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Dynamic inventory rationing for systems with multiple demand classes and general demand processes

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A R T I C L E I N F O

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A B S T R A C T

We consider the dynamic rationing problem for inventory systems with multiple demand classes and general demand processes. We assume that backorders are allowed. Our aim is to find the threshold values for this dynamic rationing policy. For single period systems, dynamic critical level policy is developed and the detailed cost approximation subject to this policy is derived. For multiperiod systems, a dynamic rationing policy with periodic review is proposed. The numerical study shows that our dynamic critical level policies are close to being optimal for various parameter settings.

1. Introduction

Inventory is an important driver in modern supply chains and has traditionally been used to provide a buffer against demand uncertainty or increased service levels. However, there are costs associated with holding inventory, such as opportunity costs, storage costs, obsolescence costs, insurance costs, and damage costs. Hence, organizations face a trade-off between incurring inventory and servicing their customers. However, inventory can serve purposes beyond its traditional role because heterogeneous customers have different service needs and priorities. This means that firms can make tactical decisions with regard to the rationing of inventory and can set different pricing and service levels according to their customer service needs. By providing a differentiated service according to customer needs, firms can benefit, because this helps to increase market size, and thereby revenue. For example, firms can charge higher prices to customers who need immediate service and can charge less for customers who only need a normal service. This practice is common in many industries, such as the airline industry, online retailing, and the services parts industry. The airline industry usually charges different prices for the same seat, and online retailers, such as Amazon.com, provide expedited and normal shipping services. The services parts industry also charges customers according to services delivery contracts.

For a firm to successfully adopt a different pricing or service level strategy for the same inventory, the main assumption is that customers can be segmented according to their different service needs and priorities. The key challenge is how to allocate the inventory to different segments of customers. For motivation, this paper uses the example of a firm that has an extensive network providing spare parts, which are used to maintain or replace failed equipment parts at the customer's site. It has a major regional distribution center, which serves its customers. Requests for spare parts are prompted by parts failure and by scheduled maintenance. Requests prompted by parts failure must be rectified immediately, whereas those prompted by scheduled maintenance can wait. Hence, in any period, the distribution center may face these two types of demand from its customers. In this situation, a firm may adopt the rationing policy that when inventory is low, only urgent demand for parts is satisfied. The inventory level at which low-priority requests are rejected is sometimes known as the critical, or threshold level. The policy of reserving stock is termed the. Many researchers have explored practical examples of inventory rationing, such as Kleijn and Dekker (1998), Deshpande et al. (2003), and Cardós and Babiloni (2011).

There are two kinds of critical-level policies: stationary and dynamic. For stationary policies, the critical levels are constant. Much research has been carried out on stationary critical level policy. For make-to-stock production systems, the stationary critical level rationing policy is optimal for specific cases (Ha, 1997a, 1997b, 2000; Gayon et al., 2004). For exogenous inventory supply problems, researchers such as Melchiors et al. (2000), Deshpande et al. (2003), and Arslan et al. (2007) propose stationary policies and then determine the optimal parameters for the critical levels that minimize inventory costs. Others such as Nahmias and Demmy (1981), Moon and Kang (1998), Cohen et al. (1988), Dekker et al. (1998), and Möllering and Thonemann (2009) determine the stationary critical levels for inventory systems operating under different service levels.

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For dynamic policies, the critical levels may change over time. Topkis (1968) considers dynamic inventory rationing policy for single period and multiple period systems with zero lead times. A dynamic programming model is proposed in which one period is divided into many small intervals. Topkis also shows that the optimal rationing policy is dynamic. However, he fails to show that the critical level is nonincreasing over time. Evans (1968) and Kaplan (1969) extend results from Topkis (1968) and explore two demand classes. Melchiors (2003) considers dynamic rationing policy under an inventory system with a Poisson demand process and an (s, Q) ordering policy in which backordering is not allowed. Lee and Hersh (1993) consider dynamic rationing policy for an airline seating problem. Teunter and Klein Haneveld (2008) develop a continuous time approach to determining the dynamic rationing policy for two Poisson demand classes under the assumption that there is no more than one outstanding order. However, its computational results are tractable only for limited settings. Fadiloglu and Bulut (2010) propose a heuristic rationing policy called "rationing with exponential replenishment flow" for continuous-review inventory systems. All except Topkis (1968) consider only two demand classes. However, the limitation of his approach is that the state spaces grow exponentially large when the number of demand classes increases. Even for two demand classes, the state space can be very large. Moreover, many researchers assume a Poisson distribution.

In this paper, we develop an approximation approach to deriving the dynamic threshold level for inventory systems with multiple demand classes and general demand processes. This approximation approach is based on comparing the marginal costs of accepting and rejecting a demand class when it arrives. It is also assumed that when this demand class is rejected, all future demands from this class will be rejected until the next replenishment arrives. Unlike existing work, this method can deal with general demand processes, and is efficient in solving problems with more than two demand classes. To illustrate the effectiveness of the proposed policy, we conduct numerical analysis. The results show that the proposed policies are close to being optimal under various parameter settings when demand follows a Poisson process. The Poisson process is used because we want to compare our solutions with the optimal solution.

Our paper is organized as follows. In Section 2, we consider a single period system with multiple demand classes. We derive the dynamic critical levels based on the concept of marginal cost. In Section 3, we consider multiperiod systems with periodic review policy. The rational policy in Section 2 is extended for multiperiod systems. In Section 4, numerical studies are conducted to investigate the performance of the proposed approaches. In Section 5, we summarize the results and discuss some possible extensions.

2. Inventory rationing for single period systems

In this section, the inventory rationing problem for a single period system is considered. The goal is to find a good dynamic rationing policy. We first examine the dynamic critical levels for only two demand classes and then extend the results for a general number of demand classes. Then, we develop an approximation method for computing the expected total costs associated with our rationing policy.

2.1. Model formulation

Consider a single period inventory system with a period length of u. There is a single product with demands from K different customer classes. We assume that the demand processes of all customer classes are independent and stationary and that these demands can be partially satisfied. We let $t=u$ denote the beginning of the period, we let $t=0$ denote the end of the period, and we let $X(t)$ denote the on-hand inventory at the remaining time $t \in [0,u]$. For each unit stored, we assume a holding charge of h per unit of time. At the beginning of the period, we assume that the initial on-hand inventory is given, and is equal to x (i.e., $X(u)=x$). During the period, each customer demand may either be satisfied or rejected according to our rationing policy. The rejected demand is backordered and a backorder cost is applied. We define the backorder cost for class i as $\pi_i+\hat{\pi}_i t$, where $t \in [0,u]$ is the remaining time to the end of the period. Note that π_i is a fixed penalty cost to reject a demand from class i, which may represent the loss of customers' loyalty. Moreover, $\hat{\pi}_i$ is the per-unit-of-time cost to hold a demand from class i to the end of the period without loss of goodwill or the order. Without loss of generality, we arrange demand classes in the order of nonincreasing backorder cost. That is, $\pi_i \geq \pi_j$ and $\hat{\pi}_i \geq \hat{\pi}_j$ for demand classes $i<j$.

