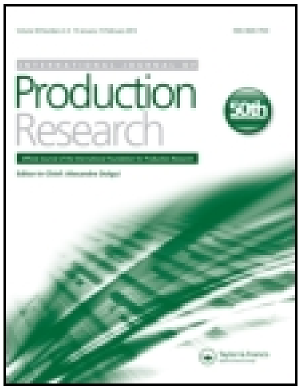

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The density control chart: a general approach for constructing a single chart for simultaneously monitoring multiple parameters

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The majority of the existing literature on simultaneous control charts, i.e. control charting mechanisms that monitor multiple population parameters such as mean and variance on a single chart, assume that the process is normally distributed. In order to adjust and maintain the overall type-I error probability, these existing charts rely largely on the property that the sample mean and sample variance are independent under the normality assumption. Furthermore, the existing charting procedures cannot be readily extended to non-normal processes. In this article, we propose and study a general charting mechanism which can be used to construct simultaneous control charts for normal and non-normal processes. The proposed control chart, which we call the density control chart, is essentially based on the premise that if a sample of observations is from an in-control process, then another sample of observations is no less likely to be also from the in-control process if the likelihood of the latter is no less than the likelihood of the former. The density control chart is developed for normal and non-normal processes where the distribution of the plotting statistic of the density control chart can be explicitly derived. Real examples are given and discussed in these cases. We also discuss how the density control chart can be constructed in cases when the distribution of the plotting statistic cannot be determined. A discussion of potential future research is also given.

Keywords: hypothesis testing; likelihood function; multiple parameters; non-normal processes; Shewhart control chart; simultaneous control chart

1. Introduction

The statistical control chart is perhaps the most widely used tool among the set of techniques, such as the Pareto chart and cause-and-effect diagram, that form the core of statistical process control (SPC). First developed by Shewhart (1925), the use of control charts to monitor process quality has become a common practice in many industrial applications.

The development of control charts generally assumes that the quality characteristic to be monitored, be it univariate or multivariate, follows a certain distribution and that various population parameters of that distribution need to be monitored. For instance, under univariate normal processes, the $\overline{X}$- and S-charts are specifically used to monitor the process mean and process standard deviation, respectively.

In the literature, there are two primary approaches for developing control charting schemes for simultaneously monitoring the process mean and variance. One approach is to combine two control charts, one for monitoring the process mean and the other for monitoring the process variance. These charts include (i) the combination of Shewhart $\bar{X}$ (or X) and R (MR or S) charts, see, e.g., Jones and Case (1981), Rahim (1989), Saniga (1989, 1991), Costa (1993, 1998), Rahim and Costa (2000), Costa and Rahim (2000, 2002), Ohta *et al*. (2002), De Magalhães and Moura Neto (2005), He and Grigoryan (2006); (ii) the combination of EWMA-type charts, see, e.g., Sweet (1986), Domangue and Patch (1991), Gan (1995), Reynolds and Stumbos (2001); (iii) the combination of CUSUM-type charts, see, e.g., Takemoto *et al*. (2003); and (iv) the combination of the generally weighted moving average (GWMA)-type charts, see, e.g., Sheu *et al*. (2009), which developed a combined scheme consisting of a two-side GWMA mean chart and a two-side GWMA variance chart for monitoring the process mean and variance, respectively.

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The other approach is to use omnibus-type test statistics, i.e. combining charting statistics designed for different parameters into a single control chart for simultaneously monitoring both process mean and variance. Cheng and Thaga (2006) gave an overview of the existing studies which use a single chart to simultaneously monitor process parameters. They summarised the control charts that have been developed into two categories. One is to simultaneously plot two quality characteristics on the same chart, called the simultaneous control chart. Examples of the simultaneous control chart include the Shewhart-type charts (White and Schroeder 1987, Spiring and Cheng 1998); the EWMA-type chart (Domangue and Patch 1991); and the synthetic control chart with two-stage testing (Costa *et al.* 2009). The other is to use a plotting variable to represent the process mean and variance, called the single control chart. Existing control charts that fall into the single control chart category include (i) the Shewhart-type charts (Cheng and Li 1993, Chao and Cheng 1996, Chen and Cheng 1998, Yeh and Lin 2002, Hawkins and Zamba 2005, and Hawkins and Deng 2009); (ii) the EWMA-type charts (Xie 1999, Gan 2000 and Chen *et al.* 2001, 2004); and (iii) the CUSUM-type charts (Thaga 2003 and Wu and Tian 2005). Other charts that also rely on a single plotting statistic include the EWMA control chart based on a non-central chi-square statistic (called NCS chart) proposed by Costa and Rahim (2004), a self-starting control chart based on the likelihood ratio test (LRT), the EWMA procedure (SSELR) proposed by Li *et al.* (2010), an EWMA control chart based on the generalised likelihood ratio test (GLRT) proposed by Zhang *et al.* (2010), and a control chart which integrates the EWMA procedure with the GLRT statistic proposed by Zhou *et al.* (2010).

In addition, the semicircle control chart (Chao and Cheng 1996) plots two statistics against each other on a semicircle, as indicated by the positions of the mean and the standard deviation. When an out-of-control signal occurs, the chart would show which parameter has shifted from its target value. However, a disadvantage of this chart is that it loses track of the time sequence of the samples. Hawkins and Deng (2009) developed a single chart for simultaneously monitoring the mean and variance of a normal process based on the GLRT for testing $H_0: \mu = \mu_0$ and $\sigma = \sigma_0$. The proposed chart, which they called the GLR chart, was shown to outperform the conventional $\overline{X}$- and S-charts.

Zhang *et al.* (2010) mentioned that all of the aforementioned single charts are shown to have satisfactory performance compared with the combination-type charts. Nevertheless, most of the existing control charts under univariate normal processes, whether a single chart or two separate charts are used, make explicit use of the property that the sample mean and sample variance are independent statistics, thus making it relatively easy to control the overall type-I error probability. However, in many applications where the process is not normal or when other non-independent statistics are used even for normal processes, it is sometimes difficult to control the overall type-I error probability. Moreover, it is usually non-trivial to extend a control charting mechanism designed for a specific underlying distribution to other types of distribution. Our main motivation in this article is to develop a general control charting mechanism which can be used to combine the monitoring of multiple parameters into a single chart. The proposed charting mechanism is general in that it can be applied to normal as well as non-normal processes, while maintaining the overall type-I error probability. In essence, the construction of the proposed chart is based on recognising that when the process is in control, the quality characteristic to be monitored follows a specified probability density function. The proposed chart is then derived from the likelihood for testing the hypothesis that the probability density function of a given sample is equal to the specified one. We will call the proposed chart the density control chart.

The rest of the article is organised as follows. We first discuss in detail the general methodology of the density control chart. A density control chart is specifically derived for normal processes. Hoping to demonstrate the flexibility of the proposed density chart, we also apply the density control charting mechanism to a negative exponential distribution which has been used to model certain processes commonly found in the semiconductor manufacturing industry. Two real examples are given for the normal and negative exponential processes. There are cases in which the distribution of the plotting statistic used in the density chart is too complex to derive. We propose using simulations to approximate the control limit in these cases. This is further discussed using a Gamma and an exponential distribution as examples. We conclude the article with some discussion of potential future research.

2. The general methodology

Let X denote the quality characteristic of interest. When the process is in control, we assume that X follows a certain distribution whose density function is denoted as $f(x; \theta_1, \theta_2, \ldots, \theta_p)$, where $\theta_1, \theta_2, \ldots, \theta_p$ are population parameters. These population parameters are assumed to be either known or can be reasonably estimated by $\widehat{\theta_1}, \widehat{\theta_2}, \ldots, \widehat{\theta_p}$

using data obtained from Phase I control when the process was in control. When the Phase II monitoring begins, independent samples of size n each are repeatedly taken from the process.

