Covering relations for coupled map networks

Leonid Bunimovich [a], Ming-Chia Li [b,*], Ming-Jiea Lyu [b]

[a] *School of Mathematics, Georgia Institute of Technology, 686 Cherry Street, Atlanta, GA 30332-0160, USA*
[b] *Department of Applied Mathematics, National Chiao Tung University, 1001 University Road, Hsinchu 300, Taiwan*

A B S T R A C T

In this paper, we study coupled map networks over arbitrary finite graphs. An estimate from below for a topological entropy of a perturbed coupled map network is obtained via a topological entropy of an unperturbed network by making use of the covering relations for coupled map networks. The result is quite general; in particular, nonlinear coupling is allowed and no assumptions of hyperbolicity of the local dynamics are made.

© 2012 Elsevier Inc. All rights reserved.

1. Introduction

A coupled map network is characterized by local dynamics operating at each node of a graph and their interaction along the edges of the graph. Coupled map networks are useful in applications: they appear naturally as electronic circuits in engineering, as chemical reactions in physics, as neuronal networks in the biological sciences, and as various agent-based models in the social sciences, etc.; refer to [1]. These networks are usually finite in size, but can be very large. Their couplings can be linear as well as nonlinear. Much attention has been paid to linear coupling and simple dynamical behaviors such as the existence of a global attractor in [2,3] and synchronization in [4,5]; see also [6] and references therein, while more complex phenomena are known to occur but are not as well recognized and understood. Recently in [7], a linear coupling of expanding circle maps was studied. It was found there that an increasing of the coupling strength leads to a cascade of bifurcations in which unstable subspaces in the coupled map systematically become stable.

We study here a topological dynamics in coupled map networks without assuming hyperbolicity of local maps and linearity of interactions. Consider a coupled map network with local dynamics having covering relations and coupling having a linear model. We give sufficient conditions for the existence of periodic points and for the existence of a positive topological entropy, in the coupled map network. Both conditions allow for a weak as well as for a strong coupling. Moreover, both results are also valid for small perturbations of the coupled map network, whose coupling might be nonlinear.

Our approach is based on the concept of covering relations, introduced by Zgliczyński [8,9]. The covering relation is a topological technique which does not require hyperbolicity (see e.g. [10–12]). Assuming that a coupling is locally topologically conjugate to a linear coupling, we show that the unperturbed coupled map network and its small perturbations both have covering relations of local dynamics as well as the existence of periodic points. To implement a topological chaos from local dynamics to perturbed coupled map networks, we introduce a notion of unified sets to guarantee the conjugacy relation between the coupling and its linear model.

---

* Corresponding author.
  E-mail addresses: bunimovh@math.gatech.edu (L. Bunimovich), mcli@math.nctu.edu.tw (M.-C. Li).

The paper is organized as follows. In Section 2, we define coupled map networks (here and in other places) of two types and state the main results for each of them. In Section 3, we begin with the formulation of known results needed in what follows and present the proofs of our results. In the Appendix, the definition of covering relations determined by a transition matrix is briefly recalled.

2. Definitions and statements of the main results

We start with the definition of a general class of coupled map networks which will be studied.

Definition 1. A coupled map network is a triple $(G, \{T_k\}, A)$ where:

1. G is a connected directed graph specified by a finite set Ω of nodes and a collection of edges $\mathcal{E} \subset \Omega \times \Omega$;
2. to each node $k \in \Omega$ there corresponds a local space X_k and a local map $T_k : X_k \to X_k$;
3. the network dynamics is defined by the iteration of $\Phi : X \to X$, where $X = \prod_{k \in \Omega} X_k$ is the product space and $\Phi = A \circ T$, where $T = \prod_{k \in \Omega} T_k$ is the (independent) application of local maps and $A : X \to X$ is the spatial interaction or coupling; for $x = (x_k)_{k \in \Omega} \in X$, the kth coordinate of $A(x)$ depends only on x_k and those x_j for which $(j, k) \in \mathcal{E}$.

Here the interaction graph is allowed to be directed, while in [7] it is restricted to being undirected.
We consider perturbations of coupled map networks in the following sense.

Definition 2. Let $(G, \{T_k\}, A)$ and $(\tilde{G}, \{\tilde{T}_k\}, \tilde{A})$ be two coupled map networks with the same set of nodes (perhaps with distinct edges) and local spaces. For $\varepsilon > 0$, we say that $(\tilde{G}, \{\tilde{T}_k\}, \tilde{A})$ is ε-close to $(G, \{T_k\}, A)$ if $|\tilde{A}(z) - A(z)| < \varepsilon$ for all $z \in X$ and $|\tilde{T}_k(x) - T_k(x)| < \varepsilon$ for all $x \in X_k$ for all nodes k, where $|\cdot|$ denotes norms on the product space and on the local spaces without ambiguity.

Given a coupled map network $(G, \{T_k\}, A)$, if $(\tilde{G}, \{\tilde{T}_k^\varepsilon\}, \tilde{A}^\varepsilon)$ is a family of coupled map networks of the same nodes and local spaces, with a real parameter ε, such that $\tilde{T}_k^\varepsilon(x_k) = T_k(x_k) + \varepsilon\alpha(x_k)$ and $\tilde{A}^\varepsilon(x) = A(x) + \varepsilon\beta(x)$, where both α and β are bounded and continuous functions, then $(\tilde{G}, \{\tilde{T}_k^\varepsilon\}, \tilde{A}^\varepsilon)$ is approaching $(G, \{T_k\}, A)$ as ε tends to zero.

For a positive integer m, let $\mathbb{R}^m$ denote the space of all m-tuples of real numbers. Let $|\cdot|$ be a given norm on $\mathbb{R}^m$, and let $\|\cdot\|$ denote the operator-norm on the space of linear maps on $\mathbb{R}^m$ induced by $|\cdot|$. For $x \in \mathbb{R}^m$ and $r > 0$, we define $B^m(x, r) = \{z \in \mathbb{R}^m : |z - x| < r\}$; for the particular case where $x = 0$ and $r = 1$, we write $B^m = B^m(0, 1)$, that is, the open unit ball in $\mathbb{R}^m$. Furthermore, for any subset S of $\mathbb{R}^m$, let $\overline{S}$, $\text{int}(S)$ and ∂S denote the closure, interior and boundary of S, respectively. For the definition of *covering relations determined by a transition matrix*, see the Appendix.

Definition 3. Let F be a continuous map on $\mathbb{R}^m$. Define the maximum stretch $\|F\|_{\max} = \max\{|F(x)| : x \in \overline{B^m}\}$ and the minimum stretch $\|F\|_{\min} = \min\{|F(x)| : x \in \partial B^m\}$.

The maximum and minimum stretches are the radii of the smallest ball with center at the origin that contains $F(\overline{B^m})$ and of the largest open ball with center at the origin not intersecting $F(\partial B^m)$. If F is a linear map, the maximum and minimum stretches are the norm and conorm of F.

From now on, we consider a coupled map network $(G, \{T_k\}, A)$ such that G is a connected directed graph with nodes $\Omega = \{1, \ldots, d\}$ and edges $\mathcal{E} \subset \Omega \times \Omega$, and for $1 \leq k \leq d$, T_k is a continuous local map on $\mathbb{R}^{u+s}$ having covering relations on h-sets $\{M_{ki}\}_{i=1}^{d_k}$ determined by a transition matrix $W_k = [w_{kij}]_{1 \leq i,j \leq d_k}$ such that $u(M_{ki}) = u$ and $s(M_{ki}) = s$.

