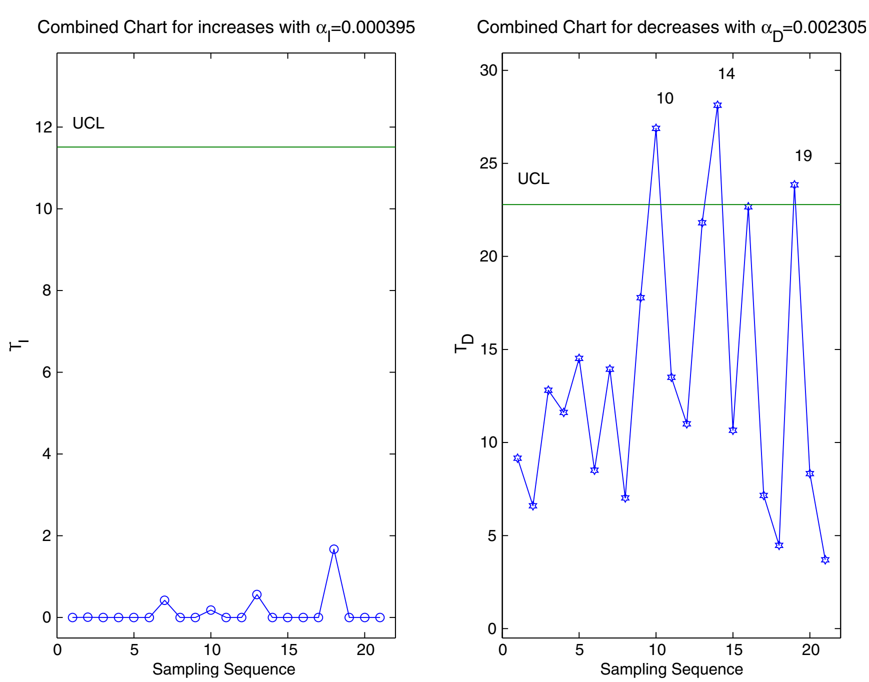

Figure 5. The combined control chart for 21 new samples of the integrated circuit component example