When the process is in-control, assuming that the true in-control parameter values are $\theta_1, \theta_2, \ldots, \theta_p$. For any given sample of size $n, x_1, x_2, \ldots, x_n$ from the in-control process, the joint likelihood is $L(x_1, x_2, \ldots, x_n;\ \theta_1, \theta_2, \ldots, \theta_p) = \prod_{i=1}^{n} f(x_i;\ \theta_1, \theta_2, \ldots, \theta_p)$. Suppose that $x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}$ are considered to be a sample from the in-control process, and $x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}$ are another sample. If $L(x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)};\ \theta_1, \theta_2, \ldots, \theta_p) \leq L(x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)};\ \theta_1, \theta_2, \ldots, \theta_p)$, then from the perspective of testing the hypothesis that H_0: the process is in control, the sample $(x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)})$ provides an even stronger evidence of supporting that the process is in control. In other words, if $(x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)})$ comes from the in-control process, there is no reason not to support the same conclusion when $(x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)})$ is observed. This line of thinking leads us to consider

$$L(X_1, X_2, \ldots, X_n;\ \theta_1, \theta_2, \ldots, \theta_p)$$

as the charting mechanism. Specifically, given a random sample $x_1, x_2, \ldots, x_n$, one calculates the joint likelihood $L(x_1, x_2, \ldots, x_n;\ \theta_1, \theta_2, \ldots, \theta_p)$ and plots it against the sampling sequence. The control chart will signal when the plotting statistic falls below the lower control limit (LCL), where the LCL is to be chosen such that

$$P(L(X_1, X_2, \ldots, X_n;\ \theta_1, \theta_2, \ldots, \theta_p) \leq \mathrm{LCL}) = \alpha, \tag{1}$$

where α is a pre-determined probability of type-I error. For example, in a typical 3-σ limit $\overline{X}$-chart for normal processes, the α is about 0.0027. Note that in the case when $\theta_1, \theta_2, \ldots, \theta_p$ are unknown and have to be estimated using Phase I data, the parameter estimates $\widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_p$ are treated as known in Phase II monitoring. Therefore, we will use $L(X_1, X_2, \ldots, X_n;\ \widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_p)$ instead of $L(X_1, X_2, \ldots, X_n; \theta_1, \theta_2, \ldots, \theta_p)$. Subsequently, in the case when the parameters have to be estimated from Phase I data, we will continue to use $\widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_p$ to denote the true parameter values in Phase II monitoring, if no confusion is caused.

We will call the proposed chart the density control chart since it relies on calculating the joint density of the sample observations. The density control charting mechanism is quite general in that it can be applied to any density function $f(x; \theta_1, \theta_2, \ldots, \theta_p)$. However, in order to explicitly identify the LCL, one needs to know the distribution of $L(X_1, X_2, \ldots, X_n;\ \theta_1, \theta_2, \ldots, \theta_p)$ (or $L(X_1, X_2, \ldots, X_n;\ \widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_p)$). In some cases, it is possible to derive the distribution of the joint density. In the subsequent sections, we will discuss two such cases. One is the most commonly used normal process and the other is based on a negative exponential distribution which has been used to model certain processes in semiconductor manufacturing applications. As pointed out by one referee, the $L(X_1, X_2, \ldots, X_n;\ \widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_p)$ is only an estimate of the true $L(X_1, X_2, \ldots, X_n;\ \theta_1, \theta_2, \ldots, \theta_p)$, and that the statement in (1) is only approximately true when the true parameters are replaced by their corresponding estimates. Further, the likelihood estimate $L(X_1, X_2, \ldots, X_n;\ \widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_p)$ could be poor if the parameter estimates $\widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_p$ obtained from Phase I data are poor, which will affect the performance of the density control chart. For a recent review of the effects of parameter estimation on the control chart properties, please see Jensen *et al.* (2006).

3. The density control chart under a normal process

We assume that when the process is in control, the quality characteristic X is distributed as $N(\mu_0, \sigma_0^2)$, where μ_0 and σ_0 are unknown. We also assume that m samples of size n each are available for estimating μ_0 and σ_0 and that $\widehat{\mu}_0 = \frac{1}{m}\sum_{j=1}^{m} \overline{X}_j$ and $\widehat{\sigma}_0^2 = \frac{1}{m}\sum_{j=1}^{m} s_j^2$, where $\overline{X}_j$ and s_j^2 are, respectively, the sample mean and sample variance of the jth sample, $j = 1, 2, \ldots, m$.

When the Phase II monitoring begins, it is customary, as in most of the existing literature, to assume that the true in-control parameters μ_0 and σ_0^2 are equal to $\widehat{\mu}_0$ and $\widehat{\sigma}_0^2$, respectively. For each sample of size n, $x_1, x_2, \ldots, x_n$, one calculates

$$L(x_1, x_2, \ldots, x_n;\ \widehat{\mu}_0, \widehat{\sigma}_0^2) = \frac{1}{(2\pi)^{n/2}(\widehat{\sigma}_0^2)^{n/2}} \exp\left\{\frac{-\sum_{i=1}^{n}(x_i - \widehat{\mu}_0)^2}{2\widehat{\sigma}_0^2}\right\}$$

and plots $L(x_1, x_2, \ldots, x_n;\ \widehat{\mu}_0, \widehat{\sigma}_0^2)$ against the sampling sequence. The chart signals when

$$L(x_1, x_2, \ldots, x_n;\ \widehat{\mu}_0, \widehat{\sigma}_0^2) \leq \mathrm{LCL}. \tag{2}$$

The LCL $= l(\widehat{\mu}_0, \widehat{\sigma}_0^2)$ is chosen such that

$$P_{\widehat{\mu}_0, \widehat{\sigma}_0^2}(L(X_1, X_2, \ldots, X_n; \widehat{\mu}_0, \widehat{\sigma}_0^2) \geq l(\widehat{\mu}_0, \widehat{\sigma}_0^2)) = 1 - \alpha, \tag{3}$$

where α is pre-determined.

Note that given $\widehat{\mu}_0$ and $\widehat{\sigma}_0^2$, X_i's are independent and identically distributed (i.i.d.) as $N(\widehat{\mu}_0, \widehat{\sigma}_0^2)$ and therefore $\sum_{i=1}^n (x_i - \widehat{\mu}_0)^2 / \widehat{\sigma}_0^2$ is distributed as χ_n^2, a chi-square distribution with n degrees of freedom. Straightforward calculations show that the restriction in (3) leads to

$$l(\widehat{\mu}_0, \widehat{\sigma}_0^2) = \frac{1}{(2\pi)^{n/2} (\widehat{\sigma}_0^2)^{n/2}} \exp\left\{ \frac{-\chi_{n,1-\alpha}^2}{2} \right\}, \tag{4}$$

where $P(\chi_n^2 \leq \chi_{n,1-\alpha}^2) = 1 - \alpha$. It should also be noted that under the normality assumption,

$$L(x_1, x_2, \ldots, x_n; \widehat{\mu}_0, \widehat{\sigma}_0^2) \leq \text{LCL}$$

if and only if

$$\frac{\sum_{i=1}^n (x_i - \widehat{\mu}_0)^2}{\widehat{\sigma}_0^2} \geq \chi_{n,1-\alpha}^2. \tag{5}$$

Therefore, one can use the plotting statistic $\sum_{i=1}^n (x_i - \widehat{\mu}_0)^2 / \widehat{\sigma}_0^2$ and the chart signals if it exceeds $\chi_{n,1-\alpha}^2$. Note that although the plotting statistic does depend on $\widehat{\mu}_0$ and $\widehat{\sigma}_0^2$, the control limit does not.

Example 1: A vane operating example

The process control of vane operating, which is an important functional parameter of a component part in jet aircraft engines, has been studied by Montgomery *et al.* (2004) who used statistical control charts to assess the statistical stability of this manufacturing process. Twenty samples each of size 5 were collected in Phase I and used to construct the $\overline{X}$- and R-charts. The $\overline{X}$-chart indicated that samples 6, 8, 11 and 19 were out of control, while the R-chart indicated that sample 9 was out of control. Assuming that assignable causes can be identified and corrected for these out-of-control samples, updated $\overline{X}$- and R-charts were calculated using the 15 remaining samples, and no additional samples were found to be out of control. For Phase II monitoring based on the 15 in-control Phase I samples, the upper and lower control limits are 36.1 and 30.33, respectively, for the $\bar{X}$- chart, and 10.57 and 0, respectively, for the R- chart.

We use this example to demonstrate how the density chart can be used in real life applications. For ease of presentation, we construct the density chart based on the logarithm of the likelihood function. Based on the 15 Phase I samples, $\widehat{\mu}_0$ and $\widehat{\sigma}_0^2$ were equal to 33.2133 and 3.3595, respectively. For Phase II monitoring, we generated 20 new samples, each of size 5. Of the 20 samples generated, sample 3 was generated from $N(\widehat{\mu}_0 + 1.75\widehat{\sigma}_0, \widehat{\sigma}_0^2)$, sample 10 from $N(\widehat{\mu}_0, (2\widehat{\sigma}_0)^2)$, and sample 18 from $N(\widehat{\mu}_0 + 1.5\widehat{\sigma}_0, (1.5\widehat{\sigma}_0)^2)$. The other 17 samples were generated from the in-control process $N(\widehat{\mu}_0, \widehat{\sigma}_0^2)$. The $\overline{X}$- and R-charts for 20 newly generated samples are shown in Figure 1. As can be seen from Figure 1, only sample 3 was out of control on the $\overline{X}$- chart and no sample was out of control on the R-chart.

Setting the type-I error probability equal to 0.0027, the corresponding density control chart calculates, for each sample $x_1, x_2, \ldots, x_5$, the plotting statistic

$$\ln\left(0.0004885 \times \exp\left\{ \frac{-\sum_{i=1}^5 (x_i - 33.2133)^2}{2 \times 3.2595} \right\} \right),$$

and the chart signals if the plotting statistic falls below

$$\text{LCL} = -\frac{5}{2} ln(2\pi\widehat{\sigma}_0^2) - \frac{1}{2}\chi_{5,0.9973}^2 = -16.7267.$$

The density chart of the 20 generated samples is shown in Figure 2, which shows that samples 3, 10, and 18 were out of control. It is interesting to point out that samples 10 and 18 were out-of-control in the density chart but not in the $\overline{X}$- and R-charts.

