XT Domineering: A new combinatorial game

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ABSTRACT

This paper introduces a new combinatorial game, named XT Domineering, together with its mathematical analysis. XT Domineering is modified from the Domineering game in which 1×2 or 2×1 dominos are allowed to be placed on empty squares in an $m \times n$ board. This new game allows a player to place a 1×1 domino on an empty square s while unable to place a 1×2 or 2×1 domino in the connected group of empty squares that includes s. After modifying the rule, each position in the game becomes an infinitesimal. This paper calculates the game values of all sub-graphs of 3×3 squares and shows that each sub-graph of 3×3 squares is a linear combination of 8 elementary infinitesimals. These pre-stored game values can be viewed as a knowledge base for playing XT Domineering. Instead of searching the whole game trees, a simple rule for determining the optimal outcome of any sum of these positions is presented.

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1. Introduction

Since the 1970s, combinatorial game theory [1,2] has become the common fundamental mathematical model for the analysis of many intelligent games. Based on the theory, playing or solving many combinatorial games such as Nim [3,4], Triangular Nim [5], Clobber [6] and Cutthroat [7] may simply become mathematical calculations, such as summation, instead of a complex tree search.

Domineering, designed by Göran Andersson (cf.[8]), is one of combinatorial games based on the model. In an $m \times n$ Domineering, two players alternatively place 1×2 and 2×1 domino at a position, if there exists such a vacancy in a board with $m \times n$ squares. One player is allowed to place 1×2 domino only, while the other is 2×1 domino only. The one who cannot place domino loses.

In the past, many Domineering problems were solved. The general Domineering problem of $2 \times n$ board for all odd n was solved by Berlekamp [9]. The researchers in [10] used the technique of transposition tables to solve the 8×8 board. Subsequently, the researchers in [11] found out the results for boards of width 2, 3, 5, and 7 and some specific cases. Recently, Bullock solved the 10×10 board Domineering [12]. Furthermore, Cincotti developed three players Domineering on a three dimensional board [13,14].

This paper introduces a new game named *XT Domineering* (named from *eXTended Domineering*). XT Domineering, modified from the Domineering game, allows players to place a 1×1 domino on an empty square s while unable to place a 1×2 or 2×1 domino in the connected group of empty squares that includes s.

A connected group of empty squares is called an *active* group. After modifying the rule, players are allowed to place 1×1 domino on any square of an active group on which players are not allowed to place any dominos in the original Domineering game. For example, in XT Domineering, all 1×1 isolated vacancies in the board are allowed to be placed by more dominos. Thus, the move lengths in the new game are normally longer than those in Domineering. Thus, the game has higher game-tree complexity, based on the definition in [15].

This paper also introduces the mathematical analysis of XT Domineering. In XT Domineering, each game position is actually an infinitesimal (as described in Section 4). In this paper, we study several interesting infinitesimals in XT Domineering. This paper calculates the game values of all sub-graphs of 3×3 squares and presents a rule to determine the outcome of any sum of these positions.

Section 2 reviews the combinational games including three subgroups of games. Section 3 reviews the game Domineering and introduces the new game, XT Domineering. Section 4 derives the game values of 3×3 XT Domineering, while Section 5 derives the outcomes of sums of 3×3 XT Domineering. Section 6 concludes this paper.

2. Combinatorial games

Combinatorial game theory [2] starts from a simple definition of game: a game is an ordered pair of sets of games. Conventionally, a game G is denoted as:

$$G = \{G^L | G^R\}, \tag{1}$$

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where G^L and G^R are sets of games. A special game is named 0, when both G^L and G^R are empty sets, $\varnothing$.

Negation, addition and comparisons are defined as follows.

$$- G = \{-G^R | - G^L\}, \tag{2}$$

$$G + H = \left\{ G^L + H, G + H^L |^R + H, G + H^R \right\}, \tag{3}$$

$$G \geqslant 0, \quad \text{if and only if there is no element in } G^R \leqslant 0, \tag{4}$$

$$G \leqslant 0, \quad \text{if and only if } - G^R \geqslant 0, \tag{5}$$

$$G \geqslant H, \quad \text{if and only if } G - H \geqslant 0. \tag{6}$$

When neither $G \geqslant H$ nor $G \leqslant H$, it is said G confused with H, denoted by $G\|H$. $G < |H$ denotes either $G < H$ or $G\|H$, and similarly for $G| > H$. Furthermore, an equivalence relation on the sets of games is defined as follows.

$$G \equiv H, \quad \text{if and only if } G \geqslant H \text{ and } G \leqslant H. \tag{7}$$

The equivalence classes of games form an algebraic group, which can be used to describe the positions of many intelligent games as follows.

