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# Wellbore flow-rate solution for a constant-head test in two-zone finite confined aquifers

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## Abstract:

The solution describing the wellbore flow rate in a constant-head test integrated with an optimization approach is commonly used to analyze observed wellbore flow-rate data for estimating the hydrogeological parameters of low-permeability aquifers. To our knowledge, the wellbore flow-rate solution for the constant-head test in a two-zone finite-extent confined aquifer has never been reported so far in the literature. This article is first to develop a mathematical model for describing the head distribution in the two-zone aquifer. The Laplace domain solutions for the head distributions and wellbore flow rate in a two-zone finite confined aquifer are derived using the Laplace transform, and their corresponding time domain solutions are then obtained using the Bromwich integral method and residue theorem. These new solutions are expressed in terms of an infinite series with Bessel functions and not straightforward to calculate numerically. A large-time solution for the wellbore flow rate is therefore developed by employing the relationship of small Laplace variable *versus* large time variable and L'Hospital's rule. The result shows that the large-time solution is identical to the steady-state solution obtained after applying the Tauberian theorem into the Laplace domain solution. This large-time solution can reduce to the Thiem equation in the case of no skin. Finally, the newly developed solution is used to investigate the effects of outer boundary distance and conductivity ratio on the wellbore flow rate. 

KEY WORDS analytical solution; aquifer test; Laplace transform; Bromwich integral; residue theorem; composite aquifer; Thiem equation

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## INTRODUCTION

The observed data of wellbore flow rate from a constant-head test is commonly analyzed to determine the aquifer hydrogeological parameters of low-permeability aquifers. The analysis of wellbore flow rate data usually relies upon an approach which includes the analytical solution of the wellbore flow rate coupled with an optimization scheme (see, e.g. Yeh *et al.*, 2007a,b; Yeh and Chen, 2007). A number of studies have been presented the wellbore flow-rate models for describing field constant-head tests (e.g. Mishra and Guyonnet, 1992; Markle *et al.*, 1995; Chen and Chang, 2002). Interestingly, the transient solution for the wellbore flow rate expressed in slightly different formats also appears in a variety of disciplines such as heat transfer (Carslaw and Jaeger, 1959) and electrochemistry (e.g. Aoki *et al.*, 1985; Szabo *et al.*, 1987; Fang *et al.*, 2009; Britz *et al.* 2010; Bieniasz, 2011). Based on the heat conduction solution of Smith (1937), Jacob and Lohman (1952) presented an analytical expression for the wellbore flow rate to a constant-head test in an infinite aquifer. Moreover, the transmissivity and storativity were determined by plotting the ratio of flow rate to the constant drawdown at the test well against time to squared well radius. Later, Carslaw and Jaeger provided a formula for the heat flux across the inner boundary in a radial heat conduction problem (1959, p. 336, Equation (8)) which can also be used to describe the groundwater flow problems of the constant-head test. Van Everdingen and Hurst (1949) developed the transient pressure head and wellbore flow rate solutions for the constant-flux and constant-head tests in finite and infinite confined aquifers without considering the skin zone. For the well surrounded by a skin, Uraiet and Raghavan (1980) examined the transient flow rate at a well and pressure behavior in finite and infinite aquifers when the test well maintains at a constant drawdown. They demonstrated the concepts of the infinitesimally thin skin and effective wellbore radius are applicable to describe the skin region around a well producing at constant pressure. In addition, they also plotted the simulation results obtained from a finite difference model to demonstrate the effects of skin thickness and permeability ratio between the skin and formation zones on the wellbore flow rate. Yang and Yeh (2002) provided an analytical solution of the wellbore flow rate for the constant-head test performed in an infinite-extent aquifer with considering the effects of the finite well radius and skin zone. They further developed the transient analytical solution of the hydraulic head distributions in the patch and outer regions for the constant-head test in a patchy aquifer of infinite extent (Yang and Yeh, 2006). To our knowledge, the transient analytical solution for the head distribution or the wellbore flow rate in a two-zone finite-extent aquifer has never been reported so far in the literature.

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The objective of this article is first to develop a mathematical model to describe the groundwater flow in the skin and formation zones for the constant-head test in two-zone finite confined aquifers. The head solution of the model is obtained by applying the Laplace transform technique, Bromwich integral method, and residue theorem. The wellbore flow rate solution is obtained by first applying Darcy's law to the Laplace-domain head solution in the skin zone and then transforming the result to the time domain using the Bromwich integral method and residue theorem. In addition, a large-time solution for the wellbore flow rate is also developed by employing the relationship of small Laplace-domain variable $p$ with a large time-domain variable $t$ and L'Hospital's rule. The result of large-time wellbore flow happens to be the steady-state solution which can reduce to the Thiem equation if neglecting the skin effect. Finally, the dimensionless forms of the solutions for hydraulic head and well bore flow rate are also presented for practical uses or engineering applications. The wellbore flow-rate solution is also used to study the effects of boundary distance and conductivity ratio on the estimated flow rate for confined aquifers of finite extent.