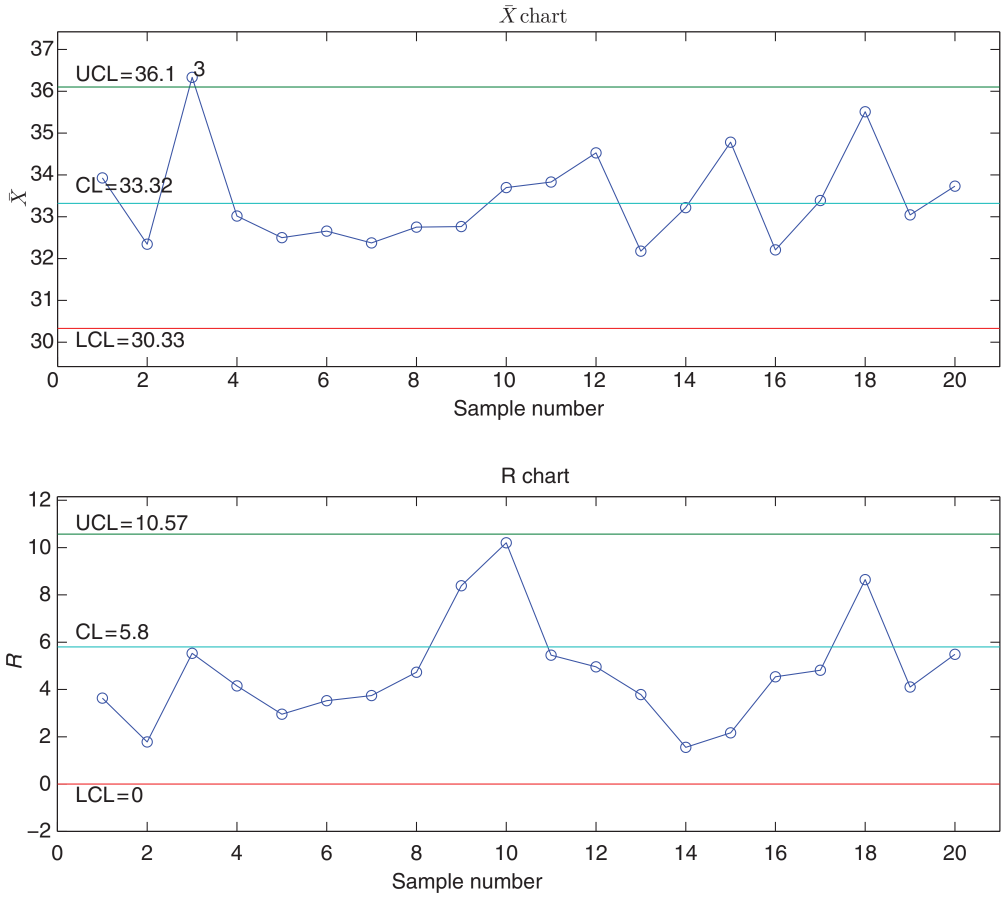

Figure 1. The $\bar{X}$- and R- charts of the vane operating example for 20 new generated samples.

In the case of normal processes, one can actually use the statistic

$$\frac{\sum_{i=1}^{n}(x_i-\widehat{\mu}_0)^2}{\widehat{\sigma}_0^2}$$

for diagnosing whether the mean or the variance or both are out of control. Rewrite the statistic as

$$\frac{\sum_{i=1}^{n}(x_i-\overline{X}+\overline{X}-\widehat{\mu}_0)^2}{\widehat{\sigma}_0^2}=\frac{(n-1)s^2}{\widehat{\sigma}_0^2}+\frac{n(\overline{X}-\widehat{\mu}_0)^2}{\widehat{\sigma}_0^2}, \tag{6}$$

where $\overline{X}=\sum_{i=1}^{n}x_i/n$ and $s^2=\sum_{i=1}^{n}(x_i-\overline{X})^2/n-1$. Given $\widehat{\mu}_0$, $\widehat{\sigma}_0^2$ and α, the region

$$\left\{(\overline{X},s^2):\frac{(n-1)s^2}{\widehat{\sigma}_0^2}+\frac{n(\overline{X}-\widehat{\mu}_0)^2}{\widehat{\sigma}_0^2}\le\chi^2_{n,1-\alpha}\right\} \tag{7}$$

can be represented by a semi-circle whose center is at $\overline{X}=\mu_0$ and $S^2=0$ as shown in Figure 3.

Any sample which produces $(\overline{X},s^2)$ that lies outside of the semi-circle indicates that the sample is out of control. If s^2 is closer to 0 while $\overline{X}$ is away from $\widehat{\mu}_0$, this is an indication that there is a possible mean shift. On the other hand, if $\overline{X}$ is closer to $\widehat{\mu}_0$ while s^2 is far away from 0, it is an indication that the variance has changed. As can be seen from Figure 3, sample 3 was likely the result of an upward mean shift and sample 10 was mostly due to an increase in variance. Only sample 18 was likely due to changes both in mean and variance. Note that the semi-circle discussed here is very similar to that studied in Chao and Cheng (1996). They will be identical if μ_0 and σ_0 are assumed to be known. However, our proposed density chart has a much broader appeal as it can be readily applied to other non-normal distributions.

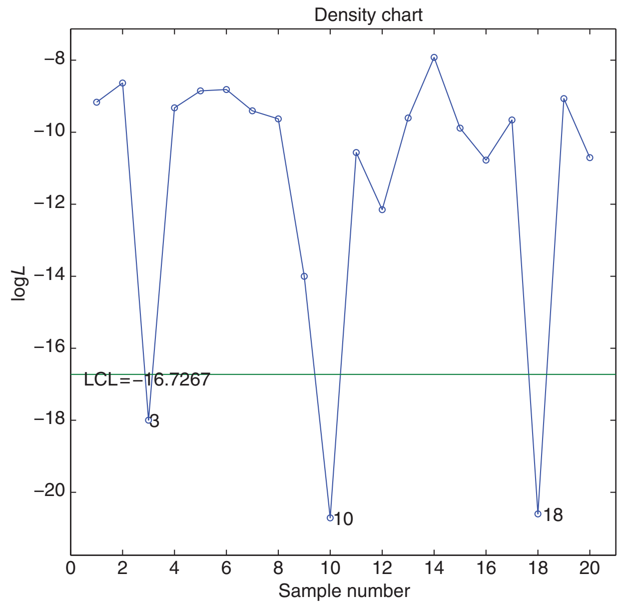

Figure 2. The density control chart of the vane operating example for 20 new generated samples.

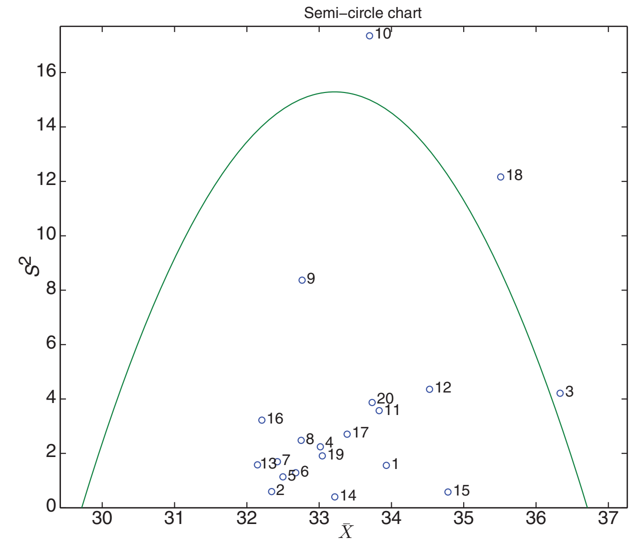

Figure 3. The semi-circle chart of the vane operating example for 20 new generated samples.

In a recent article, Hawkins and Deng (2009) proposed a simultaneous chart that is constructed based on the generalised likelihood ratio test for testing the hypothesis that H_0: $\mu = \mu_0$ and $\sigma = \sigma_0$. The plotting statistic to be calculated for each sample, assuming μ_0 and σ_0 are known under H_0, is given as

$$G = U^2 + W - nln(W) + nln(n) - n,$$

where

$$U = \frac{\sqrt{n}(\overline{X} - \mu_0)}{\sigma_0},$$

$$W = \frac{(n-1)s^2}{\sigma_0^2}.$$

They showed that the proposed chart, which they named the GLR chart, compared favourably to the conventional $\overline{X}$- and S-charts. Note that if the parameters μ_0 and σ_0 are replaced by the estimates obtained from the Phase I data, then the test statistic proposed in (6) for the density control chart is equal to $U^2 + W$. However, the two plotting statistics, (6) and G, are intrinsically different. For any given sample in Phase II, the plotting statistic of the density control chart under normality is essentially the likelihood function, while the GLR test based G, on the other hand, calculates the generalised likelihood ratio for testing $H_0 : \mu = \mu_0$ and $\sigma = \sigma_0$. In both plotting statistics, the mean and standard deviation are either assumed known or can be estimated based on Phase I data.