At the end of the period $(t=0)$, we assume that all backorders must be fulfilled. We propose using the remaining inventory to fulfill these backorders first and if this is insufficient, we propose the purchase of additional units to fulfill the remaining backorders from the open market. If the remaining inventory exceeds the backorders, the surplus is sold at salvage value on the same market. We assume that both the salvage value and the additional purchasing cost on the open market equal c_0 per unit of product at the end of the period.

To determine the dynamic rationing policy, we need to compute critical levels over time. Define $s_i(t)$ to be the dynamic critical level of class i for the remaining time $t \in [0,u]$. When $X(t)>s_i(t)$, we satisfy the demand from class i. Otherwise, the demand from class i is rejected. As we have arranged demand classes in the order of nonincreasing backorder cost, we have $s_i(t) \leq s_j(t)$ for classes $i<j$. We also define $H(t,X(t))$ as the expected cost for the remaining time t. Thus, our objective is to find the rationing policy that minimizes the expected total cost, which can be written as follows:

$$\min_{\substack{s_i(t)\\ i=1,\ldots,K}} \quad H(u,x). \tag{1}$$

Without loss of generality, let $s_i^*(t)$ denote the optimal critical level of class i and let $H^*(u,x)$ denote the optimal total expected cost. We know that $s_1^*(t)$ must be zero because class 1 has the highest backorder cost and there is no advantage in rejecting demand from class 1.

2.2. Dynamic critical levels for systems with two demand classes

Consider a single period system with two demand classes $(K=2)$ and suppose that a customer demand has just arrived when the time remaining is t. If the demand is from class 1, it must be satisfied. If the demand is from class 2, it can either be satisfied or rejected. When this demand is satisfied, the total expected cost at the remaining time t is $H^*(t,X(t)-1)$. If this demand is rejected, the total expected cost at the remaining time t is $H^*(t,X(t))+e_2(t)$, where $e_2(t)=c_0+\pi_2+\hat{\pi}_2 t$ is the backorder cost. If $H^*(t,X(t)-1)>H^*(t,X(t))+e_2(t)$, then this demand should be rejected. Hence, the optimal dynamic critical level of class 2 is

$$s_2^*(t)=\max\{X(t) \mid H^*(t,X(t))+e_2(t)-H^*(t,X(t)-1)<0\}. \tag{2}$$

It is difficult to solve Eq. (2) because the closed form for $H^*(t,X(t))-H^*(t,X(t)-1)$ cannot be found easily.

Thus, we adopt two important ideas to approximate $s_2^*(t)$. First, if a demand class is rejected, this demand class will be rejected for the remaining time until the end of the period. This is a reasonable approximation because when a demand class is rejected, it intuitively implies that there is unlikely to be sufficient stock to cater for the less important classes for the remainder of the period.

Second, we use the marginal cost of rejecting the demand to determine the critical level. The marginal cost is computed by assuming that either this and all future demand classes are rejected or that the current demand is accepted but future demands are rejected. When a demand from class 2 arrives, we use this marginal cost to decide whether this demand is accepted or rejected.

Define $\Delta J_2(t,X(t))$ as the marginal cost of rejecting demand class 2 at the remaining time t when on-hand inventory is $X(t)$.

Without loss of generality, we consider two cases: $X(t)=s$ (when the demand is rejected at the remaining time t) and $X(t)=s-1$ (when the demand is accepted at the remaining time t, but all future demands of this class are rejected).

Fig. 1 illustrates the on-hand inventory for both cases, where the solid line represents $X(t)=s$ and the dashed line represents $X(t)=s-1$. Note that, because we assume that all future demands from class 2 are rejected, the inventory decreases only when the demand of class 1 arrives. The line below the horizontal axis represents the backorder quantities of class 1.

Given on-hand inventory of s at the remaining time t, let τ_s be the time it takes to run out of inventory. Define $D_1(t)$ as the total demand of class 1 from the remaining time t to the end of the period.

The $X(t)=s$ case has higher holding costs than does the $X(t)=s-1$ case. If $D_1(t)<s$, the difference in holding costs between the two cases is $h\cdot t$. If $D_1(t)\geq s$, the difference in holding costs between the two cases is $\tau_s\cdot h$.

The $X(t)=s$ case has lower backorder cost than does the $X(t)=s-1$ case.

If $D_1(t)<s$, then $D_1(t)\leq s-1$. Thus, there is no backorder for both cases and there is a difference of one unit of product. This means that there is an extra salvage value, c_0, for the $X(t)=s$ case.

If $D_1(t)\geq s$, the sth demand of class 1 can be satisfied in the $X(t)=s$ case. However, the sth demand of class 1 cannot be satisfied in the $X(t)=s-1$ case. Thus, this demand class has an extra backorder cost of $c_0+\pi_1+\hat{\pi}_1\cdot(t-\tau_s)$ in the $X(t)=s-1$ case.

In fact, these cost differences between the $X(t)=s$ and $X(t)=s-1$ cases are the marginal costs of rejecting demand class 2 at the remaining time t. Thus

$$\begin{aligned}\Delta J_2(t,s) &= t\cdot h\cdot P(D_1(t)<s)+h\cdot E(\tau_s|D_1(t)\geq s)\cdot P(D_1(t)\geq s)\\ &\quad -\{c_0+\pi_1+\hat{\pi}_1\cdot E[(t-\tau_s)|D_1(t)\geq s]\}\cdot P(D_1(t)\geq s)\\ &\quad -c_0\cdot P(D_1(t)<s)\\ &= h\cdot t-[t\cdot(\hat{\pi}_1+h)+\pi_1]\cdot P(D_1(t)\geq s)\\ &\quad +(\hat{\pi}_1+h)\cdot E[\tau_s|D_1(t)\geq s]\cdot P(D_1(t)\geq s)-c_0, \end{aligned} \tag{3}$$