The solutions for the head distribution and wellbore flow rate developed herein are mainly for a two-zone aquifer which can also be called as patchy aquifer (Baker and Herbert, 1988), nonuniform aquifer (Butler, 1988), composite formation (Novakowski, 1989), or aquifer with a skin zone (Yang and Yeh, 2002). It is well recognized that the groundwater flow is analogous to the heat flow. Therefore, the solutions developed in this paper can naturally be regarded as an extension of the work in Carslaw and Jaeger (1959, p. 333, Equation (10) for the case that $k_1 = k_1' = 0$) for heat flow in a composite hollow cylinder.

## MATHEMATICAL DEVELOPMENT

### *Mathematical model for a two-zone confined aquifer*

The assumptions for the mathematical model describing the head distribution for the constant-head test in a two-zone confined aquifer are: (1) the test well is of a finite radius and fully penetrates the aquifer thickness; (2) the well has a finite-thickness skin zone with different hydrogeological properties from the formation zone; (3) the aquifer is homogeneous in each zone and bounded by a finite outer boundary in the formation zone; and (4) the water level in the test well is maintained constant.

Figure 1 shows the schematic diagram of the two-zone aquifer. The governing equations describing the hydraulic head $h(r, t)$ for the skin zone and formation zone are (Yang and Yeh, 2002), respectively,

$$\frac{\partial^2 h_1}{\partial r^2} + \frac{1}{r}\frac{\partial h_1}{\partial r} = \frac{S_1}{T_1}\frac{\partial h_1}{\partial t} \qquad r_w < r < r_1 \tag{1}$$

and

$$\frac{\partial^2 h_2}{\partial r^2} + \frac{1}{r}\frac{\partial h_2}{\partial r} = \frac{S_2}{T_2}\frac{\partial h_2}{\partial t} \qquad r_1 < r < R \tag{2}$$

where subscripts 1 and 2 denote the skin and formation zones, respectively, the variable $r$ is the radial distance from the central line of the test well, $r_w$ is the well radius, $r_1$ is the radial distance from the central line to the outer boundary of the skin zone, $R$ is the distance from the central line to the outer boundary of the formation zone, $t$ is the time, $S$ is the storage coefficient, and $T$ is the transmissivity.

The initial hydraulic head of the aquifer is considered to be zero. The initial head conditions for Equations (1) and (2) are therefore

$$h_1(r, 0) = h_2(r, 0) = 0 \tag{3}$$

The hydraulic head is assumed to equal zero at the outer boundary. On the other hand, a constant well water level (or hydraulic head) $h_w$ is maintained at the wellbore. Thus, the hydraulic heads at the outer and inner boundaries are given, respectively, as

$$h_2(R, t) = 0 \tag{4}$$

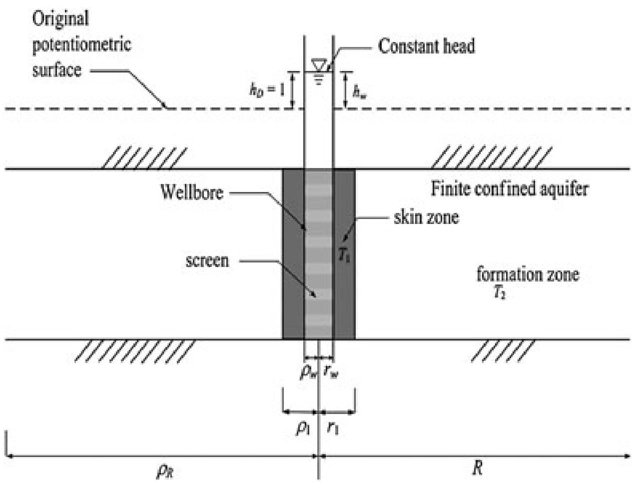


Figure 1. Schematic diagram of the constant-head test in a finite confined aquifer

 

and

$$h_1(r_w, t) = h_w \tag{5}$$

The continuity conditions for the hydraulic head and flow rate at the interface between the skin and formation zones require, respectively,

$$h_1(r_1, t) = h_2(r_1, t) \tag{6}$$

and

$$T_1 \frac{\partial h_1(r_1, t)}{\partial r} = T_2 \frac{\partial h_2(r_1, t)}{\partial r} \tag{7}$$

*The Laplace-domain solutions for head distribution and wellbore flow rate*

The Laplace-domain solution for head distributions in the skin and formation zones obtained by applying the Laplace Transform to Equations (1) and (2) with Equations (3–7) are

$$\bar{h}_1 = \frac{1}{p}\left[\frac{h_w(\Phi_1 I_0(q_1 r) - \Phi_2 K_0(q_1 r))}{\Phi_1 I_0(q_1 r_w) - \Phi_2 K_0(q_1 r_w)}\right] \tag{8}$$

and

$$\bar{h}_2 = \frac{h_w}{p}\frac{\Phi_1 I_0(q_1 r_1) - \Phi_2 K_0(q_1 r_1)}{[\Phi_1 I_0(q_1 r_w) - \Phi_2 K_0(q_1 r_w)]} \times \left[\frac{I_0(q_2 R)K_0(q_2 r) - K_0(q_2 R)I_0(q_2 r)}{I_0(q_2 R)K_0(q_2 r_1) - I_0(q_2 r_1)K_0(q_2 R)}\right] \tag{9}$$

