J. Math. Anal. Appl. 394 (2012) 592–602

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Maximizing numerical radii of weighted shifts under weight permutations

Chi-Tung Chang, Kuo-Zhong Wang *

Department of Applied Mathematics, National Chiao Tung University, Hsinchu 300, Taiwan

ARTICLE INFO

Article history:
Received 15 December 2011
Available online 30 April 2012
Submitted by J.A. Ball

Keywords:
Numerical range
Numerical radius
Weighted cyclic matrix
Unilateral weighted shift
Bilateral weighted shift

ABSTRACT

Let $w_i \in \mathbb{C}\,(1 \le i \le n)$ and $l \in S_n$, the symmetric group of all permutations of $1, 2, \ldots, n$. Suppose A_l is the weighted cyclic matrix

$$A_l \equiv \begin{pmatrix} 0 & w_{l(1)} & & \\ & 0 & \ddots & \\ & & \ddots & w_{l(n-1)} \\ w_{l(n)} & & & 0 \end{pmatrix}$$

and $w(A_l)$ denotes its numerical radius. We characterize those $\zeta \in S_n$ which satisfy $w(A_\zeta) = \max_{l \in S_n} w(A_l)$. The characterizations for unilateral and bilateral weighted (backward) shifts are also obtained.

1. Introduction

Let H be a complex Hilbert space and A be a bounded linear operator on H. The *numerical range* of A is defined by

$$W(A) \equiv \{\langle Ax, x\rangle : x \in H \text{ and } \|x\| = 1\},$$

where $\langle\cdot,\cdot\rangle$ is the inner product in H and $\|\cdot\|$ is the corresponding norm. $W(A)$ is always a nonempty, convex and bounded subset of $\mathbb{C}$. In addition, $W(A)$ is closed if A is of finite rank. The *numerical radius* of A is $w(A) \equiv \sup\{|z| : z \in W(A)\}$.

The weighted cyclic matrix with the weight $w = (w_i)_{i=1}^n$, where $w_i \in \mathbb{C}$ for all $1 \le i \le n$, is the matrix

$$\begin{pmatrix} 0 & w_1 & & \\ & 0 & \ddots & \\ & & \ddots & w_{n-1} \\ w_n & & & 0 \end{pmatrix},$$

and is regarded as a bounded linear operator on $\mathbb{C}^n$ (with the standard inner product) under the matrix multiplication. Let S_n stand for the symmetric group on $1, 2, \ldots, n$ and $l \in S_n$. The weighted cyclic matrix with the weight $(w_{l(i)})_{i=1}^n$ is denoted

* Corresponding author.
E-mail addresses: d937208@oz.nthu.edu.tw (C.-T. Chang), kzwang@math.nctu.edu.tw (K.-Z. Wang).

0022-247X/$ – see front matter Crown Copyright © 2012 Published by Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2012.04.039

by $A_{w,l}$ or simply A_l, that is,

$$A_{w,l} = A_l \equiv \begin{pmatrix} 0 & w_{l(1)} & & \\ & 0 & \ddots & \\ & & \ddots & w_{l(n-1)} \\ w_{l(n)} & & & 0 \end{pmatrix}.$$

In Section 2, we characterize those $\zeta \in S_n$ such that $w(A_\zeta) = \max_{l\in S_n} w(A_l)$. The characterizations for unilateral and bilateral weighted backward shifts are also obtained in Section 3.

Let $\operatorname{Re} A \equiv (A+A^*)/2$ and $\rho(A)$ be the *real part* and the *spectral radius* of a matrix A, respectively. Suppose A is the weighted cyclic matrix with the weight $w = (w_i)_{i=1}^n$. It is well-known that, for some real θ, $e^{i\theta}A$ is unitarily equivalent to the weighted cyclic matrix with the weight $|w| \equiv (|w_i|)_{i=1}^n$, which implies that their numerical radii coincide. When w is nonnegative, that is, $w_i \geq 0$ for all $1 \leq i \leq n$, $\langle Ax, x\rangle \leq \langle A|x|, |x|\rangle$ for each $x = (x_1 \cdots x_n)^T \in \mathbb{C}^n$, where $|x| \equiv (|x_1| \cdots |x_n|)^T$, and hence

$$w(A) = \max\{\langle Ax, x\rangle : x \text{ is a nonnegative unit vector in } \mathbb{C}^n\} = \rho(\operatorname{Re} A),$$

the largest eigenvalue of $\operatorname{Re} A$ (cf. [1, Proposition 3.3]). In addition, if at most one of these w_i's is zero, then $\operatorname{Re} A$ is nonnegative and unitarily irreducible, and there exists a unique positive unit vector y such that $w(A) = (\operatorname{Re} A)y = \langle Ay, y\rangle$ by Perron–Frobenius theorem. On the other hand, if some of these w_i's are zero, then A and $e^{i\theta}A$ are unitarily equivalent for all real θ and hence $W(A)$ is a closed circular disc centered at the origin with radius $w(A)$ (cf. [2]). Moreover, if at least two of these w_i's are zero, then A is the direct sum of two weighted cyclic matrices and its numerical range equals the largest numerical range of these summands. Therefore, it is sufficient to consider the case that at most one of these w_i's is zero when evaluating $w(A)$. For further information about numerical ranges and weighted cyclic matrices, we refer the readers to [1–8].

Let $l \in S_n$ and write $x_l \equiv (x_{l(1)} \cdots x_{l(n)})^T$, where $x = (x_1 \cdots x_n)^T \in \mathbb{C}^n$. For any weight $w = (w_i)_{i=1}^n$, define $F_w \equiv \langle \{(ij) \in S_n : 1 \leq i < j \leq n \text{ and } |w_i| = |w_j|\}\rangle$, the subgroup of S_n generated by the transpositions for which the weight $|w|$ is unchanged (we adopt that F_w consists of only the identity permutation if these $|w_i|$'s are all distinct). Notice that $\psi \in F_w$ if and only if $|w_{\psi(i)}| = |w_i|$ for all $1 \leq i \leq n$. We observe that $w(A_{w,\psi l}) = w(A_{|w|,\psi l}) = w(A_{|w|,l}) = w(A_{w,l})$ for each $\psi \in F_w$. Moreover, set $\tau_n = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1\end{pmatrix}$ and $\vartheta_n = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1\end{pmatrix}$, the permutations which shift and reverse the order of $\{1, \ldots, n\}$, respectively, and define $H_n \equiv \langle \tau_n, \vartheta_n\rangle$, the subgroup of S_n generated by τ_n and ϑ_n. Because for each $x \in \mathbb{C}^n$, $\langle A_{l\tau_n}x_{\tau_n}, x_{\tau_n}\rangle = \langle A_{l\vartheta_n}x_{\vartheta_n}, x_{\vartheta_n}\rangle = \langle A_l x, x\rangle$, we obtain that $w(A_{l\varphi}) = w(A_l)$ for all $\varphi \in H_n$. As a result, $w(A_{\psi l\varphi}) = w(A_l)$ for all $\psi \in F_w$ and $\varphi \in H_n$. In Section 2, we show that under the condition $|w_1| \geq |w_2| \geq \cdots \geq |w_n|$ with $w_{n-1} \neq 0$, $\zeta \in S_n$ satisfies $w(A_\zeta) = \max_{l\in S_n} w(A_l)$ if and only if $\zeta = \psi\sigma_n\varphi$ for some $\psi \in F_w$ and $\varphi \in H_n$ (Theorem 2.1). Here σ_n is the permutation given by $\sigma_n(i) = 2\,[n/2] - 2i + 2$ if $1 \leq i \leq [n/2]$ and $\sigma_n(i) = 2i - 2\,[n/2] - 1$ if $[n/2] + 1 \leq i \leq n$, where $[\,\cdot\,]$ denotes the greatest integer function. Therefore,

$$\sigma_n = \begin{cases} \begin{pmatrix} 1 & \cdots & \left[\frac{n}{2}\right]-1 & \left[\frac{n}{2}\right] & \left[\frac{n}{2}\right]+1 & \left[\frac{n}{2}\right]+2 & \left[\frac{n}{2}\right]+3 & \cdots & n \\ n-1 & \cdots & 4 & 2 & 1 & 3 & 5 & \cdots & n \end{pmatrix} & \text{if } n \text{ is odd}, \\ \begin{pmatrix} 1 & \cdots & \left[\frac{n}{2}\right]-1 & \left[\frac{n}{2}\right] & \left[\frac{n}{2}\right]+1 & \left[\frac{n}{2}\right]+2 & \left[\frac{n}{2}\right]+3 & \cdots & n \\ n & \cdots & 4 & 2 & 1 & 3 & 5 & \cdots & n-1 \end{pmatrix} & \text{if } n \text{ is even}. \end{cases}$$

One could interpret σ_n as a rearrangement which centralizes those w_i's with largest absolute values. Furthermore, if w is nonnegative, the unique positive unit vector y satisfying $w(A_{\sigma_n}) = \langle A_{\sigma_n}y, y\rangle$ must be of the form $y = x_{\sigma_n'}$, where $x = (x_1 \cdots x_n)^T$ with $x_1 \geq x_2 \geq \cdots \geq x_n > 0$ and

$$\sigma_n' = \begin{cases} \sigma_n\vartheta_n = \begin{pmatrix} 1 & \cdots & \left[\frac{n}{2}\right]-1 & \left[\frac{n}{2}\right] & \left[\frac{n}{2}\right]+1 & \left[\frac{n}{2}\right]+2 & \left[\frac{n}{2}\right]+3 & \cdots & n \\ n & \cdots & 5 & 3 & 1 & 2 & 4 & \cdots & n-1 \end{pmatrix} \\ \quad \text{if } n \text{ is odd}, \\ \sigma_n\vartheta_n\tau_n^{-1} = \begin{pmatrix} 1 & 2 & \cdots & \left[\frac{n}{2}\right]-1 & \left[\frac{n}{2}\right] & \left[\frac{n}{2}\right]+1 & \left[\frac{n}{2}\right]+2 & \left[\frac{n}{2}\right]+3 & \cdots & n \\ n & n-1 & \cdots & 5 & 3 & 1 & 2 & 4 & \cdots & n-2 \end{pmatrix} \\ \quad \text{if } n \text{ is even}. \end{cases}$$

