Solution to an open problem on 4-ordered Hamiltonian graphs

Lih-Hsing Hsu [a], Jimmy J.M. Tan [b], Eddie Cheng [c,*], László Lipták [c], Cheng-Kuan Lin [b], Ming Tsai [a]

[a] *Department of Computer Science and Information Engineering, Providence University, Taichung, 43301, Taiwan, ROC*

[b] *Department of Computer Science, National Chiao Tung University, Hsinchu, 30010, Taiwan, ROC*

[c] *Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309, United States*

ABSTRACT

A graph G is *k-ordered* if for any sequence of k distinct vertices of G, there exists a cycle in G containing these k vertices in the specified order. It is *k-ordered Hamiltonian* if, in addition, the required cycle is Hamiltonian. The question of the existence of an infinite class of 3-regular 4-ordered Hamiltonian graphs was posed in Ng and Schultz (1997) [10]. At the time, the only known examples were K_4 and $K_{3,3}$. Some progress was made in Mészáros (2008) [9] when the Peterson graph was found to be 4-ordered and the Heawood graph was proved to be 4-ordered Hamiltonian; moreover an infinite class of 3-regular 4-ordered graphs was found. In this paper we show that a subclass of generalized Petersen graphs are 4-ordered and give a complete classification for which of these graphs are 4-ordered Hamiltonian. In particular, this answers the open question regarding the existence of an infinite class of 3-regular 4-ordered Hamiltonian graphs. Moreover, a number of results related to other open problems are presented.

© 2012 Elsevier B.V. All rights reserved.

1. Introduction and preliminaries

Throughout this paper, we use standard graph theory terminology as in [8]. A graph G is *k-ordered* if for any sequence of k distinct vertices of G, there exists a cycle in G containing these k vertices in the specified order. It is *k-ordered Hamiltonian* if, in addition, the required cycle is Hamiltonian. This concept was introduced in [10] and the following open problem was posed: Find an infinite class of 3-regular 4-ordered Hamiltonian graphs. (Incidentally, when the concept of *k*-ordered Hamiltonian was introduced in [10], it was simply called *k*-ordered. It was later that researchers used the terms *k*-ordered and *k*-ordered Hamiltonian as defined here.) In [9], it was determined that the Petersen graph is 4-ordered and the Heawood graph is 4-ordered Hamiltonian. Moreover, an infinite class of 3-regular 4-ordered graphs was found. However, the question regarding an infinite class of 3-regular 4-ordered Hamiltonian graphs remained open. It is known that the generalized Honeycomb torus $GHT(3, n, 1)$ is 4-ordered for even $n \geq 8$ [4]. These graphs are vertex transitive. Among such known examples, there are only two non-bipartite graphs, namely, K_4 and the Petersen graph. Recently [13] showed that there exist bipartite non-vertex-transitive 4-ordered 3-regular graphs with n vertices for sufficiently large even integer n. Moreover, the same is true for vertex-transitive 4-ordered 3-regular graphs. We refer the readers to [3,5–7] for results including several sufficient conditions for *k*-orderedness; in particular, Faudree [7] provides a comprehensive survey. The goal of this paper is to answer the open question of whether there exists an infinite class of 3-regular 4-ordered Hamiltonian graphs posed in [10].

The class of generalized Petersen graphs has attracted much research throughout the years. Some recent research include [11,1,14,12,2]. The generalized Petersen graph $P(n, k)$ where $1 \leq k \leq (n-1)/2$ has $\{a_i, b_i : 0 \leq i < n\}$ as its

* Corresponding author.
E-mail addresses: lhhsu@pu.edu.tw (L.-H. Hsu), jmtan@cs.nctu.edu.tw (J.J.M. Tan), echeng@oakland.edu (E. Cheng), liptak@oakland.edu (L. Lipták), cklin@cs.nctu.edu.tw (C.-K. Lin), cater_home@hotmail.com (M. Tsai).
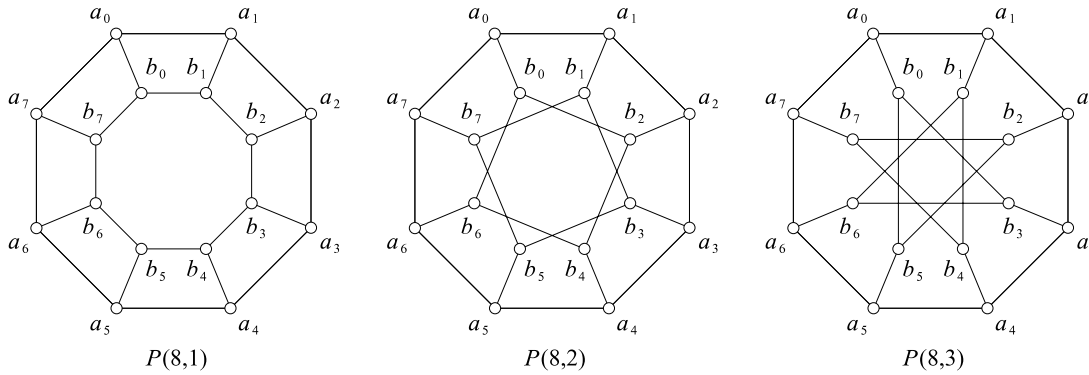
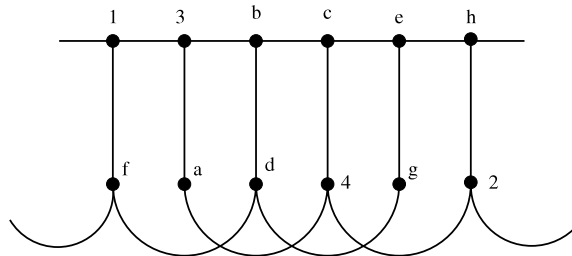**Fig. 1.** The graphs $P(8, 1)$, $P(8, 2)$, and $P(8, 3)$.

Fig. 2. $P(n, 2)$ is not 4-ordered.

vertex set. There are three types of edges. The first is of the form (a_i, a_{i+1}) (with $i + 1$ computed modulo n) for $0 \le i < n$. The second is of the form (b_i, b_{i+k}) (with $i + k$ computed modulo n) for $0 \le i < n$. The third is of the form (a_i, b_i) for $0 \le i < n$. We call the edges in the third case the *columns*. We note that the subgraph induced by vertices $a_i, 0 \le i < n$, form an n-cycle, and the subgraph induced by vertices $b_i, 0 \le i < n$, form a circulant graph. So $P(5, 2)$ is the Petersen graph. Fig. 1 gives several small examples. In this paper we study the 4-ordered and the 4-ordered Hamiltonian problems for $P(n, 1)$, $P(n, 2)$ and $P(n, 3)$. The answer for $P(n, 1)$ follows directly from the following known result.