- There are two players (say Left and Right) move alternatively.
- The game is a sum of positions; each position has two sets of next positions; one for each player.
- On each player's turn, the player can choose one position and move the position to one of its next positions.
- The player who cannot find a move is the loser.

For each game G, there are four types of possible outcomes. The corresponding relations between G and 0 are described as below:

- $G \equiv 0$: The first player cannot win the game.
- $G < 0$: Left cannot win the game.
- $G > 0$: Right cannot win the game.
- $G\|0$: The first player can win the game.

In general, players are concerned with who can win a given game G. Mathematically speaking, the question is equivalent to determining one of the above four relations between G and 0. Since we are dealing with equivalence classes, for simplicity, we shall use the symbol = to replace $\equiv$ in the following context.

There are several subgroups of combinatorial games whose addition and outcome properties are well-studied. Some of them are reviewed in the following subsections.

2.1. Numbers

A game G is called a *number* [1,2] if all the elements in G^L and G^R are numbers and there is no element in G^L greater than or equal to any element in G^R. Some numbers are illustrated as follows:

$$1 = \{0|\varnothing\}, \dots, \\ n = \{n - 1|\varnothing\}, \tag{8}$$

$$1/2 = \{0|1\}, \dots, \\ m/2^k = \{(m - 1)/2^k|(m + 1)/2^k\}. \tag{9}$$

These numbers (integers and rationals) can be added as the usual ways. Numbers are well ordered, and their relations with 0 are clear. Hence, one can easily determine the outcome for any sum of numbers.

2.2. Nimbers

A game G is a *nimber* [3,4] if all the elements in G^L and G^R are nimbers and $G^L = G^R$. Nimbers are defined as:

$$^*1 = \{0|0\},$$
$$^*2 = \{0, ^*1|0, ^*1\},$$
$$\dots$$
$$^*n = \{0, ^*1, ^*2, \dots, ^*(n - 1)|0, ^*1, ^*2, \dots, ^*(n - 1)\}. \tag{10}$$

For simplicity, ^{*1} is also denoted as * and named *star*. The special nimber with infinite options:

$$\Leftrightarrow = \{0, ^*1, ^*2, \dots | 0, ^*1, ^*2, \dots\} \tag{11}$$

is named *remote star*.

For each non-zero nimber, the first player can win a game. That is, each non-zero nimber is confused with 0. Hence one can easily determine the outcome of any sum of nimbers [4,16]. From this, two well-known properties are (1) $^*n + ^*n = 0$, and (2) $\{^*n|^*n\} = 0$.

2.3. Sumbers

For each number d, there is a corresponding *up* defined as [1,2,17].

$$\uparrow (d) = \left\{ \uparrow (d^L), ^*| \uparrow (d^R), ^* \right\}. \tag{12}$$

The negation of up is called *down*.

$$\downarrow (d) = - \uparrow (d). \tag{13}$$

A property between all ups and stars [1,2] is: for all numbers $d > 0$ and $n > 1$, we have

$$\uparrow (d) > {^*}n \quad \text{and} \quad \uparrow (d) > \Leftrightarrow \tag{14}$$

and, for all numbers d, we have

$$\uparrow (d)\|^*1 \quad (\text{or} \quad \uparrow (d)\|^*). \tag{15}$$

We use the notation $m.\uparrow(d)$ to denote the sum of m copies of $\uparrow(d)$. A *sumber* S (cf. [20]) is a sum of ups, downs and stars (*).

$$S = \sum_{k=1,n} a_k \cdot \uparrow (d_k) + a_0 \cdot ^*, \tag{16}$$

where a_k are integers and d_k are numbers, $0 < k \leqslant n$. Without loss of generality, in (16), we assume $0 < d_1 < d_2 < \cdots < d_n$ and $a_0 = 0$ or 1. Clearly, sumbers are closed under addition. We use the notation $G \ll H$ to denote that the sum of any number of copies of G is less than H. The sumbers have the following properties:

$$0 < \uparrow (d_1) < \uparrow (d_2), \tag{17}$$

$$0 \ll \uparrow (d_{n+1}) - \uparrow (d_n) \ll \uparrow (d_n) - \uparrow (d_{n-1}), \tag{18}$$