In Section 3, the comparison of numerical radii of unilateral and bilateral weighted (backward) shifts is considered. Let $w = (w_i)_{i=0}^\infty$ (resp., $w = (w_i)_{i=-\infty}^\infty = (\cdots w_{-1}\, \underline{w_0}\, w_1 \cdots)$), where the w_i's in $\mathbb{C}$ for $i \in \mathbb{N}^0 \equiv \{0\} \cup \mathbb{N}$ (resp., $i \in \mathbb{Z}$), are bounded. Here we underline the 0th component of w for the bilateral case. The unilateral (resp., bilateral) weighted backward shift with the weight w is the bounded linear operator on $\ell^2 \equiv \{(x_0\, x_1 \cdots) : \sum_{i=0}^\infty |x_i|^2 < \infty\}$ (resp.,

$\ell^2(\mathbb{Z}) \equiv \{(\cdots\ x_{-1}\ \underline{x_0}\ x_1\ \cdots) : \sum_{i=-\infty}^{\infty} |x_i|^2 < \infty\}$) with the matrix representation

$$A = \begin{pmatrix} 0 & w_0 & & \\ & 0 & w_1 & \\ & & 0 & \ddots \\ & & & \ddots \end{pmatrix} \left(\text{resp., } B = \begin{pmatrix} \ddots & \ddots & & & & \\ & \ddots & w_{-1} & & & \\ & & 0 & w_0 & & \\ & & & \underline{0} & w_1 & \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix}\right),$$

where the 0th component of a vector in $\ell^2(\mathbb{Z})$ and the $(0, 0)$-entry of an infinite matrix for the bilateral case is underlined. Its numerical range is also a circular disc centered at the origin with radius $w(A)$ (resp., $w(B)$) (cf. [2,8]). Let $S_{\mathbb{N}^0}$ (resp., $S_{\mathbb{Z}}$) stand for the collection of all rearrangements of the elements of $\mathbb{N}^0$ (resp., of $\mathbb{Z}$). If $l \in S_{\mathbb{N}^0}$ (resp., $l \in S_{\mathbb{Z}}$), the unilateral (resp., bilateral) weighted backward shift with the weight $(w_{l(i)})_{i=0}^{\infty}$ (resp., $(w_{l(i)})_{i=-\infty}^{\infty}$) is denoted by $A_{w,l}$ or simply A_l (resp., $B_{w,l}$ or simply B_l). We will show that there exists a $\zeta \in S_{\mathbb{N}^0}$ satisfying $w(A_\zeta) = \sup_{l\in S_{\mathbb{N}^0}} w(A_l)$ if and only if $\sup_{k\in\mathbb{N}^0} |w_k| = \limsup_{k\to\infty} |w_k|$ (Theorem 3.1). A corresponding result for the bilateral case is also obtained (Theorem 3.2). We also represent the values $\sup_{l\in S_{\mathbb{N}^0}} w(A_l)$ and $\sup_{l\in S_{\mathbb{Z}}} w(B_l)$ as the numerical radius of some bilateral weighted backward shift U_w.

2. Weighted cyclic matrices

Throughout this section, for each $x = (x_1\ \cdots\ x_n)^T \in \mathbb{C}^n$, we define $\hat{x} = (\hat{x}_1\ \cdots\ \hat{x}_n)^T$ and $\tilde{x} = (\tilde{x}_1\ \cdots\ \tilde{x}_n)^T$ by

$$\hat{x}_i = \begin{cases} x_1 & \text{if } i = 1, \\ x_{i-1} & \text{if } 2 \le i \le n, \end{cases} \quad \text{and} \quad \tilde{x}_i = \begin{cases} x_{i+1} & \text{if } 1 \le i \le n-1, \\ x_n & \text{if } i = n. \end{cases}$$

We also define $l(n+1) \equiv l(1)$ for each $l \in S_n$. The following theorem is the main result of this section.

Theorem 2.1. *Let $w = (w_i)_{i=1}^n$ with $|w_1| \ge |w_2| \ge \cdots \ge |w_n|$ and $w_{n-1} \ne 0$. We have*

$$\max_{l\in S_n} w(A_l) = w(A_{\sigma_n}).$$

Moreover, the following statements are true:

(a) *$w(A_\zeta) = \max_{l\in S_n} w(A_l)$ for some $\zeta \in S_n$ if and only if $\zeta = \psi\sigma_n\varphi$ for some $\psi \in F_w$ and $\varphi \in H_n$, and*
(b) *if w is nonnegative, then there exists a unique positive unit vector $x = (x_1\ \cdots\ x_n)^T$ with $x_1 \ge x_2 \ge \cdots \ge x_n > 0$ such that $w(A_{\sigma_n}) = \langle A_{\sigma_n} x_{\sigma_n'}, x_{\sigma_n'}\rangle = \sum_{i=1}^n w_i\hat{x}_i\tilde{x}_i$.*

In a word, Theorem 2.1 states that a permutation $\zeta \in S_n$ satisfies A_ζ attains the maximal numerical radius among all $l \in S_n$ if and only if there is a $\phi \in H_n$ such that the sequence $d_i \equiv |w_{\zeta\phi(i)}|$, $i = 1, 2, \ldots, n$, satisfies $d_r \ge d_{r-1} \ge d_{r+1} \ge d_{r-2} \ge d_{r+2} \ge \cdots$ with $r = [n/2] + 1$. We establish Theorem 2.1 via a series of lemmas. The first involves an interesting inequality which is quite well known; for example, see [9, Lemma 3.6]. We give a short proof for completeness.

Lemma 2.2. *Let $u = (u_i)_{i=1}^n$, $v = (v_i)_{i=1}^n$ and $w = (w_i)_{i=1}^n$ be n-tuples with nonnegative components and with w_i's nonincreasing. Suppose that $\sum_{i=1}^k u_i \le \sum_{i=1}^k v_i$ for all $1 \le k \le n$. Then*

$$\sum_{i=1}^k u_i w_i \le \sum_{i=1}^k v_i w_i$$

for all $1 \le k \le n$. Furthermore, under the condition $\sum_{i=1}^n u_i w_i = \sum_{i=1}^n v_i w_i$, we have $\sum_{i=1}^s u_i = \sum_{i=1}^s v_i$ for all $1 \le s < n$ with $w_s > w_{s+1}$, and $\sum_{i=1}^n u_i = \sum_{i=1}^n v_i$ if $w_n \ne 0$. In particular, $u_i = v_i$ for all $1 \le i \le n$ if w_i's are positive and strictly decreasing.

Proof. The fact $u_1w_1 \le v_1w_1$ is trivial. Fix $k = 2, \ldots, n$. Summing by parts, we have

$$\sum_{i=1}^k (v_i - u_i)w_i = \sum_{i=1}^{k-1} (w_i - w_{i+1}) \sum_{j=1}^i (v_j - u_j) + w_k \sum_{i=1}^k (v_i - u_i) \ge 0. \tag{1}$$

Hence the desired inequality follows. Under the condition $\sum_{i=1}^n u_i w_i = \sum_{i=1}^n v_i w_i$, the inequality in (1) is an equality when $k = n$. We get that $\sum_{i=1}^s u_i = \sum_{i=1}^s v_i$ for all $1 \le s < n$ with $w_s > w_{s+1}$, and $\sum_{i=1}^n u_i = \sum_{i=1}^n v_i$ if $w_n \ne 0$. Finally, the last statement holds since $\sum_{i=1}^s u_i = \sum_{i=1}^s v_i$ for all $1 \le s \le n$ in such a case. $\square$

Combining the inequality in Lemma 2.2 and the effect of permutations, we obtain the next lemma, which plays an important role when deriving the main theorem.

Lemma 2.3. *Let $x = (x_1 \cdots x_n)^T$ be nonnegative and nonincreasing, and let $l, \eta \in S_n$. For each $k = 1, 2, \dots, n$, there exist $\nu, l_1, l_2 \in S_n$ such that $x_{l(\eta(\nu(i)))} x_{l(\eta(\nu(i))+1)} = \tilde{x}_{l_1(i)} \hat{x}_{l_2(i)}$ for all $1 \le i \le n$, $l_1(1) > l_1(2) > \cdots > l_1(k)$ and*

$$\sum_{i=1}^{k} x_{l(\eta(i))} x_{l(\eta(i)+1)} = \sum_{i=1}^{k} \tilde{x}_{l_1(i)} \hat{x}_{l_2(i)} \le \sum_{i=1}^{k} \tilde{x}_i \hat{x}_i.$$

Proof. Let $l, \eta \in S_n$. Assume that $l(p_0) = 1$ and $l(p_1) = n$ for some $1 \le p_0, p_1 \le n$. We only consider the case $p_0 < p_1$; the case $p_0 > p_1$ can be obtained analogously. There exist $\phi, \psi \in S_n$ such that

$$\tilde{x}_{\phi(i)} = \begin{cases} x_{l(i)} & \text{if } 1 \le i < p_0 \text{ or } p_1 \le i \le n, \\ x_{l(i+1)} & \text{if } p_0 \le i < p_1, \end{cases}$$

and

$$\hat{x}_{\psi(i)} = \begin{cases} x_{l(i+1)} & \text{if } 1 \le i < p_0 \text{ or } p_1 \le i \le n, \\ x_{l(i)} & \text{if } p_0 \le i < p_1. \end{cases}$$