Proposition 1.1 ([9]). *No 4-ordered 3-regular graph with more than six vertices contains a 4-cycle.*

Proposition 1.2. $P(n, 1)$ *is neither 4-ordered nor 4-ordered Hamiltonian.*

Proof. By definition, $n \ge 3$. It is easy to check that $P(3, 1)$, isomorphic to $K_3 \square K_3$, the prism, is not 4-ordered. So we may assume $n \ge 4$ and hence $P(n, 1)$ has more than six vertices. It follows directly from Proposition 1.1 that $P(n, 1)$ is not 4-ordered as it contains a 4-cycle. Consequently, it is not 4-ordered Hamiltonian. □

2. The graph $P(n, 2)$

Since $P(5, 2)$ is the Petersen graph, which is known to be 4-ordered, it is interesting to see that for $n > 5$, $P(n, 2)$ is not 4-ordered. Consider a portion of $P(n, 2)$ as shown in Fig. 2. The labels 1–4 are the four vertices in the order that we prescribed. Suppose that there is a cycle C containing these 4 vertices in the prescribed order. Then clearly the edges $(1, 3)$ and $(2, 4)$ are not in C. But C must contain vertices 3 and 4, so the edges $(3, b)$, $(3, a)$, $(4, c)$, $(4, a)$ must be in C. Now (b, c) cannot be in C as it will complete a cycle containing the vertices 3 and 4 but not 1 and 2. Since b is on C, so (b, d) must be in C. Now c is on C, so (c, e) must be in C. Since $(1, 3)$ is not in C, the other two edges incident to 1, in particular $(1, f)$ must be in C. Similarly, since $(4, 2)$ is not in C, the other two edges incident to 2, in particular $(2, h)$ must be in C. Now (f, d) cannot be in C; otherwise, we have a path in C from vertex 1 to vertex 3 not containing 2 and 4. Since d is on C, (d, g) must be in C. Now, (e, g) cannot be in C; otherwise, it will complete a cycle containing the vertices 3 and 4 but not 1 and 2. So (e, h) must be in C. But now C has a subpath from 3 to 4 to 2, a contradiction. (Note that the proof involves at least 6 columns and so it is not applicable for $n = 5$. If $n = 6$, then the column containing 1 and the column containing 2 are "adjacent" and one can actually obtain a contradiction sooner.) Since $P(5, 2)$ is the Petersen graph, which is not Hamiltonian, thus it is not 4-ordered Hamiltonian and this completes the proof of the following result.

Theorem 2.1. $P(5, 2)$ *is 4-ordered but not 4-ordered Hamiltonian. If $n \ge 6$, then $P(n, 2)$ is not 4-ordered, and hence, not 4-ordered Hamiltonian.*
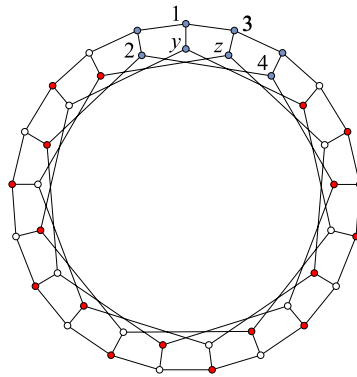
Fig. 3. Example of $P(21, 3)$ of Theorem 3.4.

3. The graph $P(n, 3)$

In this section, we consider the 4-ordered and 4-ordered Hamiltonian problems for $P(n, 3)$. By the definition of $P(n, 3)$, $n \geq 2 \cdot 3 + 1 = 7$. We start with the following two results whose validity were checked by a computer.

Lemma 3.1. *Let* $7 \leq n \leq 26$. *Then* $P(n, 3)$ *is 4-ordered if and only if* $n \notin \{7, 9, 12\}$.

Thus $P(n, 3)$ is usually 4-ordered for small n. However, this is not the case for 4-ordered Hamiltonicity:

Lemma 3.2. *Let* $7 \leq n \leq 26$. *Then* $P(n, 3)$ *is 4-ordered Hamiltonian if and only if* $n \in \{18, 24, 26\}$.

We claim that $P(n, 3)$ is 4-ordered for every n except when n is small. On the other hand, we claim that $P(n, 3)$ is not 4-ordered Hamiltonian if n is odd but it is 4-ordered Hamiltonian if n is even and large enough. We start with the first claim.

Theorem 3.3. *Let* $n \geq 7$. *Then* $P(n, 3)$ *is 4-ordered unless* $n \in \{7, 9, 12\}$.

Proof. Lemma 3.1 provides the justification for $7 \leq n \leq 26$. In particular $P(n, 3)$ is 4-ordered if $13 \leq n \leq 26$. We use induction on n. Suppose $n \geq 27$. Let x_1, x_2, x_3, x_4 be the four prescribed vertices in this specific order. We claim that there are six consecutive columns that do not contain any one of the prescribed vertices. This is true as $n \geq 25$. For notational convenience, we may assume these six columns are

$$(a_{11}, b_{11}), \qquad (a_{12}, b_{12}), \qquad (a_{13}, b_{13}), \qquad (a_{14}, b_{14}), \qquad (a_{15}, b_{15}), \qquad (a_{16}, b_{16}).$$

Now construct graph G as follows: Delete the vertices in these six columns and add the edges (a_{10}, a_{17}), (b_8, b_{17}), (b_9, b_{18}), (b_{10}, b_{19}). Then G is isomorphic to $P(n - 6, 3)$ and so it is 4-ordered by the induction hypothesis. Thus there exists a cycle C in G containing x_1, x_2, x_3, x_4 in this specific order. We now consider the 16 cases depending on whether any of the edges (a_{10}, a_{17}), (b_8, b_{17}), (b_9, b_{18}), (b_{10}, b_{19}) belongs to C. We can then extend C to a cycle in $P(n, 3)$ the following way: If edge (a_{10}, a_{17}) is in C, replace it with the edges (a_{10}, a_{11}), (a_{11}, a_{12}), (a_{12}, a_{13}), (a_{13}, a_{14}), (a_{14}, a_{15}), (a_{15}, a_{16}), (a_{16}, a_{17}). If edge (b_8, b_{17}) is in C, replace it with the edges (b_8, b_{11}), (b_{11}, b_{14}), (b_{14}, b_{17}). If edge (b_9, b_{18}) is in C, replace it with the edges (b_9, b_{12}), (b_{12}, b_{15}), (b_{15}, b_{18}). If edge (b_{10}, b_{19}) is in C, replace it with the edges (b_{10}, b_{13}), (b_{13}, b_{16}), (b_{16}, b_{19}). Now note that these four replacement paths are vertex disjoint, hence the extension is valid. $\square$

Although the proof of Theorem 3.3 is simple, it provides the recursive nature of the graph which will be used in the rest of the paper. In particular, the proof for 4-ordered Hamiltonicity will be based on the same idea except we now have to replace edges with paths that are vertex disjoint and spanning all the inserted vertices.

Theorem 3.4. *Let* $n \geq 7$ *be odd. Then* $P(n, 3)$ *is not 4-ordered Hamiltonian.*

Proof. Lemma 3.2 provides the justification for $n \leq 25$. We may assume $n \geq 27$. (The proof actually works for smaller n.) We pick the vertices labeled 1–4 as in Fig. 3. We claim that there does not exist a Hamiltonian cycle containing these 4 vertices in the order 1–4. Suppose not. Let C be such a Hamiltonian cycle. Then clearly the edge $(1, 3)$ is not in C. So we know which two edges incident to 1 and which two edges incident to 3 are in C. Thus we have to choose an edge incident to y and an edge incident to z to be in C. By symmetry, we have 3 cases as presented in Figs. 4–6. Let G be the graph obtained from $P(n, 3)$ by deleting the vertices in the columns containing 1–4. Then G is bipartite with the two classes colored red and white, respectively (see Fig. 7). We have $n - 4$ red vertices and $n - 4$ white vertices. We now consider the three cases.