We compare the performance of the density control chart with that of the GLR chart. We use the same data set as in Example 1 and treat the in-control process as having a normal distribution with $\mu_0 = 33.2133$ and $\sigma_0^2 = 3.3595$. The out-of-control process is considered to be normally distributed with $\mu = \mu_0 + \delta\ \sigma_0$ and $\sigma = \lambda\,\sigma_0$ $(\lambda > 0)$. Numerous combinations of (δ, λ) are considered. The performance measure is based on the average run length (ARL), where the run length is equal to the number of samples needed before the first out-of-control signal shows up on a control chart. The in-control ARL is approximately equal to 370.

Note that for the density chart, the ARL can be calculated as

$$ARL = \frac{1}{p},$$

where

$$p = P\left(\chi^2_{n,\frac{n\delta^2}{\lambda^2}} \geq \frac{\chi^2_{n,1-\alpha}}{\lambda^2}\right), \tag{8}$$

where $\chi^2_{k,a}$ denotes a non-central chi-square distribution with k degrees of freedom and non-centrality parameter a and $\chi^2_{n,1-\alpha}$ is the $1-\alpha$ quantile of a central chi-square distribution with n degrees of freedom. The results of the comparison are summarised in Table 1 and Table 2 for $n=5$ and 10, respectively.

As can be seen from Tables 1 and 2, the density control chart is not effective in detecting a decrease in variance (i.e. $\lambda<1$). It is also less effective than the GLR chart in detecting small shifts in mean only (i.e. $\lambda=0$, $1<\delta\leq 1.25$). On the other hand, when there is a small increase in variance only ($1<\lambda\leq 2$, $\delta=0$) or a small increase in variance and a small shift in mean ($1<\lambda\leq 2$, $0<\delta\leq 1$), the proposed density chart is more effective than the GLR chart. It should be noted that in most industrial quality control applications, it is more important that a control chart can effectively detect quality deterioration as characterised by a shift in the mean ($\delta>0$) or an increase in variance ($\lambda>1$) or both. Therefore, as far as the performance under normal process is concerned, the density control chart represents a viable alternative to the existing control charts for simultaneously monitoring process mean and variance. Furthermore, while the existing charts are limited to normal processes, the density control chart can be easily extended to non-normal processes (see Sections 4 and 5 for detailed discussion).

As mentioned earlier, replacing the parameters in the likelihood function by their estimates obtained from Phase I data is likely to affect the performance of the density control chart. Under normal processes, here we examine via simulations the effect of Phase I estimates on the performance of the density chart for $n=5$, and various total numbers of Phase I samples (m), assuming that $\mu_0=0$ and $\sigma_0=1$. Given m, m Phase I samples of size 5 each were first generated, and $\widehat{\mu}_0$ and $\widehat{\sigma}_0$ were calculated. These estimates were then used to evaluate the ARL. Such a simulation was then repeated 50,000 times, resulting in 50,000 ARLs, and the average of these 50,000 ARLs was calculated and tabulated.

Table 1. The ARLs of the density control chart (DCC) and the GLR chart when $n=5$ ($\mu=\mu_0+\delta\sigma_0$, $\sigma=\lambda\sigma_0$).

$n=5$		λ									
δ		0.87	1.00	1.15	1.52	2.00	2.15	2.30	2.55	2.75	3.00
0.00	DCC	4713.61	**370.37**	58.25	6.14	2.11	1.79	1.58	1.37	1.27	1.18
	GLR	302.65	**370.38**	242.13	20.80	3.95	2.99	2.41	1.86	1.61	1.42
0.25	DCC	2222.05	239.33	45.54	5.71	2.07	1.77	1.57	1.36	1.26	1.18
	GLR	225.47	236.30	142.30	17.41	3.79	2.90	2.36	1.84	1.60	1.41
0.50	DCC	495.44	91.75	25.35	4.71	1.96	1.70	1.52	1.34	1.25	1.17
	GLR	101.06	87.09	52.34	11.43	3.37	2.67	2.23	1.78	1.56	1.39
0.75	DCC	104.93	31.92	12.69	3.63	1.81	1.60	1.46	1.31	1.23	1.16
	GLR	35.04	28.91	19.61	7.02	2.85	2.37	2.04	1.69	1.51	1.36
1.00	DCC	26.41	12.10	6.54	2.76	1.65	1.50	1.39	1.27	1.20	1.14
	GLR	12.12	10.67	8.40	4.42	2.38	2.07	1.84	1.58	1.44	1.32
1.25	DCC	8.51	5.36	3.69	2.13	1.49	1.39	1.31	1.22	1.17	1.13
	GLR	4.90	4.71	4.21	2.95	1.99	1.80	1.66	1.48	1.37	1.28
1.50	DCC	3.59	2.84	2.33	1.71	1.36	1.30	1.25	1.18	1.14	1.11
	GLR	2.46	2.53	2.48	2.13	1.69	1.58	1.50	1.38	1.30	1.23
1.75	DCC	1.97	1.80	1.66	1.43	1.25	1.22	1.19	1.14	1.12	1.09
	GLR	1.56	1.65	1.69	1.64	1.47	1.41	1.36	1.29	1.24	1.19
2.00	DCC	1.36	1.34	1.31	1.25	1.17	1.15	1.14	1.11	1.09	1.07
	GLR	1.20	1.26	1.32	1.36	1.31	1.29	1.26	1.22	1.19	1.15
2.25	DCC	1.12	1.13	1.14	1.14	1.11	1.10	1.10	1.08	1.07	1.06
	GLR	1.06	1.10	1.13	1.20	1.20	1.19	1.18	1.16	1.14	1.12
2.50	DCC	1.03	1.04	1.05	1.07	1.07	1.07	1.06	1.06	1.05	1.04
	GLR	1.01	1.03	1.05	1.10	1.12	1.12	1.12	1.11	1.11	1.09
2.75	DCC	1.01	1.01	1.02	1.03	1.04	1.04	1.04	1.04	1.04	1.03
	GLR	1.00	1.01	1.02	1.05	1.07	1.08	1.08	1.08	1.08	1.07
3.00	DCC	1.00	1.00	1.01	1.01	1.02	1.03	1.03	1.03	1.03	1.02
	GLR	1.00	1.00	1.00	1.02	1.04	1.05	1.05	1.05	1.05	1.05

Table 2. The ARLs of the density control chart (DCC) and the GLR chart when $n = 10$ ($\mu = \mu_0 + \delta\sigma_0$, $\sigma = \lambda\sigma_0$).

$n = 10$		λ									
δ		0.87	1.00	1.15	1.52	2.00	2.15	2.30	2.55	2.75	3.00
0.00	DCC	9909.26	**370.37**	38.19	3.23	1.33	1.20	1.13	1.06	1.04	1.02
	GLR	222.01	**370.59**	141.64	6.85	1.73	1.45	1.29	1.15	1.09	1.05
0.25	DCC	3644.81	213.03	28.51	3.01	1.31	1.20	1.12	1.06	1.04	1.02
	GLR	114.84	131.64	59.49	5.78	1.69	1.43	1.28	1.14	1.08	1.05
0.50	DCC	485.09	63.13	14.30	2.51	1.27	1.17	1.11	1.05	1.03	1.02
	GLR	27.18	25.64	15.80	3.91	1.56	1.36	1.24	1.13	1.08	1.04
0.75	DCC	63.26	17.14	6.50	1.99	1.21	1.14	1.09	1.05	1.03	1.01
	GLR	6.64	6.58	5.30	2.55	1.41	1.28	1.19	1.10	1.06	1.04
1.00	DCC	11.68	5.62	3.22	1.59	1.15	1.10	1.07	1.04	1.02	1.01
	GLR	2.40	2.54	2.42	1.78	1.28	1.20	1.14	1.08	1.05	1.03
1.25	DCC	3.45	2.47	1.89	1.32	1.10	1.07	1.05	1.03	1.02	1.01
	GLR	1.35	1.45	1.49	1.37	1.17	1.13	1.09	1.06	1.04	1.02
1.50	DCC	1.64	1.47	1.34	1.16	1.06	1.04	1.03	1.02	1.01	1.01
	GLR	1.07	1.11	1.15	1.16	1.09	1.08	1.06	1.04	1.03	1.02
1.75	DCC	1.15	1.13	1.11	1.07	1.03	1.02	1.02	1.01	1.01	1.00
	GLR	1.01	1.02	1.04	1.06	1.05	1.04	1.03	1.02	1.02	1.01
2.00	DCC	1.02	1.03	1.03	1.03	1.02	1.01	1.01	1.01	1.01	1.00
	GLR	1.00	1.00	1.01	1.02	1.02	1.02	1.02	1.01	1.01	1.01
2.25	DCC	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.00	1.00	1.00
	GLR	1.00	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.00
2.50	DCC	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	GLR	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2.75	DCC	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	GLR	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 3 summarises the effects of Phase I estimates on ARL under the true parameters. The results reported in Table 3 indicate that using Phase I parameter estimates will increase the false alarm rate when the process is in control, as evidenced by the decrease in the in-control ARL. Take $m = 20$ and $n = 5$ as an example. The in-control ARL assuming that $\mu_0 = 0$ and $\sigma_0 = 1$ are known is set at approximately 370. On the other hand, if the parameters are replaced by estimates obtained from 20 Phase I samples of size 5 each, the in-control ARL is only approximately 175.68. As the total number of Phase I sample increases, thus increasing the total number of Phase I observations, the ARL becomes closer to the value obtained when assuming that $\mu_0 = 0$ and $\sigma_0 = 1$ are known. Note that these observations are consistent with what has been reported in the literature. For example, Bischak and Trietsch (2007) argue that not only is the in-control ARL affected but that the ARL itself may be a confusing performance measure when used with control charts with estimated in-control parameters.

