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## A spectral excess theorem for nonregular graphs

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### ABSTRACT

The spectral excess theorem asserts that the average excess is, at most, the spectral excess in a regular graph, and equality holds if and only if the graph is distance-regular. An example demonstrates that this theorem cannot directly apply to nonregular graphs. This paper defines average weighted excess and generalized spectral excess as generalizations of average excess and spectral excess, respectively, in nonregular graphs, and proves that for any graph the average weighted excess is at most the generalized spectral excess. Aside from distance-regular graphs, additional graphs obtain the new equality. We show that a graph is distance-regular if and only if the new equality holds and the diameter  $D$  equals the spectral diameter  $d$ . For application, we demonstrate that a graph with odd-girth  $2d + 1$  must be distance-regular, generalizing a recent result of van Dam and Haemers.

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### 1. Introduction

Throughout this paper, let  $G = (VG, EG)$  be a connected graph on  $n$  vertices, with diameter  $D$ , adjacency matrix  $A$ , and distance function  $\partial$ . Assume that  $A$  has  $d + 1$  distinct eigenvalues  $\lambda_0 > \lambda_1 > \dots > \lambda_d$  with corresponding multiplicities  $m_0 = 1, m_1, \dots, m_d$ . From the spectrum of  $G$  we then define an inner product  $\langle \cdot, \cdot \rangle_\Delta$  on the vector space  $\mathbb{R}_d[x]$  of real polynomials of degree at most  $d$ . It is well known that  $\mathbb{R}_d[x]$  has a unique orthogonal basis  $p_0(x), p_1(x), \dots, p_d(x)$ , satisfying  $\deg p_i(x) = i$  and  $\langle p_i(x), p_i(x) \rangle_\Delta = p_i(\lambda_0)$  for  $0 \leq i \leq d$ . The number  $\bar{k}_d := |\{(u, v) \mid u, v \in VG, \partial(u, v) = d\}|/n$  is called the *average excess* of  $G$ , and the number  $p_d(\lambda_0)$  is called the *spectral excess* of  $G$ . The spectral excess theorem, proposed by Fiol and Garriga [9], states that

$$\bar{k}_d \leq p_d(\lambda_0) \quad (1)$$

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if  $G$  is regular; and equality holds if and only if  $G$  is distance-regular. For short proofs, see [5,11]. Furthermore, see [4,3] for some generalizations.

Example 2.2 indicates that (1) cannot directly apply to nonregular graphs. In addition, (1) is trivial if  $D < d$ , because  $\bar{k}_d = 0$ . Consequently, we provide a generalization of the spectral excess theorem to make it applicable to nonregular graphs using a global approach. Fiol, Garriga, and Yebra [13] also considered nonregular graphs, using a local approach, however. In Section 3 we define the *average weighted excess*  $\delta_D$  and the *generalized spectral excess*  $p_{\geq D}(\lambda_0) := p_D(\lambda_0) + p_{D+1}(\lambda_0) + \dots + p_d(\lambda_0)$ , as generalizations of average excess and spectral excess, respectively, in nonregular graphs, and prove that

$$\delta_D \leq p_{\geq D}(\lambda_0). \tag{2}$$

As the spectral excess theorem, distance-regular graphs also attain equality in (2). Moreover, additional graphs obtain equality (see Remark 3.3 for more details). Furthermore, under the assumption  $D = d$ , we show in Theorem 3.5 that  $G$  is distance-regular if and only if equality in (2) holds.

The *odd-girth* of a graph is the length of its shortest odd cycle. For application, in Section 4, we demonstrate that a graph with odd-girth  $2d + 1$  must be distance-regular. The same consequence was previously proved by van Dam and Haemers [6] under the assumption that  $G$  is regular. Moreover, our result is the solution of a problem raised in the same paper (though these authors also proved this result for the case  $d + 1 = 3$  and claimed to have proofs for the cases  $d + 1 \in \{4, 5\}$ ). Because the odd-girth is determined according to the spectrum, this result is also a generalization of the spectral characterization of the generalized odd graphs [15,16].