where $p$ is the Laplace variable, $q_1 = \sqrt{pS_1/T_1}$, $q_2 = \sqrt{pS_2/T_2}$, $I_0$ and $K_0$ are the modified Bessel functions of the first and second kinds of order zero, respectively, and $I_1$ and $K_1$ are the modified Bessel functions of the first and second kinds of order first, respectively. The variables $\Phi_1$ and $\Phi_2$ in Equations (8) and (9) are defined as

$$\Phi_1 = \phi\sqrt{\frac{S_2 T_2}{S_1 T_1}} K_0(q_1 r_1)K_0(q_2 r_1) - K_1(q_1 r_1)K_0(q_2 r_1) \tag{10}$$

and

$$\Phi_2 = \phi\sqrt{\frac{S_2 T_2}{S_1 T_1}} I_0(q_1 r_1)K_0(q_2 r_1) + I_1(q_1 r_1)K_0(q_2 r_1) \tag{11}$$

with

$$\phi = \frac{I_0(q_2 R)K_1(q_2 r_1) + I_1(q_2 r_1)K_0(q_2 R)}{I_0(q_2 R)K_0(q_2 r_1) - I_0(q_2 r_1)K_0(q_2 R)} \tag{12}$$

Applying Darcy's law to Equation (8) and setting $r = r_w$, the Laplace-domain solution for the wellbore flow rate can then be obtained as

$$\overline{Q}(r_w) = 2\pi r_w T_1 \frac{q_1 h_w}{p}\left[\frac{\Phi_1 I_1(q_1 r_w) + \Phi_2 K_1(q_1 r_w)}{\Phi_2 K_0(q_1 r_w) - \Phi_1 I_0(q_1 r_w)}\right] \tag{13}$$

If the outer boundary distance approaches infinity, i.e. $R \to \infty$, Equation (12) becomes $\phi = K_1(q_2 r_1)/K_0(q_2 r_1)$. Equations (10) and (11) can then reduce to the corresponding equations given in Yang and Yeh (2002, Equations (10) and (11)). Equations (8) and (9) for the head distribution therefore reduce to the solutions represented in Yang and Yeh (2002, Equations (8) and (9)) for the infinite aquifer. In addition, Equation (13) for the wellbore flow rate reduces to the one given in Yang and Yeh (2002, Equation (12)) for the infinite aquifer.

*Time-domain solution*

The time-domain solution for the head distributions in the skin and formation zones can be obtained by applying the Bromwich integral method and residue theorem to Equations (8) and (9). The detailed derivation is shown in Appendix A, and the results are

$$h_1(r,t) = h_w\left\{\frac{\ln(r_1/r) + T_1/T_2 \ln(R/r_1)}{\ln(r_1/r_w) + T_1/T_2 \ln(R/r_1)} + \pi\sum_{n=1}^{\infty}\frac{[J_0(\alpha_n r_w)Y_0(\alpha_n r) - Y_0(\alpha_n r_w)J_0(\alpha_n r)]\exp\left(-\alpha_n^2 t T_1/S_1\right)}{\xi_n{}^2\left[(\zeta A_n)^2 + B_n^2 + \zeta((B_n C_n + A_n D_n) + A_n B_n/\alpha_n)/r_1\right] - 1}\right\} \tag{14}$$

$$h_2(r,t) = h_w\left\{\frac{T_1/T_2 \ln(R/r)}{\ln(r_1/r_w) + T_1/T_2 \ln(R/r_1)} + \pi\sum_{n=1}^{\infty}\frac{[J_0(\alpha_n r_w)Y_0(\alpha_n r_1) - Y_0(\alpha_n r_w)J_0(\alpha_n r_1)] \times [Y_0(\xi\alpha_n R)J_0(\xi\alpha_n r) - Y_0(\xi\alpha_n r)J_0(\xi\alpha_n R)]\exp\left(-\alpha_n^2 t T_1/S_1\right)}{\left\{\xi_n{}^2\left[(\zeta A_n)^2 + B_n^2 + \zeta((B_n C_n + A_n D_n) + A_n B_n/\alpha_n)/r_1\right] - 1\right\} \times (-B_n)}\right\} \tag{15}$$

These two solutions are significantly different from the solutions presented in Yang and Yeh (2002, Equations (12) and (13)) which are in terms of integrals with lower and upper limits from zone to infinity for the two-zone aquifer of infinite extent. In addition, Equations (14) and (15) are developed by the Bromwich integral along a contour of infinite poles and residue theorem, while the solutions of Yang and Yeh (2002) were obtained based on the Bromwich integral with a single branch point in the integrand. The transient solution for the wellbore flow rate can also be developed in a similar manner to the one presented in Appendix A when transforming Equation (13) into the time domain. The result is

$$Q(r_w,t)=2\pi h_w T_1\left\{\frac{1}{\ln(r_1/r_w)+T_1/T_2\ln(R/r_1)}+2\sum_{n=1}^{\infty}\frac{\exp\left(-\alpha_n^2 tT_1/S_1\right)}{\varsigma_n{}^2\left[(\zeta A_n)^2+B_n^2+\zeta((B_nC_n+A_nD_n)+A_nB_n/\alpha_n)/r_1\right]-1}\right\} \tag{16}$$

