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The Boundedness of Calderón–Zygmund Operators by Wavelet Characterization

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Abstract This article deals with the boundedness properties of Calderón–Zygmund operators on Hardy spaces $H^p(\mathbb{R}^n)$. We use wavelet characterization of $H^p(\mathbb{R}^n)$ to show that a Calderón–Zygmund operator T with $T^*1 = 0$ is bounded on $H^p(\mathbb{R}^n)$, $\frac{n}{n+\varepsilon} < p \le 1$, where ε is the regular exponent of kernel of T. This approach can be applied to the boundedness of operators on certain Hardy spaces without atomic decomposition or molecular characterization.

Keywords Calderón–Zygmund operators, Hardy spaces, para-product operators

MR(2000) Subject Classification 42B20, 42B30

1 Introduction

The original notion of a Calderón–Zygmund operator was introduced by Calderón and Zygmund in [1]. Their main object was to generalize the Hilbert transform and Riesz transforms. They showed that Calderón–Zygmund operators are bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. For Hardy spaces $H^p(\mathbb{R}^n)$, p sufficiently closed to 1, it is well known that Calderón–Zygmund operators T with $T^*1 = 0$ are bounded on $H^p(\mathbb{R}^n)$. There are some methods to gain the result in general. One is shown in terms of atomic decomposition together with the maximal function characterization of $H^p(\mathbb{R}^n)$ (see [2, p. 115, Theorem 4]). Another approach is given by molecular characterization of $H^p(\mathbb{R}^n)$ (see [3, p. 335, Theorem 7.18]). Some approaches are given by wavelets with smooth atomic and molecular decomposition (see [4, Section 6] and [5, Section 2]). In this article, we use almost orthogonal property and wavelet characterization of $H^p(\mathbb{R}^n)$ to get the $H^p(\mathbb{R}^n)$ boundedness of Calderón–Zygmund operators without atoms and molecules. The method mentioned in this article can be applied to certain types of Hardy spaces without atomic decomposition or molecular characterization, for example, flag Hardy spaces [6].

We say that T is a *singular integral operator* with regularity exponent $\varepsilon \in (0, 1]$, denoted by $T \in \mathrm{SIO}(\varepsilon)$, if T is a continuous linear operator from $\mathcal{D}$, the set of C^∞ functions with compact

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support, into its dual associated to a kernel $K(x,y)$, a continuous function on $\mathbb{R}^n\times\mathbb{R}^n\setminus\{x=y\}$, satisfying the following conditions: there exists an absolute constant $C>0$ such that

$$|K(x,y)|\le\frac{C}{|x-y|^n},\tag{1.1}$$

$$|K(x,y)-K(x',y)|\le C\frac{|x-x'|^\varepsilon}{|x-y|^{n+\varepsilon}},\quad \text{if } |x-x'|\le\frac{1}{2}|x-y|,\tag{1.2}$$

$$|K(x,y)-K(x,y')|\le C\frac{|y-y'|^\varepsilon}{|x-y|^{n+\varepsilon}},\quad \text{if } |y-y'|\le\frac{1}{2}|x-y|.\tag{1.3}$$

Moreover, the operator T can be represented by

$$\langle Tf,g\rangle=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}K(x,y)f(y)g(x)dydx$$

for $f,g\in\mathcal{D}$ with $\operatorname{supp}(f)\cap\operatorname{supp}(g)=\emptyset$. We say that a singular integral operator T is a *Calderón–Zygmund operator*, denoted by $T\in \mathrm{CZO}(\varepsilon)$, if T can be extended to be a bounded operator on $L^2(\mathbb{R}^n)$.

The Hardy space $H^p(\mathbb{R}^n)$, $0<p\le 1$, denotes the class of distribution f such that $M_\phi f\in L^p(\mathbb{R}^n)$, where

$$M_\phi(f)(x):=\sup_{t>0}|f*\phi_t(x)|=\sup_{t>0}\left|\int_{\mathbb{R}^n}f(x-y)\phi_t(y)dy\right|$$

for some ϕ is a Schwartz function on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n}\phi=1$ and $\|f\|_{H^p}:=\|M_\phi f\|_p$. We said that the local integrable function $f\in\mathrm{BMO}$ if

$$\|f\|_{\mathrm{BMO}}:=\sup_Q\frac{1}{|Q|}\int_Q|f(x)-f_Q|dx<\infty,$$

where supremum ranges over all finite cubes Q in $\mathbb{R}^n$, $|Q|$ is the Lebesgue measure of Q, and f_Q denotes the mean value of f over Q. It is well known that the space BMO is the dual of H^1 (see [7]) and we have the inequality

$$\left|\int_{\mathbb{R}^n}fgdx\right|\le C\|f\|_{H^1}\|g\|_{\mathrm{BMO}}$$

for $f\in H^1$ and $g\in\mathrm{BMO}$. For a fixed number $N\in\mathbb{N}$, the $T1=0$ means

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}K(x,y)\psi(x)dydx=0\quad\text{for any } \psi\in C^N \text{ with compact support}$$

and $T^*1=0$ means

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}K(x,y)\psi(y)dxdy=0\quad\text{for any } \psi\in C^N \text{ with compact support.}$$

We will use almost orthogonal property and wavelet characterizations to show the following well known result.

Theorem 1.1 *Let $T\in\mathrm{CZO}(\varepsilon)$, $0<\varepsilon\le 1$, with $T^*1=0$. Then T is bounded on $H^p(\mathbb{R}^n)$ for* $\frac{n}{n+\varepsilon}<p\le 1$.

2 Preliminary

Let φ and ψ in $C^N, N\in\mathbb{N}$ (depends on the regularity exponent ε), with compact supports. Denote by E the set of 2^n-1 vectors $\mathbf{e}=(e_1,e_2,\dots,e_n)$ of 0's and 1's, excluding the origin

$(0,\ldots,0)$. For $\mathbf{e}\in E$ and $\mathbf{k}=(k_1,k_2,\ldots,k_n)\in\mathbb{Z}^n$, define

$$\psi^{\mathbf{e}}_{j,\mathbf{k}}(x)=2^{-jn/2}\psi^{\mathbf{e}}(2^{-j}x-\mathbf{k})=2^{-jn/2}\psi^{e_1}(2^{-j}x_1-k_1)\cdots\psi^{e_n}(2^{-j}x_n-k_n)$$

with the convention that $\psi^0=\varphi$ and $\psi^1=\psi$. We may assume the family $\{\psi^{\mathbf{e}}_{j,\mathbf{k}}\}$ has four fundamental properties as follows (see [8, p. 108]).

(a) $\{\psi^{\mathbf{e}}_{j,\mathbf{k}}: j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^n,\mathbf{e}\in E\}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$; that is,

$$f=\sum_{\mathbf{e}\in E}\sum_{j\in\mathbb{Z}}\sum_{\mathbf{k}\in\mathbb{Z}^\mathbf{n}}\langle f,\psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\psi^{\mathbf{e}}_{j,\mathbf{k}}\quad\text{ for any } f\in L^2(\mathbb{R}^n);$$

(b) $\operatorname{supp}\psi^{\mathbf{e}}_{j,\mathbf{k}}\subset Q_{j,\mathbf{k}}$ where $Q_{j,\mathbf{k}}=\{x\in\mathbb{R}^n: 2^{-j}x-\mathbf{k}\in[0,1)^n\}$, $\mathbf{k}=(k_1,k_2,\ldots,k_n)\in\mathbb{Z}^n, j\in\mathbb{Z}$;

(c) $|\partial^\alpha\psi^{\mathbf{e}}_{j,\mathbf{k}}|\le C2^{-j|\alpha|}2^{-nj/2}$, for every multi-index $\alpha\in\mathbb{N}^n$ of order $|\alpha|\le N$;

(d) $\int x^\alpha\psi^{\mathbf{e}}_{j,\mathbf{k}}(x)dx=0$, when $|\alpha|\le N$.