$$\uparrow (d_{n+1}) + \uparrow (d_{n+1}) - \uparrow (d_n) > ^*, \tag{19}$$

where $0 < d_1 < d_2 < \cdots < d_n < d_{n+1} < \cdots$. These properties are *sufficient* to determine the outcome of any sum of sumbers. The research in [20] provides a simple rule to determine the outcome of (16):

$$S > 0 \quad \text{if and only if} \quad \sum_{k=1,n} a_k > a_0$$

$$\text{or} \left(\sum_{k=1,n} a_k = a_0, \text{ and } a_1 < 0 \right),$$

where a_0 is either 1 or 0. Note that the net number of ups is greater than the net number of *, or the net number of ups equals the net number of * and the smallest up has a negative coefficient.

For example, consider $S_A = -\uparrow(3) + 3.\uparrow(1) + ^*$. In S_A, the net number of ups (=2) is greater than the net number of * (=1), thus $S_A > 0$. Consider $S_B = -\uparrow(3) + 3.\uparrow(2) - \uparrow(1) + ^*$. In S_B, the net number of ups (=1) equals the net number of *(=1), and the smallest up (=$\uparrow(1)$) has a negative coefficient (= −1), thus $S_B > 0$. Consider $S_C = \uparrow(3) - 2.\uparrow(2) + 2.\uparrow(1) + ^*$. In S_C, the net number of ups (=1) equals the net number of * (=1), but the smallest up (=$\uparrow(1)$) has a positive coefficient (=2), thus $S_C \ngtr 0$. Let "$\ngtr$" denote "not greater than".

2.4. Infinitesimal and atomic weight

A game *G* is called an *infinitesimal* if and only if *G* is less than any positive numbers and greater than any negative numbers. Nimbers and sumbers are all infinitesimals. Researchers in [1,2] introduced the definition of atomic weight. If $G = \{G_a, G_b, G_c, \ldots, | G_d, G_e, G_f, \ldots\}$ where G_a, G_b, G_c, G_d, G_e, G_f, ... have atomic weight *a, b, c, d, e, f*, ..., then the atomic weight of *G* is

$$G_0 = \{a - 2, b - 2, c - 2, \ldots | d + 2, e + 2, f + 2, \ldots\}$$

unless G_0 is an integer and either $G > \star$ or $G < \star$. In these exceptional cases, if $G > \star$ then the atomic weight of *G* is the largest integer $< | d + 2, e + 2, f + 2, \ldots$, and if $G < \star$ then the atomic weight of *G* is the least integer $| > a - 2, b - 2, c - 2, \ldots$.

According to the above definition, each nimber has atomic weight 0; each up has atomic weight 1.

Two important properties [1,2] about atomic weights are described as follows.

1. The atomic weight of a sum of games equals to the sum of the atomic weights of the games.
2. If the atomic weight of a game is greater than or equals to 2, then Left wins the game. On the other hand, if it is less than or equals to −2, then Right wins the game. However, there are no general rules when the atomic weight is between −2 and 2.

For example, $\uparrow + \uparrow(2)$ has atomic weight 2, hence Left can win the game; $\downarrow + \uparrow(2) + \downarrow(3) + \downarrow(4) - \star + * - *(3)$ has atomic weight −2, hence Right can win the game. Thus, for some games, we can determine the winners by computing the atomic weight of sub-games, instead of searching complex trees.

3. Domineering and XT Domineering

Domineering (also called Stop-Gate or Crosscram) [8] is a mathematical game played on a board with $n \times n$ squares. Two players have a collection of 1×2 and 2×1 dominos which they place on the grid in turn, covering up squares. One player, Left, plays first and places domino vertically (1×2), while the other, Right, places horizontally (2×1). The first player who cannot place a domino loses the game.

As the game progresses, the original $n \times n$ squares may be partitioned into a set of disjoint sub-positions. Fig. 1 shows a graph in the middle of a 6×6 Domineering. It contains 5 disjoint sub-positions shown in Fig. 2.

In terms of combinatorial game theory, the game *G* in Fig. 1 is a sum of sub-positions *A, B, C, D,* and *E,* i.e., $G = A + B + C + D + E$. Note that by rotating position *D* 90° counter clockwise, one can get position *E*. In general, rotating a Domineering position 90° (either clockwise or counter clockwise) will result a negation of the original position, and reflecting a Domineering position with respect to a vertical axis or horizontal axis will not change the game value of the position. Hence, $E = -D$, and $G = A + B + C$.
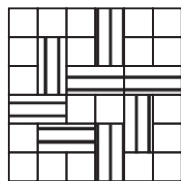

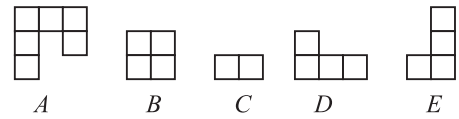**Fig. 1.** Middle game of 6×6 Domineering.