with

$$\varsigma_n=\frac{-J_0(\alpha_n r_w)}{-\zeta A_n J_0(\alpha_n r_1)-B_n J_1(\alpha_n r_1)} \tag{17}$$

$$A_n=J_1(\xi\alpha_n r_1)Y_0(\xi\alpha_n R)-J_0(\xi\alpha_n R)Y_1(\xi\alpha_n r_1) \tag{18}$$

$$B_n=J_0(\xi\alpha_n R)Y_0(\xi\alpha_n r_1)-J_0(\xi\alpha_n r_1)Y_0(\xi\alpha_n R) \tag{19}$$

$$C_n=-\xi R[J_1(\xi\alpha_n r_1)Y_1(\xi\alpha_n R)-J_1(\xi\alpha_n R)Y_1(\xi\alpha_n r_1)]-\xi r_1 B_n-\frac{A_n}{\alpha_n} \tag{20}$$

$$D_n=\xi R[J_1(\xi\alpha_n R)Y_0(\xi\alpha_n r_1)-J_0(\xi\alpha_n r_1)Y_1(\xi\alpha_n R)]-\xi r_1 A_n \tag{21}$$

where $J_0$ and $Y_0$ are the Bessel functions of the first and second kinds of order zero, respectively, $J_1$ and $Y_1$ are the Bessel functions of the first and second kinds of order first, respectively, $\xi=\sqrt{T_1S_2/T_2S_1}$, $\zeta=\sqrt{S_2T_2/S_1T_1}$, and $\alpha_n$ are the roots of

$$\begin{aligned}&[J_1(\xi\alpha r_1)Y_0(\xi\alpha R)-J_0(\xi\alpha R)Y_1(\xi\alpha r_1)]\\&\times\zeta[Y_0(\alpha r_w)J_0(\alpha r_1)-Y_0(\alpha r_1)J_0(\alpha r_w)]\\&+[J_0(\xi\alpha r_1)Y_0(\xi\alpha R)-J_0(\xi\alpha R)Y_0(\xi\alpha r_1)]\\&\times[Y_1(\alpha r_1)J_0(\alpha r_w)-J_1(\alpha r_1)Y_0(\alpha r_w)]=0\end{aligned} \tag{22}$$

By neglecting the presence of skin zone, Equation (16) reduces to

$$Q(r_w,t)=2\pi h_w T\left\{\frac{1}{\ln(R/r_w)}+2\sum_{n=1}^{\infty}\frac{\exp\left(-T\alpha_n^2 t/S\right)}{\left[J_0^2(\alpha_n r_w)-J_0^2(\alpha_n R)\right]/J_0^2(\alpha_n R)}\right\} \tag{23}$$

where $\alpha_n$ are the roots of $J_0(\alpha r_w)Y_0(\alpha R)-Y_0(\alpha r_w)J_0(\alpha R)=0$. Equation (23) is exactly the same as the formula represented in Wang and Yeh (2008, Equation (5)).

*The large-time wellbore flow-rate solution in a finite confined aquifer*

The approximations of $I_0(x)\sim 1/\Gamma(1)$, $I_1(x)\sim x/2\Gamma(2)$, $K_0(x)\sim -\ln(x)$, and $K_1(x)\sim 1/x$ can be made when the arguments of Bessel functions are small (Abramowitz and Stegun, 1979, p. 375). The Laplace domain solution for the large-time wellbore flow rate can therefore be obtained from Equation (13) after employing the relationship of small *p versus* large *t* (Yeh and Wang, 2007) and L'Hospital's rule as

$$\overline{Q}(r_w,p)=\frac{2\pi T_1 h_w}{p}\frac{-1}{\ln\left(\frac{r_w}{r_1}\right)+\frac{T_1}{T_2}\ln\left(\frac{R}{r_1}\right)} \tag{24}$$

where the negative sign in Equation (24) represents withdrawal in the test well.

The large-time solution for the wellbore flow rate is then obtained after taking the inverse Laplace transform of Equation (24) as

$$Q(r_w,t)=2\pi T_1 h_w\frac{-1}{\ln\left(\frac{r_w}{r_1}\right)+\frac{T_1}{T_2}\ln\left(\frac{R}{r_1}\right)} \tag{25}$$

which is independent of time and indeed a steady-state solution. In fact, this solution can also be obtained if applying the Tauberian theorem (Yeh and Wang, 2007) to Equation (13). This result indicates that the wellbore flow-rate solution for a two-zone confined aquifer of finite extent can reach steady state when the time is large. In addition, Equation (25) can be simplified to the Thiem equation if neglecting the skin zone, i.e. setting $r_1$ equals $r_w$.

*Dimensionless solutions*

The dimensionless variables defined for simplifying the developed solutions are $\kappa=T_2/T_1$, $\gamma=S_2/S_1$, $\tau=T_2t/S_2r_w^2$, $\rho=r/r_w$, $\rho_1=r_1/r_w$, $\rho_R=R/r_w$, $h_D=h/h_w$ and $Q_D=Q(r_w)/(2\pi T_2 h_w)$. The variable $\kappa$ represents the conductivity ratio, $\gamma$ represents the ratio of storage coefficients of the skin and formation zones, $\rho$ represents dimensionless distance, $\rho_1$

represents dimensionless distance of the outer boundary of the skin zone, $\overline{h}_D$ represents the dimensionless head distribution in the Laplace domain, $h_D$ represents the dimensionless head distribution in time domain, $\overline{Q}D$ represents the dimensionless flow rate in Laplace domain, and $Q_D$ represents the dimensionless flow rate in time domain.