**2. Preliminaries**

In this section, we review the concept of orthogonal polynomials related to  $G$ . The basic idea is to generalize the study of distance-regular graphs (see [1,2,18]). The *spectrum* of  $G$  is denoted by the multiset  $\text{sp } G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$ , and the parameter  $d$  is called the *spectral diameter* of  $G$ . It is well known that  $D \leq d$  and  $Z(x) := \prod_{i=0}^d (x - \lambda_i)$  is the *minimal polynomial* of  $G$  [1, Chapter 2]. From the spectrum  $\text{sp } G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$  we consider the  $(d + 1)$ -dimensional vector space  $\mathbb{R}_d[x] \cong \mathbb{R}[x]/(Z(x))$  of real polynomials of degree at most  $d$  with inner product defined by

$$\langle p(x), q(x) \rangle_{\Delta} := \sum_{i=0}^d \frac{m_i}{n} p(\lambda_i)q(\lambda_i) = \text{tr}(p(A)q(A))/n.$$

Using this inner product, there is a unique system of orthogonal polynomials  $p_i(x)$ ,  $i = 0, 1, \dots, d$ , where  $\text{deg } p_i(x) = i$  and  $\langle p_i(x), p_j(x) \rangle_{\Delta} = p_i(\lambda_0)$  [17]. These polynomials  $p_0(x), p_1(x), \dots, p_d(x)$  are referred to as the *predistance polynomials* of  $G$ . Moreover, the sum of all predistance polynomials gives the *Hoffman polynomial*  $H(x)$  [14]:

$$H(x) := n \prod_{i=1}^d \frac{x - \lambda_i}{\lambda_0 - \lambda_i} = p_0(x) + p_1(x) + \dots + p_d(x), \tag{3}$$

no matter whether the graph  $G$  is regular or not. For a proof, see for instance [5]. Let  $\alpha$  be the eigenvector of  $A$  corresponding to  $\lambda_0$  such that  $\alpha^t \alpha = n$  and all entries of  $\alpha$  are positive. Notice that  $\alpha = (1, 1, \dots, 1)^t$  if and only if  $G$  is regular. The following lemma generalizes [14] to nonregular graphs, which was considered in [8, p. 117].

**Lemma 2.1.** *Let  $G$  be a graph with adjacency matrix  $A$ . Then,  $H(A) = \alpha \alpha^t$ . Moreover,  $G$  is regular if and only if  $H(A) = J$ , the all 1’s matrix.  $\square$*

Recall that the spectral excess theorem states that if  $G$  is regular then (1) holds. The following example demonstrates that the regularity assumption of  $G$  is necessary.

**Example 2.2.** Let  $P_3$  be a path of three vertices, with spectrum  $\text{sp } P_3 = \{\sqrt{2}, 0, -\sqrt{2}\}$ . One can check that  $p_0(x) = 1$ ,  $p_1(x) = 3\sqrt{2}x/4$ , and  $p_2(x) = 3(x^2 - 4/3)/4$ . Notice that  $k_2 = 2/3$  and  $p_2(\lambda_0) = 1/2$ . This shows that (1) does not hold.

### 3. A weighted spectral excess theorem

Let  $A_i$  be the  $i$ -th distance matrix of  $G$ , i.e., the  $n \times n$  matrix with rows and columns indexed by the vertex set  $VG$  such that

$$(A_i)_{uv} = \begin{cases} 1 & \text{if } \partial(u, v) = i, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,  $A_0 = I$  is the identity matrix and  $A_1 = A$  is the adjacency matrix. Define  $\tilde{A}_i := A_i \circ H(A)$ , where “ $\circ$ ” is the entrywise product of matrices. Notice that  $\tilde{A}_i$  can be regarded as a “weighted” version of  $A_i$  since

$$(\tilde{A}_i)_{uv} = \begin{cases} \alpha_u \alpha_v & \text{if } \partial(u, v) = i, \\ 0 & \text{otherwise} \end{cases}$$

by Lemma 2.1. This approach of giving weights, the entries of the positive eigenvector, to the vertices of a nonregular graph has been recently used many times in the literature (see, for instance, [12,9,10,7]). For instance, this idea was crucial to introduce the concept of pseudo-distance-regularity and the corresponding ( $u$ -local)  $i$ -distance matrices in [12]. The reason that  $\tilde{A}_i$  is weighted in the above way is to have the property that

$$\tilde{A}_0 + \tilde{A}_1 + \dots + \tilde{A}_D = H(A) = p_0(A) + p_1(A) + \dots + p_d(A)$$

by Lemma 2.1 and (3). Define  $\delta_i := \langle \tilde{A}_i, \tilde{A}_i \rangle$  and  $p_{\geq D}(x) := p_D(x) + p_{D+1}(x) + \dots + p_d(x)$ . The number  $\delta_D$  is referred to as the *average weighted excess* and  $p_{\geq D}(\lambda_0)$  as the *generalized spectral excess* of  $G$ . Notice that if  $D = d$  then  $p_{\geq D}(x) = p_d(x)$ . For any two  $n \times n$  real symmetric matrices  $M$  and  $N$ , define the inner product

$$\langle M, N \rangle := \frac{1}{n} \text{tr}(MN) = \frac{1}{n} \sum_{i,j} (M \circ N)_{ij}.$$

Moreover, let

$$\text{Proj}_N(M) := \frac{\langle N, M \rangle}{\langle N, N \rangle} N$$

denote the projection of  $M$  onto  $\text{Span}\{N\}$ . The proof of the following lemma is basically the same as in [11, Lemma 1] (the only difference is the use of “weights”, the entries of  $\alpha$ , on the vertices of  $G$ ).