Hereafter, for simplicity we use $\sum_{\mathbf{e},j,\mathbf{k}}$ to denote $\sum_{\mathbf{e}\in E}\sum_{j\in\mathbb{Z}}\sum_{\mathbf{k}\in\mathbb{Z}^\mathbf{n}}$. We also have the wavelet characterization of Hardy spaces (see [8, p. 143, Theorem 1] and [9, p. 113, Theorem 4.16]).

Theorem 2.1 ([8, 9]) *Let ε be the regularity exponent associated with singular integral operator and $\frac{n}{n+\varepsilon}<p\le 1$. Suppose the family $\{\psi^{\mathbf{e}}_{j,\mathbf{k}}\}$ has properties* (a)–(d)*, then the following are equivalent*:

(i) $f\in H^p(\mathbb{R}^n)$;

(ii) $\big(\sum_{\mathbf{e},j,\mathbf{k}}\big|\langle f,\psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\big|^2|\psi^{\mathbf{e}}_{j,\mathbf{k}}(x)|^2\big)^{1/2}\in L^p(\mathbb{R}^n)$;

(iii) $\big(\sum_{\mathbf{e},j,\mathbf{k}}\big|\langle f,\psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\big|^2|Q_{j,\mathbf{k}}|^{-1}\chi_{Q_{j,\mathbf{k}}}(x)\big)^{1/2}\in L^p(\mathbb{R}^n)$.

To show the main theorem, we claim that $T\psi^{\mathbf{e}}_{j,\mathbf{k}}$ and $\psi^{\mathbf{e}'}_{j',\mathbf{k}'}$ have the almost orthogonal property.

Lemma 2.2 *Let $T\in\mathrm{CZO}(\varepsilon)$, $0<\varepsilon\le 1$, with $T1=T^*1=0$ and the family $\{\psi^{\mathbf{e}}_{j,\mathbf{k}}\}$ has properties* (a)–(d)*. Then for $0<\varepsilon'<\varepsilon$,*

$$\big|\langle T\psi^{\mathbf{e}'}_{j',\mathbf{k}'},\psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\big|\le C2^{\frac{n}{2}(j+j')}2^{-|j-j'|\varepsilon'}\frac{2^{(j\vee j')\varepsilon}}{(2^{(j\vee j')}+|y_{j,\mathbf{k}}-y_{j',\mathbf{k}'}|)^{n+\varepsilon}},\tag{2.1}$$

where $j\vee j'=\max\{j,j'\}$ and $y_{j,\mathbf{k}},y_{j',\mathbf{k}'}$ are the center points of $Q_{j,\mathbf{k}},Q_{j',\mathbf{k}'}$ which contains $\operatorname{supp}\psi^{\mathbf{e}}_{j,\mathbf{k}}$, $\operatorname{supp}\psi^{\mathbf{e}'}_{j',\mathbf{k}'}$*, respectively.*

Proof Let $\eta_0\in C^\infty(\mathbb{R}^n)$ satisfy

$$\eta_0(x)=\begin{cases}1, & \text{if } |x|\le 1,\\ 0, & \text{if } |x|>2,\end{cases}$$

and set $\eta_1=1-\eta_0$. To prove this lemma, we consider the four cases: (i) $\{j'\le j$ and $|y_{j,\mathbf{k}}-y_{j',\mathbf{k}'}|\le 4\cdot 2^j\}$, (ii) $\{j'\le j$ and $|y_{j,\mathbf{k}}-y_{j',\mathbf{k}'}|>4\cdot 2^j\}$, (iii) $\{j'>j$ and $|y_{j,\mathbf{k}}-y_{j',\mathbf{k}'}|\le 4\cdot 2^{j'}\}$ and (iv) $\{j'>j$ and $|y_{j,\mathbf{k}}-y_{j',\mathbf{k}'}|>4\cdot 2^{j'}\}$.

In Case (i), it suffices to show

$$\big|\langle T\psi^{\mathbf{e}'}_{j',\mathbf{k}'},\psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\big|\le C2^{\frac{n}{2}(j'-j)}2^{-(j-j')\varepsilon'}.$$

Using $T^*1=0$, we rewrite

$$\langle T\psi^{\mathbf{e}'}_{j',\mathbf{k}'},\psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle=\big\langle T\psi^{\mathbf{e}'}_{j',\mathbf{k}'},\psi^{\mathbf{e}}_{j,\mathbf{k}}-\psi^{\mathbf{e}}_{j,\mathbf{k}}(y_{j',\mathbf{k}'})\big\rangle$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} T\psi^{\mathbf{e}'}_{j',\mathbf{k}'}(x)\big(\psi^{\mathbf{e}}_{j,\mathbf{k}}(x) - \psi^{\mathbf{e}}_{j,\mathbf{k}}(y_{j',\mathbf{k}'})\big)dx \\
&= \int_{\mathbb{R}^n} T\psi^{\mathbf{e}'}_{j',\mathbf{k}'}(x)\big(\psi^{\mathbf{e}}_{j,\mathbf{k}}(x) - \psi^{\mathbf{e}}_{j,\mathbf{k}}(y_{j',\mathbf{k}'})\big)\eta_0\left(\frac{x-y_{j',\mathbf{k}'}}{2\cdot 2^{j'}}\right)dx \\
&\quad + \int_{\mathbb{R}^n} T\psi^{\mathbf{e}'}_{j',\mathbf{k}'}(x)\big(\psi^{\mathbf{e}}_{j,\mathbf{k}}(x) - \psi^{\mathbf{e}}_{j,\mathbf{k}}(y_{j',\mathbf{k}'})\big)\eta_1\left(\frac{x-y_{j',\mathbf{k}'}}{2\cdot 2^{j'}}\right)dx \\
&:= \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}
$$

To estimate I_1, Hölder's inequality and the L^2 boundedness of T imply

$$
\begin{aligned}
|\mathrm{I}_1| &\le \left\|T\psi^{\mathbf{e}'}_{j',\mathbf{k}'}\right\|_2 \left\|\big(\psi^{\mathbf{e}}_{j,\mathbf{k}}(\cdot) - \psi^{\mathbf{e}}_{j,\mathbf{k}}(y_{j',\mathbf{k}'})\big)\eta_0\left(\frac{\cdot - y_{j',\mathbf{k}'}}{2\cdot 2^{j'}}\right)\right\|_2 \\
&\le C\bigg(\int_{|x-y_{j',\mathbf{k}'}|\le 4\cdot 2^{j'}} \left|\psi^{\mathbf{e}}_{j,\mathbf{k}}(x) - \psi^{\mathbf{e}}_{j,\mathbf{k}}(y_{j',\mathbf{k}'})\right|^2 dx\bigg)^{1/2}.
\end{aligned}
$$