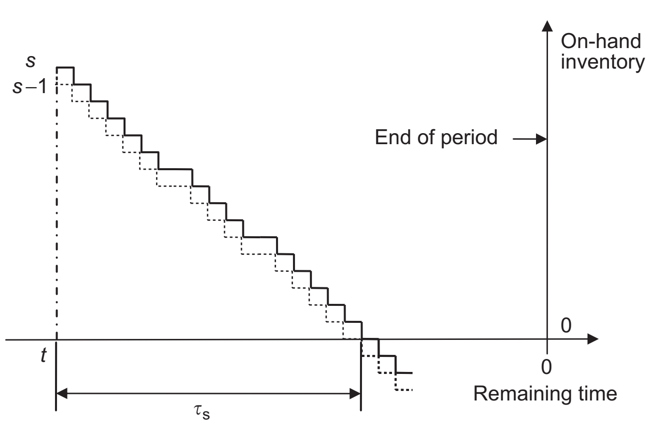

Fig. 1. On-hand inventory after rejecting some order from class 2 at remaining time t.

where $P(D_1(t)<s)$ is the probability of $D_1(t)<s$ and $P(D_1(t)\geq s)$ is the probability of $D_1(t)\geq s$.

Now, we demonstrate the monotonicity property of $\Delta J_2(t,s)$.

Lemma 1. *For the remaining time t and $s\geq 0$, $\Delta J_2(t,s)$ is nondecreasing in s.*

Proof. See Appendix. □

After deriving the marginal cost of rejecting demand class 2, we use the above two ideas to approximate $s_2^*(t)$. Define

$$s_2^a(t)=\max\{s|\Delta J_2(t,s)+e_2(t)<0\}. \tag{4}$$

We show the relationship between $s_2^a(t)$ and $\Delta J_2(t,s)$.

Theorem 1. *For demand class 2, there exists a unique critical level, $s_2^a(t)$, at the remaining time t such that*

$$\Delta J_2(t,s)+e_2(t)\geq 0 \quad for \quad s>s_2^a(t), \quad and$$
$$\Delta J_2(t,s)+e_2(t)<0 \quad for \quad s\leq s_2^a(t).$$

Proof. Theorem 1 holds from Lemma 1. □

Theorem 1 shows that once the on-hand inventory drops below the unique critical level, $s_2^a(t)$, all future demands from demand class 2 will be rejected until the next replenishment arrives. This result is consistent with our assumptions. We now establish the monotonicity property of $s_2^a(t)$ from Theorem 1 and Eqs. (3) and (4).

Theorem 2. *For $\pi_1=\pi_2=0$, $s_2^a(t)$ is nondecreasing in the remaining time t.*

Proof. See Appendix. □

Theorem 2 shows that $s_2^a(t_1)\geq s_2^a(t_2)$ for $t_1\geq t_2$ when $\pi_1=\pi_2=0$. This means that when the remaining time becomes smaller, we only need to reserve fewer inventories for class 1.

2.3. Dynamic critical levels for systems with more than two demand classes

We now consider a single period system with more than two demand classes. We assume that a demand arrives at the remaining time t. If the demand is from class 1, then it must be satisfied. Otherwise, it can either be satisfied or rejected. Similarly to the case of two demand classes in the previous subsection, we generalize $s_2^a(t)$ and approximate the dynamic critical level of class m by $s_m^a(t)$. Hence, define

$$s_m^a(t)=\max\{X(t)|\Delta J_m(t,X(t))+e_m(t)<0\}, \tag{5}$$

where $\Delta J_m(t,X(t))$ is the marginal cost of rejecting demand class m at the remaining time t.

To compute the critical levels for all demand classes, we adopt a sequential approach; i.e., we use the dynamic critical levels for demand classes 2 to $m-1$ to compute the dynamic critical level for demand class m. Note that from Subsection 2.1, we know that the critical level of class 1 is zero.

Suppose we want to compute the critical level for demand class m, assuming that the dynamic critical levels for demand classes 2, 3,…,$m-1$ are known. Similarly, we assume that if a demand class is rejected, that demand class will be rejected for the remainder of the period. Without loss of generality, we consider two cases: $X(t)=s$ (when the demand of class m is rejected at the remaining time t) and $X(t)=s-1$ (when the demand of class m is accepted at the remaining time t, but all future demands of this class will be rejected). Fig. 2 illustrates a realization of on-hand inventory for both cases, where the solid line represents $X(t)=s$ and the dashed line represents $X(t)=s-1$.

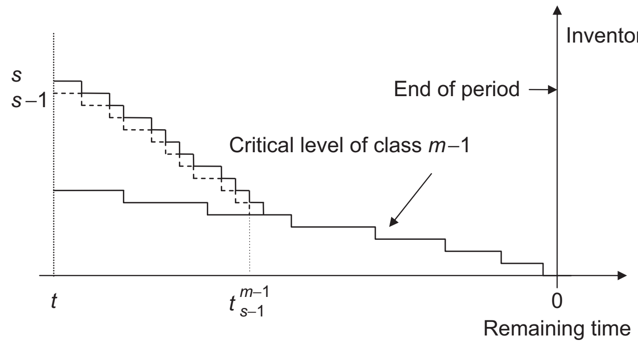

Fig. 2. On-hand inventory versus remaining time.

This figure shows that, to compute the marginal cost difference between the two cases, we should consider two different situations. In the first situation, all demands from class $m-1$ are accepted in both cases, and in the second situation, demands from class $m-1$ are rejected for the $X(t)=s-1$ case. In the first situation, the only cost difference is that in inventory holding costs. In the second situation, the cost difference is a weighted sum of the differences in inventory holding costs and backorder costs.

To compute the critical level for demand class m at the remaining time t, we assume that demand class m is rejected at the remaining time t. This implies that any demand class i, where $i>m$, will be rejected. A demand class i, where $i<m$, may be rejected in the future if the inventory level is below the respective dynamic critical level. To determine the conditions under which situation 1 applies, we must identify the time at which demand class $m-1$ is first rejected given $X(t)=s-1$; we denote this time as t_{s-1}^{m-1}. This means that t_{s-1}^{m-1} is the maximum amount of time remaining when $X(t_{s-1}^{m-1})$ is equal to the dynamic critical level of demand class $m-1$. When the remaining time is between t and t_{s-1}^{m-1}, situation 1 applies. Hence, situation 2 applies when the remaining time is between t_{s-1}^{m-1} and 0.

For subsequent derivations, we must define t_s^{m-1}, which is the maximum amount of time remaining when $X(t_s^{m-1})$ is equal to the dynamic critical level of demand class $m-1$. Note that $t_s^{m-1} \le t_{s-1}^{m-1}$.