4. The density control chart under a negative exponential distribution

Suppose that the quality characteristic of interest of a process follows a negative exponential distribution having the following density function

$$f(x; \theta_1, \theta_2) = \frac{1}{\theta_1} \exp\left\{-\frac{x - \theta_2}{\theta_1}\right\}, \quad x \geq \theta_2,$$

where $\theta_1 > 0$ and $\theta_2 \in R$ are unknown parameters. Given m samples of size n, let x_{ij}, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, denote the ith observation of the jth sample. One can estimate θ_1 and θ_2 by

$$\widehat{\theta}_1 = \frac{1}{m} \sum_{j=1}^{m} \widehat{\theta}_{1j} \tag{9}$$

Table 3. The simulated *ARL*s of the density control chart (DCC) when the estimates obtained from Phase I data are used for $n=5$, $m=20$, 25, 30 ($\mu=\mu_0+\delta\sigma_0$, $\sigma=\lambda\sigma_0$).

	λ									
δ	0.87	1.00	1.15	1.52	2.00	2.15	2.30	2.55	2.75	3.00
$m=20$	$n=5$									
0.00	1292.97	**175.68**	38.16	5.39	2.03	1.74	1.55	1.35	1.25	1.18
0.25	689.79	121.28	30.84	5.05	2.00	1.72	1.54	1.34	1.25	1.17
0.50	196.34	53.28	18.48	4.24	1.90	1.66	1.50	1.32	1.24	1.17
0.75	53.66	21.43	10.03	3.35	1.76	1.57	1.44	1.29	1.22	1.15
1.00	16.93	9.25	5.57	2.60	1.61	1.47	1.37	1.26	1.19	1.14
1.25	6.54	4.56	3.34	2.05	1.47	1.37	1.30	1.22	1.17	1.12
1.50	3.15	2.61	2.21	1.67	1.35	1.29	1.24	1.18	1.14	1.10
1.75	1.88	1.74	1.62	1.41	1.25	1.21	1.18	1.14	1.11	1.09
2.00	1.35	1.33	1.31	1.24	1.17	1.15	1.13	1.11	1.09	1.07
2.25	1.13	1.14	1.14	1.14	1.11	1.10	1.09	1.08	1.07	1.06
2.50	1.04	1.05	1.06	1.07	1.07	1.07	1.06	1.06	1.05	1.04
2.75	1.01	1.02	1.02	1.04	1.04	1.04	1.04	1.04	1.04	1.03
$m=25$	$n=5$									
0.00	1626.95	**201.05**	41.24	5.52	2.05	1.75	1.56	1.35	1.26	1.18
0.25	840.78	136.50	33.06	5.16	2.01	1.73	1.54	1.35	1.25	1.17
0.50	228.62	58.41	19.51	4.32	1.91	1.67	1.50	1.33	1.24	1.17
0.75	59.80	22.89	10.44	3.40	1.77	1.58	1.44	1.30	1.22	1.15
1.00	18.17	9.67	5.72	2.63	1.62	1.48	1.37	1.26	1.19	1.14
1.25	6.82	4.68	3.39	2.06	1.47	1.38	1.30	1.22	1.17	1.12
1.50	3.21	2.65	2.23	1.67	1.35	1.29	1.24	1.18	1.14	1.11
1.75	1.89	1.75	1.63	1.41	1.25	1.21	1.18	1.14	1.11	1.09
2.00	1.35	1.33	1.31	1.24	1.17	1.15	1.13	1.11	1.09	1.07
2.25	1.13	1.14	1.14	1.14	1.11	1.10	1.09	1.08	1.07	1.06
2.50	1.04	1.05	1.06	1.07	1.07	1.07	1.06	1.06	1.05	1.04
2.75	1.01	1.01	1.02	1.04	1.04	1.04	1.04	1.04	1.04	1.03
$m=30$	$n=5$									
0.00	1890.74	**219.66**	43.39	5.61	2.06	1.76	1.56	1.36	1.26	1.18
0.25	967.30	148.40	34.71	5.24	2.02	1.74	1.55	1.35	1.25	1.18
0.50	255.97	62.55	20.33	4.38	1.92	1.67	1.51	1.33	1.24	1.17
0.75	65.01	24.10	10.77	3.43	1.78	1.58	1.45	1.30	1.22	1.16
1.00	19.23	10.02	5.85	2.65	1.62	1.48	1.38	1.26	1.20	1.14
1.25	7.06	4.78	3.44	2.07	1.48	1.38	1.31	1.22	1.17	1.12
1.50	3.27	2.68	2.25	1.68	1.35	1.29	1.24	1.18	1.14	1.11
1.75	1.90	1.76	1.63	1.42	1.25	1.21	1.18	1.14	1.11	1.09
2.00	1.35	1.33	1.31	1.24	1.17	1.15	1.13	1.11	1.09	1.07
2.25	1.12	1.13	1.14	1.14	1.11	1.10	1.09	1.08	1.07	1.06
2.50	1.04	1.05	1.06	1.07	1.07	1.07	1.06	1.06	1.05	1.04
2.75	1.01	1.01	1.02	1.04	1.04	1.04	1.04	1.04	1.04	1.03

and

$$\widehat{\theta_2}=\min\{\widehat{\theta_{21}},\widehat{\theta_{22}},\ldots,\widehat{\theta_{2m}}\}, \tag{10}$$

where $\widehat{\theta_{2j}}=\min\{x_{1j},x_{2j},\ldots,x_{nj}\}$ and $\widehat{\theta_{1j}}=1/n\sum_{i=1}^{n}(x_{ij}-\widehat{\theta_{2j}})$, $j=1,2,\ldots,m$, are the estimates of θ_2 and θ_1, respectively, obtained from the *j*th sample.

When we consider $\widehat{\theta_1}$ and $\widehat{\theta_2}$ as the true θ_1 and θ_2, respectively, it is easy to see that for any given sample, $(X_1-\widehat{\theta_2})/\widehat{\theta_1}$, $(X_2-\widehat{\theta_2})/\widehat{\theta_1},\ldots,(X_n-\widehat{\theta_2})/\widehat{\theta_1}$ are i.i.d. as exp(1) and therefore $\sum_{i=1}^{n}(X_i-\widehat{\theta_2})/\widehat{\theta_1}\sim\Gamma(n,1)$, which is a Gamma distribution. The inequality

$$L(X_1,X_2,\ldots,X_n;\widehat{\theta_1},\widehat{\theta_2})\geq l(\widehat{\theta_1},\widehat{\theta_2})$$

subject to

$$P_{\widehat{\theta_1},\widehat{\theta_2}}(L(X_1, X_2, \ldots, X_n; \widehat{\theta_1}, \widehat{\theta_2}) \geq l(\widehat{\theta_1}, \widehat{\theta_2})) = 1 - \alpha$$

yields

$$l(\widehat{\theta_1}, \widehat{\theta_2}) = \frac{1}{\widehat{\theta}_1^n} \exp\left\{-\frac{\chi^2_{2n,1-\alpha}}{2}\right\}. \tag{11}$$

Therefore, for the negative exponential distribution, the density control chart plots, for any given sample of observations $x_1, x_2, \ldots, x_n$,

$$L(x_1, x_2, \ldots, x_n; \widehat{\theta_1}, \widehat{\theta_2}) = \frac{1}{\widehat{\theta}_1^n} \exp\left\{-\frac{\sum_{i=1}^n (x_i - \widehat{\theta_2})}{\widehat{\theta_1}}\right\} \tag{12}$$

and an out-of-control signal is detected as soon as the plotting statistic is below

$$\text{LCL} = \frac{1}{\widehat{\theta}_1^n} exp\left\{-\frac{\chi^2_{2n,1-\alpha}}{2}\right\}. \tag{13}$$