Fig. 2. Sub-positions of the graph in Fig. 1.

Domineering attracted many combinatorial game researchers because the game contains many numbers, switches of numbers, and complicated hot positions. Fig. 3 (below) shows the game values of the positions in Fig. 2. Note that the derivations are based on [9] and the details of derivations are therefore omitted in this paper. By summing up the values, we have $G = 3/4 + \{1 | -1\} - 1 = -1/4 + \{1 | -1\} = \{3/4 | -5/4\}$, thus the first player can win the game. This illustrates the power of using combinatorial theory, since we can derive the result without tree search as many board games do. A simpler example is illustrated in Appendix A.

XT Domineering is modified from the Domineering game by changing the rule to allow a player placing a small (1×1) domino on a sub-position while unable to place his big domino (1×2 or 2×1) in the sub-position in the original Domineering game. For example, consider sub-position *C* in Fig. 2. In Domineering, Left cannot place a domino vertically (1×2) at sub-position *C*, while in XT Domineering, Left is allowed to place a 1×1 domino at sub-position *C*. More specifically, sub-position *C* has the value $\{\emptyset | 0\} = -1$ in Domineering, and $\{\{0 | 0\} \| 0\} = \{* | 0\} = \downarrow$ in XT Domineering. Note that Left is not allowed to place a 1×1 domino at a position while he is able to place a 1×2 domino at that position and Right is not allowed to place a 1×1 domino at a position while he is able to place a 2×1 domino at that position. For example, both players are not allowed to place 1×1 domino at positions *A*, *B*, *D* and *E* in Fig. 2.

Since XT Domineering has at least the same number of options as Domineering and allows more moves (e.g., on 1×1 vacancies), XT Domineering has higher game-tree complexity [15].

Note that each player has at least one option at any non-empty position in XT Domineering. This nature prevents the occurrence of non-zero numbers and ensures that each position in XT Domineering is an infinitesimal. One of the major motivations of this paper is to see what kind of infinitesimals may be shown up in this game.

4. Game values of 3×3 XT Domineering

For XT Domineering with $1 \times n$ squares, the games have periodic values with period length 8, $\{0, *, \downarrow, \uparrow, *, 0, \uparrow^*, \downarrow^*\}$ [18]. This is in fact a partisan octal game [19]. In this section, we investigate a total of 2^9 sub-graphs of 3×3 squares in XT Domineering.

After excluding non-connected sub-graphs, rotated negation sub-graphs, or reflected equivalence sub-graphs, there are 34 distinct positions. The game values of these distinct positions are derived based on the above inequalities (1)–(19), and shown in Table 1. Each position in Table 1 is a linear combination of the following eight elementary games:

$$* = \{0 | 0\}, \tag{20}$$
$$\uparrow = \{0 | *\}, \tag{21}$$
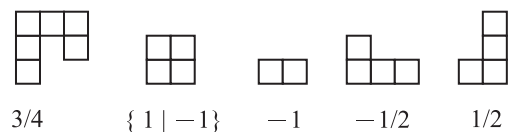$$\uparrow^+ = \{\uparrow | *\}, \tag{22}$$

Fig. 3. Some game values in Domineering.

Table 1
Game values of 3 × 3 XT Domineering.

No.	Position	Value	No.	Position	Value
$P_{1\text{-}1}$		$*$	$P_{6\text{-}1}$		$*/2$
$P_{2\text{-}1}$		$\uparrow$	$P_{6\text{-}2}$		$*/2$
$P_{3\text{-}1}$		0	$P_{6\text{-}3}$		$\uparrow\uparrow*$
$P_{3\text{-}2}$		$\downarrow$	$P_{6\text{-}4}$		$\uparrow+$
$P_{4\text{-}1}$		$*$	$P_{6\text{-}5}$		$\uparrow/2$
$P_{4\text{-}2}$		$*$	$P_{6\text{-}6}$		$\bigstar+*$
$P_{4\text{-}3}$		$\uparrow\uparrow*$	$P_{6\text{-}7}$		0
$P_{4\text{-}4}$		$*$	$P_{6\text{-}8}$		0
$P_{5\text{-}1}$		$\bigstar$	$P_{7\text{-}1}$		$*$
$P_{5\text{-}2}$		$*$	$P_{7\text{-}2}$		$\uparrow*$
$P_{5\text{-}3}$		$\uparrow\uparrow$	$P_{7\text{-}3}$		$\uparrow$
$P_{5\text{-}4}$		$\bigstar$	$P_{7\text{-}4}$		$\uparrow*$
$P_{5\text{-}5}$		$\uparrow\uparrow$	$P_{7\text{-}5}$		$*$
$P_{5\text{-}6}$		$\uparrow$	$P_{7\text{-}6}$		0
$P_{5\text{-}7}$		0	$P_{7\text{-}7}$		$*+(*/2)+$
$P_{8\text{-}1}$		$*$	$P_{8\text{-}3}$		0
$P_{8\text{-}2}$		$\diamondsuit$	$P_{9\text{-}1}$		0