The time domain solution for the dimensionless head distributions in the skin and formation zones can then be written as

$$h_{1D} = \left\{ \frac{\ln(\rho_1/\rho) + 1/\kappa \ln(\rho_R/\rho_1)}{\ln(\rho_1) + 1/\kappa \ln(\rho_R/\rho_1)} + \pi \sum_{n=1}^{\infty} \frac{[J_0(\beta_n)Y_0(\beta_n\rho) - Y_0(\beta_n)J_0(\beta_n\rho)]\exp\left(-\gamma\beta_n^2\tau/\kappa\right)}{\xi_{Dn}{}^2\left[(\zeta a_n)^2 + b_n^2 + \zeta((b_nc_n + a_nd_n) + a_nb_n/\beta_n)/\rho_1\right] - 1} \right\} \tag{26}$$

$$h_{2D} = \left\{ \frac{1/\kappa \ln(\rho_R/\rho)}{\ln(\rho_1) + 1/\kappa \ln(\rho_R/\rho_1)} \right. \tag{27}$$

$$\left. + \pi \sum_{n=1}^{\infty} \frac{[J_0(\beta_n)Y_0(\beta_n\rho) - Y_0(\beta_n)J_0(\beta_n\rho)] \times [Y_0(\xi\beta_n\rho_R)J_0(\xi\beta_n\rho) - Y_0(\xi\beta_n\rho)J_0(\xi\beta_n\rho_R)]\exp\left(-\gamma\beta_n^2\tau/\kappa\right)}{\left\{\xi_{Dn}{}^2\left[(\zeta a_n)^2 + b_n^2 + \zeta((b_nc_n + a_nd_n) + a_nb_n/\beta_n)/\rho_1\right] - 1\right\} \times (-b_n)} \right\}$$

In addition, the time-domain solution of the dimensionless wellbore flow rate is

$$Q_D = \frac{1}{\kappa} \left\{ \frac{1}{\ln(\rho_1) + 1/\kappa \ln(\rho_R/\rho_1)} + 2 \sum_{n=1}^{\infty} \frac{\exp\left(-\gamma\beta_n^2\tau/\kappa\right)}{\xi_{Dn}{}^2\left[(\zeta a_n)^2 + b_n^2 + \zeta((b_nc_n + a_nd_n) + a_nb_n/\beta_n)/\rho_1\right] - 1} \right\} \tag{28}$$

where $\beta_n = r_w\alpha_n$ are the roots of

$$\begin{aligned} &[J_1(\xi\beta\rho_1)Y_0(\xi\beta\rho_R) - J_0(\xi\beta\rho_R)Y_1(\xi\beta\rho_1)] \\ &\times\zeta[Y_0(\beta\rho_w)J_0(\beta\rho_1) - Y_0(\beta\rho_1)J_0(\beta\rho_w)] \\ &+[J_0(\xi\beta\rho_1)Y_0(\xi\beta\rho_R) - J_0(\xi\beta\rho_R)Y_0(\xi\beta\rho_1)] \\ &\times[Y_1(\beta\rho_1)J_0(\beta\rho_w) - J_1(\beta\rho_1)Y_0(\beta\rho_w)] = 0 \end{aligned} \tag{29}$$

with

$$\varsigma_{Dn} = \frac{-J_0(\beta_n)}{-\zeta a_nJ_0(\beta_n\rho_1) - b_nJ_1(\beta_n\rho_1)} \tag{30}$$

$$a_n = J_1(\xi\beta_n\rho_1)Y_0(\xi\beta_n\rho_R) - J_0(\xi\beta_n\rho_R)Y_1(\xi\beta_n\rho_1) \tag{31}$$

$$b_n = J_0(\xi\beta_n\rho_R)Y_0(\xi\beta_n\rho_1) - J_0(\xi\beta_n\rho_1)Y_0(\xi\beta_n\rho_R) \tag{32}$$

$$c_n = -\kappa\rho_R[J_1(\xi\beta_n\rho_1)Y_1(\xi\beta_n\rho_R) - J_1(\xi\beta_n\rho_R)Y_1(\xi\beta_n\rho_1)] - \xi\rho_1b_n - \frac{a_n}{\beta_n} \tag{33}$$

$$d_n = \xi\rho_R[J_1(\xi\beta_n\rho_R)Y_0(\xi\beta_n\rho_1) - J_0(\xi\beta_n\rho_1)Y_1(\xi\beta_n\rho_R)] - \xi\rho_1a_n \tag{34}$$

The numerical evaluations of Equation (28) can be achieved by finding the roots of Equation (29) first by Newton's method and then adding the summation term for $n$ up to 100. The accuracy of the results can be made at least to the fifth decimal.