We say that the coupled map network $(G, \{T_k\}, A)$ is of *type I* with *locally linear coupling* if for each nonzero entry $\prod_{k=1}^d w_{k i_k j_k}$ of the Kronecker product $\otimes_{k=1}^d W_k$, there exists a $d \times d$ invertible real matrix $[a_{lm}]$ satisfying that $(m, l) \notin \mathcal{E}$ implies $a_{lm} = 0$, and the following conditions hold:

- For $1 \leq k \leq d$ and $1 \leq i', j' \leq d_k$ with $i' \neq i_k$ and $j' \neq j_k$,

$$T_k(M_{ki_k}) \cap (M_{kj'} \cup T(M_{ki'})) = \emptyset. \tag{1}$$

- For $1 \leq k \leq d$ and $x \in \overline{B^u}, y \in \overline{B^s}$,

$$c_{M_{kj_k}} \circ T_k \circ c_{M_{ki_k}}^{-1}(x, y) = (U_{ki_kj_k}(x), V_{ki_kj_k}(y)), \tag{2}$$

where $U_{ki_kj_k}$ and $V_{ki_kj_k}$ are continuous maps on $\mathbb{R}^u$ and $\mathbb{R}^s$, respectively, such that

$$\|U_{ki_kj_k}\|_{\min} > 1, \qquad \deg(U_{ki_kj_kj}, B^u, 0) \neq 0, \quad \text{and} \quad \|V_{ki_kj_k}\|_{\max} < 1. \tag{3}$$

- For $z \in (\prod_{k=1}^d c_{M_{kj_k}}) \circ T(\prod_{k=1}^d M_{ki_k})$,

$$\left(\prod_{k=1}^d c_{M_{kj_k}}\right) \circ A \circ \left(\prod_{k=1}^d c_{M_{kj_k}}\right)^{-1}(z) = ([a_{lm}] \otimes I)z, \tag{4}$$

where I is the $(u + s) \times (u + s)$ identity matrix.

Notice that (2) and (3) are only used to specify the covering relation $M_{ki_k} \overset{T_k}{\Longrightarrow} M_{kj_k}$. From (1) it follows that (4) is well defined and this says that the restriction of A to the set $T(\prod_{k=1}^{d} M_{ki_k})$ is topologically conjugate to a linear map, by the homeomorphism $\prod_{k=1}^{d} c_{M_{kj_k}}$. In general, the map on the left-hand side of (4) is not well defined.

Now, we state the first result about covering relations and the existence of periodic points for perturbations of coupled map networks, under permutation transition matrices.

Theorem 1. *Let $(G, \{T_k\}, A)$ be a coupled map network of type I with locally linear coupling as in (4) such that each of the transition matrices W_k, $1 \leq k \leq d$, is a permutation. Suppose that for each nonzero entry $\prod_{k=1}^{d} w_{ki_k j_k}$ of the Kronecker product $\otimes_{k=1}^{d} W_k$, there exists a permutation τ on $\{1, 2, \ldots, d\}$ such that for $1 \leq k \leq d$,*

$$\|a_{k\tau(k)} U_{\tau(k) i_{\tau(k)} j_{\tau(k)}}\|_{\min} - \sum_{l=1, l \neq \tau(k)}^{d} \|a_{kl} U_{l i_l j_l}\|_{\max} > 1 \quad and \quad \sum_{l=1}^{d} \|a_{kl} V_{l i_l j_l}\|_{\max} < 1. \tag{5}$$

Then any coupled map network $(\tilde{G}, \{\tilde{T}_k\}, \tilde{A})$ sufficiently close to $(G, \{T_k\}, A)$ has covering relations determined by $\otimes_{k=1}^{d} W_k$ and has a periodic point of period $\mathtt{lcm}(\dim W_1, \ldots, \dim W_d)$, where $\mathtt{lcm}$ stands for the least common multiple.

Before further investigating topological chaos for perturbations of coupled map networks, we introduce a notion of a unified h-set.

Definition 4. A d-tuple $(M_1, \ldots, M_d)$ of disjoint h-sets in $\mathbb{R}^m$ with $u(M_i) = u$ is said to be *unified* by a subset N of $\mathbb{R}^m$ if $\cup_{i=1}^{d} M_i \subset N$, and there exists a homeomorphism $\hat{c}_N : \mathbb{R}^m \to \mathbb{R}^m = \mathbb{R}^u \times \mathbb{R}^{m-u}$ such that for $1 \leq i \leq d$,

$$\hat{c}_N(N) = \overline{B^u}(q^u, (3d-1)/2) \times \overline{B^s}(0, 1) \quad and \quad \hat{c}_N(M_i) = \overline{B^u}(q_i^u, 1) \times \overline{B^s}(q_i^s, r_i), \tag{6}$$

where $q^u = ((3d-3)/2, 0, \ldots, 0)$ and $q_i^u = (3(i-1), 0, \ldots, 0)$ belong to $\mathbb{R}^u$, and $q_i^s = (\tilde{q}_i^s, 0, \ldots, 0)$ belongs to $\mathbb{R}^s$ for some real numbers $|\tilde{q}_i^s| < 1$ and $0 < r_i \leq 1$. Here, we call q_i^u, q_i^s, and r_i the ith *unstable center*, *stable center*, and *radius of stability* of N respectively. In particular, any h-set is unified by itself.

Being unified is a topological aspect: a tuple being unified means that the union of elements in the tuple is an enlarged h-set such that under a change of coordinates such as (6), the union looks like a product of unstable and stable balls, while the choice of centers and radii is flexible.

Let A_c be the following Kronecker product:

$$A_c = [a_{lm}] \otimes I, \tag{7}$$

where $[a_{lm}]$ is a $d \times d$ invertible real matrix such that if $(m, l) \notin \mathcal{E}$ then $a_{lm} = 0$, and I is the $(u+s) \times (u+s)$ identity matrix. We say that the coupled map network $(G, \{T_k\}, A)$ is of *type II* with the *linear coupling model A_c*, if the following conditions hold:

- For $1 \leq k \leq d$, there exists a set N_k such that the tuple $(M_{k1}, \ldots, M_{kd_k})$ is unified by N_k with the jth unstable center at p_{kj}^u, stable center at p_{kj}^s, and stable radius r_{kj} for all $1 \leq j \leq d_k$.
- For $1 \leq k \leq d$, if $w_{kij} = 1$, then for $x \in \overline{B^u}, y \in \overline{B^s}$,

$$\hat{c}_{N_k} \circ T_k \circ \tilde{c}_{M_{ki}}^{-1}(x, y) = (U_{ki}(x), V_{ki}(y)), \tag{8}$$

where

$$g_{ki}(x, y) = (x - p_{ki}^u, (y - p_{ki}^s)/r_{ki}) \quad and \quad \tilde{c}_{M_{ki}} = g_{ki} \circ \hat{c}_{N_k}, \tag{9}$$

and U_{ki}, V_{ki} are continuous maps on $\mathbb{R}^u$, $\mathbb{R}^s$, respectively, such that

$$\|U_{ki} - p_{kj}^u\|_{\min} > 1, \qquad \deg(U_{ki}, B^u, p_{kj}^u) \neq 0, \quad and \quad \|V_{ki} - p_{kj}^s\|_{\max} < r_{kj}. \tag{10}$$