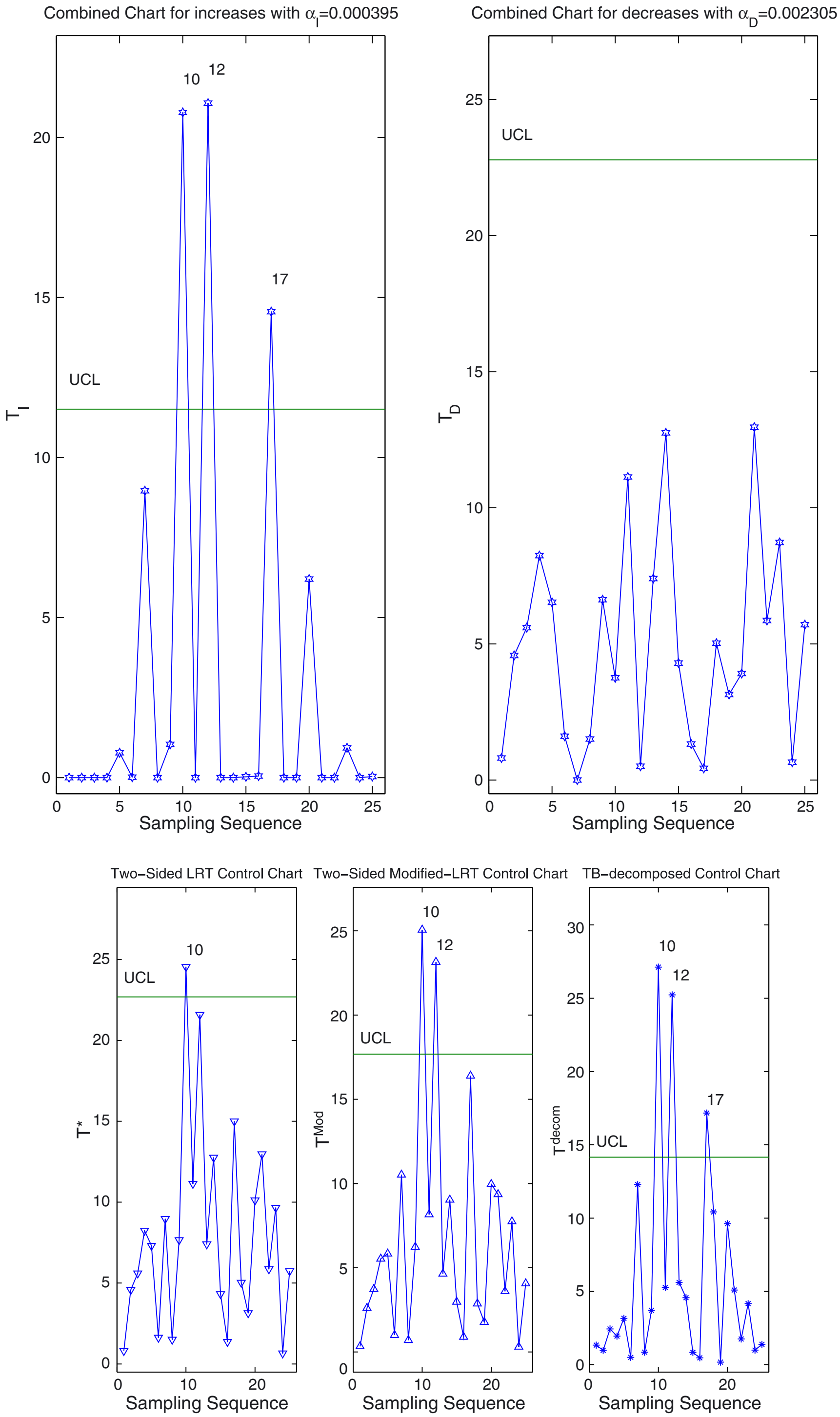

Figure 6. The combined, two-sided likelihood ratio test, two-sided modified likelihood ratio test, and TB-decomposed control charts for 25 new samples of the metal layer process example

The first data set is related to the integrated circuit (IC) components failure rates of the wafer sort (WS). The WS is a process after wafer fabrication that performs on each die in a wafer, during which the electrical parameters of ICs are tested for functionality. Probes contact the pads of the circuit to conduct the test. If a die does not pass the test, it will not be packaged. The failure rate of a test is defined as the ratio of the failed dies over all tested dies. The two most common IC parameters to test are Open and Short. These two values are strongly related to the process stability. The two quality characteristics to be monitored, X_1 and X_2, are the failure rates (in percent) of Open and Short by lot, respectively, that is, the failure rate within each lot of 25 wafers. Denote $\boldsymbol{X} = (X_1, X_2)'$. Fifty subgroups of random samples, each of size 5, were taken from the presumably in-control process. The sample mean is

$\bar{\bar{\boldsymbol{X}}} = \begin{pmatrix} 1.98920 \\ 6.14052 \end{pmatrix}$ and the sample covariance matrix is $\mathbf{S} = \frac{1}{(50\times5-1)}\sum_{i=1}^{50}\sum_{j=1}^{5}\left(\boldsymbol{X}_{ij} - \bar{\bar{\boldsymbol{X}}}\right)\left(\boldsymbol{X}_{ij} - \bar{\bar{\boldsymbol{X}}}\right)' = \begin{pmatrix} 0.84598 & 0.54288 \\ 0.54288 & 5.46428 \end{pmatrix}$. We take $\mathbf{S}$ as the in-control covariance matrix $\boldsymbol{\Sigma}_0$ for our phase II process monitoring.