## ADVANTAGES AND APPLICATIONS OF THE SOLUTIONS

### *Advantages over the existing solutions*

To our knowledge, there are only two articles, i.e. Yang and Yeh (2002) and the present one, in the groundwater literature to provide the transient analytical solutions (in time domain) of the wellbore flow rate for the constant-head test in two-zone aquifer systems. The present article has following three advantages over Yang and Yeh (2002). First, the present solutions can reduce to those given in Yang and Yeh (2002) when the outer boundary goes infinity. In other words, the solutions presented in Yang and Yeh (2002) can be considered as a special case of the present solutions. Second, the solution in Yang and

Yeh (2002) for the head distributions in the skin and formation zones is only in Laplace domain while that in the present article is in time domain. Finally and most importantly, the wellbore flow-rate solution in Yang and Yeh (2002, Equation (19)) is in terms of an integral from zero to infinity with the variable $u$ in the denominator of the integrand, posing the problem of singularity at the origin for the integration as indicated in Yang and Yeh (2002, p. 178, Figure 2). Due to the presence of singular point, the results of numerical evaluation for the integral are accurate only to the second decimal as shown in Yang and Yeh (2002, p. 179, Tables 1 and 2). On the other hand, the present wellbore flow solution is composed of infinite series and can be easily evaluated with accuracy to at least five digits after the decimal.

*Potential applications*

The properties of an aquifer with the presence of skin zone may be characterized by five parameters, i.e. the outer radius of the skin zone and the transmissivity and storage coefficient for each of the skin and aquifer zones. If those parameters are known, the presented solution can be used to predict the wellbore flow rate and head distributions in both the skin and formation zones and explore the physical insight of the constant-head test in a two-zone aquifer system. On the other hand, if the parameters are not available, the determination of those five parameters from analyzing measured data is a subject of inverse problem. It may not be possible or too complicate to develop type curves for parameter estimation because the unknowns are too many. Feasible ways of solving such an inverse problem of involving five unknown parameters are to adopt the presented solution and couple it with the algorithm of extended Kalman filter (e.g. Leng and Yeh 2003; Yeh and Huang 2005) or with an optimization approach such as the nonlinear least-squares (e.g. Yeh, 1987) or simulated annealing (e.g. Lin and Yeh, 2005; Yeh *et al.*, 2007a,b).

The present solution can also be used as a tool to design a field constant-head or to verify newly developed numerical codes for simulating the flow in two-zone aquifer systems. Generally speaking, the sensitivity analysis (Liou and Yeh, 1997) can be performed to investigate the effect of changing input parameters (i.e. the five parameters) on the output (i.e. head distribution or wellbore flow rate). It works as an indicator in assessing the influences of parameter uncertainty on the predicted head or wellbore flow rate. If the head (or wellbore flow rate) is very sensitive to a specific parameter, a small change in that parameter will then markedly affect the predicted head (or wellbore flow rate). In contrast, the change in a less sensitive parameter will have little impact on the predicted result. This indicates that a less sensitive parameter is much more difficult to be estimated. With the present solution, the sensitivity analysis can be easily performed to the targeted parameters (e.g. Huang and Yeh, 2007), and the results will provide useful information about the degrees of sensitivity among targeted parameters.

## RESULTS AND DISCUSSION

Figure 2 shows the curves of the dimensionless wellbore flow rate *versus* dimensionless time $\tau$ for various conductivity ratios $\kappa$ and outer boundary distance $\rho$ with dimensionless radial distance $\rho_1 = 5$. Note that $\kappa < 1$ denotes for the negative, skin case while $\kappa > 1$ for the positive skin case. The figure indicates that the dimensionless wellbore flow rate for an aquifer with a positive skin is always smaller than that with a negative one. For small values of $\kappa$, the wellbore flow-rate solution for an aquifer of finite extent is equal to that of infinite extent

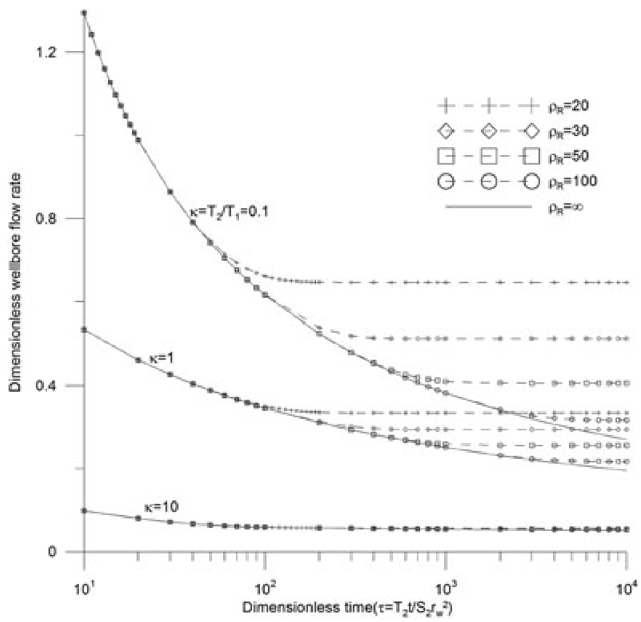


Figure 2. The curves of the dimensionless wellbore flow rate *versus* dimensionless pumping time for various conductivity ratio $\kappa$ and outer boundary distance $\rho_R$ with $\rho_1 = 3$. The solid line represents the solution for the infinite-extent aquifer, while the dash line with the symbols represents the solution for the finite-extent aquifer