- For $z \in (\prod_{k=1}^{d} \hat{c}_{N_k}) \circ T(\prod_{k=1}^{d} M_{ki})$,

$$\left(\prod_{k=1}^{d} \hat{c}_{N_k}\right) \circ A \circ \left(\prod_{k=1}^{d} \hat{c}_{N_k}\right)^{-1}(z) = A_c z. \tag{11}$$

Notice that (8) and (10) are only used to specify the covering relation $M_{ki} \overset{T_k}{\Longrightarrow} M_{kj}$ under the unified structure. With a help of (9), each quadruple $(M_{ki}, \tilde{c}_{M_{ki}}, u, s)$ is now an h-set in $\mathbb{R}^{u+s}$. Moreover, since $(\prod_{k=1}^{d} \hat{c}_{N_k})$ is independent of i and j, (11) is always well defined and this says that the restriction of A to the set $T(\prod_{k=1}^{d} M_{ki})$ is topologically conjugate to the linear map A_c, by the homeomorphism $(\prod_{k=1}^{d} \hat{c}_{N_k})$.

We give examples of coupled map networks of type II: one for $A = A_c$ and the other for $A \neq A_c$. For simplicity, linear local maps are given; they can be easily modified to nonlinear ones for the purpose of giving type II cases.

Example 5. Define the local dynamics by

$$T_1(x) = \begin{cases} 3.5x + 1.5, & \text{if } x \le 3/2, \\ 2x - 6, & \text{if } x > 3/2, \end{cases} \quad \text{and} \quad T_2(x) = \begin{cases} 2x + 3, & \text{if } x \le 3/2, \\ 3.5x - 9, & \text{if } x > 3/2. \end{cases}$$

Then T_1 has covering relations determined by $W_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ on the h-set tuple ($M_{11} = [-1, 1]$, $M_{12} = [2, 4]$), with $c_{M_{11}}(x) = x$, $c_{M_{12}}(x) = x - 3$ and $u = 1, s = 0$, which is unified by $N_1 = [-1, 4]$ with $\hat{c}_{N_1}(x) = x$ and unstable centers at $p_{11}^u = 0$ and $p_{12}^u = 3$. Also, T_2 has covering relations determined by $W_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ on the h-set tuple ($M_{21} = [-1, 1]$, $M_{22} = [2, 4]$), with $c_{M_{21}}(x) = x$, $c_{M_{22}}(x) = x - 3$ and $u = 1, s = 0$, which is unified by $N_2 = [-1, 4]$ with $\hat{c}_{N_2}(x) = x$ and unstable centers at $p_{11}^u = 0$ and $p_{12}^u = 3$. For $-1 \le x \le 1$, let $U_{11}(x) = 3.5x + 1.5, U_{12}(x) = 2x, U_{21}(x) = 2x + 3$, and $U_{22}(x) = 3.5x + 1.5$. Then (8) and (10) hold. Let G be a complete graph with two nodes and define a coupling by $A(x, y) = (a_{11}x + a_{12}y, a_{21}x + a_{22}y)$, where the a_{ij}'s are real numbers. Then, $(G, \{T_k\}, A)$ is of type II with the linear coupling model $A_c = A$.

Example 6. Define the local dynamics by

$$T_1(x) = \begin{cases} 3.5x - 1, & \text{if } x \le 5/2, \\ 2x - 7, & \text{if } x > 5/2, \end{cases} \quad \text{and} \quad T_2(x) = \begin{cases} 2x + 1, & \text{if } x \le 7/2, \\ 3.5x - 14, & \text{if } x > 7/2. \end{cases}$$

Then T_1 has covering relations determined by $W_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ on the h-set tuple ($M_{11} = [0, 2]$, $M_{12} = [3, 5]$), with $c_{M_{11}}(x) = x - 1$, $c_{M_{12}}(x) = x - 4$ and $u = 1, s = 0$, which is unified by $N_1 = [0, 5]$ with $\hat{c}_{N_1}(x) = x - 1$ and unstable centers at $p_{11}^u = 0$ and $p_{12}^u = 3$. Also, T_2 has covering relations determined by $W_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ on the h-set tuple ($M_{21} = [1, 3]$, $M_{22} = [4, 6]$), with $c_{M_{21}}(x) = x - 2$, $c_{M_{22}}(x) = x - 5$ and $u = 1, s = 0$, which is unified by $N_2 = [1, 6]$ with $\hat{c}_{N_2}(x) = x - 2$ and unstable centers at $p_{11}^u = 0$ and $p_{12}^u = 3$. For $-1 \le x \le 1$, let $U_{11}(x) = 3.5x + 1.5, U_{12}(x) = 2x, U_{21}(x) = 2x + 3$, and $U_{22}(x) = 3.5x + 1.5$. Then (8) and (10) hold. Let G be a complete graph with two nodes and define a coupling by $A(x, y) = (a_{11}(x - 1) + a_{12}(y - 2) + 1, a_{21}(x - 1) + a_{22}(y - 2) + 2)$, where the a_{ij}'s are real numbers. Then, $(G, \{T_k\}, A)$ is of type II with the linear coupling model $A_c(x, y) = (a_{11}x + a_{12}y, a_{21}x + a_{22}y)$.

Now, we state our result on covering relations and the topological entropy of perturbed coupled map networks.

Theorem 2. *Let $(G, \{T_k\}, A)$ be a coupled map network of type II with the linear coupling model A_c as in (7). Suppose that for each nonzero entry $\prod_{k=1}^d w_{k i_j j_k}$ of the Kronecker product $\otimes_{k=1}^d W_k$, there exists a permutation τ on $\{1, \ldots, d\}$ such that for $1 \le k \le d$, $p_{k j_k}^u \in a_{k\tau(k)} U_{\tau(k) i_{\tau(k)}}(B^u)$, and*

$$\|a_{k\tau(k)} U_{\tau(k) i_{\tau(k)}} - p_{k j_k}^u\|_{\min} - \sum_{l=1, l \ne \tau(k)}^d \|a_{kl} U_{l i_l}\|_{\max} > 1, \quad \text{and} \quad \sum_{l=1}^d \|a_{kl} V_{l i_l} - p_{k j_k}^s\|_{\max} < r_{k j_k}. \tag{12}$$

Then any coupled map network $(\tilde{G}, \{\tilde{T}_k\}, \tilde{A})$ sufficiently close to $(G, \{T_k\}, A)$ has covering relations determined by $\otimes_{k=1}^d W_k$ and has topological entropy bounded below by $\log(\prod_{k=1}^d \rho(W_k))$.

Remark 7. Let $\{A^\varepsilon\}$ and $\{T^\varepsilon\}$ be one-parameter families of maps on $\mathbb{R}^{(u+s)d}$, where $\varepsilon \in \mathbb{R}$ is a parameter, such that $A^0 = A, T^0 = \prod_{k \in \Omega} T_k$, and $A^\varepsilon(z)$ and $T^\varepsilon(z)$ are both continuous jointly in ε and z; then Theorem 2 holds for $A^\varepsilon \circ T^\varepsilon$ if ε is sufficiently small.

3. Proofs of Theorems 1 and 2

First, we list some known results [11,13] which will be needed in the proofs. The following one ensures the persistence of covering relations for C^0 perturbations.