We use the mean value theorem to get

$$
|\mathrm{I}_1| \le C2^{\frac{n}{2}(j'-j)}2^{-(j-j')\varepsilon'}.
$$

To estimate I_2, the cancellation $\int \psi^{\mathbf{e}'} = 0$ and $x \notin B(y_{j',\mathbf{k}'}, 2^{j'})$ show

$$
\begin{aligned}
|\mathrm{I}_2| = \bigg| &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\psi^{\mathbf{e}}_{j,\mathbf{k}}(x) - \psi^{\mathbf{e}}_{j,\mathbf{k}}(y_{j',\mathbf{k}'})\big) \\
&\times \eta_1\left(\frac{x-y_{j',\mathbf{k}'}}{2\cdot 2^{j'}}\right)\big(K(x,y) - K(x,y_{j',\mathbf{k}'})\big)\psi^{\mathbf{e}'}_{j',\mathbf{k}'}(y)dydx\bigg|.
\end{aligned}
$$

By an easy estimate

$$
\left|\psi^{\mathbf{e}}_{j,\mathbf{k}}(x) - \psi^{\mathbf{e}}_{j,\mathbf{k}}(y_{j',\mathbf{k}'})\right| \le C\bigg(\frac{|x-y_{j',\mathbf{k}'}|}{2^j + |x-y_{j',\mathbf{k}'}|}\bigg)2^{-\frac{jn}{2}}.
$$

Since $y \in B(y_{j',\mathbf{k}'}, 2^{j'})$, we have $|y - y_{j',\mathbf{k}'}| \le 2^{j'} \le \frac{1}{2}|x - y_{j',\mathbf{k}'}|$ and hence the condition of standard kernel implies

$$
\begin{aligned}
|\mathrm{I}_2| &\le C\int_{\mathbb{R}^n}\int_{|x-y_{j',\mathbf{k}'}|\ge 2\cdot 2^{j'}} \left(\frac{|x-y_{j',\mathbf{k}'}|}{2^j + |x-y_{j',\mathbf{k}'}|}\right)^{\varepsilon} 2^{-\frac{jn}{2}} \frac{|y-y_{j',\mathbf{k}'}|^{\varepsilon}}{|x-y_{j',\mathbf{k}'}|^{n+\varepsilon}}|\psi^{\mathbf{e}'}_{j',\mathbf{k}'}(y)|dxdy \\
&\le C\int_{\mathbb{R}^n}\int_{|x-y_{j',\mathbf{k}'}|\ge 2\cdot 2^{j'}} \frac{1}{(2^j + |x-y_{j',\mathbf{k}'}|)^{\varepsilon}} 2^{-\frac{jn}{2}} \frac{2^{j'\varepsilon}}{|x-y_{j',\mathbf{k}'}|^{n}}|\psi^{\mathbf{e}'}_{j',\mathbf{k}'}(y)|dxdy \\
&\le C2^{-\frac{jn}{2}}2^{j'\varepsilon}2^{\frac{j'n}{2}}\int_{|x-y_{j',\mathbf{k}'}|\ge 2\cdot 2^{j'}} \frac{1}{(2^j + |x-y_{j',\mathbf{k}'}|)^{\varepsilon}}|x-y_{j',\mathbf{k}'}|^{-n}dx \\
&\le C2^{-\frac{jn}{2}}2^{j'\varepsilon}2^{\frac{j'n}{2}}\bigg(\int_{|x-y_{j',\mathbf{k}'}|\ge 2\cdot 2^{j'}} |x-y_{j',\mathbf{k}'}|^{-n-\varepsilon}dx \\
&\qquad\qquad + \int_{2\cdot 2^j > |x-y_{j',\mathbf{k}'}|\ge 2\cdot 2^{j'}} |x-y_{j',\mathbf{k}'}|^{-n}2^{-j\varepsilon}dx\bigg) \\
&\le C2^{-\frac{jn}{2}}2^{j'\varepsilon}2^{\frac{j'n}{2}}2^{-j\varepsilon}(1+\log 2^{j-j'}) \le C2^{\frac{n}{2}(j'-j)}2^{-(j-j')\varepsilon'}.
\end{aligned}
$$

Next, we consider Case (ii). We only need to show

$$
\left|\langle T\psi^{\mathbf{e}'}_{j',\mathbf{k}'}, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\right| \le C2^{\frac{n}{2}(j+j')}\frac{2^{j'\varepsilon}}{|y_{j,\mathbf{k}} - y_{j',\mathbf{k}'}|^{n+\varepsilon}}.
$$

Since

$$|x - y_{j,\mathbf{k}}| \geq |y_{j',\mathbf{k}'} - y_{j,\mathbf{k}}| - |y_{j',\mathbf{k}'} - x| > 4 \cdot 2^j - 2^{j'} > 2^j$$

for any $|x - y_{j',\mathbf{k}'}| < 2^{j'}$, we have $\operatorname{supp} \psi^{\mathbf{e}'}_{j',\mathbf{k}'} \cap \operatorname{supp} \psi^{\mathbf{e}}_{j,\mathbf{k}} = \emptyset$. Thus $\int \psi^{\mathbf{e}'} = 0$ implies

$$\big|\langle T\psi^{\mathbf{e}'}_{j',\mathbf{k}'}, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\big| = \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (K(x,y) - K(x, y_{j',\mathbf{k}'}))\psi^{\mathbf{e}'}_{j',\mathbf{k}'}(y)\psi^{\mathbf{e}}_{j,\mathbf{k}}(x)dydx\right|.$$

By

$$|x - y_{j,\mathbf{k}}| < 2^j < \frac{1}{4}|y_{j,\mathbf{k}} - y_{j',\mathbf{k}'}| \quad \text{and} \quad |y - y_{j',\mathbf{k}'}| < 2^{j'} \leq 2^j < \frac{1}{4}|y_{j,\mathbf{k}} - y_{j',\mathbf{k}'}|,$$

we have

$$|y_{j,\mathbf{k}} - y_{j',\mathbf{k}'}| \leq \frac{1}{2}|y_{j,\mathbf{k}} - y_{j',\mathbf{k}'}| + |x - y|.$$

It follows that

$$|x - y| \geq \frac{1}{2}|y_{j,\mathbf{k}} - y_{j',\mathbf{k}'}| \quad \text{and} \quad |y - y_{j',\mathbf{k}'}| \leq \frac{1}{2}|x - y|. \tag{2.2}$$

The condition of the standard kernel and inequality (2.2) yield

$$\begin{aligned}\big|\langle T\psi^{\mathbf{e}'}_{j',\mathbf{k}'}, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\big| &\leq C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\psi^{\mathbf{e}}_{j,\mathbf{k}}(x)|\frac{|y - y_{j',\mathbf{k}'}|^{\varepsilon}}{|x - y|^{n+\varepsilon}}|\psi^{\mathbf{e}'}_{j',\mathbf{k}'}(y)|dxdy \\ &\leq C\frac{2^{j'\varepsilon}}{|y_{j,\mathbf{k}} - y_{j',\mathbf{k}'}|^{n+\varepsilon}}\int_{\mathbb{R}^n}|\psi^{\mathbf{e}}_{j,\mathbf{k}}(x)|dx\int_{\mathbb{R}^n}|\psi^{\mathbf{e}'}_{j',\mathbf{k}'}(y)|dy \\ &\leq C\frac{2^{j'\varepsilon}}{|y_{j,\mathbf{k}} - y_{j',\mathbf{k}'}|^{n+\varepsilon}}2^{\frac{jn}{2}}2^{\frac{j'n}{2}}.\end{aligned}$$

The cases (iii) and (iv) are similar to (i) and (ii), and the proof is completed. □

3 Proof of Theorem 1.1

In this section, we will show the H^p boundedness of Calderón–Zygmund operators. To reach the goal, we first consider an easier case.