We obtain that $x_{l(i)} x_{l(i+1)} = \tilde{x}_{\phi(i)} \hat{x}_{\psi(i)}$ for all $1 \le i \le n$. Fix $k = 1, 2, \dots, n$. There is a $\nu \in S_n$ satisfying $1 \le \nu(i) \le k$ for all $1 \le i \le k$ and $\phi(\eta(\nu(1))) > \phi(\eta(\nu(2))) > \cdots > \phi(\eta(\nu(k)))$. Let $l_1 = \phi\eta\nu$ and $l_2 = \psi\eta\nu$. It is obvious that $x_{l(\eta(\nu(i)))} x_{l(\eta(\nu(i))+1)} = \tilde{x}_{l_1(i)} \hat{x}_{l_2(i)}$ for all $1 \le i \le n$. Moreover, $\tilde{x}_i \ge \tilde{x}_{l_1(i)}$ for all $1 \le i \le k$ since $\tilde{x}$ is nonincreasing. This shows that

$$\sum_{i=1}^{k} x_{l(\eta(i))} x_{l(\eta(i)+1)} = \sum_{i=1}^{k} \tilde{x}_{\phi(\eta(i))} \hat{x}_{\psi(\eta(i))} = \sum_{i=1}^{k} \tilde{x}_{\phi(\eta(\nu(i)))} \hat{x}_{\psi(\eta(\nu(i)))} = \sum_{i=1}^{k} \tilde{x}_{l_1(i)} \hat{x}_{l_2(i)} \le \sum_{i=1}^{k} \tilde{x}_i \hat{x}_{l_2(i)}. \tag{2}$$

Besides, the fact that $\hat{x}$ is nonincreasing implies that $\sum_{i=1}^{j} \hat{x}_{l_2(i)} \le \sum_{i=1}^{j} \hat{x}_i$ for all $1 \le j \le k$. Therefore, $\sum_{i=1}^{j} \tilde{x}_i \hat{x}_{l_2(i)} \le \sum_{i=1}^{j} \tilde{x}_i \hat{x}_i$ for all $1 \le j \le k$ by Lemma 2.2. Combining this with (2), we obtain that $\sum_{i=1}^{k} x_{l(\eta(i))} x_{l(\eta(i)+1)} \le \sum_{i=1}^{k} \tilde{x}_i \hat{x}_{l_2(i)} \le \sum_{i=1}^{k} \tilde{x}_i \hat{x}_i$. Thus the proof is completed. □

In brief, the preceding lemma says that $(\tilde{x}_i \hat{x}_i)_{i=1}^n$ weakly majorizes $(x_{l(\eta(i))} x_{l(\eta(i)+1)})_{i=1}^n$ (cf. [10, Definition 4.3.24]).

Lemma 2.4. *Let $0 < \alpha < y$. There is a $\beta > 0$ such that $\beta < \alpha$ and*

$$2y^2 = (y + \beta)^2 + (y - \alpha)^2. \tag{3}$$

Moreover, $\beta/\alpha \to 1$ as $\alpha \to 0^+$.

Proof. Since $y^2 > (y - \alpha)^2$, (3) holds for some $\beta > 0$. Expanding (3) and dividing it by α, we obtain $2y = 2y(\beta/\alpha) + \beta(\beta/\alpha) + \alpha$. This implies that $\beta/\alpha < 1$ and $\beta/\alpha \to 1$ as $\alpha \to 0^+$. □

We apply Lemma 2.4 to derive Lemma 2.5, which describes the relation between the weight w and the unit vector x such that $w(A_{\sigma_n}) = \langle A_{\sigma_n} x_{\sigma_n'}, x_{\sigma_n'} \rangle$.

Lemma 2.5. *Let $w_1 \ge w_2 \ge \cdots \ge w_n \ge 0$ with $w_{n-1} \ne 0$ and let $x = (x_1 \cdots x_n)^T$ with $x_1 \ge x_2 \ge \cdots \ge x_n > 0$ be a positive unit vector such that $\langle A_{\sigma_n} x_{\sigma_n'}, x_{\sigma_n'} \rangle = w(A_{\sigma_n})$. Suppose that $x_{i_0} = x_{i_0+1}$ for some $1 \le i_0 < n$.*

(a) *If i_0 is even, then $w_i = w_{i+1}$ for all odd indices i and $x_i = x_{i+1}$ for all even indices i.*
(b) *If i_0 is odd, then $w_i = w_{i+1}$ for all even indices i and $x_i = x_{i+1}$ for all odd indices i.*

Proof. We only derive (a); (b) can be obtained analogously. (a) will be established if the following statement is true: if i_0 is even, then $w_{i_0-1} = w_{i_0}$, $w_{i_0+1} = w_{i_0+2}$ (for $i_0 \le n-2$), $x_{i_0-2} = x_{i_0-1}$ (for $i_0 \ge 4$) and $x_{i_0+2} = x_{i_0+3}$ (for $i_0 \le n-3$). We prove the cases $4 \le i_0 \le n-3$ and $i_0 = n-1$. The cases $i_0 = 2$ and $i_0 = n-2$ can be proven in a similar way. If $4 \le i_0 \le n-3$ and the assertion fails, then $(w_{i_0-1} x_{i_0-2} + w_{i_0+1} x_{i_0+2}) - (w_{i_0} x_{i_0-1} + w_{i_0+2} x_{i_0+3}) > 0$. Set $y \equiv x_{i_0} = x_{i_0+1}$ and let α, β be defined as in Lemma 2.4. We have $0 < \beta < \alpha < y$ and $2y^2 = (y+\beta)^2 + (y-\alpha)^2$. Because $\beta/\alpha \to 1$ as $\alpha \to 0^+$, we may assume that α is sufficiently small such that $(w_{i_0-1} x_{i_0-2} + w_{i_0+1} x_{i_0+2})(\beta/\alpha) - (w_{i_0} x_{i_0-1} + w_{i_0+2} x_{i_0+3}) > 0$. Define $z = (z_1 \cdots z_n)^T$ by

$$z_i = \begin{cases} x_i & \text{if } i \ne i_0, i_0+1, \\ y + \beta & \text{if } i = i_0, \\ y - \alpha & \text{if } i = i_0 + 1. \end{cases}$$

Then z is a unit vector. We get that

$$\begin{aligned} w(A_{\sigma_n}) &\geq \langle A_{\sigma_n} z_{\sigma_n'}, z_{\sigma_n'} \rangle = \sum_{i=1}^n w_i \hat{z}_i \tilde{z}_i \\ &= \sum_{i=1}^n w_i \hat{x}_i \tilde{x}_i + (w_{i_0-1} x_{i_0-2} + w_{i_0+1} x_{i_0+2})\beta - (w_{i_0} x_{i_0-1} + w_{i_0+2} x_{i_0+3})\alpha \\ &> \sum_{i=1}^n w_i \hat{x}_i \tilde{x}_i = \langle A_{\sigma_n} x_{\sigma_n'}, x_{\sigma_n'} \rangle = w(A_{\sigma_n}), \end{aligned}$$

a contradiction. Hence our assertion holds. Suppose $i_0 = n-1$. If $w_{n-2} \neq w_{n-1}$ or $x_{n-3} \neq x_{n-2}$, then $w_{n-2}x_{n-3} > w_{n-1}x_{n-2}$. Set $y \equiv x_{n-1} = x_n$. By Lemma 2.4, there exist α and β such that $0 < \beta < \alpha < y$, $2y^2 = (y+\beta)^2 + (y-\alpha)^2$ and $w_{n-2}x_{n-3}(\beta/\alpha) - w_{n-1}x_{n-2} + w_n(((\beta/\alpha) - 1)y - \beta) > 0$. Define $z' = (z_1' \cdots z_n')^T$ by

$$z_i' = \begin{cases} x_i & \text{if } 1 \leq i \leq n-2, \\ y + \beta & \text{if } i = n-1, \\ y - \alpha & \text{if } i = n. \end{cases}$$

Then z' is a unit vector and

$$\begin{aligned} w(A_{\sigma_n}) &\geq \langle A_{\sigma_n} z'_{\sigma_n'}, z'_{\sigma_n'} \rangle = \sum_{i=1}^n w_i \hat{z'}_i \tilde{z'}_i \\ &= \sum_{i=1}^n w_i \hat{x}_i \tilde{x}_i + w_{n-2} x_{n-3} \beta - w_{n-1} x_{n-2} \alpha + w_n((\beta - \alpha)y - \alpha\beta) \\ &> \sum_{i=1}^n w_i \hat{x}_i \tilde{x}_i = \langle A_{\sigma_n} x_{\sigma_n'}, x_{\sigma_n'} \rangle = w(A_{\sigma_n}). \end{aligned}$$

This is impossible and thus $w_{n-2} = w_{n-1}$ and $x_{n-3} = x_{n-2}$. □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Without loss of generality, we may assume that w is nonnegative. Let $y = (y_1 \cdots y_n)^T$ be a nonnegative unit vector with $y_1 \geq y_2 \geq \cdots \geq y_n$. Suppose that l and l_1 are in S_n. Notice that $\langle A_l y_{l_1}, y_{l_1} \rangle = \sum_{i=1}^n w_{l(i)} y_{l_1(i)} y_{l_1(i+1)} = \sum_{i=1}^n w_i y_{l_1(l^{-1}(i))} y_{l_1(l^{-1}(i)+1)}$. Since $\sum_{i=1}^k y_{l_1(l^{-1}(i))} y_{l_1(l^{-1}(i)+1)} \leq \sum_{i=1}^k \hat{y}_i \tilde{y}_i$ for all $1 \leq k \leq n$ by Lemma 2.3 and $w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$, we have

$$\langle A_l y_{l_1}, y_{l_1} \rangle \leq \sum_{i=1}^n w_i \hat{y}_i \tilde{y}_i = \langle A_{\sigma_n} y_{\sigma_n'}, y_{\sigma_n'} \rangle \tag{4}$$

by Lemma 2.2. Consequently, $\max_{l \in S_n} w(A_l) = w(A_{\sigma_n})$.

(b) The irreducibility of $\operatorname{Re} A_{\sigma_n}$ and (4) guarantee the existence of a unique positive unit vector $x = (x_1 \cdots x_n)^T$ with $x_1 \geq x_2 \geq \cdots \geq x_n > 0$ such that $w(A_{\sigma_n}) = \langle A_{\sigma_n} x_{\sigma_n'}, x_{\sigma_n'} \rangle = \sum_{i=1}^n w_i \hat{x}_i \tilde{x}_i$.