Proposition 8 (*[11, Proposition 14]*). *Let M and N be h-sets in $\mathbb{R}^m$ with $u(M) = u(N) = u$ and $s(M) = s(N) = s$ and let $f, g : M \to \mathbb{R}^m$ be continuous. Assume that $M \overset{f}{\Longrightarrow} N$. Then there exists $\delta > 0$ such that if $|f(x) - g(x)| < \delta$ for all $x \in M$ then $M \overset{g}{\Longrightarrow} N$.*

The next statement says that a closed loop of covering relations implies the existence of a periodic point.

Proposition 9 (*[13, Theorem 9]*). *Let $\{f_i\}_{i=1}^k$ be a collection of continuous maps on $\mathbb{R}^m$ and $\{M_i\}_{i=1}^k$ be a collection of h-sets in $\mathbb{R}^m$ such that $M_{k+1} = M_1$ and $M_i \overset{f_i}{\Longrightarrow} M_{i+1}$ for $1 \le i \le k$. Then there exists a point $x \in \text{int}(M_1)$ such that*

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(x) \in \text{int}(M_{i+1}) \quad \text{for } i = 1, \ldots, k, \quad \text{and}$$
$$f_k \circ f_{k-1} \circ \cdots \circ f_1(x) = x.$$

It is known that a continuous map having covering relations determined by a transition matrix is topologically semi-conjugate to a one-sided subshift of finite type.

Proposition 10 ([11, Proposition 15]). *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous map which has covering relations determined by a transition matrix W. Then there exists a compact subset Λ of $\mathbb{R}^m$ such that Λ is maximal positively invariant for f in the union of the h-sets (with respect to W) and $f|\Lambda$ is topologically semi-conjugate to σ_W^+.*

Finally, we summarize basic properties of the local Brouwer degree; refer to [14, Chapter III] for the proof.

Proposition 11. *Let S be an open and bounded subset of $\mathbb{R}^m$ with $m \geq 1$, and let $\varphi : \bar{S} \to \mathbb{R}^m$ be continuous and $q \in \mathbb{R}^m$ such that $q \notin \varphi(\partial S)$. Then the following holds:*

1. *If φ is C^1 and for each $x \in \varphi^{-1}(q) \cap S$ the Jacobian matrix of φ at x, denoted by $D\varphi_x$, is nonsingular, then*

$$\deg(\varphi, S, q) = \sum_{x \in \varphi^{-1}(q) \cap S} \text{sgn}(\det D\varphi_x),$$

   where sgn is the sign function.
2. *Let $\psi : \mathbb{R}^m \to \mathbb{R}^m$ be a C^1 map and $p \in \mathbb{R}^m$ such that $\psi^{-1}(p)$ consists of a single point and lies in a bounded connected component Δ of $\mathbb{R}^m \setminus \varphi(\partial S)$, and $D\psi_{\psi^{-1}(p)}$ is nonsingular. Then*

$$\deg(\psi \circ \varphi, S, p) = \text{sgn}(\det D\psi_{\psi^{-1}(p)}) \deg(\varphi, S, v),$$

   for any $v \in \Delta$.
3. *Let S' be an open and bounded subset of $\mathbb{R}^n$ with $n \geq 1$, and let $\psi : \bar{S}' \to \mathbb{R}^n$ be continuous and $q' \in \mathbb{R}^n$ such that $q' \notin \psi(\partial S')$. Define a map $(\varphi, \psi) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ by $(\varphi, \psi)(x, y) = (\varphi(x), \psi(y))$ for $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. Then*

$$\deg((\varphi, \psi), S \times S', (q, q')) = \deg(\varphi, S, q) \deg(\psi, S', q').$$

Now, we are in a position to prove our main results.

Proof of Theorem 1. Let $\prod_{k=1}^d w_{k i_k j_k}$ be a nonzero entry of $\otimes_{k=1}^d W_k$. We shall prove that the following covering relation holds:

$$\prod_{k=1}^d M_{k i_k} \overset{A \circ T}{\Longrightarrow} \prod_{k=1}^d M_{k j_k}.$$

In the sequel, we use the following notation: $x_k \in \mathbb{R}^u$ and $y_k \in \mathbb{R}^s$ for $1 \leq k \leq d$, $x = \prod_{k=1}^d x_k \in (\mathbb{R}^u)^d = \mathbb{R}^{ud}$, $y = \prod_{k=1}^d y_k \in (\mathbb{R}^s)^d = \mathbb{R}^{sd}$, $\prod_{k=1}^d (x_k, y_k) \in (\mathbb{R}^{u+s})^d = \mathbb{R}^{(u+s)d}$, and $(x, y) \in \mathbb{R}^{ud} \times \mathbb{R}^{sd} = \mathbb{R}^{(u+s)d}$.

First, we check conditions on h-sets. Let $M = \prod_{k=1}^d M_{k i_k}$ and $N = \prod_{k=1}^d M_{k j_k}$. Then M and N are h-sets, with constants $u(M) = u(N) = ud$ and $s(M) = s(N) = sd$, and homeomorphisms $c_M, c_N : \mathbb{R}^{(u+s)d} \to \mathbb{R}^{ud} \times \mathbb{R}^{sd}$, defined as follows:

$$c_M \left(\prod_{k=1}^d (x_k, y_k) \right) = \varsigma \circ \prod_{k=1}^d c_{M_{k i_k}}(x_k, y_k), \quad \text{and} \quad c_N \left(\prod_{k=1}^d (x_k, y_k) \right) = \varsigma \circ \prod_{k=1}^d c_{M_{k j_k}}(x_k, y_k),$$

where $\varsigma : \mathbb{R}^{(u+s)d} \to \mathbb{R}^{ud} \times \mathbb{R}^{sd}$ is defined by $\varsigma(\prod_{k=1}^d (x_k, y_k)) = (x, y)$.

Second, we construct a homotopy such that (15)–(17) holds. Define a homotopy $H : [0, 1] \times \overline{B^{ud}} \times \overline{B^{sd}} \to \mathbb{R}^{(u+s)d}$ as

$$H(t, x, y) = (1 - t) c_N \circ A \circ T \circ c_M^{-1}(x, y) + t \pi_1 \circ c_N \circ A \circ T \circ c_M^{-1}(x, y)$$

where $\pi_1 : \mathbb{R}^{(u+s)d} \to \mathbb{R}^{(u+s)d}$ is defined by $\pi_1(x, y) = (x, 0)$ for all $x \in \mathbb{R}^{ud}$ and $y \in \mathbb{R}^{sd}$. Clearly, (15) holds.