Now, we examine the cost differences between the $X(t)=s$ case and the $X(t)=s-1$ case under two different situations; i.e., the remaining time intervals $[t_{s-1}^{m-1},t]$ and $[0,t_{s-1}^{m-1})$.

(i) Situation 1: **Remaining time interval** $(t_{s-1}^{m-1},t]$
The difference in costs between the two cases in this interval is one unit of inventory cost. Thus, the cost difference is $h\cdot(t-t_{s-1}^{m-1})$.

(ii) Situation 2: **Remaining time interval** $[0,t_{s-1}^{m-1}]$
To derive the cost difference between the two cases, we assume that the approximated dynamic critical level of demand class $m-1$ remains unchanged in the remaining time interval $[t_s^{m-1},t_{s-1}^{m-1}]$. That is, $s_{m-1}^a\,(t_{s-1}^{m-1})=s_{m-1}^a(t_s^{m-1})$. Thus, for the $X(t)=s-1$ case, we have $X(t_{s-1}^{m-1})=s_{m-1}^a(t_{s-1}^{m-1})=s_{m-1}^a(t_s^{m-1})$. For the $X(t)=s$ case, we have $X(t_{s-1}^{m-1})=s_{m-1}^a(t_{s-1}^{m-1})+1=s_{m-1}^a(t_s^{m-1})+1$.

Consider the $X(t)=s$ case. Demand class i, where $i\ge m-1$, will be rejected when the remaining time is less than t_s^{m-1}. This is because $X(t_s^{m-1})$ is less than or equal to the dynamic critical level of demand class $m-1$.

Similarly, for the $X(t)=s-1$ case, when the remaining time is less than t_{s-1}^{m-1}, demand class i, where $i\ge m-1$, will be rejected because $X(t_{s-1}^{m-1})$ is less than or equal to the dynamic critical level of demand class $m-1$.

Hence, in the interval $[t_s^{m-1},t_{s-1}^{m-1}]$, there is one extra unit of inventory in the $X(t)=s$ case. The marginal cost over the $[0,t_s^{m-1}]$ interval depends on the realization of the demand class at the remaining time t_s^{m-1}. (Note that by definition, this demand class cannot come from classes m and above because all these will be rejected and, thus, there will be no corresponding fall in inventory for the $X(t)=s$ case.) If the demand comes from classes 1 to $m-2$, that demand will be accepted in both cases, and so the marginal cost will be $\Delta J_{m-1}(t_s^{m-1},s_{m-1}^a(t_{s-1}^{m-1}))$. This is because the starting inventory at the remaining time t_s^{m-1} for the $X(t)=s$ case is always one more than the starting inventory for the $X(t)=s-1$ case, and it is equal to the critical level for demand class $m-1$; i.e., $s_{m-1}^a(t_{s-1}^{m-1})$. However, if the demand comes from class $m-1$, this demand will be accepted for the $X(t)=s$ case but will be rejected for the $X(t)=s-1$ case. Hence, the cost difference is $e_{m-1}(t_s^{m-1})$.

Thus, the difference in average costs between the two cases in situation 2 is

$$h\cdot(t_{s-1}^{m-1}-t_s^{m-1})+\Delta J_{m-1}(t_s^{m-1},s_{m-1}^a(t_{s-1}^{m-1}))\cdot\left(\frac{\sum_{i=1}^{m-2}\lambda_i}{\sum_{i=1}^{m-1}\lambda_i}\right)$$
$$-e_{m-1}(t_s^{m-1})\cdot\left(\frac{\lambda_{m-1}}{\sum_{i=1}^{m-1}\lambda_i}\right), \tag{6}$$

where $(\sum_{i=1}^{m-2}\lambda_i)/(\sum_{i=1}^{m-1}\lambda_i)$ is the probability that the next demand is from class 1 to $m-2$, and $\lambda_{m-1}/(\sum_{i=1}^{m-1}\lambda_i)$ is the probability that the next demand is from class $m-1$.

By combining situations 1 and 2, given the realization of t_s^{m-1}, the total marginal cost of rejecting demand class m at the remaining time t when the inventory level is s is

$$\Delta J_m(t,s\,|\,t_s^{m-1})=h\cdot(t-t_s^{m-1})+\Delta J_m(t_s^{m-1},s_{m-1}^a(t_s^{m-1}))\cdot\left(\frac{\sum_{i=1}^{m-2}\lambda_i}{\sum_{i=1}^{m-1}\lambda_i}\right)$$
$$-e_{m-1}(t_s^{m-1})\cdot\left(\frac{\lambda_{m-1}}{\sum_{i=1}^{m-1}\lambda_i}\right). \tag{7}$$

Note that t_s^{m-1} is a random variable that depends on the on-hand inventory $X(t)$, the dynamic critical level $s_{m-1}^a(t)$, and the demand arrival process. Let $p(\cdot)$ denote the probability density function of this random variable. It is possible that $X(0)>s_{m-1}^a(0)$. If $X(0)>s_{m-1}^a(0)$, then inventory always exceeds the dynamic critical level of demand class $m-1$. In this case, the marginal cost of rejecting demand from class $m-1$ is $ht-c_0$. By defining $\overline{P}=1-\int_0^t p(t_s^{m-1})dt_s^{m-1}$, which is the probability of $X(0)>s_{m-1}^a(0)$, we have

$$\Delta J_m(t_c,s)\approx\int_0^t[\Delta J_m(t,s\,|\,t_s^{m-1})]\cdot p(t_s^{m-1})dt_s^{m-1}+\overline{P}\cdot(ht-c_0). \tag{8}$$

Eq. (8) can be reduced to Eq. (4) when $m=2$.

Because $s_{m-1}^a(t_s^{m-1})=\max\{s\,|\,\Delta J_{m-1}(t_s^{m-1},X(t_s^{m-1}))+e_{m-1}(t_s^{m-1})<0\}$, it follows that $\Delta J_X^{m-1}(t,s)+e_{m-1}(t)$ is approximately zero. Moreover, because of our assumption that the dynamic critical level for class $m-1$ remains unchanged in the remaining time interval $[t_s^{m-1},t_{s-1}^{m-1}]$ (i.e., $X(t_s^{m-1})\approx s_{m-1}^a(t)$), we can approximate $\Delta J_{m-1}(t_s^{m-1},s_{m-1}^a(t_s^{m-1}))$ by $-e_{m-1}(t_s^{m-1})$. Hence, Eq. (8) can be approximated by

$$\Delta J_m(t,s)\approx\int_0^t[h\cdot(t-t_s^{m-1})-e_{m-1}(t_s^{m-1})]\cdot p(t_s^{m-1})dt_s^{m-1}+\overline{P}(ht-c_0). \tag{9}$$

From Eqs. (5) and (9), we can generate the approximate optimal dynamic critical level of class m from $s_2^a(t),s_3^a(t),\ldots,s_{m-1}^a(t)$. Therefore, all dynamic critical levels can be generated sequentially starting from the lowest demand class index.