(a) We only need to prove the sufficiency. Suppose that $\zeta \in S_n$ satisfies $w(A_\zeta) = \max_{l \in S_n} w(A_l)$. Because of the irreducibility of $\operatorname{Re} A_\zeta$, there exist a positive unit vector $z = (z_1 \cdots z_n)^T$ with $z_1 \geq z_2 \geq \cdots \geq z_n > 0$ and an $l_2 \in S_n$ such that $w(A_\zeta) = \langle A_\zeta z_{l_2}, z_{l_2} \rangle$. From (4),

$$\begin{aligned} w(A_\zeta) &= \langle A_\zeta z_{l_2}, z_{l_2} \rangle = \sum_{i=1}^n w_{\zeta(i)} z_{l_2(i)} z_{l_2(i+1)} = \sum_{i=1}^n w_i z_{l_2(\zeta^{-1}(i))} z_{l_2(\zeta^{-1}(i)+1)} \\ &\leq \sum_{i=1}^n w_i \hat{z}_i \tilde{z}_i = \langle A_{\sigma_n} z_{\sigma_n'}, z_{\sigma_n'} \rangle \leq w(A_{\sigma_n}) = \max_{l \in S_n} w(A_l) = w(A_\zeta). \end{aligned}$$

This shows that the inequalities involved are actually equalities and hence $\sum_{i=1}^n w_i z_{l_2(\zeta^{-1}(i))} z_{l_2(\zeta^{-1}(i)+1)} = \sum_{i=1}^n w_i \hat{z}_i \tilde{z}_i$. We show that $\sum_{i=1}^n z_{l_2(\zeta^{-1}(i))} z_{l_2(\zeta^{-1}(i)+1)} = \sum_{i=1}^n \tilde{z}_i \hat{z}_i$. If $w_n \neq 0$, then this equality is confirmed by Lemma 2.2. Suppose next that $w_n = 0$. Since $w_{n-1} \neq 0$, $\sum_{i=1}^{n-1} z_{l_2(\zeta^{-1}(i))} z_{l_2(\zeta^{-1}(i)+1)} = \sum_{i=1}^{n-1} \tilde{z}_i \hat{z}_i$ by Lemma 2.2 again and we obtain $z_{l_2(\zeta^{-1}(n))} z_{l_2(\zeta^{-1}(n)+1)} \leq \tilde{z}_n \hat{z}_n = z_n z_{n-1}$. The situation that $z_{l_2(\zeta^{-1}(n))} z_{l_2(\zeta^{-1}(n)+1)} < z_n z_{n-1}$ only occurs when $z_{l_2(\zeta^{-1}(n))} z_{l_2(\zeta^{-1}(n)+1)} = z_n^2$. Because of $l_2(\zeta^{-1}(n)) \neq l_2(\zeta^{-1}(n)+1)$, we get that $z_{n-1} = z_n$, a contradiction. This ensures that $\sum_{i=1}^n z_{l_2(\zeta^{-1}(i))} z_{l_2(\zeta^{-1}(i)+1)} = \sum_{i=1}^n \hat{z}_i \tilde{z}_i$. From Lemma 2.3, there exist $\nu, \rho \in S_n$ such that $z_{l_2(\zeta^{-1}(\nu(i)))} z_{l_2(\zeta^{-1}(\nu(i))+1)} = \tilde{z}_i \hat{z}_{\rho(i)}$ for all $1 \leq i \leq n$ and

$$\sum_{i=1}^n \tilde{z}_i \hat{z}_{\rho(i)} = \sum_{i=1}^n \tilde{z}_i \hat{z}_i. \tag{5}$$

We claim that $l_2 = \sigma_n'\varphi$ for some $\varphi \in H_n$. First consider the case that the z_i's are distinct. Since $\tilde{z}_1 > \tilde{z}_2 > \cdots > \tilde{z}_{n-1} = \tilde{z}_n > 0$, we get that $\sum_{i=1}^{s} \hat{z}_{\rho(i)} = \sum_{i=1}^{s} \hat{z}_i$ for all $1 \le s \le n$ with $s \ne n-1$ by (5) and Lemma 2.2. This is equivalent to $\hat{z}_{\rho(i)} = \hat{z}_i$ for $1 \le i \le n-2$ and $\hat{z}_{\rho(n-1)}, \hat{z}_{\rho(n)} \in \{\hat{z}_{n-1}, \hat{z}_n\} = \{z_{n-2}, z_{n-1}\}$. We derive that $(z_{l_2(i)} z_{l_2(i+1)})_{i=1}^n$ and $(\tilde{z}_i \hat{z}_{\rho(i)})_{i=1}^n$ consist of the same components (up to permutation), which are

$$z_1z_2,\, z_1z_3,\, z_2z_4,\, \ldots,\, z_iz_{i+2},\, \ldots,\, z_{n-3}z_{n-1},\, z_{\rho(n-1)-1}z_n,\, z_{\rho(n)-1}z_n$$

with $z_{\rho(n-1)-1}, z_{\rho(n)-1} \in \{z_{n-2}, z_{n-1}\}$. This only occurs when $z_{l_2} = (z_{l_2(1)} \cdots z_{l_2(n)})^T$ is such that z_2 and z_3 are in the positions near z_1, and z_4 is next to z_2 and on the opposite side of z_1, etc. Hence, we get that $l_2 = \sigma_n'\varphi$ for some $\varphi \in H_n$. Now suppose that $z_{i_0} = z_{i_0+1}$ for some even index i_0. By Lemma 2.5, $z_i = z_{i+1}$ for all even indices i. We may assume that $z_i > z_{i+1}$ for all odd indices i since otherwise, by Lemma 2.5 again, w and z are both constant vectors and this theorem obviously holds. From (5) and Lemma 2.2, we have $\sum_{i=1}^{s} \hat{z}_{\rho(i)} = \sum_{i=1}^{s} \hat{z}_i$ for $s = n$ and for all even indices s with $2 \le s < n-1$. Hence, $\hat{z}_{\rho(i)} = \hat{z}_{\rho(i+1)} = \hat{z}_i = \hat{z}_{i+1}$ for all odd indices i with $1 \le i < n-2$, and, in addition, $\hat{z}_{\rho(n-1)} = \hat{z}_{\rho(n)} = \hat{z}_{n-1} = \hat{z}_n$ when n is even, and $\hat{z}_{\rho(n-2)} + \hat{z}_{\rho(n-1)} + \hat{z}_{\rho(n)} = \hat{z}_{n-2} + \hat{z}_{n-1} + \hat{z}_n$ when n is odd. Under the condition that n is even, we obtain $\hat{z}_{\rho(i)} = \hat{z}_i$ for all $1 \le i \le n$ and the same argument above leads us to $l_2 = \sigma_n'\varphi$ for some $\varphi \in H_n$ (here we do not distinguish z_i and z_{i+1} for even indices i since $z_i = z_{i+1}$ in this case). When n is odd, we have $\hat{z}_{\rho(n-2)}, \hat{z}_{\rho(n-1)}, \hat{z}_{\rho(n)} \in \{\hat{z}_{n-2}, \hat{z}_{n-1}, \hat{z}_n\} = \{z_{n-3} = z_{n-2}, z_{n-1}\}$, and the components of $(z_{l_2(i)} z_{l_2(i+1)})_{i=1}^n$ and of $(\tilde{z}_i \hat{z}_{\rho(i)})_{i=1}^n$ are (up to permutation) both

$$z_1z_2,\, z_1z_3,\, z_2z_4,\, \ldots,\, z_iz_{i+2},\, \ldots,\, z_{n-4}z_{n-2},\, z_{\rho(n-2)-1}z_{n-1},\, z_{\rho(n-1)-1}z_n,\, z_{\rho(n)-1}z_n$$

with $z_{\rho(n-2)-1}, z_{\rho(n-1)-1}, z_{\rho(n)-1} \in \{z_{n-3} = z_{n-2}, z_{n-1}\}$. Because $l_2(i) \ne l_2(i+1)$ for all $1 \le i \le n$, we observe that the third term from the right implies that $\rho(n-2) - 1 \ne n-1$ and hence, $z_{\rho(n-2)-1} = z_{n-3}$. Applying the same arguments as above, we get $l_2 = \sigma_n'\varphi$ for some $\varphi \in H_n$ (we do not distinguish z_i and z_{i+1} for even indices i). For the case that $z_{i_0} = z_{i_0+1}$ for some odd index i_0, the same result can be obtained analogously. Therefore,

$$\begin{aligned}\langle A_\zeta z_{l_2}, z_{l_2}\rangle &= \langle A_\zeta z_{\sigma_n'\varphi}, z_{\sigma_n'\varphi}\rangle = \langle A_{\zeta\varphi^{-1}} z_{\sigma_n'}, z_{\sigma_n'}\rangle \\ &= \sum_{i=1}^{n} w_{\zeta(\varphi^{-1}(i))} z_{\sigma_n'(i)} z_{\sigma_n'(i+1)} = \sum_{i=1}^{n} w_{\zeta(\varphi^{-1}(\sigma_n^{-1}(i)))} z_{\sigma_n'(\sigma_n^{-1}(i))} z_{\sigma_n'(\sigma_n^{-1}(i)+1)}.\end{aligned}$$

It can be verified that $z_{\sigma_n'(\sigma_n^{-1}(i))} z_{\sigma_n'(\sigma_n^{-1}(i)+1)} = \hat{z}_i \tilde{z}_i$ for all $1 \le i \le n$. This yields $\sum_{i=1}^n w_{\psi(i)} \hat{z}_i \tilde{z}_i = \sum_{i=1}^n w_i \hat{z}_i \tilde{z}_i$, where $\psi = \zeta\varphi^{-1}\sigma_n^{-1}$. With the help of Lemmas 2.2 and 2.5, we get $w_{\psi(i)} = w_i$ for all $1 \le i \le n$. Hence $\psi \in F_w$ and $\zeta = \psi\sigma_n\varphi$ as asserted. $\square$