Before checking (16) and (17) (see the Appendix), we derive a new form for the homotopy. Define $\bar{c}_M = \prod_{k=1}^d c_{M_{k i_k}}$ and $\bar{c}_N = \prod_{k=1}^d c_{M_{k j_k}}$. Then $c_M = \varsigma \circ \bar{c}_M$ and $c_N = \varsigma \circ \bar{c}_N$. Let $(x, y) \in \overline{B^{ud}} \times \overline{B^{sd}} = M_c$. Then $\bar{c}_N \circ T \circ \bar{c}_M^{-1}(\prod_{k=1}^d (x_k, y_k)) \in \bar{c}_N(T(M))$ and $\bar{c}_N \circ T \circ \bar{c}_M^{-1}(\prod_{k=1}^d (x_k, y_k)) = \prod_{k=1}^d (U_{k i_k j_k}(x_k), V_{k i_k j_k}(y_k))$. Moreover, by the definition of A_c, we obtain that

$$c_N \circ A \circ T \circ c_M^{-1}(x, y) = \varsigma \circ \bar{c}_N \circ A \circ T \circ \bar{c}_M^{-1} \circ \varsigma^{-1}(x, y)$$

$$= \varsigma \circ \bar{c}_N \circ A \circ \bar{c}_N^{-1} \circ \bar{c}_N \circ T \circ \bar{c}_M^{-1} \left(\prod_{k=1}^d (x_k, y_k) \right)$$

$$= \varsigma \circ A_c \circ \bar{c}_N \circ T \circ \bar{c}_M^{-1} \left(\prod_{k=1}^d (x_k, y_k) \right) = \varsigma \circ A_c \circ \prod_{k=1}^d (U_{k i_k j_k}(x_k), V_{k i_k j_k}(y_k))$$

$$= \varsigma \circ ([a_{lk}] \otimes I) \circ \prod_{k=1}^d (U_{k i_k j_k}(x_k), V_{k i_k j_k}(y_k))$$

$$= \varsigma \circ \prod_{l=1}^{d} \left(\sum_{k=1}^{d} a_{lk} U_{k i_k j_k}(x_k), \sum_{k=1}^{d} a_{lk} V_{k i_k j_k}(y_k) \right)$$

$$= \left(\prod_{l=1}^{d} \left(\sum_{k=1}^{d} a_{lk} U_{k i_k j_k}(x_k) \right), \prod_{l=1}^{d} \left(\sum_{k=1}^{d} a_{lk} V_{k i_k j_k}(y_k) \right) \right).$$

Therefore,

$$H(t, x, y) = \left(\prod_{l=1}^{d} \left(\sum_{k=1}^{d} a_{lk} U_{k i_k j_k}(x_k) \right), \prod_{l=1}^{d} \left((1-t) \sum_{k=1}^{d} a_{lk} V_{k i_k j_k}(y_k) \right) \right).$$

To prove (16), consider $(x, y) \in M_c^-$. Then there exists $1 \leq \beta \leq d$ such that $|x_\beta| = 1$. Since τ is a permutation one can find γ, $1 \leq \gamma \leq d$ such that $\tau(\gamma) = \beta$. By (5), we have that

$$\|a_{\gamma \beta} U_{\beta i_\beta j_\beta}\|_{\min} - \sum_{k=1, k \neq \beta}^{d} \|a_{\gamma k} U_{k i_k j_k}\|_{\max} > 1.$$

Hence,

$$\left| \sum_{k=1}^{d} a_{\gamma k} U_{k i_k j_k}(x_k) \right| \geq |a_{\gamma \beta} U_{\beta i_\beta j_\beta}(x_\beta)| - \sum_{k=1, k \neq \beta}^{d} |a_{\gamma k} U_{k i_k j_k}(x_k)|$$

$$\geq \|a_{\gamma \beta} U_{\beta i_\beta j_\beta}\|_{\min} - \sum_{k=1, k \neq \beta}^{d} \|a_{\gamma k} U_{k i_k j_k}\|_{\max}$$

$$> 1. \tag{13}$$

This implies that $H(t, x, y) \notin N_c$ and thus (16) holds.

For checking (17), consider $(x, y) \in M_c$. Then we get that

$$\left| (1-t) \sum_{k=1}^{d} a_{lk} V_{k i_k j_k}(y_k) \right| \leq \left| \sum_{k=1}^{d} a_{lk} V_{k i_k j_k}(y_k) \right|$$

$$\leq \sum_{k=1}^{d} \|a_{lk} V_{k i_k j_k}\|_{\max}$$

$$< 1,$$

where the last inequality follows from (5). Therefore $H(t, x, y) \notin N_c^+$ and hence (17) is true.

Next, we check item 2 in Definition 13 (see the Appendix). Consider a map $\varphi : \mathbb{R}^{ud} \to \mathbb{R}^{ud}$, where

$$\varphi(x) = \prod_{l=1}^{d} \left(\sum_{k=1}^{d} a_{lk} U_{k i_k j_k}(x_k) \right).$$

Then $H(1, x, y) = (\prod_{l=1}^{d} (\sum_{k=1}^{d} a_{lk} U_{k i_k j_k}(x_k)), 0) = (\varphi(x), 0)$. By (13), we have $\varphi(\partial B^{ud}) \subset \mathbb{R}^{ud} \setminus \overline{B^{ud}}$.

Finally, we show that the local Brouwer degree $\deg(\varphi, B^{ud}, 0)$ is nonzero. Observe that we can rewrite the above expression as

$$\varphi(x) = ([a_{lk}] \otimes I_u) \circ \prod_{k=1}^{d} U_{k i_k j_k}(x_k),$$

where I_u is the $u \times u$ identity matrix. Since the matrix $[a_{lk}]$ is invertible, $[a_{lk}] \otimes I_u$ is also invertible. Since $\|U_{k i_k j_k}\|_{\min} > 1$ and $\deg(U_{k i_k j_k}, B^u, 0) \neq 0$ for all $1 \leq k \leq d$, we have $0 \in U_{k i_k j_k}(B^u)$, and hence $0 = ([a_{lk}] \otimes I_u)^{-1}(0)$ lies in a bounded connected component of $\mathbb{R}^{ud} \setminus (\prod_{k=1}^{d} U_{k i_k j_k})(\partial B^{ud})$. By Proposition 11, we obtain that

$$\deg(\varphi, B^{ud}, 0) = \operatorname{sgn}(\det([a_{lk}] \otimes I_u)) \prod_{k=1}^{d} \deg(U_{k i_k j_k}, B^u, 0)$$

$$= \operatorname{sgn}(\det([a_{lk}] \otimes I_u)) \prod_{k=1}^{d} \deg(U_{k i_k j_k}, B^u, 0)$$

$$\neq 0.$$

We have proved that the needed covering relation holds. If $\tilde{T}_k$ and $\tilde{A}$ are both C^0 close enough to T_k and A respectively, then by Proposition 8, the following covering relation holds, for all nonzero entries $\prod_{k=1}^{d} w_{ki_kj_k}$ of $\otimes_{k=1}^{d} W_k$:

$$\prod_{k=1}^{d} M_{ki_k} \overset{\tilde{A} \circ \tilde{T}}{\Longrightarrow} \prod_{k=1}^{d} M_{kj_k}.$$

Therefore, $(\tilde{G}, \{\tilde{T}_k\}, \tilde{A})$ has covering relations determined by $\otimes_{k=1}^{d} W_k$. Since each W_k is a permutation, there exists a closed loop of covering relations for $\tilde{A} \circ \tilde{T}$ with loop length $\mathtt{lcm}(\dim W_1, \ldots, \dim W_d)$. By Proposition 9, $\tilde{A} \circ \tilde{T}$ has a periodic point of period $\mathtt{lcm}(\dim W_1, \ldots, \dim W_d)$. $\quad\square$

Next, we prove the second main result.