Alternatively, one may choose to use $1/n \sum_{i=1}^n (x_i - \widehat{\theta_2})$ as the plotting statistics. Note that given $\widehat{\theta_1}$ and $\widehat{\theta_2}$, $\chi_0^2 = 2\sum_{i=1}^n (x_i - \widehat{\theta_2})/\widehat{\theta_1}$ is distributed as χ^2_{2n}, and that $L(x_1, x_2, \ldots, x_n; \widehat{\theta_1}, \widehat{\theta_2}) \leq \text{LCL}$ if and only if $\chi_0^2 \geq \chi^2_{2n,1-\alpha}$. That is, the density control chart in this case is equivalent to the chi-square control chart where the plotting statistic is $1/n \sum_{i=1}^n (x_i - \widehat{\theta_2})$ and

$$\text{UCL} = \frac{\widehat{\theta_1}}{2n} \chi^2_{2n,1-\alpha}. \tag{14}$$

Given $\widehat{\theta_1}$ and $\widehat{\theta_2}$, and specific out-of-control parameters θ_1 and θ_2, it is also quite straightforward to calculate the ARL of the density chart as $ARL = 1/p$, where the probability p is defined as

$$P\left(\chi^2_{2n} \geq \frac{\widehat{\theta_1}}{\theta_1} \chi^2_{2n,1-\alpha} - \frac{2n(\theta_2 - \widehat{\theta_2})}{\theta_1}\right). \tag{15}$$

Here we look at simple cases when $n = 5$ and 10. When the process is in control, we assume $\widehat{\theta_1} = 1.0$ and $\widehat{\theta_2} = 0$, and the in-control ARL is set at 370. The ARLs of different out-of-control scenarios are summarised in Table 4.

The ARL decreases when either the location parameter (θ_2) shifts or the scale parameter (θ_1) changes, or both. Not surprisingly, the ARL decreases as the sample size increases. Moreover, the density chart seems to be more effective in detecting changes in the scale parameter than shifts in the location parameter.

Example 2: A particle counts example

The semiconductor manufacturing processes often involve quality characteristics whose distributions are non-normal. Levinson and Polny (1999) studied a set of particle counts data generated from an Applied Materials etcher. See Levinson and Polny (1999) for a detailed account of the data set and its generation. A three-parameter Gamma distribution with the following density function

$$f(x; \theta_1, \theta_2, \theta_3) = \frac{1}{\Gamma(\theta_3)\theta_1^{\theta_3}} (x - \theta_2)^{\theta_3 - 1} \exp\left\{-\frac{x - \theta_2}{\theta_1}\right\}, \quad x \geq \theta_2$$

was fit and shown to be appropriate for the data set. The maximum likelihood estimates of the three parameters are $\widehat{\theta_1} = 222.32$, $\widehat{\theta_2} = 34$, and $\widehat{\theta_3} = 1.172$. Levinson and Polny (1999) also constructed a Shewhart chart for the mean $\mu = \theta_1\ \theta_3 + \theta_2$ with $n = 1$, and the resulting control limits are $\text{LCL} = -362.63$ and $\text{UCL} = 951.79$. They mentioned that this classical X-chart has a serious problem since negative particle counts do not exist in practice.

Since the estimated $\widehat{\theta_3}$ is close to 1, the negative exponential distribution may provide a simpler but also appropriate distribution to model the particle counts data. Here we use this example to demonstrate how the density

Table 4. The *ARL* of the density control chart for the negative exponential distribution (θ_1 is the scale parameter and θ_2 is the location parameter).

		θ_2					
	θ_1	0.0	0.2	0.5	1.0	2.0	3.0
$n=5$	1.00	**370.37**	180.62	64.03	13.06	1.36	1.00
	1.25	56.37	33.32	15.72	5.10	1.17	1.00
	1.50	17.83	11.95	6.79	2.97	1.09	1.00
	1.75	8.40	6.14	3.97	2.12	1.05	1.00
	2.00	5.01	3.91	2.77	1.71	1.03	1.00
	2.25	3.47	2.84	2.15	1.48	1.02	1.00
	2.50	2.66	2.25	1.80	1.34	1.01	1.00
	2.75	2.17	1.90	1.58	1.25	1.01	1.00
	3.00	1.87	1.67	1.44	1.18	1.01	1.00
$n=10$	1.00	**370.37**	115.49	23.55	2.97	1.00	1.00
	1.25	35.10	15.95	5.64	1.64	1.00	1.00
	1.50	9.25	5.35	2.67	1.26	1.00	1.00
	1.75	4.16	2.83	1.77	1.12	1.00	1.00
	2.00	2.53	1.93	1.40	1.06	1.00	1.00
	2.25	1.85	1.52	1.22	1.03	1.00	1.00
	2.50	1.51	1.31	1.13	1.02	1.00	1.00
	2.75	1.32	1.19	1.08	1.01	1.00	1.00
	3.00	1.21	1.12	1.05	1.00	1.00	1.00

control chart can be readily applied to a negative exponential distribution. We carried out both the chi-square goodness-of-fit test and the Kolmogorov–Smirnov test for testing the hypothesis that the particle counts data follow a negative exponential distribution, and the p-values were 0.2466 and 0.7889, respectively. Both tests did not provide strong evidence to reject the hypothesis that the particle counts follow a negative exponential distribution.

A total of 12 samples each with size 4 were collected and the resulting estimates for θ_1 and θ_2 were $\widehat{\theta_1} = 160.25$ and $\widehat{\theta_2} = 35.5$, respectively. Based on the density chart described in (12) and (13), the plotting statistic in (12), for each of the 12 samples with size 4, was calculated by

$$\frac{1}{(160.25)^4} \times \exp\left\{ -\frac{\sum_{i=1}^{4}(x_i - 35.5)}{160.25} \right\},$$

and the chart signals if the plotting statistic falls below

$$\mathrm{LCL} = 160.25^{-4} - exp\left(\frac{\chi^2_{8,0.9973}}{2} \right) = 0.0001153 \times 10^{10}.$$

Since the 12 plotting statistics obtained from (12) and LCL are all very large (in the magnitude of 10^{10}), for the chart to be visible to the reader, these values are divided by 10^{10} and then transformed on log-scale. The density control chart of the 12 samples (with the transformed LCL $= ln(\frac{0.0001153 \times 10^{10}}{10^{10}}) = -9.0683$) is shown in Figure 4. It is evident from Figure 4 that samples 2 and 6 are out of control.

In this case, we can use the plotting statistic described in (14) to help us diagnose the possible causes of any out-of-control signal. Specifically, for a given sample $x_1, x_2, \ldots, x_n$, we denote $v = 1/n \sum_{i=1}^{n}(x_i - x_{(1)})$ and $u = x_{(1)}$, where $x_{(1)} = \min \{x_1, x_2, \ldots, x_n\}$. Given $\widehat{\theta_1}$ and $\widehat{\theta_2}$, the diagnostics can be carried out by looking at the equation

$$v + (u - \widehat{\theta_2}) = \frac{\widehat{\theta_1}}{2n} \chi^2_{2n,1-\alpha}. \tag{16}$$

Here the v measures the variation of the sample and $u - \widehat{\theta_2}$ measures possible location shift. The diagonal line with axes u and v from the above equation represents the control limit. Any sample producing a point lying above the diagonal line is considered as an out-of-control sample. If the point is away from the line $u = \widehat{\theta_2}$, it is an indication

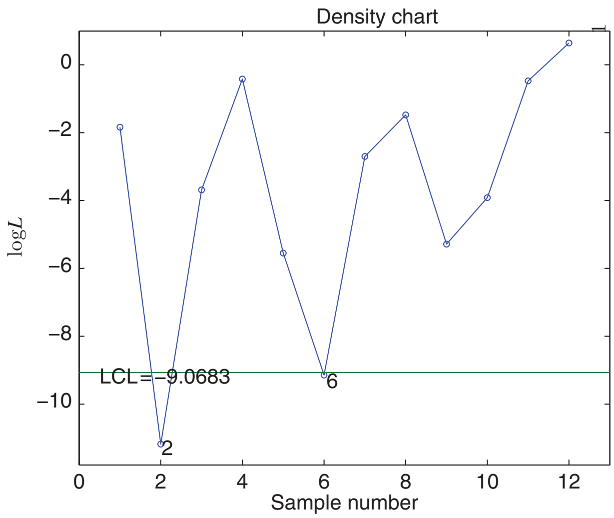

Figure 4. The density control chart of the particle counts example with the plotting statistics.