3. Unilateral and bilateral weighted backward shifts

In the preceding section, we deal with finite weighted cyclic matrices. Related theory for infinite matrices, that is, unilateral and bilateral weighted backward shifts, is developed in this section. Before exhibiting the main theorems, we introduce a bilateral weighted backward shift which is frequently used latter. Suppose that $w = (w_i)_{i=0}^\infty$ (resp., $w = (w_i)_{i=-\infty}^\infty$) is bounded. We define a nonnegative sequence $(u_i)_{i=0}^\infty$ as follows. Let $s \equiv \limsup_{k\to\infty} |w_k|$ (resp., $s \equiv \max\{\limsup_{k\to\infty} |w_k|, \limsup_{k\to-\infty} |w_k|\}$). If $\sup_{k\in\mathbb{N}^0} |w_k| = s$ (resp., $\sup_{k\in\mathbb{Z}} |w_k| = s$), then select $u_i = s$ for all $i \ge 0$. If the set $\Omega \equiv \{i \in \mathbb{N}^0 : |w_i| > s\}$ (resp., $\Omega \equiv \{i \in \mathbb{Z} : |w_i| > s\}$) is finite, we can arrange $\{|w_i|\}_{i\in\Omega}$ in nonincreasing order, say $|w_{j_0}| \ge |w_{j_1}| \ge \cdots \ge |w_{j_n}|$, and select

$$u_i = \begin{cases} |w_{j_i}| & \text{if } 0 \le i \le n, \\ s & \text{if } i > n. \end{cases}$$

If Ω is infinite, then there exists a one-to-one correspondence $\lambda : \mathbb{N}^0 \to \Omega$ such that $|w_{\lambda(0)}| \ge |w_{\lambda(1)}| \ge \cdots$ and we select $u_i = |w_{\lambda(i)}|$ for all $i \ge 0$. Obviously, $(u_i)_{i=0}^\infty$ is nonincreasing. Let $\sigma : \mathbb{Z} \to \mathbb{N}^0$ be the one-to-one correspondence satisfying $\sigma(i) = 2i$ if $i \ge 0$ and $\sigma(i) = -2i-1$ if $i < 0$, that is, $\sigma(0) = 0, \sigma(-1) = 1, \sigma(1) = 2, \sigma(-2) = 3, \sigma(2) = 4, \ldots$, etc. We define U_w as the bilateral weighted backward shift with the weight $(u_{\sigma(i)})_{i=-\infty}^\infty$, that is,

$$U_w = \begin{pmatrix} \ddots & \ddots & & & & \\ & \ddots & u_{\sigma(-1)} & & & \\ & & 0 & u_{\sigma(0)} & & \\ & & & \underline{0} & u_{\sigma(1)} & \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix}.$$

Recall that for each $l \in S_{\mathbb{N}^0}$ (resp., $l \in S_{\mathbb{Z}}$), A_l (resp., B_l) is the unilateral (resp., bilateral) weighted backward shift with the weight $(w_{l(i)})_{i=0}^\infty$ (resp., $(w_{l(i)})_{i=-\infty}^\infty$). Our main results are Theorems 3.1 and 3.2. They deal with the unilateral and bilateral weighted backward shifts, respectively.

Theorem 3.1. *Let $w = (w_i)_{i=0}^{\infty}$ be bounded. We have*

$$\sup_{l \in S_{\mathbb{N}^0}} w(A_l) = w(U_w).$$

In addition, $w(A_\zeta) = \sup_{l \in S_{\mathbb{N}^0}} w(A_l)$ for some $\zeta \in S_{\mathbb{N}^0}$ if and only if $\sup_{k \in \mathbb{N}^0} |w_k| = \limsup_{k \to \infty} |w_k| \equiv s$, and $\sup_{l \in S_{\mathbb{N}^0}} w(A_l) = s$ in this case.

Theorem 3.2. *Let $w = (w_i)_{i=-\infty}^{\infty}$ be bounded. We have*

$$\sup_{l \in S_{\mathbb{Z}}} w(B_l) = w(U_w).$$

In addition, $w(B_\zeta) = \sup_{l \in S_{\mathbb{Z}}} w(B_l)$ for some $\zeta \in S_{\mathbb{Z}}$ if and only if one of the following conditions holds:

(1) $\sup_{k \in \mathbb{Z}} |w_k| = \max\{\limsup_{k \to \infty} |w_k|, \limsup_{k \to -\infty} |w_k|\} \equiv s$,
(2) $|w_i| > \limsup_{k \to \infty} |w_k| = \limsup_{k \to -\infty} |w_k|$ *for all* $i \in \mathbb{Z}$, *and*
(3) $|w_i| \geq \limsup_{k \to \infty} |w_k| = \limsup_{k \to -\infty} |w_k| = s$ *for all* $i \in \mathbb{Z}$ *and there is an* $i_0 \in \mathbb{N}$ *such that* $|w_i| = s$ *for all* $|i| \geq i_0$.

In case (1), *we have* $\sup_{l \in S_{\mathbb{Z}}} w(B_l) = s$.

To establish Theorems 3.1 and 3.2, we introduce several lemmas, some of which are interesting on their own. Because the numerical ranges of unilateral (resp., bilateral) weighted backward shifts with the weight $(w_i)_{i=0}^{\infty}$ (resp., $(w_i)_{i=-\infty}^{\infty}$) and with the weight $(|w_i|)_{i=0}^{\infty}$ (resp., $(|w_i|)_{i=-\infty}^{\infty}$) are the same, we only need to consider the unilateral and bilateral weighted backward shifts with nonnegative weights. We may further assume that all the w_i's are nonzero. Otherwise, they are the direct sum of weighted cyclic matrices or unilateral weighted backward shifts, and their numerical ranges are the largest numerical ranges of these summands. For any $B = (B_{ij})_{i,j=0}^{\infty}$ (resp., $B = (B_{ij})_{i,j=-\infty}^{\infty}$) and $0 \leq m \leq n \leq \infty$ (resp., $-\infty \leq m \leq n \leq \infty$), we let $B[m, n]$ denote the matrix $(B_{ij})_{i,j=m}^{n}$.

Lemma 3.3. *Let $w = (w_i)_{i=0}^{\infty}$ (resp., $w = (w_i)_{i=-\infty}^{\infty}$) be positive and bounded. We have*

$$\sup_{l \in S_{\mathbb{N}^0}} w(A_l) = w(U_w) \quad \left(\text{resp., } \sup_{l \in S_{\mathbb{Z}}} w(B_l) = w(U_w)\right).$$

Proof. Suppose $l \in S_{\mathbb{N}^0}$ (resp., $l \in S_{\mathbb{Z}}$). For a fixed $n \in \mathbb{N}$, $A_l[0, 2n]$ (resp., $B_l[-n, n]$) is of the form

$$A_l = \begin{pmatrix} 0 & w_{l(0)} & & \\ & 0 & \ddots & \\ & & \ddots & w_{l(2n-1)} \\ & & & 0 \end{pmatrix} \left(\text{resp., } B_l = \begin{pmatrix} 0 & w_{l(-n+1)} & & & & \\ & 0 & \ddots & & & \\ & & \ddots & w_{l(0)} & & \\ & & & \underline{0} & \ddots & \\ & & & & \ddots & w_{l(n)} \\ & & & & & 0 \end{pmatrix}\right).$$

Let $(c_i)_{i=1}^{2n+1}$ be the rearrangement of $\{w_{l(i)}\}_{i=0}^{2n-1} \cup \{0\}$ (resp., $\{w_{l(i)}\}_{i=-n+1}^{n} \cup \{0\}$) in nonincreasing order and set

$$M_{2n+1} \equiv \begin{pmatrix} 0 & c_{\sigma_{2n+1}(1)} & & \\ & 0 & \ddots & \\ & & \ddots & c_{\sigma_{2n+1}(2n)} \\ c_{\sigma_{2n+1}(2n+1)} & & & 0 \end{pmatrix},$$

where $\sigma_{2n+1} \in S_{2n+1}$ is defined as in Section 1. Notice that $c_{\sigma_{2n+1}(2n+1)} = c_{2n+1} = 0$. From Theorem 2.1, $w(A_l[0, 2n]) \leq w(M_{2n+1})$ (resp., $w(B_l[-n, n]) \leq w(M_{2n+1})$). Since each entry of M_{2n+1} is less than or equal to the corresponding one of $U_w[-n, n]$, [1, Corollary 3.6] implies that $w(M_{2n+1}) \leq w(U_w[-n, n])$. Hence,

$$w(A_l) = w\left(\bigcup_{n=1}^{\infty} A_l[0, 2n]\right) \leq w\left(\bigcup_{n=1}^{\infty} U_w[-n, n]\right) = w(U_w)$$
$$\left(\text{resp., } w(B_l) = w\left(\bigcup_{n=1}^{\infty} B_l[-n, n]\right) \leq w\left(\bigcup_{n=1}^{\infty} U_w[-n, n]\right) = w(U_w)\right)$$

by Wang and Wu [8, Lemma 2.6(a)]. We conclude that $\sup_{l\in S_{\mathbb{N}^0}} w(A_l) \le w(U_w)$ (resp., $\sup_{l\in S_{\mathbb{Z}}} w(B_l) \le w(U_w)$). It remains to prove the converse. Let $\epsilon > 0$. We may pick an $n_0 \in \mathbb{N}$ such that $w(U_w[-n_0, n_0]) > w(U_w) - \epsilon$ because $w(U_w) = w(\bigcup_{n=1}^{\infty} U_w[-n, n])$. We observe that the way to construct U_w guarantees the existence of some $l' \in S_{\mathbb{N}^0}$ (resp., $l' \in S_{\mathbb{Z}}$) which satisfies $w(A_{l'}[0, 2n_0]) > w(U_w[-n_0, n_0]) - \epsilon$ (resp., $w(B_{l'}[-n_0, n_0]) > w(U_w[-n_0, n_0]) - \epsilon$) since the eigenvalues of the real part of a finite matrix depends continuously on its entries (cf. [11]). Consequently,

$$w(A_{l'}) \ge w(A_{l'}[0, 2n_0]) > w(U_w[-n_0, n_0]) - \epsilon > w(U_w) - 2\epsilon$$
$$(\text{resp., } w(B_{l'}) \ge w(B_{l'}[-n_0, n_0]) > w(U_w[-n_0, n_0]) - \epsilon > w(U_w) - 2\epsilon)$$

and we obtain that $\sup_{l\in S_{\mathbb{N}^0}} w(A_l) \ge w(U_w)$ (resp., $\sup_{l\in S_{\mathbb{Z}}} w(B_l) \ge w(U_w)$). This completes the proof. □

The next lemma provides a sufficient condition under which $w(A_\zeta) = \sup_{l\in S_{\mathbb{N}^0}} w(A_l)$ (resp., $w(B_\zeta) = \sup_{l\in S_{\mathbb{Z}}} w(B_l)$) for some $\zeta \in S_{\mathbb{N}^0}$ (resp., $\zeta \in S_{\mathbb{Z}}$). In order to simplify the notations, for each $s \ge 0$, let $A(s)$ and $B(s)$ be the unilateral and bilateral weighted backward shifts with the constant weight $w_i \equiv s$, respectively, that is,

$$A(s) \equiv \begin{pmatrix} 0 & s & & \\ & 0 & s & \\ & & 0 & \ddots \\ & & & \ddots \end{pmatrix} \quad \text{and} \quad B(s) \equiv \begin{pmatrix} \ddots & \ddots & & & & \\ & \ddots & s & & & \\ & & 0 & s & & \\ & & & \underline{0} & s & \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix}.$$

We have $w(A(s)) = w(B(s)) = s$ by [8, Corollary 4.7 and Theorem 4.9].