only at early test time (say, roughly $\tau < 40$ when $\kappa = 0.1$ and $\tau < 100$ when $\kappa = 1$) indicating that wellbore flow rate will reach steady state quickly for aquifers with negative skins. In the period of moderate time ($100 < \tau < 1000$), these two flow-rate solutions deviate from one another, indicating that the finite-extent solution can no longer be used to approximate to the infinite-extent solution. In other words, the outer boundary distance has an effect on the wellbore flow rate when the pumping time is not short. Finally, the finite-extent solution tends to reach an asymptotic limit, the steady-state solution, revealing the significance of boundary effect on the wellbore flow rate while the infinite-extent solution decreases endlessly with dimensionless time. The effect of boundary distance on the wellbore flow rate is very small for a high value of $\kappa$ (say, 10). Figure 2 also indicates that the dimensionless time when the outer boundary starts to affect the wellbore flow rate increases with the dimensionless outer boundary distance $\rho_R$.

## CONCLUSIONS

A mathematical model for describing the head distributions in a two-zone confined aquifer bounded by a finite outer boundary for a constant-head test has been presented. The Laplace-domain solution for the head distributions in the skin and formation zones are first developed using the Laplace transform, and the Laplace-domain solution for the wellbore flow rate is then developed based on the head solution of the skin zone and Darcy's law. Both the Laplace-domain solutions for the head distribution and wellbore flow rate can reduce to the solutions for the two-zone aquifer with an infinite boundary when the outer boundary distance approaches infinity. The transient solutions of the head distribution and wellbore flow rate are then developed from their Laplace-domain solutions by the Bromwich integral method and residue theorem. Finally, the relationship of small Laplace variable *versus* large time variable and L'Hospital's rule are used to develop the large-time wellbore flow rate which turns out to be the steady-state solution. This result indicates that the wellbore flow rate can reach steady state quickly for a finite-domain aquifer. In addition, this steady-state result reduces to the Thiem equation if the skin effect is negligible.

The wellbore flow-rate solution for the constant-head test is used to investigate the effects of conductivity ratio and outer boundary distance on the flow rate across the wellbore in a two-zone confined aquifer bounded by a finite outer boundary. The result indicates that the dimensionless flow rate for an aquifer with a positive skin is always smaller than that with a negative skin. For small conductivity ratios, the flow-rate solution for an aquifer of finite extent equals that of infinite extent only at early test time, indicating that wellbore flow rate can reach steady state quickly for aquifers with the negative skin. The wellbore flow-rate solution for a finite aquifer tends to an asymptotic limit at large time while that solution for an infinite aquifer decreases endlessly with dimensionless time. The effect of outer boundary distance on the wellbore flow rate is small for an aquifer of high conductivity ratio and the dimensionless time when the outer boundary begins to influence the wellbore flow rate increases with the dimensionless boundary distance.

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## APPENDIX: DERIVATION OF EQUATION (14)

The head distributions in time domain, $h_1$, can be obtained by Bromwich integral method (Carslaw and Jaeger, 1959) as

$$h_1(t) = L^{-1}[\bar{h_1}(p)] = \frac{1}{2\pi i}\int_{r_e-i\infty}^{r_e+i\infty} e^{pt}\bar{h_1}(p)dp \quad \text{(A1)}$$

where $i$ is an imaginary unit and $r_e$ is a very large real constant that all of the real parts of the poles are smaller than it. The graph of the Bromwich integral contains a close contour with a straight line parallel to the imaginary axis and a semicircle. According to Jordan's Lemma, the value of the integration for the semicircle tends to zero when its radius approaches infinity. Based on the residue theorem, the head distribution in the skin zone (Equation (A1)) can be expressed as

$$h_1(t) = \sum_{n=1}^{\infty} \mathrm{Re}\, s[e^{pt}\bar{h_1}(p); g_n] \quad \text{(A2)}$$

where $g_n$ are the poles in the complex plane.

There are infinite poles in $\bar{h_1}(p)$ and obviously one pole at $p=0$. To determine other poles, the denominator and numerator in the brackets of Equation (8) are written, respectively, as

$$\Delta = \Phi_1 I_0(q_1 r_w) - \Phi_2 K_0(q_1 r_w) \quad \text{(A3)}$$

$$\Psi = h_w[\Phi_1 I_0(q_1 r) - \Phi_2 K_0(q_1 r)] \quad \text{(A4)}$$

Let $\Delta = 0$, the roots $\alpha_n$ in $p = p_n = \left(-T_1\alpha_n^2\right)/S_1$ can be determined from Equation (A3) with $q_1 = i\alpha_n$ and $q_2 = -\xi\alpha_n^2$. Substituting $p_n = \left(-T_1\alpha_n^2\right)/S_1$ into Equation (A3) yields Equation (22). The residue of the pole at $p=0$ can then be obtained from the following formula (Kreyszig, 1999)

$$\mathrm{Res}[e^{pt}\bar{h_1}(p); 0] = \lim_{p\to 0}\bar{h_1}(p)e^{pt}(p-0) \quad \text{(A5)}$$