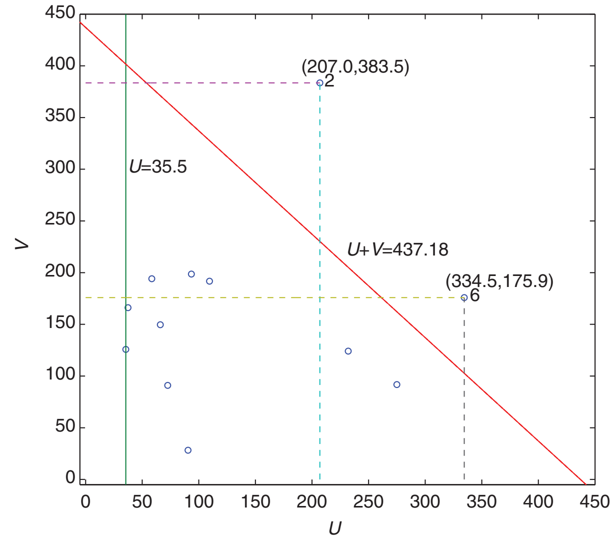

Figure 5. The chi-square control chart of the particle counts example.

that there is a possible mean shift. On the other hand, if the point is away from the line $v = 0$, it is an indication that there is a possible change in the variance.

For the particle counts data, the equation turned out to be

$$v + (u - 35.35) = 401.83.$$

The diagonal line with axes u and v, along with all 12 samples are shown in Figure 5. The two out-of-control samples, sample 2 and sample 6 produced $(u, v) = (207.0, 383.5)$ and $(334.5, 175.9)$, respectively. Figure 5 indicates that both samples 2 and 6 are due to mean shift and variance change. On the other hand, the level of mean shift seems to be larger in sample 6 than in sample 2, while the degree of change in variance seems to be greater in sample 2 than in sample 6.

5. Approximate density control chart

In previous sections, we discussed situations in which the distribution of the plotting statistic of the density control chart can be readily derived. However, in other applications it may be too complicated to derive such a distribution. In general, given $\widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_p$, if the distribution of $L(X_1, X_2, \ldots, X_n; \widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_p)$ is too complicated, one can repeatedly generate samples from $f(x; \widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_p)$ and calculate the likelihood of each of the, say, k generated samples, which are denoted by $l_1, l_2, \ldots, l_k$. Then the LCL can be estimated as $\widehat{\mathrm{LCL}} = l_{([k\alpha]+1)}$, where $l_{(1)} \le l_{(2)} \le \cdots \le l_{(k)}$ are the ordered statistics and $[a]$ is the largest integer smaller than or equal to a.

Here we look at another distribution to illustrate the use of simulations to approximate the LCL. Suppose that a quality characteristic follows a Gamma distribution having the following density function

$$f(x; \theta_1, \theta_2) = \frac{1}{\Gamma(\theta_2)\theta_1^{\theta_2}} x^{\theta_2 - 1} \exp\left\{-\frac{x}{\theta_1}\right\}, \quad x > 0.$$

The Gamma distribution has been used to model a variety of lifetime data, including some situations involving the negative exponential distributions. For the Gamma distribution, it is rather difficult to derive the distribution of the likelihood function. Suppose that, based on the Phase I data, both of two parameters, θ_1 and θ_2, are estimated to be equal to 2, i.e. $\widehat{\theta}_1 = \widehat{\theta}_2 = 2$. Based on these estimates and assume that the subgroup size is 5, we then generated 10 million samples of size 5 each and calculated, for each sample, the likelihood function

$$f(x_1, x_2, \ldots, x_5; \widehat{\theta}_1 = \widehat{\theta}_2) = \frac{1}{(\Gamma(2))^5 2^{5\times 2}} \left(\prod_{i=1}^{5} x_i\right)^{2-1} \exp\left\{-\frac{\sum_{i=1}^{5} x_i}{2}\right\}.$$

From the 10 million likelihoods, we obtained $\widehat{\mathrm{LCL}} = l_{(27{,}000)}$. Using this $\widehat{\mathrm{LCL}}$, we then simulated the *ARL*s for a variety of combinations of θ_1 and θ_2, including the in-control case when $\theta_1 = \theta_2 = 2$. The results are summarised in Table 5. From Table 5, the results seem to indicate that the approximate density control chart is more effective in detecting changes in the shape parameter (θ_2) than in the scale parameter (θ_1) in the case of the Gamma distribution.

As an additional example of using simulation to obtain the control limit for the approximate density control chart, we assume that the estimates of θ_1 and θ_2 from Phase I data are both equal to 1, i.e. $\widehat{\theta_1} = \widehat{\theta_2} = 1$. Note that if we treat these estimates as the true parameter values in Phase II, the corresponding Gamma distribution is equivalent to a negative exponential distribution with scale parameter equal to 1 and location parameter equal to 0. The simulated *ARL* values for this example are summarised in Table 6. The $\widehat{\mathrm{LCL}}$ for this case was also obtained based on 10 million simulated likelihood values. The approximate density control chart seems to be more effective when the in-control parameter $\theta_1 = \theta_2 = 1$ than when the in-control parameters $\theta_1 = \theta_2 = 2$, in terms of producing smaller *ARL* value for the same level of change in each parameter, in the case of the negative exponential distribution than in the case of the Gamma distribution.

Table 5. The *ARL*s of the approximate density control chart for the Gamma distribution ($n = 5$, θ_1 is the scale parameter, and θ_2 is the shape parameter).

	θ_2												
θ_1	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00
2.00	**370.59**	100.81	37.52	17.80	9.98	6.35	4.45	3.35	2.68	2.23	1.93	1.71	1.56
2.25	160.95	47.19	19.10	9.74	5.84	3.96	2.94	2.33	1.94	1.69	1.51	1.39	1.30
2.50	74.22	23.89	10.55	5.83	3.75	2.71	2.12	1.77	1.54	1.39	1.28	1.21	1.16
2.75	36.90	13.13	6.36	3.80	2.63	2.01	1.66	1.44	1.31	1.21	1.15	1.11	1.08
3.00	19.83	7.85	4.16	2.69	1.98	1.61	1.39	1.25	1.17	1.11	1.08	1.05	1.04
3.25	11.48	5.05	2.93	2.04	1.60	1.36	1.23	1.14	1.09	1.06	1.04	1.02	1.02
3.50	7.18	3.50	2.21	1.65	1.37	1.21	1.13	1.08	1.05	1.03	1.02	1.01	1.01
3.75	4.79	2.58	1.77	1.40	1.22	1.12	1.07	1.04	1.02	1.01	1.01	1.00	1.00
4.00	3.40	2.02	1.49	1.25	1.13	1.07	1.04	1.02	1.01	1.01	1.00	1.00	1.00
4.25	2.57	1.66	1.31	1.15	1.07	1.04	1.02	1.01	1.00	1.00	1.00	1.00	1.00
4.50	2.03	1.43	1.19	1.09	1.04	1.02	1.01	1.00	1.00	1.00	1.00	1.00	1.00
4.75	1.69	1.28	1.12	1.05	1.02	1.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00
5.00	1.46	1.18	1.07	1.03	1.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 6. The *ARL*s of the approximate density control chart for the negative exponential distribution ($n = 5$, θ_1 is the scale parameter, and θ_2 is the shape parameter).

	θ_2												
θ_1	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00
1.00	**370.70**	56.46	17.82	8.38	5.01	3.47	2.66	2.17	1.87	1.66	1.52	1.41	1.33
1.25	97.83	18.98	7.24	3.96	2.67	2.04	1.69	1.48	1.35	1.26	1.19	1.15	1.11
1.50	33.74	8.26	3.75	2.35	1.76	1.47	1.30	1.20	1.14	1.10	1.07	1.05	1.04
1.75	14.33	4.38	2.34	1.66	1.36	1.21	1.13	1.08	1.05	1.03	1.02	1.01	1.01
2.00	7.22	2.72	1.69	1.32	1.17	1.09	1.05	1.03	1.02	1.01	1.01	1.00	1.00
2.25	4.21	1.93	1.36	1.16	1.07	1.04	1.02	1.01	1.01	1.00	1.00	1.00	1.00
2.50	2.77	1.51	1.18	1.07	1.03	1.01	1.01	1.00	1.00	1.00	1.00	1.00	1.00
2.75	2.01	1.28	1.09	1.03	1.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
3.00	1.59	1.15	1.04	1.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
3.25	1.34	1.08	1.02	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
3.50	1.20	1.04	1.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
3.75	1.11	1.02	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
4.00	1.06	1.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

6. Conclusion and discussion

We have proposed and studied the density control chart which can be used to monitor multiple parameters simultaneously on a single chart for any given type-I error probability. The approach of the density chart is quite general which allows easy extensions to non-normal processes. We gave examples of non-normal processes and demonstrated how the density chart can be constructed and applied in those cases.

Under the normality assumption, we compared the performance of the density chart with that of the GLR chart of Hawkins and Deng (2009). The density chart performs better than the GLR chart when there is a small increase in the variance. On the other hand, when there is a decrease in the variance and a small shift in the mean, the density chart is ineffective. When only mean shifts but variance remains unchanged, the two charts have comparable performance.

The density chart studied in the current article is a Shewhart type control chart. It is rather straightforward to calculate the EWMA of the likelihoods in very much the same manner as those discussed in Yeh *et al*. (2005). It would be worthwhile to study to what extent the EWMA density chart improves the performance of the Shewhart type density chart.