We propose the following algorithm to generate the dynamic critical levels for all demand classes.

Step 1: Set the optimal dynamic critical level of class 1 to be zero: $s_1^*(t)=0$.
Step 2: Considering only classes 1 and 2, use Theorem 1 in Section 2.2 to estimate $s_2^a(t)$, $t\in[0,u]$. Set $m=3$.
Step 3: Given $s_i^a(t_c)$, $i\in\{2,\ldots,m-1\}$, use Eqs. (5) and (9) to find $s_m^a(t)$, $t\in[0,u]$. Set $m=m+1$.
Step 4: Stop if $m\geq K+1$. Otherwise, go to Step 3.

2.4. Expected total cost

In this subsection, we estimate the expected total cost by using an iterative method based on

$$H(t,0)=\sum_{i=1}^{K}\pi_i+\hat{\pi}_i(t), \tag{10a}$$

$$H(t,x)\approx H(t,0)+\sum_{j=1}^{x}\Delta J_2(t,j)\quad\text{for}\quad x\in(0,s_2^a(t)], \tag{10b}$$

$$H(t,x)\approx H(t,s_m^a(t))+\sum_{j=s_m^a(t)+1}^{x}\Delta J_{m+1}(t,j)\quad\text{for}\quad x\in(s_m^a(t_c),s_{m+1}^a(t_c)]\quad\text{and}\quad 2\leq m<K, \tag{10c}$$

$$H(t,x)\approx H(t,s_K^a(t))+\sum_{j=s_K^a(t)+1}^{x}\Delta J_{K+1}(t,j)\quad\text{for}\quad x\in(s_K^a(t_c),\infty). \tag{10d}$$

Eq. (10a) represents the case in which there is no initial on-hand inventory, i.e., $X(t)=x=0$. In this case, all demands should be rejected. The total cost will be equal to the sum of the total expected backorder costs of all the classes.

Eq. (10b) represents the case in which the initial on-hand inventory is below the critical level of class 2. In this case, only class 1 demand is accepted, and so the expected total cost can be approximated by $\sum_{j=1}^{x}\Delta J_2(t,j)$, where $\Delta J_2(t,j)$ is given by Eq. (4).

Eqs. (10c) and (10d) represent cases in which the initial on-hand inventory is between the critical levels of class m and class $m+1$, where m is between two and $K-1$ and the initial on-hand inventory is above the critical level of class K. Similarly, the expected total cost can be approximated by the sum of the marginal costs given in Eq. (9).

3. Inventory rationing in multiperiod systems

In this section, we consider dynamic inventory rationing models for multiperiod systems with general demand process in which backordering is allowed.

3.1. Notation and model formulation

Consider an inventory system with a single type of product and K demand classes for an infinite time horizon. We assume that the demand process, the backorder cost, and the holding cost are the same as those of the single period model in Section 2. We adopt a policy of dynamic critical level rationing with periodic review ordering (R,S).

Under this ordering policy, the ordering opportunities, $m=0$, 1, 2,..., occur at fixed intervals of time and the amount of time between two successive order opportunities is u, where $R=u$. In addition, the deterministic lead time taken to replenish orders is L. The order placed at the mth order opportunity will arrive at time $l_m=mu+L$. The time interval (l_m,l_{m+1}) is termed the mth replenishment period. When the mth replenishment order arrives at time l_m, backorders are instantaneously fulfilled according to the following mechanism. Backorders from the most important (lowest index) demand class are always fulfilled first. Second, if replenishment is sufficient, less important demand classes are fulfilled.

With backorders fulfilled, we define y_m as the net inventory and B_m as the remaining backorders at time l_m. Note that $B_m=\{b_1^m,\ldots,b_K^m\}$ is the vector of backorders for all demand classes, where $b_i^m\geq 0$ is the amount of backorders for demand class i. Under the above backorder fulfillment mechanism, y_m may be negative if some backorders remain unfulfilled. If $y_m\geq 0$, then $b_i^m=0$ for all i. Otherwise, there is no on-hand inventory and $\sum_{i=1}^{K}b_i^m=-y_m$. Thus, (y_m,B_m) is the state variable at the beginning of the mth replenishment period and its distribution depends on the rationing policy and the base stock level S. Let $P_{S,v}(y_m,B_m)$ denote the probability distribution of the state variable (y_m,B_m) at the beginning of the mth replenishment period subject to the rationing policy v and the base stock level S.

Similarly, let $C_v(y_m,B_m)$ be the expected inventory cost incurred in the mth replenishment period subject to the state variable (y_m,B_m) and the rationing policy v. For $y_m<0$, $C_v(y_m,B_m)$ consists of backorder costs from B_m and from new backorders arising in this period.

For the multiperiod system, define $AC(S,v)$ as the expected average cost incurred during a period in which the base stock level is S and the prevailing dynamic rationing policy is v. We can now model the optimization problem for the dynamic critical level rationing policy as follows:

$$\min_{S,v}AC(S,v)=\lim_{M\to\infty}\frac{1}{M}\sum_{m=0}^{M-1}\sum_{y_m,B_m}C_v(y_m,B_m)\cdot P_{S,v}(y_m,B_m). \tag{11}$$

The goal is to minimize the expected average cost in Eq. (11), which consists of inventory holding costs and backorder costs. Note that all rejected or unfulfilled orders are backlogged. Hence, we can ignore the variable ordering cost and set the salvage value to zero: $c_0=0$.

3.2. A solution to the optimization problem

In solving the rationing problem given by Eq. (11), the main issue is that the probability distribution for backorders, B_m, depends on the rationing policy v and the base stock level S. It is difficult to derive a closed form solution for the probability mass function for B_m. Hence, we propose a heuristic method to solve the rationing problem given by Eq. (11).

In Section 2, we proposed an approximate dynamic rationing policy for a single period problem. We now propose a rationing policy for a multiperiod problem. Let v_a be the rationing policy for the multiperiod system, where each stock ration in each period is determined according to our corresponding approximate dynamic critical level. The goal is to locally minimize inventory costs in each period, ignoring subsequent periods.