In Fig. 4, we have two subpaths in C, namely, one between a and d, and one between b and c. But in G, C becomes two paths which together span all the vertices of G. The end vertices of these two paths are a, b, c, d. This is impossible as all these four vertices are red and G is bipartite with the same number of red vertices and white vertices.
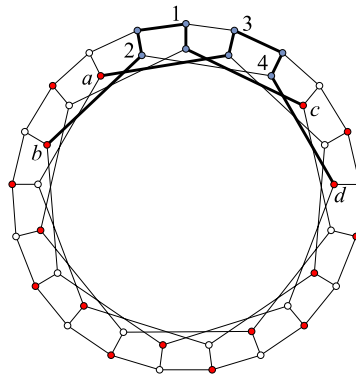
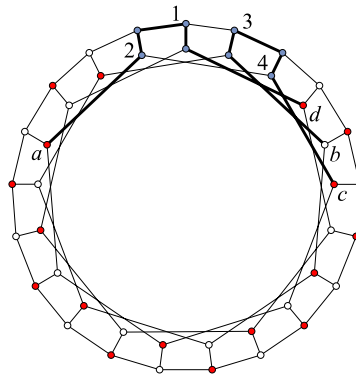**Fig. 4.** Example of $P(21, 3)$ of Case 1 in Theorem 3.4.

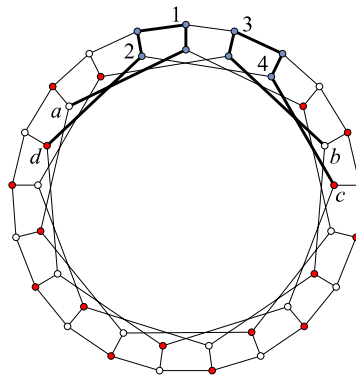**Fig. 5.** Example of $P(21, 3)$ of Case 2 in Theorem 3.4.

Fig. 6. Example of $P(21, 3)$ of Case 3 in Theorem 3.4.

In Fig. 5, we have two subpaths in C, namely, one between a and d, and one between b and c. But in G, C becomes two paths which together span all the vertices of G. The end vertices of these two paths are a, b, c, d. This is impossible as three of them are red and G is bipartite with the same number of red vertices and white vertices.

In Fig. 6, we have two subpaths in C, namely, one between a and d, and one between b and c. But in G, C becomes two paths which together span all the vertices of G. The end vertices of these two paths are a, b, c, d. Here the parity argument does not work as exactly two of these vertices are red. We will use a different argument. Indeed, we will prove something stronger, we will prove that regardless of the parity of n, this case cannot happen. So we can reduce "modulo 3" rather than "modulo 6". (By virtue of the definition of $P(n, 3)$, it is natural to partition the columns into 3 sets, that is, working in "modulo 3" is natural. However, because of the parity condition in the statement, one expects to work in "modulo 6". For this case, we can work in "modulo 3" because of the stronger claim.) Consider the path on the left. There are three additional edges that must be in C; see Fig. 8. Clearly, C must contain a column since it contains both paths. We start with the configuration on the left, proceed counterclockwise and consider the first time a column is in C. Since we are working "modulo 3", we have three cases, depending on the location of this column. We proceed in several steps.
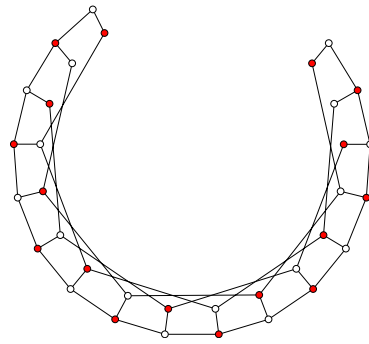
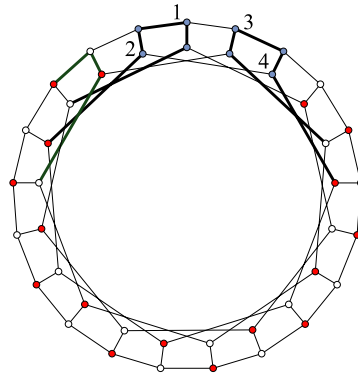**Fig. 7.** *P*(21, 3) with the vertices in the columns containing 1–4 deleted.

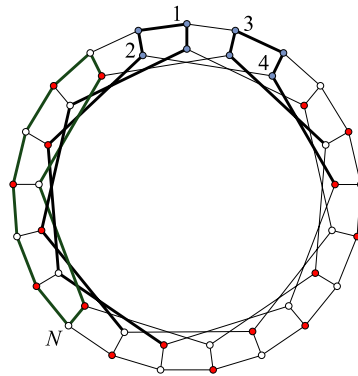**Fig. 8.** Three additional edges.

Fig. 9. Case 1 of Step 1 in Theorem 3.4.

Step 1: Case 1 is when the first column is the (3*q*)th column counting counterclockwise from the column containing 1 for some *q*. It is presented in Fig. 9, where the first column in *C*, counterclockwise, from the left configuration is in green, and it is marked by *N*. This forces additional edges to be in *C* and they are in green. Observe that in this case, the column completes a cycle that is not Hamiltonian. Consequently, the case is eliminated.

Case 2 is when the first column is the (3*q* + 2)nd column counting counterclockwise from the column containing 1 for some *q*. This is presented in Fig. 10, where the first column in *C*, counterclockwise, from the left configuration is in green, and is marked by *N*. Several additional edges, including a column, are forced to be in *C*, up to the vertices labeled by 1′ and 2′. These edges are in green or blue. Note that vertex 1 is on the outer cycle, 2 is on the inner circulant graph, 1′ is on the inner circulant graph and 2′ is on the outer cycle. So the set of edges identified to be in *C* contains a path between 1′ and 2′ that spans all the vertices of the columns that we have visited starting from the left configuration. More importantly, the path is of the form 1′ to 1 to 2 to 2′. Note that *C* cannot contain the edge (1′, 2′), otherwise it completes a cycle that is not Hamiltonian. So one more edge incident to 2′, one more edge incident to 1′ and a few more additional edges are forced to be in *C*. Hence we arrive in a configuration that we will refer to as the *dead-end configuration* as in Fig. 11. Case 3 is when the
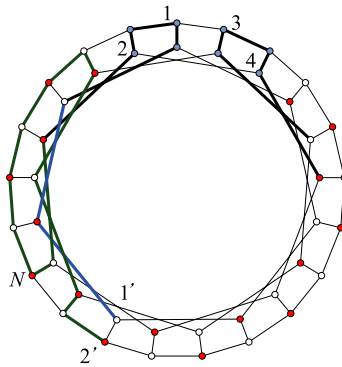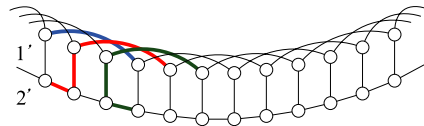
Fig. 10. Case 2 of Step 1 in Theorem 3.4.
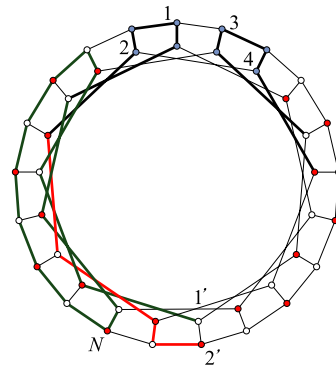

Fig. 11. The dead-end configuration.