Proof of Theorem 2. Let $\prod_{k=1}^{d} w_{ki_kj_k}$ be a nonzero entry of $\otimes_{k=1}^{d} W_k$. We shall prove that the following covering relation holds:

$$\prod_{k=1}^{d} M_{ki_k} \overset{A \circ T}{\Longrightarrow} \prod_{k=1}^{d} M_{kj_k}.$$

We shall retain the use of the following notation: $x_k \in \mathbb{R}^u$ and $y_k \in \mathbb{R}^s$ for $1 \leq k \leq d$, $x = \prod_{k=1}^{d} x_k \in (\mathbb{R}^u)^d = \mathbb{R}^{ud}$, $y = \prod_{k=1}^{d} y_k \in (\mathbb{R}^s)^d = \mathbb{R}^{sd}$, $\prod_{k=1}^{d} (x_k, y_k) \in (\mathbb{R}^{u+s})^d = \mathbb{R}^{(u+s)d}$, and $(x, y) \in \mathbb{R}^{ud} \times \mathbb{R}^{sd} = \mathbb{R}^{(u+s)d}$.

First, we check conditions on h-sets. For convenience, we define $M = \prod_{k=1}^{d} M_{ki_k}$ and $M' = \prod_{k=1}^{d} M_{kj_k}$. Then M and M' are h-sets, and we also have constants $u(M) = u(M') = ud$ and $s(M) = s(M') = sd$, and homeomorphisms $c_M, c_{M'} : \mathbb{R}^{(u+s)d} \to \mathbb{R}^{ud} \times \mathbb{R}^{sd}$, defined as follows, for all $x_k \in \mathbb{R}^u$ and $y_k \in \mathbb{R}^s$, $1 \leq k \leq d$:

$$c_M \left(\prod_{k=1}^{d} (x_k, y_k) \right) = \varsigma \circ \prod_{k=1}^{d} \tilde{c}_{M_{ki_k}}(x_k, y_k), \quad \text{and} \quad c_{M'} \left(\prod_{k=1}^{d} (x_k, y_k) \right) = \varsigma \circ \prod_{k=1}^{d} \tilde{c}_{M_{kj_k}}(x_k, y_k),$$

where $\varsigma : \mathbb{R}^{(u+s)d} \to \mathbb{R}^{ud} \times \mathbb{R}^{sd}$ is defined by $\varsigma(\prod_{k=1}^{d}(x_k, y_k)) = (x, y)$.

Second, we construct a homotopy such that Eqs. (15)–(17) holds. Define a homotopy $H : [0, 1] \times \overline{B^{ud}} \times \overline{B^{sd}} \to \mathbb{R}^{(u+s)d}$ by, if $0 \leq t \leq 1/2$, then

$$H(t, x, y) = (1 - 2t)c_{M'} \circ A \circ T \circ c_M^{-1}(x, y) + 2t\pi_1 \circ c_{M'} \circ A \circ T \circ c_M^{-1}(x, y),$$

and if $1/2 < t \leq 1$, then

$$H(t, x, y) = (2 - 2t)\pi_1 \circ c_{M'} \circ A \circ T \circ c_M^{-1}(x, y) + (2t - 1) \left(\prod_{k=1}^{d} (a_{k\tau(k)} U_{\tau(k)i_{\tau(k)}}(x_{\tau(k)}) - p_{kj_k}^u), 0 \right)$$

where $\pi_1 : \mathbb{R}^{(u+s)d} \to \mathbb{R}^{(u+s)d}$ is defined by $\pi_1(x, y) = (x, 0)$ for all $x \in \mathbb{R}^{ud}$ and $y \in \mathbb{R}^{sd}$. Clearly, (15) holds.

Before checking (16) and (17), we derive a new form for the homotopy. Define $\check{c}_N = (\prod_{k=1}^{d} \hat{c}_{N_k})$. Let $(x, y) \in \overline{B^{ud}} \times \overline{B^{sd}} = M_c$. Then by the definitions of $\tilde{c}_{N_{kj_k}}$, we get that

$$\check{c}_N \circ T \circ \prod_{k=1}^{d} \tilde{c}_{N_{kj_k}}^{-1}(x_k, y_k) = \prod_{k=1}^{d} (U_{ki_k}(x_k), V_{ki_k}(y_k)) \in c_N(T(M)).$$

Thus,

$$\begin{aligned}
c_{M'} \circ A \circ T \circ c_M^{-1}(x, y) &= \varsigma \circ \left(\prod_{l=1}^{d} \tilde{c}_{N_{lj_l}} \right) \circ A \circ T \circ \left(\prod_{k=1}^{d} \tilde{c}_{M_{kj_k}}^{-1} \right) \circ \varsigma^{-1}(x, y) \\
&= \varsigma \circ \left(\prod_{l=1}^{d} \tilde{c}_{N_{lj_l}} \right) \circ A \circ \check{c}_N^{-1} \circ \check{c}_N \circ T \circ \prod_{k=1}^{d} \tilde{c}_{M_{kj_k}}^{-1}(x_k, y_k) \\
&= \varsigma \circ \left(\prod_{l=1}^{d} g_{lj_l} \circ \hat{c}_{N_l} \right) \circ A \circ \check{c}_N^{-1} \circ \prod_{k=1}^{d} (U_{ki_k}(x_k), V_{ki_k}(y_k)) \\
&= \varsigma \circ \left(\prod_{l=1}^{d} g_{lj_l} \right) \circ \check{c}_N \circ A \circ \check{c}_N^{-1} \circ \prod_{k=1}^{d} (U_{ki_k}(x_k), V_{ki_k}(y_k)).
\end{aligned}$$

Moreover, by the definitions of A_c and g_{lj_l}, we obtain that

$$
\begin{aligned}
c_{M'} \circ A \circ T \circ c_M^{-1}(x, y) &= \varsigma \circ \left(\prod_{l=1}^{d} g_{lj_l} \right) \circ A_c \circ \prod_{k=1}^{d} (U_{ki_k}(x_k), V_{ki_k}(y_k)) \\
&= \varsigma \circ \left(\prod_{l=1}^{d} g_{lj_l} \right) \circ ([a_{lk}] \otimes I) \circ \prod_{k=1}^{d} (U_{ki_k}(x_k), V_{ki_k}(y_k)) \\
&= \varsigma \circ \prod_{l=1}^{d} \left(\sum_{k=1}^{d} a_{lk} U_{ki_k}(x_k) - p_{lj_l}^u, \frac{\sum_{k=1}^{d} a_{lk} V_{ki_k}(y_k) - p_{lj_l}^s}{r_{lj_l}} \right) \\
&= \left(\prod_{l=1}^{d} \left(\sum_{k=1}^{d} a_{lk} U_{ki_k}(x_k) - p_{lj_l}^u \right), \prod_{l=1}^{d} \left(\frac{\sum_{k=1}^{d} a_{lk} V_{ki_k}(y_k) - p_{lj_l}^s}{r_{lj_l}} \right) \right).
\end{aligned}
$$

Therefore, for $0 \le t \le 1/2$,

$$
H(t, x, y) = \left(\prod_{l=1}^{d} \left(\sum_{k=1}^{d} a_{lk} U_{ki_k}(x_k) - p_{lj_l}^u \right), (1 - 2t) \prod_{l=1}^{d} \left(\frac{\sum_{k=1}^{d} a_{lk} V_{ki_k}(y_k) - p_{lj}^s}{r_{lj_l}} \right) \right),
$$

and, for $1/2 < t \le 1$,

$$
H(t, x, y) = \left(\prod_{l=1}^{d} \left(a_{l\tau(l)} U_{\tau(l)i_{\tau(l)}}(x_{\tau(l)}) - p_{lj_l}^u + (2 - 2t) \sum_{k=1, k \ne \tau(l)}^{d} a_{lk} U_{ki_k}(x_k) \right), 0 \right).
$$