We now develop an approximate expression for $AC(S,v_a)$. Given the base stock level S and the rationing policy v_a, the probability mass function of the state variable (y_m,B_m) is $P_{S,v_a}(y_m,B_m)=P(D_L=S-y_m)$, for $y_m\geq 0$:

$$\sum_{|B_m|=-y}P_{S,v_a}(y_m,B_m)=P(D_L=S-y_m)\quad\text{for}\quad y_m<0, \tag{12}$$

where $|B_m|=\sum_{i=1}^{K}b_i^m$ and D_L is the demand that arrives during the lead time L. Note that the probability mass functions in Eq. (12) are independent of m and v_a.

Recall that $C_{v_a}(y,B)$ is the expected inventory cost incurred in the mth replenishment period subject to the state variable (y_m,B_m)

and the rationing policy v_a. We now approximate $C_{v_a}(y_m,B_m)$ for two different cases: $y_m \geq 0$ and $y_m < 0$.

For $y_m \geq 0$:

$$C_{v_a}(y_m,B_m) \approx H(u,y_m), \tag{13}$$

where $H(u,y_m)$ is the expected total cost in a single period of length u, given zero salvage cost ($c_0=0$) and initial on-hand inventory of y_m.

For $y_m < 0$:

$$C_{v_a}(y_m,B_m) \approx H(u,0)+\sum_{i=1}^{K} b_i^m \cdot \hat{\pi}_i \cdot u, \tag{14}$$

where $C_{v_a}(y_m,B_m)$ consists of backorder costs from B_m and new backorders arising during the period.

In a multiperiod system, one can preserve inventory for future use for more important demand classes by rejecting other demand classes during each period. It is assumed that backorders from more important demand classes are fulfilled using the mechanism described in the previous section. Therefore, for the mth replenishment period, given remaining backorders of y_m, on average, there should be more backorders from less important demand classes. Moreover, in practice, $y_m < 0$ is unlikely. Thus, for $y_m < 0$, we can rewrite Eq. (14) as

$$C_{v_a}(y_m,B_m) \approx H(u,0)-y_m \cdot \hat{\pi}_K \cdot u, \tag{15}$$

where we assume that all backorders in y_m are from the least important demand class K.

Based on Eqs. (12)–(15), we can approximate the average cost as

$$\begin{aligned} AC(S,v_a) &\approx \frac{1}{M}\sum_{m=0}^{M-1}\left(\sum_{y_m=0}^{S} H(u,y_m)\cdot P(D_L=S-y_m)\right. \\ &\quad \left. +\sum_{y_m=-\infty}^{-1}(H(u,0)-y_m\cdot\hat{\pi}_K\cdot u)\cdot P(D_L=S-y_m)\right) \\ &= \sum_{y=0}^{S} H(u,y)\cdot P(D_L=S-y) \\ &\quad +\sum_{y=-\infty}^{-1}(H(u,0)-y\cdot\hat{\pi}_K\cdot u)\cdot P(D_L=S-y). \end{aligned}$$

For a given rationing policy v_a, all replenishment periods are the same. By replacing y_m with y, we can rewrite Eq. (11) as

$$\begin{aligned} \min_S\ AC(S,v_a) &\approx \min_S \sum_{y=0}^{S} H(u,y)\cdot P(D_L=S-y) \\ &\quad +\sum_{y=-\infty}^{-1}(H_{dy}(u,0)-y\cdot\hat{\pi}_K\cdot u)\cdot P(D_L=S-y). \end{aligned} \tag{16}$$

4. Numerical study

In this section, we conduct numerical studies to investigate the effectiveness of our proposed methods. We develop bounds to evaluate the quality of our proposed solutions under different scenarios when the demands follow a Poisson process. These bounds are valid for both single and multiperiod problems.

We first consider the single period problem. Let v_a denote the dynamic critical level implied by the proposed method and let $v*$ denote the optimal dynamic critical level, which can be derived when the demand follows a Poisson process (Chew et al., 2011). Also, let $H_v(u,x)$ denote the expected total cost incurred during a single period of length u under rationing policy v starting with on-hand inventory of x. We define the relative error for the expected total cost as

$$\Delta H(v_a,x)=\frac{H_{v_a}(u,x)-H_{v*}(u,x)}{H_{v*}(u,x)}\times 100\%,$$

where $\Delta H(v_a,x)$ is determined by v_a and x.

In this numerical study, we consider a base case in which there are three demand classes. In this base case, for the length of the period, we choose $u=0.1$, we choose the holding cost $h=1$, and we choose the salvage value $c_0=0$. For the backorder costs, we assume $\pi_1=\pi_2=\pi_3=0$ and $\hat{\pi}_3=1.5$, where $\hat{\pi}_1:\hat{\pi}_2:\hat{\pi}_3=20:5:1.5$. For the demand arrival rates, we let $\lambda_1=\lambda_2=\lambda_3=300$. We also varied some parameters; different parameter settings are listed in Table 1.

For each new parameter setting, we compute the optimal dynamic critical level and the approximate dynamic critical level of the proposed method. Then, we use simulation to estimate their costs. To ensure the accuracy of the estimated costs, we repeat the simulation 20,000 times for each parameter setting. The results are illustrated in Figs. 3 and 4 and reported in Table 1.

Table 1
Worst relative error under different operational conditions.

Factors	Parameters			$\Delta H(v_a,x^*)$ (%)	x^*
	u	$\lambda_1:\lambda_2:\lambda_3$	$\hat{\pi}_1:\hat{\pi}_2:\hat{\pi}_3$		
Base case	0.1	1:1:1	20:5:1.5	0.16	60
Changing ratios of backorder costs	0.1	1:1:1	5:2:1.5	0.03	63
			10:3:1.5	0.08	45
			40:8:1.5	0.50	53
			100:10:1.5	1.31	55
Changing ratios of arrival rates	0.1	1:2:3	20:5:1.5	0.24	45
		1:3:6		0.24	38
		1:10:100		0.29	10
		3:2:1		0.10	58
		6:3:1		0.08	58
		100:10:1		0.01	70
		3:6:1		0.23	58
		10:100:1		0.37	43
		1:6:3		0.32	30
		1:100:10		0.44	20
Changing time horizon	0.05	1:1:1	20:5:1.5	0.85	32
	0.15			0.13	80
	0.2			0.10	118
	0.3			0.06	160

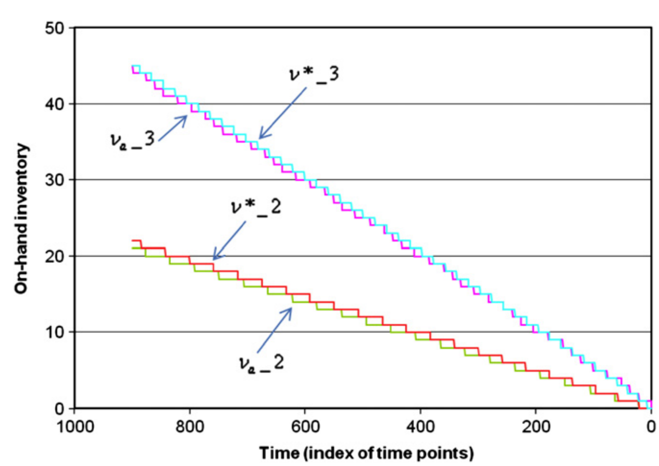

Fig. 3. Optimal and proposed critical levels in the base case.