**Lemma 3.1.**  $\text{Proj}_{\tilde{A}_D}(p_{\geq D}(A)) = \tilde{A}_D$ .  $\square$

From Lemma 3.1, we immediately have the following theorem.

**Theorem 3.2.** Let  $G$  be a connected graph with diameter  $D$ . Then  $\delta_D \leq p_{\geq D}(\lambda_0)$  with equality if and only if  $\tilde{A}_D = p_{\geq D}(A)$ .  $\square$

Revisiting the case when  $G$  is the path on three vertices  $P_3$  described in Example 2.2. Notice that  $D = d = 2$ ,  $\alpha = (\sqrt{3}/2, \sqrt{6}/2, \sqrt{3}/2)^t$  and

$$\tilde{A}_D = \begin{pmatrix} 0 & 0 & 3/4 \\ 0 & 0 & 0 \\ 3/4 & 0 & 0 \end{pmatrix}.$$

Then  $\delta_D = 3/8 \leq 1/2 = p_{\geq D}(\lambda_0)$  satisfies inequality in Theorem 3.2.

**Remark 3.3.** If  $G$  is regular with diameter  $D = 2$ , then equality in Theorem 3.2 holds. Indeed,  $\tilde{A}_2 = A_2 = J - I - A = H(A) - I - A = p_{\geq 2}(A)$ .

The graph described in Remark 3.3 is a special case of *distance-polynomial graphs* [19]. It would be interesting to characterize the graphs which satisfy equality in Theorem 3.2. We complete this characterization under the assumption  $D = d$  in Theorem 3.5. Since  $p_0(A) = I$ , the following result is simple, but plays a crucial role in proving the regularity of a graph.

**Lemma 3.4.**  $\tilde{A}_0 = p_0(A)$  if and only if  $G$  is regular.  $\square$

The following theorem is a “weighted” version of [11, Proposition 2]. The proof is essentially the same except that here the use of weights on vertices is taken into account.

**Theorem 3.5.** Let  $G$  be a connected graph of diameter  $D$  which is equal to the spectral diameter  $d$ . Then  $\tilde{A}_D = p_D(A)$  if and only if  $\tilde{A}_i = p_i(A)$  for  $0 \leq i \leq D - 1$ . Moreover, if  $\tilde{A}_D = p_D(A)$  then  $G$  is distance-regular.

**Proof.** The first part of the proof is essentially the same as in [11, Proposition 2]. Suppose  $\tilde{A}_D = p_D(A)$ . By the first part we conclude that  $\tilde{A}_0 = p_0(A)$ , and thus  $G$  is regular by Lemma 3.4. Hence  $G$  is distance-regular by [11, Proposition 2].  $\square$

#### 4. Graphs with odd-girth $2d + 1$

In this section, assume that  $G$  has odd-girth  $2d + 1$ . For application of Theorem 3.5, we demonstrate that  $G$  has diameter  $D = d$  and  $G$  must be distance-regular. The proof is basically identical to [6, pp. 487–488], except for the weights on the vertices. Let  $c = n / \prod_{i=1}^d (\lambda_0 - \lambda_i)$ , which is the leading coefficient of the Hoffman polynomial  $H(x)$ . Notice that the trace  $\text{tr}(A^{2d+1})$  of  $A^{2d+1}$  is nonzero since  $G$  has odd-girth  $2d + 1$ . The following lemma determines the average weighted excess  $\delta_D$  and the generalized spectral excess  $p_{\geq D}(\lambda_0)$ .

**Lemma 4.1.** (See [6].)  $\delta_D = c^2 \text{tr}(A^{2d+1}) / (n \sum_{i=0}^d \lambda_i) = p_d(\lambda_0)$ . In particular,  $D = d$ .  $\square$

From Lemma 4.1 and Theorem 3.5, we immediately have the following theorem.

**Theorem 4.2.** A connected graph with  $d + 1$  distinct eigenvalues and odd-girth  $2d + 1$  is distance-regular.  $\square$

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