Lemma 3.4. *Let $w = (w_i)_{i=0}^{\infty}$ (resp., $w = (w_i)_{i=-\infty}^{\infty}$) be positive and bounded. If $\sup_{k\in\mathbb{N}^0} w_k = \limsup_{k\to\infty} w_k \equiv s$ (resp., $\sup_{k\in\mathbb{Z}} w_k = \max\{\limsup_{k\to\infty} w_k, \limsup_{k\to-\infty} w_k\} \equiv s$), then there exists a $\zeta \in S_{\mathbb{N}^0}$ (resp., $\zeta \in S_{\mathbb{Z}}$) such that $w(A_\zeta) = \sup_{l\in S_{\mathbb{N}^0}} w(A_l) = w(U_w) = s$ (resp., $w(B_\zeta) = \sup_{l\in B_{\mathbb{Z}}} w(B_l) = w(U_w) = s$).*

Proof. We only prove the unilateral case; the bilateral case can be done analogously. Let $w = (w_i)_{i=0}^{\infty}$ be positive, bounded and satisfy $\sup_{k\in\mathbb{N}^0} w_k = \limsup_{k\to\infty} w_k \equiv s$. We get that $U_w = A(s)$ and therefore, $\sup_{l\in S_{\mathbb{N}^0}} w(A_l) = w(U_w) = s$ by Lemma 3.3. Now pick a subsequence $(n_i)_{i=0}^{\infty}$ of $\mathbb{N}^0$ such that $w_{n_i} \to s$ as $i \to \infty$. Let $\zeta : \mathbb{N}^0 \to \mathbb{N}^0$ be any one-to-one correspondence with $\zeta(2^{k^2} + j) = n_{2^{k^2}+j}$ for all $k = 0, 1, \dots$ and all j satisfying $0 \le j < 2^k$. For each $p = 1, 2, \dots$, define $x^{(p)} = (x_i^{(p)})_{i=0}^{\infty}$ by

$$x_i^{(p)} = \begin{cases} 2^{-p/2} & \text{if } 2^{p^2} \le i < 2^{p^2} + 2^p - 1, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\|x^{(p)}\| = 1$ and

$$\langle A_\zeta x^{(p)}, x^{(p)} \rangle = \frac{1}{2^p}(w_{n_{2^{p^2}}} + w_{n_{2^{p^2}+1}} + \dots + w_{n_{2^{p^2}+2^p-2}}) \to s \quad \text{as} \quad p \to \infty.$$

Hence $w(A_\zeta) = s$ as required. □

For any complex sequence $x = (x_i)_{i=0}^{\infty}$, we define $\hat{x} = (\hat{x}_i)_{i=0}^{\infty}$ and $\tilde{x} = (\tilde{x}_i)_{i=0}^{\infty}$ by

$$\hat{x}_i = \begin{cases} x_0 & \text{if } i = 0, \\ x_{i-1} & \text{if } i \ge 1, \end{cases} \quad \text{and} \quad \tilde{x}_i = x_{i+1} \quad \text{for all } i \ge 0.$$

A result which is parallel to Lemma 2.3 is also obtained.

Lemma 3.5. *Let $x = (x_i)_{i=0}^{\infty}$ be nonnegative and nonincreasing, and let $l, \eta \in S_{\mathbb{N}^0}$. For each $k = 0, 1, \dots,$ we have*

$$\sum_{i=0}^{k} x_{l(\eta(i))} x_{l(\eta(i)+1)} \le \sum_{i=0}^{k} \tilde{x}_i \hat{x}_i.$$

In addition, if x is positive and $x_i \to 0$ as $i \to \infty$, then there exist $k_0 \in \mathbb{N}$ and $\mu > 0$ such that $\sum_{i=0}^{k} x_{l(\eta(i))} x_{l(\eta(i)+1)} + \mu < \sum_{i=0}^{k} \tilde{x}_i \hat{x}_i$ for all $k \ge k_0$.

Proof. The first part of this proof is similar to the one for Lemma 2.3. Let $l, \eta \in S_{\mathbb{N}^0}$ and suppose $l(p_0) = 0$. Define $y = (y_i)_{i=0}^{\infty}$ by $y \equiv x$ if $p_0 = 0$, and $y \equiv (x_0\, x_0\, x_1\, x_2\, \cdots\, x_{l(0)-1}\, x_{l(0)+1}\, \cdots)$ if $p_0 > 0$. That is, y is obtained by deleting the $l(0) + 1$st component of $\hat{x}$. There exist $\phi, \psi \in S_{\mathbb{N}^0}$ satisfying

$$\tilde{x}_{\phi(i)} = \begin{cases} x_{l(i)} & \text{if } 0 \le i < p_0, \\ x_{l(i+1)} & \text{if } i \ge p_0, \end{cases} \quad \text{and} \quad y_{\psi(i)} = \begin{cases} x_{l(i+1)} & \text{if } 0 \le i < p_0, \\ x_{l(i)} & \text{if } i \ge p_0. \end{cases}$$

It is trivial that $x_{l(i)}x_{l(i+1)} = \tilde{x}_{\phi(i)}y_{\psi(i)}$ for all $i \in \mathbb{N}^0$. Fix $k = 0, 1, \ldots$. We may choose a $\nu \in \mathbb{N}^0$ satisfying $0 \le \nu(i) \le k$ for all $0 \le i \le k$ and $\phi(\eta(\nu(0))) > \phi(\eta(\nu(1))) > \cdots > \phi(\eta(\nu(k)))$. Because $\tilde{x}_i \ge \tilde{x}_{\phi(\eta(\nu(i)))}$ for all $0 \le i \le k$, we have

$$\sum_{i=0}^{k} x_{l(\eta(i))}x_{l(\eta(i)+1)} = \sum_{i=0}^{k} \tilde{x}_{\phi(\eta(i))}y_{\psi(\eta(i))} = \sum_{i=0}^{k} \tilde{x}_{l_1(i)}y_{l_2(i)} \le \sum_{i=0}^{k} \tilde{x}_i y_{l_2(i)},$$

where $l_1 = \phi\eta\nu$ and $l_2 = \psi\eta\nu$. Obviously, $y_i \le \hat{x}_i$ for all $i \in \mathbb{N}^0$ and this implies $\sum_{i=0}^{j} y_{l_2(i)} \le \sum_{i=0}^{j} \hat{x}_i$ for all $0 \le j \le k$. By Lemma 2.2, we obtain

$$\sum_{i=0}^{k} x_{l(\eta(i))}x_{l(\eta(i)+1)} \le \sum_{i=0}^{k} \tilde{x}_i y_{l_2(i)} \le \sum_{i=0}^{k} \tilde{x}_i\hat{x}_i. \tag{6}$$

Now assume that $x_i > 0$ for all $i \in \mathbb{N}^0$ and $x_i \to 0$ as $i \to \infty$. There are $i_0, i_1 \in \mathbb{N}$ such that $l(0) < i_0 \le i_1 - 2$, $x_{i_0-1} > x_{i_0}$ and $x_{i_1-1} > x_{i_1}$. If $k \ge i_1$, then from (1) and (6),

$$\begin{aligned} \sum_{i=0}^{k} \tilde{x}_i\hat{x}_i - \sum_{i=0}^{k} x_{l(\eta(i))}x_{l(\eta(i)+1)} &\ge \sum_{i=0}^{k} \tilde{x}_i\hat{x}_i - \sum_{i=0}^{k} \tilde{x}_i y_{l_2(i)} \\ &= \sum_{i=0}^{k-1} (\tilde{x}_i - \tilde{x}_{i+1}) \sum_{j=0}^{i} (\hat{x}_j - y_{l_2(j)}) + \tilde{x}_k \sum_{i=0}^{k} (\hat{x}_i - y_{l_2(i)}) \\ &\ge (\tilde{x}_{i_1-2} - \tilde{x}_{i_1-1}) \sum_{j=0}^{i_1-2} (\hat{x}_j - y_j) = (x_{i_1-1} - x_{i_1}) \sum_{j=l(0)+1}^{i_1-2} (x_{j-1} - x_j) \\ &\ge (x_{i_1-1} - x_{i_1})(x_{i_0-1} - x_{i_0}) > 0. \end{aligned}$$

This completes the proof. □

Lemma 3.5 tells us that for a nonnegative and nonincreasing sequence $x = (x_i)_{i=0}^{\infty}$, $(\tilde{x}_i\hat{x}_i)_{i=0}^{\infty}$ weakly majorizes $(x_{l(\eta(i))}x_{l(\eta(i)+1)})_{i=0}^{\infty}$. In addition, if x is positive and $x_i \to 0$ as $i \to \infty$, $\sum_{i=0}^{k}(\tilde{x}_i\hat{x}_i - x_{l(\eta(i))}x_{l(\eta(i)+1)})$ is away from 0 for sufficiently large k. In Lemma 3.6, we describe a necessary condition for the existence of some $\zeta \in S_{\mathbb{N}^0}$ such that $w(A_\zeta) = \sup_{l \in S_{\mathbb{N}^0}} w(A_l)$.