Theorem 3.1 *Let* $T \in \mathrm{CZO}(\varepsilon)$, $0 < \varepsilon \leq 1$, *with* $T1 = T^*1 = 0$. *Then* T *is bounded on* $H^p(\mathbb{R}^n)$, $\frac{n}{n+\varepsilon} < p \leq 1$.

Proof We know that the subspace $H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $H^p(\mathbb{R}^n)$. It suffices to show that T is bounded from $H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$. Given $f \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $Tf \in L^2(\mathbb{R}^n)$ since T is a Calderón–Zygmund operator. Property (a) implies

$$Tf = \sum_{\mathbf{e},j,\mathbf{k}}\langle Tf, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\psi^{\mathbf{e}}_{j,\mathbf{k}} \quad \text{and} \quad f = \sum_{\mathbf{e}',j',\mathbf{k}'}\langle f, \psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\psi^{\mathbf{e}'}_{j',\mathbf{k}'}.$$

Fixing $\mathbf{e}, j, \mathbf{k}$, we estimate that

$$\big|\langle Tf, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\big| = \left|\sum_{\mathbf{e}',j',\mathbf{k}'}\langle f, \psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\langle T\psi^{\mathbf{e}'}_{j',\mathbf{k}'}, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\right| \leq \sum_{\mathbf{e}',j',\mathbf{k}'}\big|\langle f, \psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\big|\big|\langle T\psi^{\mathbf{e}'}_{j',\mathbf{k}'}, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\big|.$$

For j, j' and $\mathbf{k}$ are fixed, we define the sets

$$A_0 = \big\{Q_{j',\mathbf{k}'} : |y_{j,\mathbf{k}} - y_{j',\mathbf{k}'}| \leq 2^{(j\vee j')}\big\}, \quad A_i = \big\{Q_{j',\mathbf{k}'} : 2^{i-1}2^{(j\vee j')} < |y_{j,\mathbf{k}} - y_{j',\mathbf{k}'}| \leq 2^i 2^{(j\vee j')}\big\}$$

for all $i \in \mathbb{N}$ and $y_{j,\mathbf{k}}, y_{j',\mathbf{k}'}$ are the center points of $Q_{j,\mathbf{k}}, Q_{j',\mathbf{k}'}$ which contains $\operatorname{supp} \psi^{\mathbf{e}}_{j,\mathbf{k}}$, $\operatorname{supp} \psi^{\mathbf{e}'}_{j',\mathbf{k}'}$, respectively. Choose r satisfying $\frac{n}{n+\varepsilon} < r < p$, (2.1) gives

$$\sum_{\mathbf{e}',j',\mathbf{k}'}\big|\langle f, \psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\big|\big|\langle T\psi^{\mathbf{e}'}_{j',\mathbf{k}'}, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\big|$$

$$
\begin{aligned}
&\le C\sum_{\mathbf{e}',j',\mathbf{k}'} 2^{\frac{n}{2}(j+j')}2^{-|j-j'|\varepsilon'}\frac{2^{(j\vee j')\varepsilon}}{(2^{(j\vee j')}+|y_{j,\mathbf{k}}-y_{j',\mathbf{k}'}|)^{n+\varepsilon}}\left|\langle f,\psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\right| \\
&= C\sum_{\mathbf{e}',j'} 2^{\frac{n}{2}(j+j')}2^{-|j-j'|\varepsilon'}\sum_{i\ge 0}\sum_{\{\mathbf{k}':Q_{j',\mathbf{k}'}\in A_i\}}\frac{2^{(j\vee j')\varepsilon}}{(2^{(j\vee j')}+|y_{j,\mathbf{k}}-y_{j',\mathbf{k}'}|)^{n+\varepsilon}}\left|\langle f,\psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\right| \\
&\le C\sum_{\mathbf{e}',j'} 2^{\frac{n}{2}(j+j')}2^{-|j-j'|\varepsilon'}2^{-(j\vee j')n}\sum_{i\ge 0}2^{-i(n+\varepsilon)}\Bigg(\sum_{\{\mathbf{k}':Q_{j',\mathbf{k}'}\in A_i\}}\left|\langle f,\psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\right|\Bigg) \\
&\le C\sum_{\mathbf{e}',j'} 2^{\frac{n}{2}(j+j')}2^{-|j-j'|\varepsilon'}2^{-(j\vee j')n}\sum_{i\ge 0}2^{-i(n+\varepsilon)}\Bigg(\sum_{\{\mathbf{k}':Q_{j',\mathbf{k}'}\in A_i\}}\left|\langle f,\psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\right|^r\Bigg)^{1/r}.
\end{aligned}
$$

Therefore, the property that all cubes in A_i are nonoverlapping for all i shows that

$$
\begin{aligned}
&\sum_{\mathbf{e}',j',\mathbf{k}'}\left|\langle f,\psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\right|\left|\langle T\psi^{\mathbf{e}'}_{j',\mathbf{k}'},\psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\right| \\
&\quad\le C\sum_{\mathbf{e}',j'} 2^{\frac{n}{2}(j+j')}2^{-|j-j'|\varepsilon'}2^{-(j\vee j')n} \\
&\qquad\times\sum_{i\ge 0}2^{-i(n+\varepsilon)}\Bigg(\sum_{\{\mathbf{k}':Q_{j',\mathbf{k}'}\in A_i\}}\int_{\mathbb{R}^n}\left|\langle f,\psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\right|^r|Q_{j',\mathbf{k}'}|^{-1}\chi_{Q_{j',\mathbf{k}'}}\,dx\Bigg)^{1/r} \\
&\quad= C\sum_{\mathbf{e}',j'} 2^{\frac{n}{2}(j+j')}2^{-|j-j'|\varepsilon'}2^{-(j\vee j')n}2^{-\frac{j'n}{r}} \\
&\qquad\times\sum_{i\ge 0}2^{-i(n+\varepsilon)}\Bigg(\int_{\mathbb{R}^n}\Bigg(\sum_{\{\mathbf{k}':Q_{j',\mathbf{k}'}\in A_i\}}\left|\langle f,\psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\right|\chi_{Q_{j',\mathbf{k}'}}\Bigg)^r dx\Bigg)^{1/r}.
\end{aligned}
$$