It should be noted that, unlike most of the simultaneous charts that exist in the literature which assume $n>1$, the proposed density chart is also applicable when $n=1$. It would be interesting to develop the density chart for individual observations and compare it with some of the existing control charts such as the exponentially weighted moving squared deviations chart studied in MacGregor and Harris (1993).

The generality of the density control charting mechanism lends itself nicely for multivariate processes, provided that the density function is known. There have been some recent studies on simultaneously monitoring multiple parameters for multivariate normal processes (see, for example, Cheng and Thaga 2005, Chen *et al*. 2005, Reynolds and Chao 2006, and Machado *et al*. 2009). It would be worthwhile to extend the density chart to multivariate normal and non-normal processes.

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References

Bischak, D.P. and Trietsch, D., 2007. The rate of false signals in $\bar{X}$ control charts with estimated limits. *Journal of Quality Technology*, 39, 54–65.

Chao, M.T. and Cheng, S.W., 1996. Semicircle control chart for variables data. *Quality Engineering*, 8, 441–446.

Chen, G. and Cheng, S.W., 1998. Max-chart: Combining $\bar{X}$ chart and S chart. *Statistica Sinica*, 8, 263–271.

Chen, G., Cheng, S.W., and Xie, H., 2001. Monitoring process mean and variability with one EWMA chart. *Journal of Quality Technology*, 33, 223–233.

Chen, G., Cheng, S.W., and Xie, H., 2004. A new EWMA control chart for monitoring both location and dispersion. *International Journal of Quality and Quantitative Management*, 1, 217V–231.

Chen, G., Cheng, S.W., and Xie, H., 2005. A new multivariate control chart for monitoring both location and dispersion. *Communications in Statistics: Computations and Simulations*, 34, 203–217.

Cheng, S.W. and Li, G.Y., 1993. A single variables control chart, Technical Report, University of Manitoba, Winnipeg, Canada.

Cheng, S.W. and Thaga, K., 2005. Multivariate Max-CUSUM chart. *Quality Technology and Quantitative Management International*, 2, 191–206.

Cheng, S.W. and Thaga, K., 2006. On single variables control charts: an overview. *Quality and Reliability Engineering International*, 22, 811–820.

Costa, A.F.B., 1993. Joint economic design of $\bar{X}$ and R control charts for processes subject to two independent assignable causes. *IIE Transactions*, 25, 27–33.

Costa, A.F.B., 1998. Joint $\bar{X}$ and R charts with variable parameters. *IIE Transactions*, 30, 505–514.

Costa, A.F.B. and Rahim, M.A., 2000. Economic design of $\bar{X}$ and R charts under Weibull shock models. *Quality and Reliability Engineering International*, 16, 143–156.

Costa, A.F.B. and Rahim, M. A., 2002. Joint $\bar{X}$ and R charts with two stage sampling. *In: IIE Research Conference*, Orlando.

Costa, A.F.B. and Rahim, M.A., 2004. Joint $\bar{X}$ and R charts with two stage samplings. *Quality and Reliability Engineering International*, 20, 699–708.

Costa, A.F.B., De Magalhães, M.S., and Epprecht, E.K., 2009. Monitoring the process mean and variance using a synthetic control chart with two-stage testing. *International Journal of Production Research*, 47 (18), 5067–5086.

De Magalhaes, M.S. and Moura Neto, F., 2005. Joint economic model for totally adaptive and R charts. *European Journal of Operational Research*, 161, 148–161.

Domangue, R. and Patch, S.C., 1991. Some omnibus exponentially weighted moving average statistical process monitoring schemes. *Technometrics*, 33, 299–313.

Gan, F.F., 1995. Joint monitoring of mean and variance using exponentially weighted moving average control charts. *Technometrics*, 37, 446–453.

Gan, F.F., 2000. Joint monitoring of process mean and variance based on the exponentially weighted moving averages. *Statistical Process Monitoring and Optimisation*. New York: Marcel Dekker, 189–208.

Hawkins, D.M. and Deng, Q., 2009. Combined charts for mean and variance information. *Journal of Quality Technology*, 41, 415–425.

Hawkins, D.M. and Zamba, K.D., 2005. Statistical process control for shifts in mean or variance using a change point formulation. *Technometrics*, 47, 164–173.

He, D. and Grigoryan, A., 2006. Joint statistical design of double sampling $\bar{X}$ and S charts. *European Journal of Operational Research*, 168, 122–142.

Levinson, W.A. and Polny, A., 1999. SPC for tool particle counts. *Semiconductor International*, 22, 117–122.

Li, Z., Zhang, J., and Wang, Z., 2010. Self-starting control chart for simultaneously monitoring process mean and variance. *International Journal of Production Research*, 48, 4537–4553.

Jensen, W.A., *et al.*, 2006. Effects of parameter estimation on control chart properties: A literature review. *Journal of Quality Technology*, 38, 349–364.

Jones, L.L. and Case, K.E., 1981. Economic design of a joint $\bar{X}$ and R-control chart. *IIE Transactions*, 13, 182V–195.

MacGregor, J.F. and Harris, T.J., 1993. The exponentially weighted moving variance. *Journal of Quality Technology*, 25, 106–118.

Machado, M.A. G., Costa, A.F. B., and Marins, F.A. S., 2009. Control charts for monitoring the mean vector and the covariance matrix of bivariate processes. *The International Journal of Advanced Manufacturing Technology*, 45, 772–785.

Montgomery, D.C., Runger, G.C., and Hubele, N.F., 2004. *Engineering statistics*. New York: Wiley.

Ohta, H., Kimura, A., and Rahim, M.A., 2002. An economic model for $\bar{X}$ and R charts with timevarying parameters. *Quality and Reliability Engineering International*, 18, 131V–139.

Rahim, M.A., 1989. Determination of optimal design parameters of joint X and R charts. *Journal of Quality Technology*, 21, 65V–70.

Rahim, M.A. and Costa, A.F. B., 2000. Joint economic design of $\bar{X}$ and R charts under Weibull shock models. *International Journal of Production Research*, 38 (13), 2871–2889.

Reynolds Jr, M.R. and Chao, G.-Y., 2006. Multivariate control charts for monitoring the mean vector and covariance matrix. *Journal of Quality Technology*, 38, 230–253.

Reynolds Jr, M.R. and Stoumbos, Z.G., 2001. Monitoring the process mean and variance using individual observations and variable sampling intervals. *Journal of Quality Technology*, 33, 181–205.

Saniga, E.M., 1989. Economic statistical control charts designs with an application on $\bar{X}$ and R charts. *Technometrics*, 31, 313–320.

Saniga, E.M., 1991. Joint statistical design of $\bar{X}$ and R control charts. *Journal of Quality Technology*, 23, 156–162.

Sheu, S.-H., *et al.*, 2009. Monitoring process mean and variability with generally weighted moving average control charts. *Computers and Industrial Engineering*, 57, 401–407.

Shewhart, W.A., 1925. The applications of statistics as an aid in maintaining quality of a manufacturing product. *Journal of American Statistical Association*, 20, 546–548.

Spiring, F.A. and Cheng, S.W., 1998. An alternative variables control chart: the univariate and multivariate cases. *Statistica Sinica*, 8, 273–287.

Sweet, A.L., 1986. Control charts using coupled exponential weighted moving average. *IIE Transactions*, 18, 26–33.

Takemoto, Y., Watakabe, K., and Arizono, I., 2003. A study of cumulative sum control charts. *International Journal of Production Research*, 41 (9), 1873–1886.

Thaga, K., 2003. *Contributions to statistical process control tools*. Thesis (PhD)., University of Manitoba, Winnipeg, Canada.

White, E.M. and Schroeder, R., 1987. A simultaneous control chart. *Journal of Quality Technology*, 19, 1–10.

Wu, Z. and Tian, Y., 2005. Weighted-loss-function CUSUM chart for monitoring mean and variance of a production process. *International Journal of Production Research*, 43 (14), 3027–3044.

Xie, H., 1999. *Contributions to qualimetry*. Thesis (PhD)., University of Manitoba, Winnipeg, Canada.

Yeh, A.B., Huwang, L., and Wu, Y. -F., 2005. A likelihood ratio based EWMA control chart for monitoring variability of multivariate normal processes. *IIE Transactions on Quality and Reliability Engineering*, 36, 865–879.

Yeh, A.B. and Lin, D.K.-J., 2002. A new variables control chart for simultaneously monitoring multivariate process mean and variability. *International Journal of Reliability, Quality and Safety Engineering*, 9, 41–59.

Zhang, J., Zou, C., and Wang, Z., 2010. A Control Chart Based on Likelihood Ratio Test for Monitoring Process Mean and Variability. *Quality and Reliability Engineering International*, 26, 63–73.

Zhou, Q., Luo, Y., and Wang, Z., 2010. A control chart based on likelihood ratio test for detecting patterned mean and variance shifts. *Computational Statistics and Data Analysis*, 54, 1634–1645.