Lemma 3.6. *Let $w = (w_i)_{i=0}^{\infty}$ be positive and bounded. If $w(A_\zeta) = \sup_{l \in S_{\mathbb{N}^0}} w(A_l)$ for some $\zeta \in S_{\mathbb{N}^0}$, then $W(A_\zeta)$ is open.*

Proof. We show that if $W(A_\zeta)$ is closed, then there exists a $\zeta' \in S_{\mathbb{N}^0}$ such that $w(A_\zeta) < w(A_{\zeta'})$. From [8, Proposition 2.1(b)], there exist a positive unit vector $x = (x_i)_{i=0}^{\infty}$ with $x_0 \ge x_1 \ge \cdots$ and $l_1 \in S_{\mathbb{N}^0}$ satisfying $w(A_\zeta) = \langle A_\zeta x_{l_1}, x_{l_1} \rangle = \sum_{i=0}^{\infty} w_{\zeta(i)}x_{l_1(i)}x_{l_1(i+1)}$. Since $x_i \to 0$ as $i \to \infty$, we have $x_{l_1(i)}x_{l_1(i+1)} \to 0$ as $i \to \infty$. This guarantees the existence of some $t \in S_{\mathbb{N}^0}$ such that $(x_{l_1(t(i))}x_{l_1(t(i)+1)})_{i=0}^{\infty}$ is nonincreasing. By Lemma 3.5, $\sum_{i=0}^{k} x_{l_1(t(i))}x_{l_1(t(i)+1)} \le \sum_{i=0}^{k} \hat{x}_i\tilde{x}_i$ for all $k = 0, 1, \ldots$, and there exist $k_0 \in \mathbb{N}$ and $\mu > 0$ such that $\sum_{i=0}^{k} x_{l_1(t(i))}x_{l_1(t(i)+1)} + \mu < \sum_{i=0}^{k} \hat{x}_i\tilde{x}_i$ for all $k \ge k_0$. Now we may pick an $N \in \mathbb{N}$ satisfying $N > k_0$ and $\sum_{i=N+1}^{\infty} w_{\zeta(i)}x_{l_1(i)}x_{l_1(i+1)} < \left(\min_{0 \le i \le k_0} w_{\zeta(i)}\right)\mu$. Let $\nu \in S_{\mathbb{N}^0}$ be such that $0 \le \nu(i) \le N$ for all $0 \le i \le N$ and $w_{\zeta(\nu(0))} \ge w_{\zeta(\nu(1))} \ge \cdots \ge w_{\zeta(\nu(N))}$. We have $w_{\zeta(\nu(k_0))} \ge \min_{0 \le i \le k_0} w_{\zeta(i)}$ and $\sum_{i=0}^{k} x_{l_1(\nu(i))}x_{l_1(\nu(i)+1)} \le \sum_{i=0}^{k} x_{l_1(t(i))}x_{l_1(t(i)+1)}$ for all $k = 0, 1, \ldots$. From (1),

$$\begin{aligned} \sum_{i=0}^{N} w_{\zeta(\nu(i))}\hat{x}_i\tilde{x}_i - \sum_{i=0}^{N} w_{\zeta(i)}x_{l_1(i)}x_{l_1(i+1)} &= \sum_{i=0}^{N} w_{\zeta(\nu(i))}\hat{x}_i\tilde{x}_i - \sum_{i=0}^{N} w_{\zeta(\nu(i))}x_{l_1(\nu(i))}x_{l_1(\nu(i)+1)} \\ &= \sum_{i=0}^{N-1} (w_{\zeta(\nu(i))} - w_{\zeta(\nu(i+1))}) \sum_{j=0}^{i} (\hat{x}_i\tilde{x}_i - x_{l_1(\nu(i))}x_{l_1(\nu(i)+1)}) \\ &\quad + w_{\zeta(\nu(N))} \sum_{i=0}^{N} (\hat{x}_i\tilde{x}_i - x_{l_1(\nu(i))}x_{l_1(\nu(i)+1)}) \\ &\ge \sum_{i=k_0}^{N-1} (w_{\zeta(\nu(i))} - w_{\zeta(\nu(i+1))}) \sum_{j=0}^{i} (\hat{x}_i\tilde{x}_i - x_{l_1(t(i))}x_{l_1(t(i)+1)}) \end{aligned}$$

$$+ w_{\zeta(\nu(N))} \sum_{i=0}^{N} (\hat{x}_i \tilde{x}_i - x_{l_1(t(i))} x_{l_1(t(i)+1)})$$

$$> \sum_{i=k_0}^{N-1} (w_{\zeta(\nu(i))} - w_{\zeta(\nu(i+1))})\mu + w_{\zeta(\nu(N))}\mu$$

$$= w_{\zeta(\nu(k_0))}\mu \geq \left(\min_{0 \leq i \leq k_0} w_{\zeta(i)}\right)\mu > \sum_{i=N+1}^{\infty} w_{\zeta(i)} x_{l_1(i)} x_{l_1(i+1)}.$$

Therefore, if $\zeta' \in S_{\mathbb{N}^0}$ is the rearrangement such that $A_{w', \sigma_{N+1}}$, where $w' = (w_{\zeta(\nu(i))})_{i=0}^{N}$, is a principal submatrix of $A_{\zeta'}$, then $w(A_{\zeta'}) \geq \sum_{i=0}^{N} w_{\zeta(\nu(i))} \hat{x}_i \tilde{x}_i > w(A_\zeta)$ as asserted. □

Proof of Theorem 3.1. We may suppose that w is positive. With the help of Lemmas 3.3 and 3.4, we only need to show that if $w(A_\zeta) = \sup_{l \in S_{\mathbb{N}^0}} w(A_l)$ for some $\zeta \in S_{\mathbb{N}^0}$, then $\sup_{k \in \mathbb{N}^0} w_k = \limsup_{k \to \infty} w_k$. If otherwise, then there exists an $i_0 \in \mathbb{N}^0$ such that $w_{i_0} = \sup_{k \in \mathbb{N}^0} w_k > \limsup_{k \to \infty} w_k \equiv s$. By multiplying a suitable scalar, we may assume that $w(A_\zeta) = 1$. This implies $1 = w(A_\zeta) = w(U_w) \geq w(B(s)) = s$. Suppose that $s = 1$. Let $w' = (w'_i)_{i=0}^{\infty}$ be defined by $w'_0 = w_{i_0}$ and $w'_i = 1$ for all $i \geq 1$. We get $1 = w(U_w) \geq w(U_{w'}) = (w_{i_0}^2 + 1)/2w_{i_0} > 1$ by Wang and Wu [8, Theorem 4.9(b)], which is impossible. Suppose $s < 1$ and let $s_0 \in (s, 1)$. From Lemma 3.6, $W(A_\zeta)$ is open and hence $W(A_\zeta) = W(A_\zeta[m, \infty])$ for all $m \in \mathbb{N}$ by Wang and Wu [8, Proposition 2.4]. We get $w(A_\zeta) = w(A_\zeta[m_0, \infty]) \leq w(A(s_0)) = s_0 < 1$ for sufficiently large m_0, a contradiction. This finishes the proof. □

To prove Theorem 3.2, we need the necessary and sufficient conditions for the existence of some $\zeta \in S_{\mathbb{Z}}$ which satisfies $U_w = B_\zeta$.

Lemma 3.7. *Let $w = (w_i)_{i=-\infty}^{\infty}$ be positive and bounded. The following statements are equivalent:*

(a) *$U_w = B_\zeta$ for some $\zeta \in S_{\mathbb{Z}}$,*
(b) *there exists a one-to-one correspondence $\phi : \mathbb{N}^0 \to \mathbb{Z}$ such that $w_{\phi(0)} \geq w_{\phi(1)} \geq \cdots$, and*
(c) *the condition (2) or (3) in Theorem 3.2 holds.*

Proof. (a) $\Longleftrightarrow$ (b) If $U_w = B_\zeta$ for some $\zeta \in S_{\mathbb{Z}}$, we have $w_{\phi(0)} \geq w_{\phi(1)} \geq \cdots$, where $\phi = \zeta\sigma^{-1}$ and σ is defined in the first paragraph in this section, and ϕ is the desired one-to-one correspondence from $\mathbb{N}^0$ to $\mathbb{Z}$. Conversely, if (b) is true, then we select $\zeta = \phi\sigma \in S_{\mathbb{Z}}$ and get $B_\zeta = U_w$.