Fig. 12. Case 3 of Step 1 in Theorem 3.4.

first column is the $(3q + 1)$st column counting counterclockwise from the column containing 1 for some q. This is presented in Fig. 12, where the first column in C, counterclockwise, from the left configuration is in green, and it is marked by N. Again, additional edges are forced, which will also lead to the dead-end configuration as in Fig. 11, using a similar argument as in Case 2.

Step 2: We look for the next column to be used going counterclockwise from the dead-end configuration. (Again, the configuration on the right and the dead-end configuration imply such a column must be used in C.) So, again, we consider three cases under "modulo 3".

Case 1 is presented in Fig. 13, where the first column in C counterclockwise from the left configuration is the $(3q + 2)$nd column from the column containing 1', and it is marked by N.

This forces some additional edges to be in C, and it will extend to the same dead-end configuration with 1″ and 2″ playing the roles of 1' and 2', respectively.

Case 2 is presented in Fig. 14, where the first column in C counterclockwise from the left configuration is the $(3q)$th column from the column containing 1', and it is marked by N. Observe that in this case the column completes a cycle that is not Hamiltonian.

Case 3 is presented in Fig. 15, where the first column in C counterclockwise from the left configuration is the $(3q + 1)$th column from the column containing 1', and it is marked by N.

This will extend to the same dead-end configuration as in Case 1.

Step 3: It follows from the previous steps that as we extend the configuration on the left, we either get a contradiction or we always arrive to the same dead-end configuration. This process must stop, so we can consider the final column used in C as we extend from each successive dead-end configuration before reaching vertex 4. Recall that the set of edges identified to be in C contains a path between 1' and 2' that spans all the vertices on the columns that we have visited starting from the
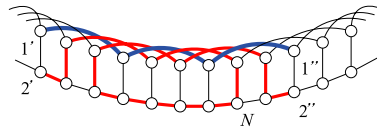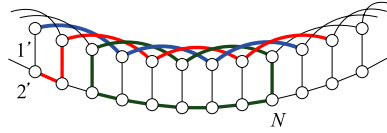
Fig. 13. Case 1 of Step 2 in Theorem 3.4.

Fig. 14. Case 2 of Step 2 in Theorem 3.4.
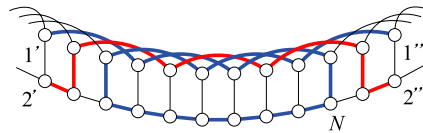

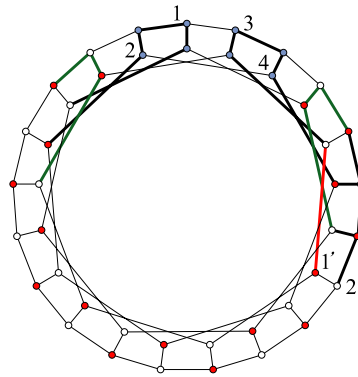**Fig. 15.** Case 3 of Step 2 in Theorem 3.4.

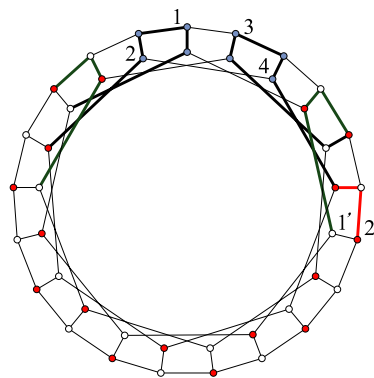
Fig. 16. Step 3 Case 1 in the proof of Theorem 3.4.

Fig. 17. Step 3 Case 2 in the proof of Theorem 3.4.

left configuration. More importantly, the path is of the form $1'$ to 1 to 2 to $2'$. Again, for $P(n, 3)$, we can work in modulo 3 and consider three cases as in Figs. 16–18. (The labels $1'$ and $2'$ correspond to this final column.)

Again, some additional edges are forced to be in C, and this should complete the Hamiltonian cycle.

In the first two cases, the completed Hamiltonian cycle C is of the form 1 to $1'$ to 3 to 4 to $2'$ to 2 to 1, a contradiction. (See Figs. 16 and 17.) In the third case we arrive at a contradiction as the vertex next to $2'$ on the outer cycle (marked by t in the picture) is incident to three edges in C. (See Fig. 18.) This completes the proof. □
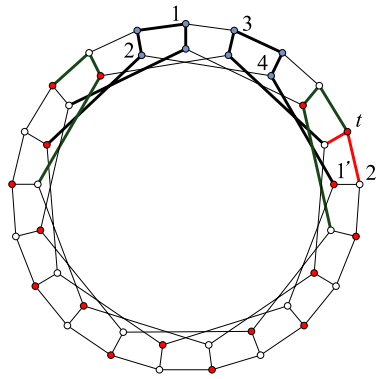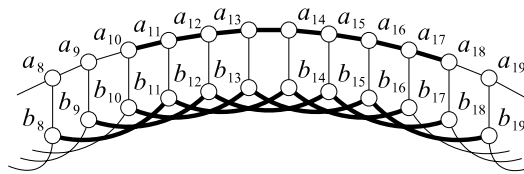
Fig. 18. Step 3 Case 3 in the proof of Theorem 3.4.

Fig. 19. Solution to Case 1.

Table 1
Table of reductions: reductions in "()" are valid but only used as remarks; the reduction in "()" and smaller font is valid by not applicable.