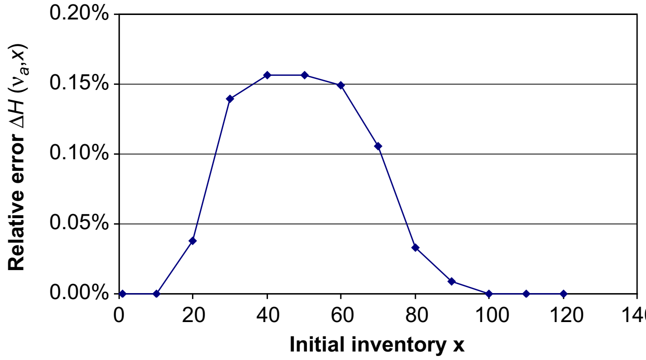

Fig. 4. Relative error, $\Delta H(v_a,x)$, in the base case.

Fig. 3 shows both optimal and proposed dynamic critical levels for classes 2 and 3 in the base case. Note that both the optimal and proposed dynamic critical level for class 1 is zero. Curves $v*\_2$ and $v*\_3$ represent the optimal dynamic critical levels for classes 2 and 3, respectively. Curves v_a_2 and v_a_3 represent the proposed dynamic critical levels for classes 2 and 3, respectively. The time horizon $u = 0.1$ is divided into 900 intervals. Fig. 3 shows that the proposed critical levels are very close to the optimal critical levels.

Fig. 4 illustrates the relative error, $\Delta H(v_a,x)$, for different initial levels of on-hand inventory of x in the base case. Note that $\Delta H(v_a,x)$ is no higher than 0.16% and may be monotonic in x. Moreover, $\Delta H(v_a,x)$ is close to zero when x is either very small or very large. When x is large, there is always enough initial inventory to satisfy demands of all classes. Thus, the relative error induced by the proposed policy is negligible. Similarly, when x is small, most demand classes cannot be satisfied for any policy in a single period problem. Thus, the best strategy is to reserve all inventory for demand class 1. Hence, the relative error induced by the proposed policy is again negligible.

Define $\Delta H(v_a,x^*) = \max_{x \geq 0} \Delta H(v_a,x)$ as the worst relative error. Table 1 shows how several factors affect the worst relative error. In all cases reported in Table 1, $\Delta H(v_a,x^*)$ is small. Few factors significantly affect the worst relative error.

The factor that most significantly affects $\Delta H(v_a,x^*)$ is the *backorder cost ratio*. As backorder costs for the more important classes increase, $\Delta H(v_a,x^*)$ increases. This follows from our assumption that if some demand classes are rejected, these demand classes will be rejected for the remainder of the time period.

Note that when changing the ratios of arrival rates, we fix the total arrival rate at 900. Thus, the arrival rate ratio of 1:2:3 is implemented as 150:300:450 and the ratio of 1:3:6 is implemented as 100:300:600. We approximate the arrival rate ratio of 1:10:100 by 9:81:810.

The numerical study shows that the proposed dynamic critical levels are very close to the optimal critical levels and that the relative errors for expected total cost are also small. However, when the number of demand classes increases, we may expect the relative errors to increase. This is because we determine the dynamic critical levels sequentially.

Note that the derived bounds reported in Table 1 are also valid for the multiperiod problem. This is because the results for the simulated cases reported in Table 1 are based on a starting inventory that represents the worst-case scenario and because the probability of having negative inventory at the beginning of each period is negligible (around 10^{-4}). Given that we have considered extreme cases, in which the penalty cost is as much as 100 times higher than the inventory cost, these results are reasonable.

5. Conclusions and extensions

In this paper, we developed a heuristic approach to computing the dynamic critical levels for systems with general demand arrival processes. We first considered a single period problem and then extended this to a multiperiod system. The heuristic approach is based on two ideas. The first idea is that any demand class that is rejected in one period will be rejected for the remainder of the period. The second idea is that dynamic critical levels can be derived based on the difference in the marginal costs of accepting and rejecting a demand class. These two ideas have enabled us to deal with more general demand processes. Our numerical study shows that the outcomes generated by our proposed approach compare favorably with the optimal solutions under most parameter settings.

For future work, we will consider relaxing some of our model assumptions. For example, we could allow a demand from a class that is initially rejected to be accepted in the future. We could also consider backorders being satisfied before replenishment orders arrive.

Appendix

Proof of Lemma 1. From Eq. (3), we have

$$\Delta J_2(t,s+1) = h\cdot t - c_0 - [t\cdot(\hat{\pi}_1+h)+\pi_1]\cdot P(D_1(t)\geq s+1) + (\hat{\pi}_1+h)\cdot E[\tau_{s+1}|D_1(t)\geq s+1]\cdot P(D_1(t)\geq s+1), \quad (A.1)$$

and

$$\Delta J_2(t,s) = h\cdot t - c_0 - [t\cdot(\hat{\pi}_1+h)+\pi_1]\cdot P(D_1(t)\geq s) + (\hat{\pi}_1+h)\cdot E[\tau_s|D_1(t)\geq s]\cdot P(D_1(t)\geq s). \quad (A.2)$$

From (A.1) and (A.2) we have

$$\Delta J_2(t,s+1) - \Delta J_2(t,s) = [t\cdot(\hat{\pi}_1+h)+\pi_1]\cdot P(D_1(t)=s) + (\hat{\pi}_1+h)\cdot\{E[\tau_{s+1}|D_1(t)\geq s+1]\cdot P(D_1(t)\geq s+1) - E[\tau_s|D_1(t)\geq s]\cdot P(D_1(t)\geq s)\}. \quad (A.3)$$