For checking (16), consider $(x, y) \in M_c^-$. Then there exists $1 \le \beta \le d$ such that $|x_\beta| = 1$. Since τ is a permutation, there exists $1 \le \gamma \le d$ such that $\tau(\gamma) = \beta$. By (12), we have that

$$
\| a_{\gamma\beta} U_{\beta i_\beta} - p_{\gamma j_\gamma}^u \|_{\min} - \sum_{k=1, k \ne \beta}^{d} \| a_{\gamma k} U_{ki_k} \|_{\max} > 1.
$$

This implies that

$$
\begin{aligned}
\left| \sum_{k=1}^{d} a_{\gamma k} U_{ki_k}(x_k) - p_{\gamma j_\gamma}^u \right| &\ge | a_{\gamma\beta} U_{\beta i_\beta}(x_\beta) - p_{\gamma j_\gamma}^u | - \sum_{k=1, k \ne \beta}^{d} | a_{\gamma k} U_{ki_k}(x_k) | \\
&\ge \| a_{\gamma\beta} U_{\beta i_\beta} - p_{\gamma j_\gamma}^u \|_{\min} - \sum_{k=1, k \ne \beta}^{d} \| a_{\gamma k} U_{ki_k} \|_{\max} \\
&> 1
\end{aligned}
\tag{14}
$$

and, for $1/2 < t \le 1$,

$$
\begin{aligned}
\left| a_{\gamma\beta} U_{\beta i_\beta}(x_\beta) - p_{\gamma j_\gamma}^s + (2 - 2t) \sum_{k=1, k \ne \beta}^{d} a_{\gamma k} U_{ki_k}(x_k) \right| &\ge | a_{\gamma\beta} U_{\beta i_\beta}(x_\beta) - p_{\gamma j_\gamma}^u | - (2 - 2t) \sum_{k=1, k \ne \beta}^{d} | a_{\gamma k} U_{ki_k}(x_k) | \\
&\ge \| a_{\gamma\beta} U_{\beta i_\beta} - p_{\gamma j_\gamma}^u \|_{\min} - \sum_{k=1, k \ne \beta}^{d} \| a_{\gamma k} U_{ki_k} \|_{\max} \\
&> 1.
\end{aligned}
$$

Thus $H(t, x, y) \notin M_c'$ and hence (16) holds.

For checking (17), consider $(x, y) \in M_c$. Then, for $0 \le t \le 1/2$, we have that

$$
\left| (1 - 2t) \frac{\sum_{k=1}^{d} a_{lk} V_{ki_k}(y_k) - p_{lj}^s}{r_{lj_l}} \right| \le \frac{1}{r_{lj_l}} \left| \sum_{k=1}^{d} a_{lk} V_{ki_k}(y_k) - p_{lj_l}^s \right|
$$

$$
\le \frac{1}{r_{lj_l}} \sum_{k=1}^{d} \| a_{lk} V_{ki_k} - p_{lj_l}^s \|_{\max}
$$

$$
< 1
$$

where the last inequality follows from (12). Thus, $H(t, x, y) \notin N_c^+$ and hence (17) holds.

Next, we check item 2 in Definition 13. Define a map $\varphi : \mathbb{R}^{ud} \to \mathbb{R}^{ud}$ by

$$
\varphi(x) = \prod_{l=1}^{d} (a_{l\tau(l)} U_{\tau(l)i_{\tau(l)}}(x_{\tau(l)}) - p_{lj_l}^u).
$$

Then $H(1, x, y) = (\prod_{l=1}^{d} (a_{l\tau(l)} U_{\tau(l)i_{\tau(l)}}(x_{\tau(l)}) - p_{lj_l}^u), 0) = (\varphi(x), 0)$. By Eq. (14), we have $\varphi(\partial B^{ud}) \subset \mathbb{R}^{ud} \setminus \overline{B^{ud}}$.

Finally, we prove that the local Brouwer degree $\deg(\varphi, B^{ud}, 0)$ is nonzero. Define a function $g(x) = \prod_{l=1}^{d} (x_l - p_{lj_l}^u)$ for $x \in \mathbb{R}^{ud}$, and a $d \times d$ matrix $[b_{lk}]$ such that $b_{l\tau(l)} = a_{l\tau(l)}$ and $b_{lk} = 0$ for all $k \ne \tau(l)$. Then

$$
\varphi(x) = g \circ ([b_{lk}] \otimes I_u) \circ \prod_{k=1}^{d} U_{ki_k}(x_k),
$$

where I_u is the $u \times u$ identity matrix. In order to apply Proposition 11 for $\deg(\varphi, B^{ud}, 0)$, we need to check conditions on the affine maps g and $[b_{lk}] \otimes I_u$. By the definition of g, we get that $g^{-1}(0)$ consists of a single point $p \equiv \prod_{l=1}^{d} p_{lj_l}^u$. By the definition of $[b_{lk}]$ and $b_{l\tau(l)} = a_{l\tau(l)}$, the hypothesis $p_{lj_l}^u \in a_{l\tau(l)} U_{\tau(l)i_{\tau(l)}}(B^u)$, together with (12), implies that the matrix $[b_{lk}]$ is invertible; otherwise, $\| a_{l\tau(l)} U_{\tau(l)i_{\tau(l)}} - p_{lj_l}^u \|_{\min} = 0$ leads to a contradiction. Hence, $([b_{lk}] \otimes I_u)^{-1}(p)$ lies in a bounded connected component of $\mathbb{R}^{ud} \setminus (\prod_{k=1}^{d} U_{ki_k})(\partial B^{ud})$. Since $\deg(U_{li_l}, B^u, p_{lj_l}^u) \ne 0$, we have $p_{lj_l}^u \in U_{li_l}(B^u)$ and hence $p \in \prod_{l=1}^{d} U_{li_l}(B^u)$. By applying Proposition 11, since $\deg(U_{li_l}, B^u, p_{lj_l}^u) \ne 0$ for all $1 \le l \le d$, we obtain that

$$
\deg(\varphi, B^{ud}, 0) = \mathrm{sgn}(\det(Dg_p)) \mathrm{sgn}(\det([b_{lk}] \otimes I_u)) \deg \left(\prod_{l=1}^{d} U_{li_l}, B^{ud}, p \right)
$$

$$
= \mathrm{sgn}(\det(Dg_p)) \mathrm{sgn}(\det([b_{lk}] \otimes I_u)) \prod_{l=1}^{d} \deg(U_{li_l}, B^u, p_{lj_l}^u)
$$

$$
\ne 0.
$$

This concludes the proof of the covering relation needed. If $\tilde{T}_k$ and $\tilde{A}$ are both C^0 close enough to T_k and A respectively, then by Proposition 8, the following covering relation holds, for all nonzero entries $\prod_{l=1}^{d} w_{li_l j_l}$ of $W \equiv \otimes W_l$:

$$
\prod_{l=1}^{d} M_{li_l} \overset{\tilde{A} \circ \tilde{T}}{\Longrightarrow} \prod_{l=1}^{d} M_{lj_l}.
$$

Therefore, $(\tilde{G}, \{\tilde{T}_k\}, \tilde{A})$ has covering relations determined by W. By Proposition 10, there exists a compact subset $\tilde{\Lambda}$ of $\mathbb{R}^{(u+s)d}$ such that $\tilde{\Lambda}$ is a maximal positive invariant for $\tilde{A} \circ \tilde{T}$ in the union of the h-sets (with respect to W) and $\tilde{A} \circ \tilde{T}|\tilde{\Lambda}$ is topologically semi-conjugate to σ_W^+. Therefore,

$$
h_{\mathrm{top}}(\tilde{A} \circ \tilde{T}) \ge h_{\mathrm{top}}(\sigma_W^+) = \log(\rho(W)) = \log \left(\prod_{l=1}^{d} \rho(W_l) \right). \quad \square
$$

Acknowledgments

Appendix

First, we briefly recall some definitions from [13] concerning covering relations.