Case	Description	Reduction
1	$(a_{10}, a_{17}), (b_8, b_{17}), (b_9, b_{18}), (b_{10}, b_{19}) \in C$	
2	$(b_8, b_{17}), (b_9, b_{18}), (b_{10}, b_{19}) \notin C$, and $(a_{10}, a_{17}) \in C$	(Left shift to 4)
3	$(a_{10}, a_{17}), (b_8, b_{17}), (b_9, b_{18}) \notin C$, and $(b_{10}, b_{19}) \in C$	
4	$(a_{10}, a_{17}), (b_9, b_{18}), (b_{10}, b_{19}) \notin C$, and $(b_8, b_{17}) \in C$	Symmetric to 3
5	$(a_{10}, a_{17}), (b_8, b_{17}), (b_{10}, b_{19}) \notin C$, and $(b_9, b_{18}) \in C$	(Left shift to 14)
6	$(a_{10}, a_{17}), (b_9, b_{18}) \notin C$, and $(b_8, b_{17}), (b_{10}, b_{19}) \in C$	
7	$(b_8, b_{17}), (b_9, b_{18}) \notin C$, and $(a_{10}, a_{17}), (b_{10}, b_{19}) \in C$	(Left shift to 8 or 11)
8	$(b_9, b_{18}), (b_{10}, b_{19}) \notin C$, and $(a_{10}, a_{17}), (b_8, b_{17}) \in C$	Symmetric to 7
9	$(a_{10}, a_{17}), (b_8, b_{17}) \notin C$, and $(b_9, b_{18}), (b_{10}, b_{19}) \in C$	Left shift to 7
10	$(a_{10}, a_{17}), (b_{10}, b_{19}) \notin C$, and $(b_8, b_{17}), (b_9, b_{18}) \in C$	Left shift to 1
11	$(a_{10}, a_{17}), (b_8, b_{17}), (b_9, b_{18}), (b_{10}, b_{19}) \notin C$	Left shift to 8
12	$(b_8, b_{17}), (b_{10}, b_{19}) \notin C$, and $(a_{10}, a_{17}), (b_9, b_{18}) \in C$	Left shift to 6
13	$(b_8, b_{17}) \notin C$, and $(a_{10}, a_{17}), (b_9, b_{18}), (b_{10}, b_{19}) \in C$	(Left shift to 3 or 14)
14	$(b_9, b_{18}) \notin C$, and $(a_{10}, a_{17}), (b_8, b_{17}), (b_{10}, b_{19}) \in C$	
15	$(b_{10}, b_{19}) \notin C$, and $(a_{10}, a_{17}), (b_8, b_{17}), (b_9, b_{18}) \in C$	Symmetric to 13
16	$(a_{10}, a_{17}) \notin C$, and $(b_8, b_{17}), (b_9, b_{18}), (b_{10}, b_{19}) \in C$	Left shift to 13

We now turn our attention to the 4-ordered Hamiltonian problem, which requires an extension of Lemma 3.2. Its validity is again verified via a computer search.

Lemma 3.5. *Suppose n is even and $28 \leq n \leq 50$. Then $P(n, 3)$ is 4-ordered Hamiltonian.*

Theorem 3.6. *Suppose $n \geq 7$ is even. If $n = 18$ or $n \geq 24$, then $P(n, 3)$ is 4-ordered Hamiltonian.*

Proof. We use induction on n. Lemmas 3.2 and 3.5 provide the justification for $n \leq 50$. Suppose $n \geq 52$. Let x_1, x_2, x_3, x_4 be the four prescribed vertices in this specific order. Similarly to the proof of Theorem 3.3, there are six consecutive columns that do not contain any one of the prescribed vertices. Again, we may assume these six columns are (a_{11}, b_{11}), (a_{12}, b_{12}), ..., (a_{16}, b_{16}). Now construct the graph G as follows: Delete the vertices in these six columns and add the edges (a_{10}, a_{17}), (b_8, b_{17}), (b_9, b_{18}), (b_{10}, b_{19}). Then G is isomorphic to $P(n - 6, 3)$ and so it is 4-ordered Hamiltonian by the induction hypothesis. So there exists a Hamiltonian cycle C in G containing x_1, x_2, x_3, x_4 in this specific order. We now consider the 16 cases depending on which of the edges (a_{10}, a_{17}), (b_8, b_{17}), (b_9, b_{18}), (b_{10}, b_{19}) belong to C (see Table 1).

We want to extend C to a Hamiltonian cycle in $P(n, 3)$. Figs. 19–24 provide the solution to the first eight cases, each case corresponding to inserting six columns back except for Case 6, which inserts four columns back. (We note that Cases 4 and 8 are not given as their solutions are symmetric to Cases 3 and 7 respectively via obvious reflections.)
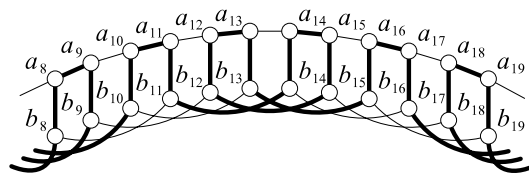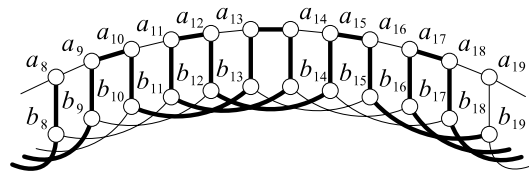
Fig. 20. Solution to Case 2.

Fig. 21. Solution to Case 3.
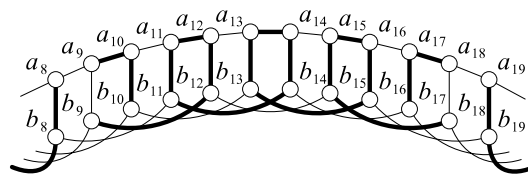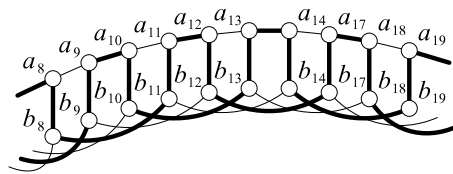

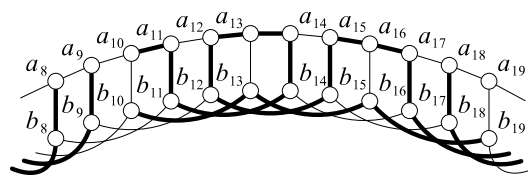**Fig. 22.** Solution to Case 5.

Fig. 23. Solution to Case 6.

Fig. 24. Solution to Case 7.

However, we need to insert the same number of columns that we deleted. Thus we will delete 12 columns instead, and then insert the solutions to these cases as many times as necessary to get 12 more columns (so we insert each case twice except for Case 6, which we insert three times). Since $n \geq 48$, we are still guaranteed to have 12 consecutive columns not containing the prescribed vertices.

Now consider the other eight cases. We observe that if we actually have seven consecutive columns that do not contain any one of the prescribed vertices, then we may assume that they are $(a_{11}, b_{11}), (a_{12}, b_{12}), \ldots, (a_{17}, b_{17})$. Moreover, if we construct G by deleting the first six columns and G' by deleting the last six columns, then G and G' are isomorphic. Indeed, more can be said since the positions of the four prescribed vertices are at the same places in both G and G'. So in G' we have three consecutive columns $(a_{10}, b_{10}), (a_{11}, b_{11}), (a_{18}, b_{18})$. The usual trick is to extend C by "inserting" the six columns between the columns (a_{11}, b_{11}) and (a_{18}, b_{18}). However, we can insert them between (a_{10}, b_{10}) and (a_{11}, b_{11}) as well. Then Case 9 is reduced to Case 7 in this way. Similarly, Case 10 is reduced to Case 1, Case 11 is reduced to Cases 8 and 12 is reduced to Case 6. (These are all shifts to the left. The cases are presented pictorially in Appendix A for easy reference.) Since we delete 12 columns, we want to insert these solutions twice, so we need 13 consecutive columns that do not contain any one of the prescribed vertices. Since $n \geq 52$, this is guaranteed.
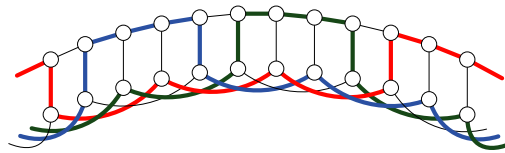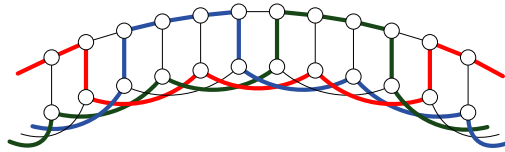
Fig. 25. Solution to Case 13.