Note that

$$E[\tau_s|D_1(t)\geq s]\cdot P(D_1(t)\geq s) = \sum_{i=s}^{\infty} E[\tau_s|D_1(t)=i]\cdot P(D_1(t)=i) = E[\tau_s|D_1(t)\geq s+1]\cdot P(D_1(t)\geq s+1) + E[\tau_s|D_1(t)=s]\cdot P(D_1(t)=s). \quad (A.4)$$

By substituting (A.4) into (A.3), we obtain

$$\begin{aligned}\Delta J_2(t,s+1) - \Delta J_2(t,s) &= [t\cdot(\hat{\pi}_1+h)+\pi_1]\cdot P(D_1(t)=s) + (\hat{\pi}_1+h)\cdot\{E[\tau_{s+1}|D_1(t)\geq s+1] \\ &\cdot P(D_1(t)\geq s+1) - E[\tau_s|D_1(t)\geq s+1] \\ &\cdot P(D_1(t)\geq s+1)\} - (\hat{\pi}_1+h)\cdot E[\tau_s|D_1(t)=s] \\ &\cdot P(D_1(t)=s) = \{[t\cdot(\hat{\pi}_1+h)+\pi_1] - (\hat{\pi}_1+h) \\ &\cdot E[\tau_s|D_1(t)=s]\}\cdot P(D_1(t)=s) + (\hat{\pi}_1+h) \\ &\cdot\{E[\tau_{s+1}|D_1(t)\geq s+1]\cdot P(D_1(t)\geq s+1) \\ &- E[\tau_s|D_1(t)\geq s+1]\cdot P(D_1(t)\geq s+1)\}. \end{aligned} \quad (A.5)$$

For $E[\tau_s|D_1(t)=s]\leq t$, we have

$$[t\cdot(\hat{\pi}_1+h)+\pi_1] - (\hat{\pi}_1+h)\cdot E[\tau_s|D_1(t)=s] = (\hat{\pi}_1+h)\cdot\{t - E[\tau_s|D_1(t)=s]\} + \pi_1 \quad \geq 0.$$

Thus, $E[\tau_{s+1}|D_1(t)\geq s+1]\cdot P(D_1(t)\geq s+1) > E[\tau_s|D_1(t)\geq s+1]\cdot P(D_1(t)\geq s+1)$.

From (A.5), we have $\Delta J_2(t,s+1)-\Delta J_2(t,s)>0$. □

Proof of Theorem 2. Lemma 1 implies that $\Delta J_2(t,s)$ is nondecreasing in s. Thus, $\Delta J_2(t,s)+e_2(t)$ is nondecreasing in s.

Next, we show the continuity of $\Delta J_2(t,s)+e_2(t)$. Given $s>0$, $J_2(t,s)$ is a continuous function of remaining time t. Moreover, $e_2(t)=c_0+\pi_2+\hat{\pi}_2 t$ is a continuous function of remaining time t. Thus, $\Delta J_2(t,s)+e_2(t)$ is a continuous function of remaining time t.

Given a remaining time of $t=0^+$, no more orders will arrive in this arbitrarily small remaining time interval. Thus, all orders from demand class 2 can be accepted at a remaining time of $t=0^+$. This implies:

$$\Delta J_2(0^+,s)+e_2(0^+)>0 \tag{A.6}$$

For given $s<\infty$, if the remaining time $t=\infty$, then we must reject demands from class 2 because of limited on-hand inventory and infinite remaining time. Thus:

$$\Delta J_2(\infty,s)+e_2(\infty)<0 \tag{A.7}$$

From (A.6) and (A.7), and the continuity of $\Delta J_2(t,s)+e_2(t)$, there exists some remaining time t_0 such that $\Delta J_2(t_0,s)+e_2(t_0)=0$ for given on-hand inventory s. Hence, define $t_2^s=\min\{t_0|\Delta J_2(t_0,s)+e_2(t_0)=0\}$.

From the continuity property and the definition of t_2^s, we have

$$\Delta J_2(t,s)+e_2(t)<0 \quad \text{for} \quad t\in[t_2^s,\infty) \tag{A.8}$$

and

$$\frac{\partial[\Delta J_2(t_2^s,s)+e_2(t_2^s)]}{\partial t}\le 0. \tag{A.9}$$

Next, we show that inequality (A.9) holds for $t\in[t_2^s,\infty)$. We know that τ_s is a continuous random variable. Let $p(\tau_s)$ be its probability density function. For the cumulative distribution function, we have

$$F(t)=P(\tau_s\le t)=P(D_1(t)\ge s). \tag{A.10}$$

Thus

$$\frac{dP(D_1(t)\ge s)}{dt}=p(\tau_s=t)\ge 0 \tag{A.11}$$

and

$$\frac{dE[\tau_s|D_1(t)\ge s]\cdot P(D_1(t)\ge s)}{dt}=\frac{d}{dt}\int_0^t \tau_s\cdot p(\tau_s)\cdot d\tau_s=t\cdot p(\tau_s=t). \tag{A.12}$$

From Eqs. (A.2) and (A.12), we have

$$\begin{aligned}\frac{\partial[\Delta J_2(t,s)+e_2(t)]}{\partial t}&=h-(\hat{\pi}_1+h)\cdot P(D_1(t)\ge s)-t\cdot(\hat{\pi}_1+h)\cdot p(\tau_s=t)\\&\quad-\pi_1\cdot p(\tau_s=t)+(\hat{\pi}_1+h)\cdot t\cdot p(\tau_s=t)+\hat{\pi}_2\\&=h-(\hat{\pi}_1+h)\cdot P(D_1(t)\ge s)-\pi_1\cdot p(\tau_s=t)+\hat{\pi}_2\\&=h-(\hat{\pi}_1+h)\cdot P(D_1(t)\ge s)+\hat{\pi}_2,\end{aligned}$$

which holds because $\pi_1=\pi_2=0$.

From (A.10), $P(D_1(t)\ge s)=F(t)$ is a nondecreasing function of t. Thus, for any remaining time $t\in[t_2^s,\infty)$

$$\frac{\partial[\Delta J_2(t,s)+e_2(t)]}{\partial t}\le\frac{\partial[\Delta J_2(t_2^s,s)+e_2(t_2^s)]}{\partial t}\le 0. \tag{A.13}$$

From (A.8) and (A.13), it follows that $\Delta J_2(t,s)+e_2(t)<0$ and that $\Delta J_2(t,s)+e_2(t)$ is a nonincreasing function of remaining time $t\in(t_2^s,\infty)$. Given that $s_2^a(t)=\max\{s|\Delta J_2(t,s)+e_2(t)<0\}$ from Eq. (4), $s_2^a(t)$ is nondecreasing in remaining time t. □

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