Since

$$
\Bigg|\bigcup_{Q_{j',\mathbf{k}'}\in A_i}Q_{j',\mathbf{k}'}\Bigg|\le C\big(\left|y_{j,\mathbf{k}}-y_{j',\mathbf{k}'}\right|+\sqrt{n}\cdot 2^{(j\vee j')}\big)^n\le C2^{in}2^{(j\vee j')n}\quad\text{for any } i\in\mathbb{Z},
$$

we obtain

$$
\begin{aligned}
&\sum_{\mathbf{e}',j',\mathbf{k}'}\left|\langle f,\psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\right|\left|\langle T\psi^{\mathbf{e}'}_{j',\mathbf{k}'},\psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\right| \\
&\quad\le C\sum_{\mathbf{e}',j'} 2^{\frac{n}{2}(j+j')}2^{-|j-j'|\varepsilon'}2^{-(j\vee j')n}2^{((j\vee j')-j')\frac{n}{r}} \\
&\qquad\times\sum_{i\ge 0}2^{-i(n+\varepsilon)}2^{\frac{in}{r}}\Bigg(\Bigg|\bigcup_{Q_{j',\mathbf{k}'}\in A_i}Q_{j',\mathbf{k}'}\Bigg|^{-1}\int_{\mathbb{R}^n}\Bigg(\sum_{Q_{j',\mathbf{k}'}\in A_i}\left|\langle f,\psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\right|\chi_{Q_{j',\mathbf{k}'}}\Bigg)^r dx\Bigg)^{1/r}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\sum_{\mathbf{e}',j',\mathbf{k}'}\left|\langle f,\psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\right|\left|\langle T\psi^{\mathbf{e}'}_{j',\mathbf{k}'},\psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\right| \\
&\quad\le C\sum_{\mathbf{e}',j'} 2^{\frac{n}{2}(j+j')}2^{-|j-j'|\varepsilon'}2^{-(j\vee j')n}2^{((j\vee j')-j')\frac{n}{r}}\Bigg\{M\Bigg(\Bigg(\sum_{\mathbf{k}'}\left|\langle f,\psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\right|\chi_{Q_{j',\mathbf{k}'}}\Bigg)^r\Bigg)\Bigg\}^{\frac{1}{r}},
\end{aligned}
$$

where M is the Hardy–Littlewood maximal function. This implies that

$$
\sum_{\mathbf{e},j,\mathbf{k}}\left|\langle Tf,\psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\right|^2\left|Q_{j,\mathbf{k}}\right|^{-1}\chi_{Q_{j,\mathbf{k}}}
$$

$$\leq C\sum_{\mathbf{e},j}\Bigg(\sum_{\mathbf{e}',j'}2^{j'n}2^{-|j-j'|\varepsilon'}2^{-(j\vee j')n}2^{((j\vee j')-j')\frac{n}{r}}$$

$$\cdot\left\{M\Bigg(\Bigg(\sum_{\mathbf{k}'}\big|\langle f,\psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\big|2^{-\frac{j'n}{2}}\chi_{Q_{j',\mathbf{k}'}}\Bigg)^r\Bigg)\right\}^{\frac{1}{r}}\Bigg)^2$$

$$\leq C\sum_{\mathbf{e},j}\sum_{\mathbf{e}',j'}2^{j'n}2^{-|j-j'|\varepsilon'}2^{-(j\vee j')n}2^{((j\vee j')-j')\frac{n}{r}}$$

$$\cdot\left\{M\Bigg(\Bigg(\sum_{\mathbf{k}'}\big|\langle f,\psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\big|2^{-\frac{j'n}{2}}\chi_{Q_{j',\mathbf{k}'}}\Bigg)^r\Bigg)\right\}^{\frac{2}{r}}$$

$$\leq C\sum_{\mathbf{e}',j'}\left\{M\Bigg(\Bigg(\sum_{\mathbf{k}'}\big|\langle f,\psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\big|2^{-\frac{j'n}{2}}\chi_{Q_{j',\mathbf{k}'}}\Bigg)^r\Bigg)\right\}^{\frac{2}{r}}$$

since

$$\sup_{j}\sum_{j'}2^{j'n}2^{-|j-j'|\varepsilon'}2^{-(j\vee j')n}2^{((j\vee j')-j')\frac{n}{r}}<\infty$$

and

$$\sup_{j'}\sum_{j}2^{j'n}2^{-|j-j'|\varepsilon'}2^{-(j\vee j')n}2^{((j\vee j')-j')\frac{n}{r}}<\infty.$$

Using the Fefferman–Stein vector-valued maximal function inequality [10, Theorem 1.1] with $r<p$, we have

$$\left\|\left\{\sum_{\mathbf{e},j,\mathbf{k}}\big|\langle Tf,\psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\big|^2\big|Q_{j,\mathbf{k}}\big|^{-1}\chi_{Q_{j,\mathbf{k}}}\right\}^{1/2}\right\|_p$$

$$\leq C\left\|\left\{\sum_{\mathbf{e}',j'}\left[M\Bigg(\Bigg(\sum_{\mathbf{k}'}\big|\langle f,\psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\big|2^{-\frac{j'n}{2}}\chi_{Q_{j',\mathbf{k}'}}\Bigg)^r\Bigg)\right]^{\frac{2}{r}}\right\}^{1/2}\right\|_p$$

$$= C\left\|\left\{\sum_{\mathbf{e}',j'}\left[M\Bigg(\Bigg(\sum_{\mathbf{k}'}\big|\langle f,\psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\big|2^{-\frac{j'n}{2}}\chi_{Q_{j',\mathbf{k}'}}\Bigg)^r\Bigg)\right]^{\frac{2}{r}}\right\}^{r/2}\right\|_{p/r}^{1/r}$$

$$\leq C\left\|\left\{\sum_{\mathbf{e}',j'}\Bigg(\sum_{\mathbf{k}'}\big|\langle f,\psi^{\mathbf{e}'}_{j',\mathbf{k}'}\rangle\big|2^{-\frac{j'n}{2}}\chi_{Q_{j',\mathbf{k}'}}\Bigg)^2\right\}^{r/2}\right\|_{p/r}^{1/r}$$

$$= C\left\|\left\{\sum_{\mathbf{e},j,\mathbf{k}}\big|\langle f,\psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\big|^2\big|Q_{j,\mathbf{k}}\big|^{-1}\chi_{Q_{j,\mathbf{k}}}\right\}^{1/2}\right\|_p.$$

The last equality holds since each $Q_{j',\mathbf{k}'}$ on the right-hand side is nonoverlapping. The proof is completed by Theorem 2.1. □

To finish the proof of Theorem 1.1, we have to study para-product operators. Given $f\in L^2(\mathbb{R}^n)\cap H^p(\mathbb{R}^n)$, there exists a Schwartz function ϕ on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n}\phi=1$ such that $M_\phi f\in L^p(\mathbb{R}^n)$. For $i\in\mathbb{N}$, define

$$\Omega_i=\left\{x\in\mathbb{R}^n: M_\phi(f)(x)>2^i\right\}$$

and

$$B_i=\left\{Q \text{ is a dyadic cube}: |Q\cap\Omega_i|>\frac{1}{2}|Q|, |Q\cap\Omega_{i+1}|\leq\frac{1}{2}|Q|\right\}. \tag{3.1}$$