Fig. 26. Solution to Case 14.

Now for the remaining four cases we have a construction that is similar to the first eight cases except that it uses all 12 columns that we deleted. The solutions are given in Figs. 25–26 with Case 15 omitted as it is symmetric to Cases 13 and 16 omitted as it is a left shift to Case 13. We note that the left shift is valid as we have 13 consecutive columns that do not contain any one of the prescribed vertices. The proof is now complete. (We note that we can actually eliminate Cases 2 and 5 by left shifting them to Cases 4 and 14, respectively. Moreover Case 13 can be left shifted to either Cases 3 or 14. In fact, Case 7 can be left shifted to either Case 8 or 11; however, Case 11 itself was reduced to Case 8 by left shifting and hence we cannot apply this reduction unless we have 14 consecutive columns that do not contain any one of the prescribed vertices. So Case 7 cannot be eliminated without checking more base cases.) □

So Theorem 3.6 gives an infinite class of 3-regular 4-ordered Hamiltonian graphs, thus answering the open question. It is interesting to note that Cases 13–16 could be avoided in our computer search, that is, for small n checked by the computer program, whenever $P(n, 3)$ is 4-ordered Hamiltonian, we get that for every four vertices and for every i, there exists a Hamiltonian cycle containing these four vertices in the prescribed order such that the edges (a_i, a_{i+7}), (b_{i-2}, b_{i+7}), (b_{i-1}, b_{i+8}), (b_i, b_{i+9}) form one of Cases 1–12. We conjecture that this pattern holds in general. (We note that from the table in the proof of Theorem 3.6, we see that Cases 13, 15 and 16 can be reduced to other cases. So the only question is Case 14.) We now have a complete classification of the 4-ordered and the 4-ordered Hamiltonian properties for $P(n, 3)$ which we will summarize.

Theorem 3.7. *Let $n \geq 7$ be even. Then $P(n, 3)$ is 4-ordered if and only if $n \notin \{7, 9, 12\}$. In addition, $P(n, 3)$ is 4-ordered Hamiltonian if and only if n is even and either $n = 18$ or $n \geq 24$.*

4. Related results

The induction step given in the proof of Theorem 3.6 can be used for other related results. A graph G is *k-ordered Hamiltonian connected* if for any sequence of k distinct vertices $t_1, t_2, \ldots, t_k$ of G, there exists a Hamiltonian path between t_1 and t_k in G containing these k vertices in the specified order. If G is bipartite, then it is *k-ordered Hamiltonian laceable* if for any sequence of k distinct vertices $t_1, t_2, \ldots, t_k$ of G where t_1 and t_2 are in different partite sets, there exists a Hamiltonian path between t_1 and t_k in G containing these k vertices in the specified order. It is known that $P(n, k)$ is bipartite if and only if k is odd and n is even. Indeed, Ng and Schultz [10] posed the following question: Study the existence of small degree k-ordered Hamiltonian-connected graphs. The next result provides an infinite class of small degree 4-ordered Hamiltonian connected graphs as well as 4-ordered Hamiltonian connected graphs; moreover, it extends the result given in [1] for $P(n, 3)$. The induction step is exactly the same as the one given in Theorem 3.6, so the only remaining task is to check the result for $n \leq 51$, which was checked by a computer.

Theorem 4.1. *If $n \geq 7$ and n is even, then $P(n, 3)$ is 4-ordered Hamiltonian laceable if and only if $n \geq 10$. If $n \geq 7$ and n is odd, then $P(n, 3)$ is 4-ordered Hamiltonian connected if and only if $n = 15$ or $n \geq 19$.*

The classification of Hamiltonian connected graphs and Hamiltonian laceable graphs for $P(n, 1)$, $P(n, 2)$, $P(n, 3)$ were done in [1]. Theorem 4.1 extends the classification for $P(n, 3)$ in terms of 4-orderedness. One may ask the same for $P(n, 1)$ and $P(n, 2)$. Recall that $P(n, 1)$ is bipartite if and only if n is even. Consider the vertices given in Fig. 27. Note that vertex 1 and vertex 4 are in different partite sets when n is even. If there is a desired Hamiltonian path P between 1 and 4, then $(1, 4)$, $(1, 3)$, $(2, 4)$ are not in P. So $(1, a)$ and $(4, c)$ are in P. Now $(1, 3)$ is not in P implies $(2, 3)$ and $(3, b)$ are in P. Similarly, $(2, d)$ is in P. Now (a, b) is not in P as otherwise, we will have a path from 1 to 3 without using 2. Similarly (c, d) is not in P. Now, the argument repeats and P has 2 components (the inner n-cycle and a Hamiltonian path on the outer n-cycle), a contradiction. This gives the following result.

Theorem 4.2. *$P(n, 1)$ is neither 4-ordered Hamiltonian connected nor 4-ordered Hamiltonian laceable.*
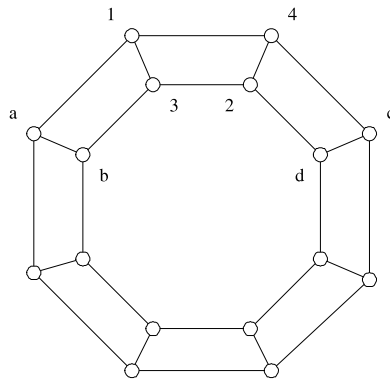
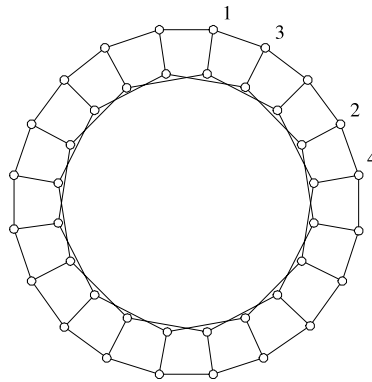**Fig. 27.** $P(n, 1)$ is not 4-ordered Hamiltonian connected/laceable.

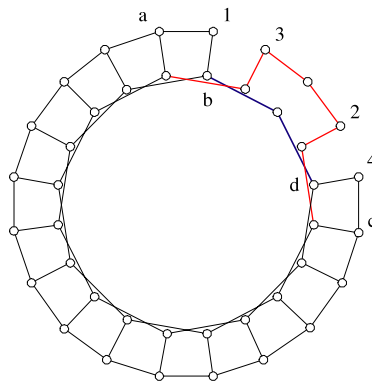**Fig. 28.** 4 prescribed vertices in $P(n, 2)$.

Fig. 29. Forced configuration.