Definition 12 ([13, Definition 6]). An *h-set* in $\mathbb{R}^m$ is a quadruple consisting of the following data:

- a nonempty compact subset M of $\mathbb{R}^m$,
- a pair of numbers $u(M), s(M) \in \{0, 1, \ldots, m\}$ with $u(M) + s(M) = m$,
- a homeomorphism $c_M : \mathbb{R}^m \to \mathbb{R}^m = \mathbb{R}^{u(M)} \times \mathbb{R}^{s(M)}$ with $c_M(M) = \overline{B^{u(M)}} \times \overline{B^{s(M)}}$, where $S \times T$ is the Cartesian product of sets S and T.

For simplicity, we will denote such an h-set by M and call c_M the *coordinate chart* of M; furthermore, we use the following notation:

$$M_c = \overline{B^{u(M)}} \times \overline{B^{s(M)}}, \qquad M_c^- = \partial B^{u(M)} \times \overline{B^{s(M)}}, \qquad M_c^+ = \overline{B^{u(M)}} \times \partial B^{s(M)},$$

$$M^- = c_M^{-1}(M_c^-), \quad \text{and} \quad M^+ = c_M^{-1}(M_c^+).$$

A covering relation between two h-sets is defined as follow.

Definition 13 ([13, Definition 7]). Let M, N be h-sets in $\mathbb{R}^m$ with $u(M) = u(N) = u$ and $s(M) = s(N) = s$, $f : M \to \mathbb{R}^m$ be a continuous map, and $f_c = c_N \circ f \circ c_M^{-1} : M_c \to \mathbb{R}^u \times \mathbb{R}^s$. We say that M f-*covers* N, and write

$$M \xLongrightarrow{f} N,$$

if the following conditions are satisfied:

1. there exists a homotopy $h : [0, 1] \times M_c \to \mathbb{R}^u \times \mathbb{R}^s$ such that

$$h(0, x) = f_c(x) \quad \text{for } x \in M_c, \tag{15}$$

$$h([0, 1], M_c^-) \cap N_c = \emptyset, \tag{16}$$

$$h([0, 1], M_c) \cap N_c^+ = \emptyset; \tag{17}$$

2. there exists a map $\varphi : \mathbb{R}^u \to \mathbb{R}^u$ such that

$$h(1, p, q) = (\varphi(p), 0) \quad \text{for any } p \in \overline{B^u} \quad \text{and} \quad q \in \overline{B^s},$$

$$\varphi(\partial B^u) \subset \mathbb{R}^u \setminus \overline{B^u}; \quad \text{and}$$

3. there exists a nonzero integer w such that the local Brouwer degree $\deg(\varphi, B^u, 0)$ of φ at 0 in B^u is w; refer to [13, Appendix] for its properties.

A *transition matrix* is a square matrix which satisfies the following conditions:

(i) all entries are either 0 or 1,
(ii) all row sums and column sums are greater than or equal to 1.

For a transition matrix W, let $\rho(W)$ denote the spectral radius of W. Then $\rho(W) \geq 1$ and, moreover, if W is irreducible and not a permutation, then $\rho(W) > 1$. Let Σ_W^+ (resp. Σ_W) be the space of all allowable one-sided (resp. two-sided) sequences generated by the transition matrix W with a usual metric, and let $\sigma_W^+ : \Sigma_W^+ \to \Sigma_W^+$ (resp. $\sigma_W : \Sigma_W \to \Sigma_W$) be the one-sided (resp. two-sided) subshift of finite type for W. Then $h_{\text{top}}(\sigma_W^+) = h_{\text{top}}(\sigma_W) = \log(\rho(W))$ (refer to [15] for more background).

Definition 14. Let $W = [w_{ij}]_{1 \leq i, j \leq \gamma}$ be a transition matrix and f be a continuous map on $\mathbb{R}^m$. We say that f has *covering relations determined by* W if the following conditions are satisfied:

1. there are γ pairwise disjoint h-sets $\{M_i\}_{i=1}^\gamma$ in $\mathbb{R}^m$;
2. if $w_{ij} = 1$ then the covering relation $M_i \xLongrightarrow{f} M_j$ holds.

References

[1] L.-S. Young, Chaotic phenomena in three settings: large, noisy and out of equilibrium, Nonlinearity 21 (2008) T245–T252.
[2] V.S. Afraimovich, L.A. Bunimovich, Dynamical networks: interplay of topology, interactions and local dynamics, Nonlinearity 20 (2007) 1761–1771.
[3] V.S. Afraimovich, L.A. Bunimovich, S.V. Moreno, Dynamical networks: continuous time and general discrete time models, Regul. Chaotic Dyn. 15 (2010) 127–145.
[4] F. Bauer, F.M. Atay, J. Jost, Synchronization in discrete-time networks with general pairwise coupling, Nonlinearity 22 (2009) 2333–2351.
[5] B. Fernandez, Global synchronization in translation invariant coupled map lattices, Int. J. Bifurcation Chaos Appl. Sci. Eng. 18 (2008) 3455–3459.
[6] J.-R. Chazottes, B. Fernandez (Eds.), Dynamics of Coupled Map Lattices and of Related Spatially Extended Systems, in: Lecture Notes in Physics, vol. 671, Springer, Berlin, 2005.
[7] J. Koiller, L.-S. Young, Coupled map networks, Nonlinearity 23 (2010) 1121–1141.
[8] P. Zgliczyński, Fixed point index for iterations, topological horseshoe and Chaos, Topol. Methods Nonlinear Anal. 8 (1996) 169–177.
[9] P. Zgliczyński, Computer assisted proof of chaos in the Rössler equations and in the Hénon map, Nonlinearity 10 (1997) 243–252.
[10] M. Misiurewicz, P. Zgliczyński, Topological entropy for multidimensional perturbations of one-dimensional maps, Int. J. Bifurcation Chaos Appl. Sci. Eng. 11 (2001) 1443–1446.
[11] M.-C. Li, M.-J. Lyu, Topological dynamics for multidimensional perturbations of maps with covering relations and strong Liapunov condition, J. Differential Equations 250 (2011) 799–812.
[12] M.-C. Li, M.-J. Lyu, P. Zgliczyński, Topological entropy for multidimensional perturbations of snap-back repellers and one-dimensional maps, Nonlinearity 21 (2008) 2555–2567.
[13] P. Zgliczyński, M. Gidea, Covering relations for multidimensional dynamical systems I, J. Differential Equations 202 (2004) 32–58.
[14] J.T. Schwartz, Nonlinear Functional Analysis, Gordon and Breach Science Publishers, New York, 1969.
[15] C. Robinson, Dynamical Systems: Stability, Symbolic Dynamics, and Chaos, CRC Press, Boca Raton, FL, 1999.