For any $Q_{j,\mathbf{k}}$, we can choose $x_{j,\mathbf{k}} \in Q_{j,\mathbf{k}} \cap (\Omega_i \setminus \Omega_{i+1})$ since $Q_{j,\mathbf{k}} \in B_i$ for some i. Then we obtain

$$M_\phi(f)(x_{j,\mathbf{k}}) \le 2^{i+1}. \tag{3.2}$$

Now, for any $b \in \mathrm{BMO}$, we define the *para-product operator*

$$\Pi_b(f)(x) = \sum_{\mathbf{e},j,\mathbf{k}} \langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle \big(\phi_{2^j} * f\big(x_{j,\mathbf{k}}\big)\big)\psi^{\mathbf{e}}_{j,\mathbf{k}}(x) \quad \text{for } x \in \mathbb{R}^n.$$

Its adjoint operator can be written as

$$\Pi^*_b(f)(x) = \sum_{\mathbf{e},j,\mathbf{k}} \langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle \langle \psi^{\mathbf{e}}_{j,\mathbf{k}}, f\rangle \phi_{2^j}(x_{j,\mathbf{k}} - x) \quad \text{for } x \in \mathbb{R}^n.$$

Then

$$\Pi_b 1 = \sum_{\mathbf{e},j,\mathbf{k}} \langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle ((\phi_{2^j} * 1)(x_{j,\mathbf{k}}))\psi^{\mathbf{e}}_{j,\mathbf{k}} = \sum_{\mathbf{e},j,\mathbf{k}} \langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle \bigg(\int_{\mathbb{R}^n} \phi_{2^j}(y)dy\bigg)\psi^{\mathbf{e}}_{j,\mathbf{k}} = b$$

and

$$\Pi^*_b 1 = \sum_{\mathbf{e},j,\mathbf{k}} \langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle \langle \psi^{\mathbf{e}}_{j,\mathbf{k}}, 1\rangle \phi_{2^j}(x_{j,\mathbf{k}} - \cdot) = \sum_{\mathbf{e},j,\mathbf{k}} \langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle \bigg(\int_{\mathbb{R}^n} \psi^{\mathbf{e}}_{j,\mathbf{k}}(y)dy\bigg)\phi_{2^j}(x_{j,\mathbf{k}} - \cdot) = 0.$$

Moreover, we have the H^p boundedness of Π_b.

Theorem 3.2 *Let ε be the regularity exponent associated with singular integral operator and $\frac{n}{n+\varepsilon} < p \le 1$. Suppose $b \in \mathrm{BMO}$, then Π_b is bounded on H^p.*

Proof By Theorem 2.1, it suffices to show that

$$\bigg\| \bigg\{ \sum_{\mathbf{e},j,\mathbf{k}} \big|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\big|^2 \big|\phi_{2^j} * f\big(x_{j,\mathbf{k}}\big)\big|^2 |Q_{j,\mathbf{k}}|^{-1} \chi_{Q_{j,\mathbf{k}}} \bigg\}^{\frac12} \bigg\|_p \le C\|f\|_{H^p}.$$

Note that

$$\begin{aligned} &\bigg\| \bigg\{ \sum_{\mathbf{e},j,\mathbf{k}} \big|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\big|^2 \big|\phi_{2^j} * f(x_{j,\mathbf{k}})\big|^2 \big|Q_{j,\mathbf{k}}\big|^{-1} \chi_{Q_{j,\mathbf{k}}} \bigg\}^{\frac12} \bigg\|_p^p \\ &= \int_{\mathbb{R}^n} \bigg(\sum_{\mathbf{e}} \sum_i \sum_{\{\mathbf{k}: \widetilde{Q}_{j,\mathbf{k}} \in B_i\}} \sum_{\substack{\{j,\mathbf{k}: Q_{j,\mathbf{k}} \subseteq \widetilde{Q}_{j,\mathbf{k}},\\ Q_{j,\mathbf{k}} \in B_i\}}} \big|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\big|^2 \big|\phi_{2^j} * f(x_{j,\mathbf{k}})\big|^2 \big|Q_{j,\mathbf{k}}\big|^{-1} \chi_{Q_{j,\mathbf{k}}} \bigg)^{\frac p2} dx, \end{aligned}$$

where $\widetilde{Q}_{j,\mathbf{k}}$ is the largest cube containing the cube $Q_{j,\mathbf{k}}$ in B_i. Hölder's inequality and (3.2) yield

$$\begin{aligned} &\bigg\| \bigg\{ \sum_{\mathbf{e},j,\mathbf{k}} \big|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\big|^2 \big|\phi_{2^j} * f(x_{j,\mathbf{k}})\big|^2 \big|Q_{j,\mathbf{k}}\big|^{-1} \chi_{Q_{j,\mathbf{k}}} \bigg\}^{\frac12} \bigg\|_p^p \\ &\le \sum_{\mathbf{e}} \sum_i \sum_{\{j,\mathbf{k}: \widetilde{Q}_{j,\mathbf{k}} \in B_i\}} \int_{\mathbb{R}^n} \bigg(\sum_{\substack{\{j,\mathbf{k}: Q_{j,\mathbf{k}} \subseteq \widetilde{Q}_{j,\mathbf{k}},\\ Q_{j,\mathbf{k}} \in B_i\}}} \big|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\big|^2 \big|\phi_{2^j} * f(x_{j,\mathbf{k}})\big|^2 \big|Q_{j,\mathbf{k}}\big|^{-1} \chi_{Q_{j,\mathbf{k}}} \bigg)^{\frac p2} dx \\ &\le \sum_{\mathbf{e}} \sum_i \sum_{\{j,\mathbf{k}: \widetilde{Q}_{j,\mathbf{k}} \in B_i\}} \big|\widetilde{Q}_{j,\mathbf{k}}\big|^{1-\frac p2} \bigg(\sum_{\substack{\{j,\mathbf{k}: Q_{j,\mathbf{k}} \subseteq \widetilde{Q}_{j,\mathbf{k}},\\ Q_{j,\mathbf{k}} \in B_i\}}} \big|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\big|^2 \big|\phi_{2^j} * f(x_{j,\mathbf{k}})\big|^2 \bigg)^{\frac p2} \end{aligned}$$

$$\leq \sum_{\mathbf{e}} \sum_{i} \Big(\sum_{\{j,\mathbf{k}:\widetilde{Q}_{j,\mathbf{k}} \in B_i\}} |\widetilde{Q}_{j,\mathbf{k}}| \Big)^{1-\frac{p}{2}} \Big(\sum_{\{j,\mathbf{k}:Q_{j,\mathbf{k}} \in B_i\}} \big|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}} \rangle\big|^2 |\phi_{2^j} * f(x_{j,\mathbf{k}})\big|^2 \Big)^{\frac{p}{2}}$$

$$\leq C \sum_{\mathbf{e}} \sum_{i} 2^{ip} \Big(\sum_{\{j,\mathbf{k}:\widetilde{Q}_{j,\mathbf{k}} \in B_i\}} 2|\widetilde{Q}_{j,\mathbf{k}} \cap \Omega_i| \Big)^{1-\frac{p}{2}} \Big(\sum_{\{j,\mathbf{k}:Q_{j,\mathbf{k}} \in B_i\}} \big|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}} \rangle\big|^2 \Big)^{\frac{p}{2}}.$$