For $p=2$, $n=5$, and $\alpha=0.0027$, the control limits of the one-sided T_D-based, two-sided LRT, two-sided modified-LRT, and TB-decomposed control charts are given earlier in Section 3.1. With these control limits, we monitor 21 subgroups of samples (each of size 5) taken on-line from the process, for which the dispersion was suspected to be decreased. The control charts are displayed in Figure 4. There are four out-of-control signals on the T_D-based one-sided chart with the first signal showing up on the 10th sample. The two-sided LRT and the two-sided modified-LRT charts have three and two out-of-control signals, respectively, with the first signal showing up on the 10th sample on both charts. On the other hand, only one out-of-control signal (the 14th sample) shows up on the TB-decomposed chart. This confirms that the T_D-based one-sided chart is more sensitive than the other charts. Also, note that the two-sided LRT chart picks up more out-of-control points than the two-sided modified-LRT chart, which could be attributable to the fact that the former outperforms the latter in detecting dispersion decreases.

Next, we use the same data set to demonstrate the proposed combined chart. Figure 5 displays the result. Setting $\alpha=0.0027$, the control limits of the proposed combined chart for $\alpha_I=0.000395$ and $\alpha_D=0.002305$ are 11.7444 and 22.7055, respectively. There are three out-of-control signals on the T_D-based one-sided chart with the first signal showing up on the 10th sample and no out-of-control signal on the T_I-based one-sided chart.

The second example, taken from Yen and Shiau,[6] is related to a metal layer process for the semiconductor elements of a wafer. The two quality characteristics being monitored are after-develop-inspection-critical-dimension (ADICD) and after-etch-inspection-critical-dimension (AEICD). The two critical dimensions are measured at five points on each wafer after the develop-action and etch-action. Let X_1 and X_2 be the averages of the five ADICD and AEICD measurements on a wafer, respectively. Denote $\boldsymbol{X}=(X_1, X_2)'$. Fifty sets of random samples, each of size 5, were taken from the in-control process. The sample mean and sample covariance matrix are $\begin{pmatrix} 0.79966 \\ 0.85744 \end{pmatrix}$ and $\begin{pmatrix} 3.70395\times10^{-4} & 1.38183\times10^{-4} \\ 1.38183\times10^{-4} & 4.95859\times10^{-4} \end{pmatrix}$, respectively. Twenty-five additional on-line samples, each of size 5, are monitored using the proposed combined chart (with $\alpha_I=0.000395$) and the other three existing charts. Figure 6 displays these charts. There are three out-of-control signals on the T_I-based one-sided chart with the first signal showing up on the 10th sample, and no out-of-control signal shows up on the T_D-based one-sided chart. The two-sided modified-LRT and TB-decomposed charts have two and three out-of-control signals, respectively, with the first signal showing up on the 10th sample on both charts. On the other hand, only one out-of-control signal (the 10th sample) shows up on the two-sided LRT chart, which confirms its less sensitivity than the other charts in detecting dispersion increases.

6. Conclusions

In this paper, we have proposed and studied a control chart based on the one-sided LRT that is specifically designed for detecting dispersion decreases in multivariate normal processes. The performance study showed that the proposed one-sided control chart indeed outperforms various existing two-sided control charts in terms of the ARL, when process dispersion decreases. The proposed control chart is analogous to that of Yen and Shiau,[6] which was designed for detecting multivariate dispersion increases. For more effective monitoring, one can consider using the control chart of Yen and Shiau[6] for dispersion increases and the proposed chart in this paper for dispersion decreases.

Furthermore, when aiming at detecting both increases and decreases in dispersion, we proposed a combined control chart by combining the two effective one-sided LRT-based control charts. We demonstrated that for the combined chart to be effective, the two individual charts need to have unequal type I error probabilities. Simulations demonstrated that, with appropriately chosen false-alarm rates for individual charts, the proposed combined control chart outperforms various existing two-sided control charts in terms of the ARL, when the process dispersion increases or decreases.

The proposed one-sided T_D-based control chart and the combined chart are Shewhart-type chart. It is well known that EWMA and CUSUM charts are more sensitive to small changes. A combination of a Shewhart and an EWMA (or CUSUM) chart in the univariate case usually provides a more effective control charting mechanism because a wider range of variance increases/decreases will be covered. As for the multivariate case, how to extend the proposed schemes to an EWMA or CUSUM version and how the proposed one-sided chart in the current paper can be combined with an EWMA or a CUSUM chart should be worthy of further investigations.

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