(b) $\Longleftrightarrow$ (c) Let $s \equiv \max\{\limsup_{k \to \infty} w_k, \limsup_{k \to -\infty} w_k\}$. Because the set $\{i \in \mathbb{Z} : w_i > s'\}$ is finite for any $s' > s$, the implication (c) $\Rightarrow$ (b) is trivial. Now suppose that (b) holds. We first claim that $w_i \geq s$ for all $i \in \mathbb{Z}$. If otherwise, then we may choose a $p \in \mathbb{Z}$ such that $w_p < s$ and $p = \phi(t)$ for some $t \in \mathbb{N}^0$. Since there are infinitely many w_i's greater than w_p, we can find some $t' \in \mathbb{N}^0$ satisfying $t' > t$ and $w_{\phi(t')} > w_p = w_{\phi(t)}$, a contradiction. Hence, $w_i \geq s$ for all $i \in \mathbb{Z}$. This implies that $\limsup_{k \to \infty} w_k = \limsup_{k \to -\infty} w_k = s$. Moreover, if $w_{i_0} = s$ for some $i_0 \in \mathbb{Z}$ and the set $\{i \in \mathbb{Z} : w_i > s\}$ is infinite, the same argument also leads us to a contradiction. Hence, (c) is true. □

Lemma 3.8. *Let B be the bilateral weighted backward shift with the positive and bounded weight $w = (w_i)_{i=-\infty}^{\infty}$. Suppose that $x^{(p)} = (x_i^{(p)})_{i=-\infty}^{\infty}$, $p = 1, 2, \ldots$, are nonnegative unit vectors such that $\langle Bx^{(p)}, x^{(p)} \rangle \to w(B)$ as $p \to \infty$, and suppose that, for each $i \in \mathbb{N}^0$, $x_i^{(p)} \to \alpha_i$ as $p \to \infty$ for some $\alpha_i \geq 0$. We have*

(a) *$\alpha_i \to 0$ as $i \to \pm\infty$, and*
(b) *if $\alpha_{i_0} = 0$ for some i_0, then $W(B)$ is open.*

Proof. (a) By Fatou's lemma, $\sum_{i=-\infty}^{\infty} \alpha_i^2 = \sum_{i=-\infty}^{\infty} \liminf_{p \to \infty} (x_i^{(p)})^2 \leq \liminf_{p \to \infty} \sum_{i=-\infty}^{\infty} (x_i^{(p)})^2 = 1$ (cf. [12, Theorems 5.17 and 10.29]). Therefore, $\sum_{i=-\infty}^{\infty} \alpha_i^2$ converges and $\alpha_i \to 0$ as $i \to \pm\infty$.

(b) Suppose $\alpha_{i_0} = 0$ for some i_0. From $\langle Bx^{(p)}, x^{(p)} \rangle \to w(B)$ and $x_{i_0}^{(p)} \to 0$ as $p \to \infty$, we get that $\langle \hat{B}x^{(p)}, x^{(p)} \rangle = \sum_{i=-\infty}^{i_0-2} w_i x_i^{(p)} x_{i+1}^{(p)} + \sum_{i=i_0+1}^{\infty} w_i x_i^{(p)} x_{i+1}^{(p)} \to w(B)$ as $p \to \infty$, where $\hat{B}$ is the bilateral weighted backward shift with the weight $(\cdots \underline{w_0} \cdots w_{i_0-2}\, \epsilon\, \delta\, w_{i_0+1} \cdots)$ with $0 < \epsilon < w_{i_0-1}$ and $0 < \delta < w_{i_0}$. This implies that $w(\hat{B}) \geq w(B)$. By [8, Proposition 2.5(a)], we have $w(\hat{B}) \leq w(B)$ and hence $w(\hat{B}) = w(B)$. Therefore, $W(B)$ is open by Wang and Wu [8, Proposition 2.5(b)]. □

We remark that the preceding lemma also holds for the unilateral case and the proof can be obtained analogously. It is now time to derive Theorem 3.2.

Proof of Theorem 3.2. We may assume that w is positive. From Lemmas 3.3, 3.4 and 3.7, we only need to show that if none of (1), (2) and (3) holds, then $w(B_\zeta) < \sup_{l\in S_{\mathbb{Z}}} w(B_l)$ for all $\zeta \in S_{\mathbb{Z}}$. Suppose that this is the case, and $w(B_\zeta) = \sup_{l\in S_{\mathbb{Z}}} w(B_l)$ for some $\zeta \in S_{\mathbb{Z}}$. Set $s \equiv \max\{\limsup_{k\to\infty} w_k, \limsup_{k\to-\infty} w_k\}$. Then $s > 0$, and there exist indices $i_1, i_2 \in \mathbb{N}^0$ and a subsequence $(n_j)_{j=0}^{\infty}$ of $\mathbb{N}^0$ such that $w_{\zeta(i_1)} > s \geq w_{\zeta(i_2)}$ and $w_{\zeta(n_j)} > w_{\zeta(i_2)}$ for all $j = 0, 1, \ldots$. Let $x^{(p)} = (x_i^{(p)})_{i=-\infty}^{\infty}$, $p = 1, 2, \ldots$, be nonnegative unit vectors satisfying $\langle B_\zeta x^{(p)}, x^{(p)}\rangle \to w(B_\zeta)$ as $p \to \infty$. Applying the diagonal process, we may assume that, for each $i \in \mathbb{N}^0$, $x_i^{(p)} \to \alpha_i$ as $p \to \infty$ for some $\alpha_i \geq 0$. Lemma 3.8(a) ensures that $\alpha_i \to 0$ as $i \to \pm\infty$. If $\alpha_{i_0} = 0$ for some $i_0 \in \mathbb{N}^0$, then Lemma 3.8(b) says that $W(B_\zeta)$ is open, and $w(B_\zeta) = w(B_{\zeta_m})$ for all $m \geq 1$ by [8, Proposition 2.4], where B_{ζ_m} is the bilateral weighted backward shift with the weight $(\cdots\ w_{-m-1}\ w_{-m}\ \underline{w_m}\ w_{m+1}\ \cdots)$. Let s_0 be such that $s < s_0 < (w_{\zeta(i_1)} + (s^2/w_{\zeta(i_1)}))/2$. We have $w(B_\zeta) = w(B_{\zeta_{m_0}}) \leq w(B(s_0)) = s_0$ for sufficiently large m_0. On the other hand, define $w' = (w_i')_{i=-\infty}^{\infty}$ by $w_0' = w_{\zeta(i_1)}$ and $w_i' = s$ for all $i \neq 0$. By Lemma 3.3 and [8, Theorem 4.9(b)], we have

$$w(B_\zeta) = w(U_w) \geq w(U_{w'}) = s \cdot \left[\left(\left(\frac{w_{\zeta(i_1)}}{s}\right)^2 + 1\right) / \left(2\left(\frac{w_{\zeta(i_1)}}{s}\right)\right)\right] = \frac{1}{2}\left(w_{\zeta(i_1)} + \frac{s^2}{w_{\zeta(i_1)}}\right) > s_0,$$

which is impossible. We conclude that $\alpha_i > 0$ for all $i \in \mathbb{Z}$. Let j_0 be the index such that $\alpha_{n_{j_0}}\alpha_{n_{j_0}+1} < \alpha_{i_2}\alpha_{i_2+1}$. Define $\zeta' \in S_{\mathbb{Z}}$ by $\zeta'(i_2) = \zeta(n_{j_0})$, $\zeta'(n_{j_0}) = \zeta(i_2)$ and $\zeta'(i) = \zeta(i)$ for all $i \neq i_2, n_{j_0}$. From elementary calculation, we obtain

$$\begin{aligned}\langle B_{\zeta'} x^{(p)}, x^{(p)}\rangle &= \langle B_\zeta x^{(p)}, x^{(p)}\rangle + (w_{\zeta'(i_2)} - w_{\zeta(i_2)})x_{i_2}^{(p)}x_{i_2+1}^{(p)} + (w_{\zeta'(n_{j_0})} - w_{\zeta(n_{j_0})})x_{n_{j_0}}^{(p)}x_{n_{j_0}+1}^{(p)} \\ &= \langle B_\zeta x^{(p)}, x^{(p)}\rangle + (w_{\zeta(n_{j_0})} - w_{\zeta(i_2)})(x_{i_2}^{(p)}x_{i_2+1}^{(p)} - x_{n_{j_0}}^{(p)}x_{n_{j_0}+1}^{(p)}) \\ &\to w(B_\zeta) + (w_{\zeta(n_{j_0})} - w_{\zeta(i_2)})(\alpha_{i_2}\alpha_{i_2+1} - \alpha_{n_{j_0}}\alpha_{n_{j_0}+1}) \quad \text{as} \quad p \to \infty.\end{aligned}$$

Therefore, $w(B_{\zeta'}) > w(B_\zeta)$, a contradiction. This completes the proof. □

Acknowledgments

The authors appreciate the advice from Professor Pei Yuan Wu. He first observed that if $\zeta \in S_n$ satisfies $w(A_{w,\zeta}) = \max_{l\in S_n} w(A_{w,l})$, where $w = (w_i)_{i=1}^n$ with $w_1 \geq w_2 \geq \cdots \geq w_{n-1} > w_n = 0$, then $\zeta = \psi\sigma_n\varphi$ for some $\psi \in F_w$ and $\varphi \in H_n$. He also verified this assertion for $n \leq 7$ by examining the characteristic polynomials of weighted cyclic matrices. The authors thank the referee for his/her helpful suggestions, too.

The first author's research was supported by the National Science Council of the Republic of China under the post-doctoral fellowship NSC 100-2811-M-009-054. The second author's research was supported by the National Science Council of the Republic of China under the projects NSC 99-2115-M-009-013-MY2.

References

[1] C.-K. Li, B.-S. Tam, P.Y. Wu, The numerical range of a nonnegative matrix, Linear Algebra Appl. 350 (2002) 1–23.
[2] Q.F. Stout, The numerical range of a weighted shift, Proc. Amer. Math. Soc. 88 (1983) 495–502.
[3] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge Univ. Press, Cambridge, 1991.
[4] W. Ridge, Numerical range of a weighted shift with periodic weights, Proc. Amer. Math. Soc. 55 (1976) 107–110.
[5] T.-Y. Tam, On a conjecture of Ridge, Proc. Amer. Math. Soc. 125 (1997) 3581–3592.
[6] M.C. Tsai, Numerical ranges of weighted shift matrices with periodic weights, Linear Algebra Appl. 435 (2011) 2296–2302.
[7] M.C. Tsai, P.Y. Wu, Numerical ranges of weighted shift matrices, Linear Algebra Appl. 435 (2011) 243–254.
[8] K.-Z. Wang, P.Y. Wu, Numerical ranges of weighted shifts, J. Math. Anal. Appl. 381 (2011) 897–909.
[9] G. Bennett, Factorizing the classical inequalities, Mem. Amer. Math. Soc. 120 (1996) 1–130.
[10] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge Univ. Press, Cambridge, 1985.
[11] L. Elsner, On the variation of the spectra of matrices, Linear Algebra Appl. 47 (1982) 127–138.
[12] R.L. Wheeden, A. Zygmund, Measure and Integral—An Introduction to Real Analysis, Marcel Dekker, 1977.