Write

$$\widetilde{\Omega}_i = \{x \in \mathbb{R}^n : M(\chi_{\Omega_i})(x) > 1/2\}.$$

Given $Q_{j,\mathbf{k}} \in B_i$, we have $Q_{j,\mathbf{k}} \subseteq \widetilde{\Omega}_i$. Then

$$\sum_{\mathbf{e}} \sum_{i} 2^{ip} \Big(\sum_{\{j,\mathbf{k}:\widetilde{Q}_{j,\mathbf{k}} \in B_i\}} 2|\widetilde{Q}_{j,\mathbf{k}} \cap \Omega_i| \Big)^{1-\frac{p}{2}} \Big(\sum_{\{j,\mathbf{k}:Q_{j,\mathbf{k}} \in B_i\}} \big|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}} \rangle\big|^2 \Big)^{\frac{p}{2}}$$

$$\leq C \sum_{\mathbf{e}} \sum_{i} 2^{ip} \big|\Omega_i\big|^{1-\frac{p}{2}} \Big(\sum_{\{j,\mathbf{k}:Q_{j,\mathbf{k}} \subseteq \widetilde{\Omega}_i\}} \big|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}} \rangle\big|^2 \Big)^{\frac{p}{2}}.$$

By [8, p. 154, Theorem 4],

$$\sum_{\mathbf{e}} \sum_{i} 2^{ip} \big|\Omega_i\big|^{1-\frac{p}{2}} \Big(\sum_{\{j,\mathbf{k}:Q_{j,\mathbf{k}} \subseteq \widetilde{\Omega}_i\}} \big|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}} \rangle\big|^2 \Big)^{\frac{p}{2}} \leq C \sum_{\mathbf{e}} \sum_{i} 2^{ip} \big|\Omega_i\big|^{1-\frac{p}{2}} \big|\widetilde{\Omega}_i\big|^{\frac{p}{2}}.$$

Since M is of weak-type $(1,1)$,

$$\Big\| \Big\{ \sum_{\mathbf{e},j,\mathbf{k}} \big|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}} \rangle\big|^2 \big|\phi_{2^j} * f(x_{j,\mathbf{k}})\big|^2 \big|Q_{j,\mathbf{k}}\big|^{-1} \chi_{Q_{j,\mathbf{k}}} \Big\}^{\frac{1}{2}} \Big\|_p^p \leq C \sum_{\mathbf{e}} \sum_{i} 2^{ip} \big|\Omega_i\big| \leq C \big\|M_\phi f\big\|_p^p,$$

which completes the proof. □

Also, Π_b is a Calderón–Zygmund operator.

Theorem 3.3 *Let $b \in \mathrm{BMO}$. Then the operator Π_b is a Calderón–Zygmund operator.*

Proof First, we claim that Π_b is bounded on $L^2(\mathbb{R}^n)$. Given $f \in L^2(\mathbb{R}^n)$,

$$\|\Pi_b f\|_2 = \sup_{\|g\|_2 \leq 1} \big|\langle \Pi_b f, g \rangle\big| \leq \sup_{\|g\|_2 \leq 1} \sum_{\mathbf{e},j,\mathbf{k}} \big|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}} \rangle\big| \big|\phi_{2^j} * f(x_{j,\mathbf{k}})\big| \big|\langle \psi^{\mathbf{e}}_{j,\mathbf{k}}, g \rangle\big|.$$

Using Hölder's inequality and Parseval's formula, we obtain

$$\|\Pi_b f\|_2 \leq \sup_{\|g\|_2 \leq 1} \Big(\sum_{\mathbf{e},j,\mathbf{k}} \big|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}} \rangle\big|^2 \big|\phi_{2^j} * f(x_{j,\mathbf{k}})\big|^2 \Big)^{1/2} \Big(\sum_{\mathbf{e},j,\mathbf{k}} \big|\langle \psi^{\mathbf{e}}_{j,\mathbf{k}}, g \rangle\big|^2 \Big)^{1/2}$$

$$\leq \Big(\sum_{\mathbf{e},j,\mathbf{k}} \big|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}} \rangle\big|^2 \big|\phi_{2^j} * f(x_{j,\mathbf{k}})\big|^2 \Big)^{1/2}$$

$$= \Big(\sum_{\mathbf{e}} \sum_{i} \sum_{\{j,\mathbf{k}:Q_{j,\mathbf{k}} \in B_i\}} \big|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}} \rangle\big|^2 \big|\phi_{2^j} * f(x_{j,\mathbf{k}})\big|^2 \Big)^{1/2},$$

where B_i is defined as (3.1). By (3.2) and Carleson's condition,

$$\|\Pi_b f\|_2 \leq \Big(\sum_{\mathbf{e}} \sum_{i} 2^{2(i+1)} \sum_{\{j,\mathbf{k}:Q_{j,\mathbf{k}} \in B_i\}} \big|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}} \rangle\big|^2 \Big)^{1/2}$$

$$
\begin{aligned}
&\leq \bigg(\sum_{\mathbf{e}} \sum_i 2^{2(i+1)} \sum_{\{j,\mathbf{k}: Q_{j,\mathbf{k}} \subset \widetilde{\Omega}_i\}} \big|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}} \rangle\big|^2 \bigg)^{1/2} \\
&\leq \bigg(\sum_{\mathbf{e}} \sum_i 2^{2(i+1)} |\widetilde{\Omega}_i| \bigg)^{1/2} \leq C \bigg(\sum_{\mathbf{e}} \sum_i 2^{2i} |\Omega_i| \bigg)^{1/2} \\
&\leq C \|M_\phi f\|_2 \leq C \|f\|_2.
\end{aligned}
$$

Obviously, the function

$$
K_b(x,y) = \sum_{\mathbf{e},j,\mathbf{k}} \langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}} \rangle \phi_{2^j}(x_{j,\mathbf{k}} - y) \psi^{\mathbf{e}}_{j,\mathbf{k}}(x), \quad x, y \in \mathbb{R}^n,
$$

is the kernel of Π_b. We check that K_b satisfies (1.1)–(1.3). Since

$$
|\langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}} \rangle| \leq \|b\|_{\mathrm{BMO}} \|\psi^{\mathbf{e}}_{j,\mathbf{k}}\|_{H^1} \leq C 2^{nj/2} \|b\|_{\mathrm{BMO}},
$$

we have

$$
\begin{aligned}
|K_b(x,y)| &= \bigg| \sum_{\mathbf{e},j,\mathbf{k}} \langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}} \rangle \phi_{2^j}(x_{j,\mathbf{k}} - y) \psi^{\mathbf{e}}_{j,\mathbf{k}}(x) \bigg| \\
&\leq C \|b\|_{\mathrm{BMO}} \sum_{\mathbf{e},j,\mathbf{k}} \|\psi^{\mathbf{e}}_{j,\mathbf{k}}\|_{H^1} \frac{2^j}{(2^j + |x_{j,\mathbf{k}} - y|)^{n+1}} |\psi^{\mathbf{e}}_{j,\mathbf{k}}(x)| \\
&\leq C \|b\|_{\mathrm{BMO}} \sum_{\mathbf{e},j,\mathbf{k}} 2^{jn/2} \frac{2^j}{(2^j + |x_{j,\mathbf{k}} - y|)^{n+1}} |\psi^{\mathbf{e}}_{j,\mathbf{k}}(x)| \quad \text{for } x, y \in \mathbb{R}^n.
\end{aligned}
$$