Substituting Equation (8) into Equation (A5) and applying L'Hospital's rule, the result is

$$\mathrm{Res}[e^{pt}\bar{h_1}(p); 0] = h_w\frac{\ln(r_1/r) + T_1/T_2\ln(R/r_1)}{\ln(r_1/r_w) + T_1/T_2\ln(R/r_1)} \quad \text{(A6)}$$

The other residues at the poles $p_n = -T_1\alpha_n^2/S_1$ can be written as

$$\mathrm{Res}[e^{pt}\bar{h_1}(p); p_n] = \lim_{p\to p_n}\bar{h_1}(p)e^{pt}(p-p_n) \quad \text{(A7)}$$

Applying L'Hospital's rule to Equation (A7), the denominator of Equation (8) becomes

$$\left[p\frac{d\Delta}{dp}\right]_{p=-T_1\alpha_n^2/S_1} = \left[\frac{1}{2}q\frac{d\Delta}{dq}\right]_{q_1=i\alpha_n, q_2=i\kappa\alpha_n} = \frac{1}{2}q_1\left\{\Phi_1' I_0(q_1 r_w) - \Phi_2' K_0(q_1 r_w) + r_w[\Phi_1 I_1(q_1 r_w) + \Phi_2 K_1(q_1 r_w)\right) \quad \text{(A8)}$$

where the variables $\Phi_1$ and $\Phi_2$ are defined in Equations (10) and (11), respectively, and $\Phi_1'$ and $\Phi_2'$ are the differentiations of $\Phi_1$ and $\Phi_2$, respectively.

A variable $\varsigma_n$ introduced based on Equation (A3) and $\Delta = 0$ to simplify Equation (A8) is defined as

$$\varsigma_n = \frac{I_0(q_1 r_w)K_0(\xi q_1 r_1)}{\Phi_2[(I_0(\xi q_1 R)K_0(\xi q_1 r_1) - I_0(\xi q_1 r_1)K_0(\xi q_1 R)]} = \frac{K_0(q_1 r_w)K_0(\xi q_1 r_1)}{\Phi_1[(I_0(\xi q_1 R)K_1(\xi q_1 r_1) + I_1(\xi q_1 r_1)K_0(\xi q_1 R)]} \quad \text{(A9)}$$

$$\left[p\frac{d\Delta}{dp}\right]_{p=-T_1\alpha_n^2/S_1} = \left[\frac{1}{2}q\frac{d\Delta}{dq}\right]_{q_1=i\alpha_n,q_2=i\kappa\alpha_n} = \frac{1}{2\varsigma_n}\left\{\varsigma_n{}^2\left[(\zeta A_n)^2 + B_n^2 + \zeta((B_nC_n + A_nD_n) + A_nB_n/\alpha_n)/r_1\right] - 1\right\} \quad \text{(A13)}$$

In addition, following two recurrence formulas (Carslaw and Jaeger, 1959, p. 490) are used to eliminate the imaginary unit in Equation (8):

$$K_v\left(ze^{\pm\frac{1}{2}\pi i}\right) = \pm\frac{1}{2}\pi i e^{\mp\frac{1}{2}v\pi i}[-J_v(z) \pm iY_v(z)] \quad \text{(A10)}$$

and

$$I_v\left(ze^{\pm\frac{1}{2}\pi i}\right) = e^{\pm\frac{1}{2}v\pi i}J_v(z) \quad \text{(A11)}$$

Substituting Equations (A10) and (A11) into Equation (A9) results in

$$\frac{-J_0(\alpha_n r_w)}{-\zeta A_n J_0(\alpha_n r_1) - B_n J_1(\alpha_n r_1)} = \frac{Y_0(\alpha_n r_w)}{-\zeta A_n Y_0(\alpha_n r_1) + B_n Y_1(\alpha_n r_1)} = \varsigma_n \quad \text{(A12)}$$

With Equations (A9) and (A12), Equation (A8) is given as

where the constants shown on the right-hand side of Equation (A13) have been defined in Equations (17–21). Similarly, the numerator of Equation (8) can also be obtained as

$$\Psi = \frac{1}{2\varsigma_n}\{\pi h_w[-Y_0(\alpha_n r_w)J_0(\alpha_n r) + J_0(\alpha_n r_w)Y_0(\alpha_n r)]\} \quad \text{(A14)}$$

The residues at the poles $p_n = -T_1\alpha_n^2/S_1$ can be obtained from Equations (A13) and (A14) as

$$\text{Res}[e^{pt}\bar{h}_1(p);p_n] = h_w\pi\sum_{n=1}^{\infty}\frac{[J_0(\alpha_n r_w)Y_0(\alpha_n r) - Y_0(\alpha_n r_w)J_0(\alpha_n r)]\exp\left(-\alpha_n^2 tT_1/S_1\right)}{\xi_n{}^2\left[(\zeta A_n)^2 + B_n^2 + \zeta((B_nC_n + A_nD_n) + A_nB_n/\alpha_n)/r_1\right] - 1}\} \quad \text{(A15)}$$

Therefore, Equation (A2) can be expressed as

$$h(t) = \text{Res}[e^{pt}\bar{h}_1(p);0] + \text{Res}[e^{pt}\bar{h}_1(p);p_n] \quad \text{(A16)}$$

Finally, the head distribution in the skin zone can be obtained from Equations (A6) and (A15) and given as Equation (14).

 