Since $P(n, 2)$ is not bipartite, we only have to consider whether it is 4-ordered Hamiltonian connected. Consider the vertices given in Fig. 28. If there is a desired Hamiltonian path P between 1 and 4, then $(1, 3)$ and $(2, 4)$ are not in P. Hence some edges are forced to be in P as in Fig. 29. Since 1 is an end of P, either $(1, a)$ or $(1, b)$ belongs to P. By the same token, either $(4, c)$ or $(4, d)$ is in P. So there are four cases to consider. The proof is along the same line as the proof of Theorem 3.4 although much simpler. We state the result without proof.

Theorem 4.3. $P(n, 2)$ is not 4-ordered Hamiltonian connected.

5. Conclusion

In this paper we gave a complete classification of the 4-ordered and the 4-ordered Hamiltonian properties for $P(n, 3)$. This contains a solution to an open problem regarding the existence of an infinite class of 3-regular 4-ordered Hamiltonian graphs. Note that the class we gave has only bipartite graphs, so it is natural to ask whether there is an infinite class of
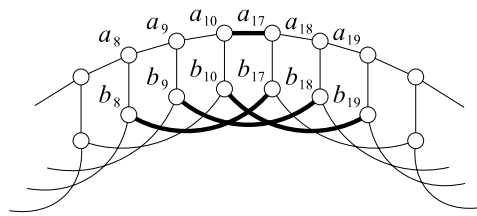
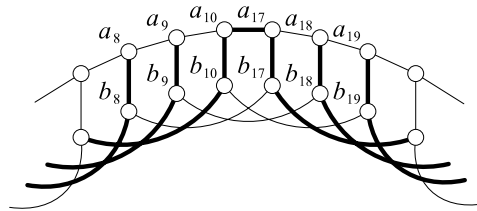**Fig. 30.** Case 1: (a_{10}, a_{17}), (b_8, b_{17}), (b_9, b_{18}), $(b_{10}, b_{19}) \in C$.

Fig. 31. Case 2: (b_8, b_{17}), (b_9, b_{18}), $(b_{10}, b_{19}) \notin C$, and $(a_{10}, a_{17}) \in C$.
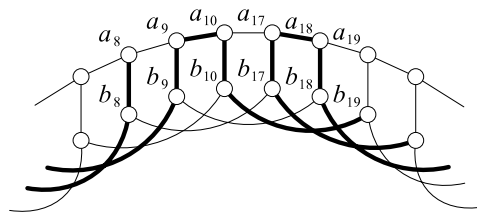

Fig. 32. Case 3: (a_{10}, a_{17}), (b_8, b_{17}), $(b_9, b_{18}) \notin C$, and $(b_{10}, b_{19}) \in C$.

nonbipartite 3-regular 4-ordered Hamiltonian graphs. (Indeed, we conjecture that $P(n, 4)$ will provide a solution. The main difficulty here is to find a good induction step construction so that it is feasible to check all the initial cases.) We also gave an infinite class of 3-regular 4-ordered Hamiltonian connected graphs as well as 3-regular 4-ordered Hamiltonian laceable graphs.

Since part of the proof relies on using a computer, we will now briefly mention the computation involved. All lemmas related to initial cases were checked using a PC with a Pentium dual core processor running Windows XP using codes written in C. More importantly, we will now address how the computer results can be verified independently. The computer verifications contain both positive and negative results. The program and the output files are available at http://www.cs.pu.edu.tw/~lhhsu/FourOrderedForGP/. In terms of the positive results, that is, $P(n, 3)$ is 4-ordered Hamiltonian if $n = 18$ or $n = 24, 26, 28, \ldots, 50$, it is not important how the program finds the Hamiltonian cycles. The output files contain the Hamiltonian cycles that certify the corresponding graph being 4-ordered Hamiltonian. Interestingly, the number of Hamiltonian cycles required for such a certification is not too big. For example, 384 Hamiltonian cycles are enough to certify that $P(50, 3)$ is 4-ordered Hamiltonian. So one can check this positive result using simply the output files. For the negative portion of the computer verification, we note that we have proved a general negative result in Theorem 3.4. As noted in the (non-inductive) proof, the proof works for smaller n as well.

The same program was used to examine $P(n, k)$ for larger k. We end this paper with the following conjecture: Let $k \geq 3$. Then $P(n, k)$ is 4-ordered for sufficiently large n. Indeed, the data suggests that if $k \geq 4$, then $P(n, k)$ is 4-ordered if and only if $n \geq 2k + 2$, $n \neq 3k$ and $n \neq 4k$.

Acknowledgments

Appendix. 16 cases

In this section, we list the 16 cases pictorially for easy reference. The darkened edges are in C. We note that some additional edges are forced to be in C according to whether any of the edges (a_{10}, a_{17}), (b_8, b_{17}), (b_9, b_{18}), (b_{10}, b_{19}) belongs to C; these edges are also darkened. (We are only adding some of these forced edges so that it is evident how one case is a shift from another) (see Figs. 30–45).
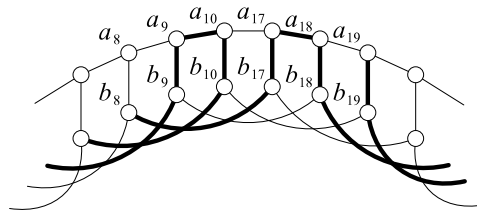
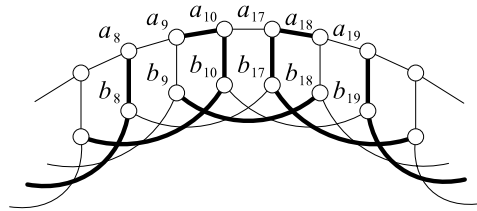**Fig. 33.** Case 4: (a_{10}, a_{17}), (b_9, b_{18}), $(b_{10}, b_{19}) \notin C$, and $(b_8, b_{17}) \in C$.

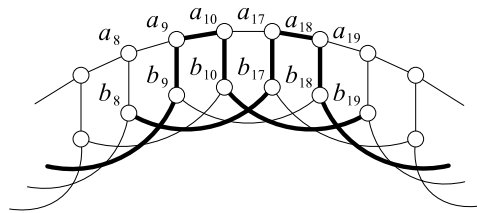**Fig. 34.** Case 5: (a_{10}, a_{17}), (b_8, b_{17}), $(b_{10}, b_{19}) \notin C$, and $(b_9, b_{18}) \in C$.

Fig. 35. Case 6: (a_{10}, a_{17}), $(b_9, b_{18}) \notin C$, and (b_8, b_{17}), $(b_{10}, b_{19}) \in C$.
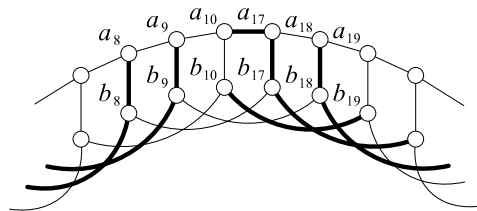

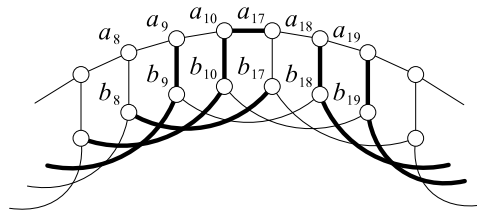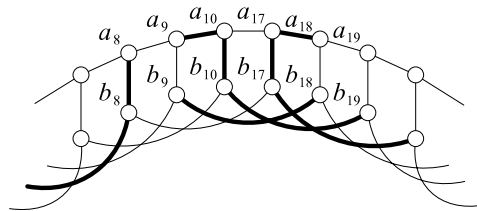**Fig. 36.** Case 7: (b_8, b_{17}), $(b_9, b_{18}) \notin C$, and (a_{10}, a_{17}), $(b_{10}, b_{19}) \in C$.