For any j, if $|x-y| > 2\sqrt{n}2^j$, then

$$
\frac{2^j}{(2^j + |x_{j,\mathbf{k}} - y|)^{n+1}} \leq C \frac{2^j}{(2^j + |x-y|)^{n+1}}
$$

since

$$
|x_{j,\mathbf{k}} - y| \geq |x-y| - |x_{j,\mathbf{k}} - x| \geq |x-y| - \sqrt{n}2^j \geq \frac{1}{2}|x-y|.
$$

If $|x-y| \leq 2\sqrt{n}2^j$, then

$$
\frac{2^j}{(2^j + |x_{j,\mathbf{k}} - y|)^{n+1}} \leq 2^{-nj} \leq C \frac{2^j}{(2^j + |x-y|)^{n+1}}.
$$

Hence

$$
|K_b(x,y)| \leq C \sum_{\mathbf{e},j} \frac{2^j}{(2^j + |x-y|)^{n+1}} \leq C|x-y|^{-n}.
$$

Next, we estimate

$$
|K_b(x,y) - K_b(x',y)| \leq C \frac{|x-x'|^\varepsilon}{|x-y|^{n+\varepsilon}} \quad \text{for } 0 < \varepsilon \leq 1.
$$

To see that, we have

$$
\begin{aligned}
|K_b(x,y) - K_b(x',y)| &= \bigg| \sum_{\mathbf{e},j,\mathbf{k}} \langle b, \psi^{\mathbf{e}}_{j,\mathbf{k}} \rangle \phi_{2^j}(x_{j,\mathbf{k}} - y) \big(\psi^{\mathbf{e}}_{j,\mathbf{k}}(x) - \psi^{\mathbf{e}}_{j,\mathbf{k}}(x')\big) \bigg| \\
&\leq C \|b\|_{\mathrm{BMO}} \sum_{\mathbf{e},j} \frac{2^{2j}}{(2^j + |x-y|)^{n+2}} \bigg(\frac{|x-x'|}{2^j + |x-x'|} \bigg)^\varepsilon
\end{aligned}
$$

$$= C\|b\|_{\mathrm{BMO}}|x-x'|^{\varepsilon}\sum_{\mathbf{e},j}\frac{2^{j(2-\varepsilon)}}{(2^j+|x-y|)^{n+2}}$$
$$\leq C\frac{|x-x'|^{\varepsilon}}{|x-y|^{n+\varepsilon}}.$$

Finally, we estimate

$$|K_b(x,y)-K_b(x,y')|\leq C\frac{|y-y'|^{\varepsilon}}{|x-y|^{n+\varepsilon}}\quad \text{for } 0<\varepsilon\leq 1 \text{ and } |y-y'|\leq\frac{1}{2}|x-y|.$$

By the definition,

$$\begin{aligned}
&|K_b(x,y)-K_b(x,y')|\\
&\quad=\Big|\sum_{\mathbf{e},j,\mathbf{k}}\langle b,\psi^{\mathbf{e}}_{j,\mathbf{k}}\rangle\big(\phi_{2^j}(x_{j,\mathbf{k}}-y)-\phi_{2^j}(x_{j,\mathbf{k}}-y')\big)\psi^{\mathbf{e}}_{j,\mathbf{k}}(x)\Big|\\
&\quad\leq C\sum_{\mathbf{e},j,\mathbf{k}}\|b\|_{\mathrm{BMO}}\|\psi^{\mathbf{e}}_{j,\mathbf{k}}\|_{H^1}\big|\phi_{2^j}(x_{j,\mathbf{k}}-y)-\phi_{2^j}(x_{j,\mathbf{k}}-y')\big||\psi^{\mathbf{e}}_{j,\mathbf{k}}(x)|\\
&\quad\leq C\sum_{\mathbf{e}\in E}\sum_{\{j,\mathbf{k}:|y-y'|\leq 2^{j-1}\}}\|\psi^{\mathbf{e}}_{j,\mathbf{k}}\|_{H^1}\big|\phi_{2^j}(x_{j,\mathbf{k}}-y)-\phi_{2^j}(x_{j,\mathbf{k}}-y')\big||\psi^{\mathbf{e}}_{j,\mathbf{k}}(x)|\\
&\qquad+C\sum_{\mathbf{e}\in E}\sum_{\{j,\mathbf{k}:|y-y'|>2^{j-1}\}}\|\psi^{\mathbf{e}}_{j,\mathbf{k}}\|_{H^1}\big|\phi_{2^j}(x_{j,\mathbf{k}}-y)-\phi_{2^j}(x_{j,\mathbf{k}}-y')\big||\psi^{\mathbf{e}}_{j,\mathbf{k}}(x)|\\
&\quad:=\mathrm{II}_1+\mathrm{II}_2.
\end{aligned}$$

For II_1, we have

$$\big|\phi_{2^j}(x_{j,\mathbf{k}}-y)-\phi_{2^j}(x_{j,\mathbf{k}}-y')\big|\leq C\Big(\frac{|y-y'|}{2^j}\Big)^{\varepsilon}\frac{2^j}{(2^j+|x-y|)^{n+1}}.$$

Hence,

$$\mathrm{II}_1\leq C|y-y'|^{\varepsilon}\sum_{\{j:|y-y'|\leq 2^j\}}\frac{2^{j(1-\varepsilon)}}{(2^j+|x-y|)^{n+1}}\leq C\frac{|y-y'|^{\varepsilon}}{|x-y|^{n+\varepsilon}}.$$

For II_2, we have

$$\begin{aligned}
\big|\phi_{2^j}(x_{j,\mathbf{k}}-y)-\phi_{2^j}(x_{j,\mathbf{k}}-y')\big|&\leq C\Big(\frac{2^j}{(2^j+|x-y|)^{n+1}}+\frac{2^j}{(2^j+|x-y'|)^{n+1}}\Big)\\
&\leq C\frac{2^j}{(2^j+|x-y|)^{n+1}}
\end{aligned}$$

since $|x-y'|\geq|x-y|-|y-y'|\geq\frac{1}{2}|x-y|$. Hence, we obtain

$$\mathrm{II}_2\leq C\sum_{\{j:|y-y'|>2^{j-1}\}}\frac{2^j}{(2^j+|x-y|)^{n+1}}\leq C\frac{|y-y'|}{|x-y|^{n+1}}$$

and the proof is completed. □

We are ready to show the main theorem.

Proof of Theorem 1.1 We define the operator $\widetilde{T}$ by

$$\widetilde{T}=T-\Pi_{T1}.$$

Since T is bounded on $L^2(\mathbb{R}^n)$, $T1$ theorem shows that $T1\in\mathrm{BMO}$ (see [11]). It is easy to check

$$\widetilde{T}1=T1-\Pi_{T1}1=T1-T1=0$$

and

$$\widetilde{T}^*1 = T^*1 - \Pi^*_{T1}1 = 0.$$

The operator $\widetilde{T}$ is Calderón–Zygmund operator because Π_b and T are Calderón–Zygmund operators. By Theorem 3.1, $\widetilde{T}$ is bounded on $H^p(\mathbb{R}^n)$, $\frac{n}{n+\varepsilon} < p \leq 1$. By Theorem 3.2, the proof is completed. □

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