Fig. 37. Case 8: (b_9, b_{18}), $(b_{10}, b_{19}) \notin C$, and (a_{10}, a_{17}), $(b_8, b_{17}) \in C$.

Fig. 38. Case 9: (a_{10}, a_{17}), $(b_8, b_{17}) \notin C$, and (b_9, b_{18}), $(b_{10}, b_{19}) \in C$.
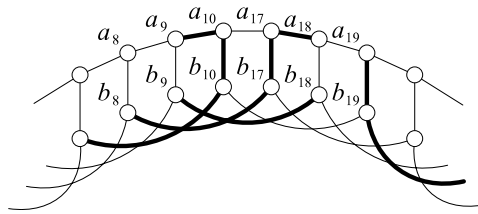
Fig. 39. Case 10: (a_{10}, a_{17}), $(b_{10}, b_{19}) \notin C$, and (b_8, b_{17}), $(b_9, b_{18}) \in C$.
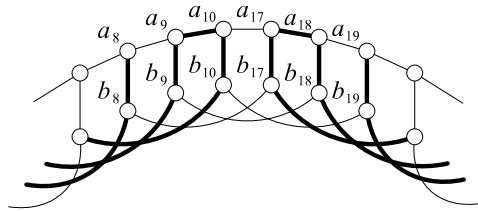

Fig. 40. Case 11: (a_{10}, a_{17}), (b_8, b_{17}), (b_9, b_{18}), $(b_{10}, b_{19}) \notin C$.
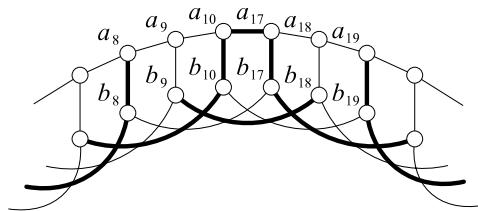

Fig. 41. Case 12: (b_8, b_{17}), $(b_{10}, b_{19}) \notin C$, and (a_{10}, a_{17}), $(b_9, b_{18}) \in C$.
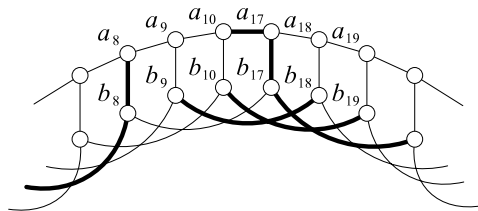

Fig. 42. Case 13: $(b_8, b_{17}) \notin C$, and (a_{10}, a_{17}), (b_9, b_{18}), $(b_{10}, b_{19}) \in C$.
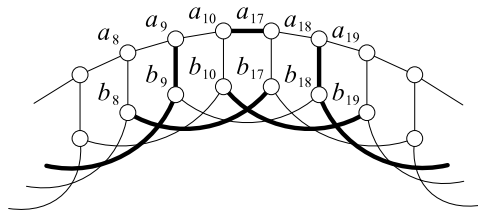

Fig. 43. Case 14: $(b_9, b_{18}) \notin C$, and (a_{10}, a_{17}), (b_8, b_{17}), $(b_{10}, b_{19}) \in C$.
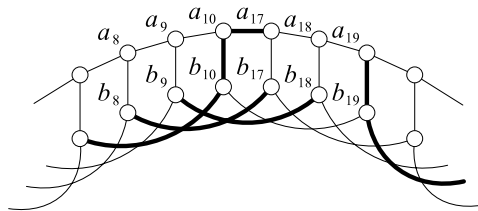

Fig. 44. Case 15: $(b_{10}, b_{19}) \notin C$, and (a_{10}, a_{17}), (b_8, b_{17}), $(b_9, b_{18}) \in C$.
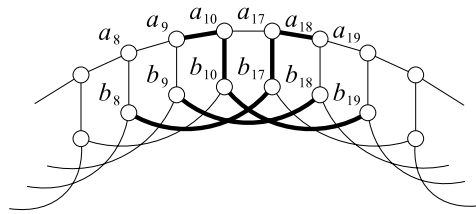
Fig. 45. Case 16: $(a_{10}, a_{17}) \notin C$, and (b_8, b_{17}), (b_9, b_{18}), $(b_{10}, b_{19}) \in C$.

References

[1] B. Alspach, J. Liu, On the Hamilton connectivity of generalized Petersen graphs, Discrete Mathematics 309 (2009) 5461–5473.
[2] M. Behzad, P. Hatami, E.S. Mahmoodian, Minimum vertex covers in the generalized Petersen graphs $p(n, 2)$, Bulletin of the Institute of Combinatorics and its Applications 56 (2009) 98–102.
[3] G. Chen, R.J. Gould, F. Pfender, New conditions for ordered k-Hamiltonian graphs, Ars Combinatoria 70 (2003) 245–255.
[4] H. Cho, L. Hsu, Generalized honeycomb torus, Information Processing Letters 86 (2003) 185–190.
[5] J.R. Faundree, R.J. Faundree, Forbidden subgraphs that imply k-ordered and k-ordered Hamiltonian, Discrete Mathematics 243 (2002) 91–108.
[6] J.R. Faundree, R.J. Gould, F. Pfender, A. Wolf, On k-ordered bipartite graphs, Electronic Journal of Combinatorics 10 (2003) R11.
[7] R.J. Faundree, Survey of results on k-ordered graphs, Discrete Mathematics 229 (2001) 73–87.
[8] L.H. Hsu, C.K. Lin, Graph Theory and Interconnection Networks, CRC Press, 2008.
[9] K. Mészáros, On 3-regular 4-ordered graphs, Discrete Mathematics 308 (2008) 2149–2155.
[10] L. Ng, M. Schultz, k-ordered Hamiltonian graphs, Journal of Graph Theory 24 (1997) 45–47.
[11] D.R. Silabanm, A. Parestu, B.N. Herawati, K.A. Sugeng, Slamin, Vertex-magic total labelings of union of generalized Petersen graphs and union of special circulant graphs, Journal of Combinatorial Mathematics and Combinatorial Computing 71 (2009) 201–207.
[12] C. Tong, X. Lin, Y. Yang, M. Luo, 2-rainbow domination in generalized Petersen graphs $P(n, 2)$, Discrete Applied Mathematics 157 (2009) 1932–1937.
[13] M. Tsai, T.-H. Tsai, J.M. Tan, L.-H. Hsu, On 4-ordered 3-regular graphs (manuscript), 2009.
[14] G. Xu, 2-rainbow domination in generalized Petersen graphs $P(n, 3)$, Discrete Applied Mathematics 157 (2009) 2570–2573.