$$\uparrow/2 = \{\uparrow\uparrow* | \downarrow*\}, \tag{23}$$
$$\bigstar = \{0, \uparrow* | \downarrow*, 0\}, \tag{24}$$
$$*/2 = \{\uparrow\uparrow | \downarrow\downarrow*\}, \tag{25}$$
$$(*/2)^+ = \{\uparrow\uparrow, \uparrow\uparrow* | \downarrow\downarrow*\}, \tag{26}$$
$$\diamondsuit = \{\uparrow\uparrow\uparrow* | \downarrow\downarrow\downarrow*\}. \tag{27}$$

For simplicity, let $\uparrow\uparrow$ indicate $\uparrow + \uparrow$, and similarly for $\uparrow\uparrow*$, $\uparrow\uparrow\uparrow*$, etc.

The games $*$, $\uparrow$ and $\uparrow^+$ (=$\uparrow(2)$) have been introduced in Section 3. $*$ has atomic weight 0 (as described in Section 2.4), while $\uparrow$ and $\uparrow^+$ have atomic weight 1 each. We use the symbol $\uparrow^2$ to denote $\uparrow^+ - \uparrow$.

$$\uparrow^2 = \uparrow^- - \uparrow. \tag{28}$$

From inequality (18), we have

$$\uparrow \gg \uparrow^2 > 0. \tag{29}$$

The game $\uparrow/2$ (*half up*), as the name suggested, has atomic weight 1/2 and the following properties:

$$\uparrow/2 + \uparrow/2 = \uparrow, \tag{30}$$
$$\uparrow/2 > \uparrow^2. \tag{31}$$

The game $\bigstar$ (*black star*) has atomic weight 0 and with property similar to nimbers:

$$\bigstar + \bigstar = 0, \tag{32}$$
$$\bigstar \| *(n), \quad \text{for integer } n > 0. \tag{33}$$

The game $*/2$ (*half star*), as the name suggested, has the following property:

$$*/2 + */2 = *. \tag{34}$$

The game $*/2$ has atomic weight $\{0|0\} = *$, since the atomic weight of $\uparrow\uparrow$ is +2 and that of $\downarrow\downarrow*$ is −2.

The game $(*/2)^+$ (*half star plus*), as the name suggested, is just slightly greater than $*/2$ and has atomic weight $\{0,0|0\} = *$. The difference between $(*/2)^+$ and $*/2$ is named $\triangle_*$:

$$\triangle_* = (*/2)^+ - */2 > 0. \tag{35}$$

Since the atomic weight of both $(*/2)^+$ and $*/2$ are $*$, the atomic weight of $\triangle_*$ equals $* - * = 0$.

The game $\diamondsuit$ (*diamond*) has atomic weight $\{1|-1\}$. Since the incentive of $\diamondsuit$ (*diamond*) is greater than the ones of all the other 7 elementary games, $\diamondsuit$ should always be played first among the 8 elementary games. Diamond also has the property below:

$$\diamondsuit + \diamondsuit = 0. \tag{36}$$

The calculation for the values of positions in Table 1 is a tedious process. In general, one first derives a position expression according to the rule and then simplifies the expression by removing the *dominated* options and replacing with the *reversible* options (c.f. [1,2]). For example, considering P_{4-3}, according the rule $P_{4-3} = \{0, \downarrow | \uparrow\}$. After eliminating the dominated option $\downarrow$ ($\downarrow < 0$), one can get $P_{4-3} = \{0 | \uparrow\}$. Considering P_{5-7}, according the rule $P_{5-7} = \{*|*\}$. After replacing P_{5-7} with reversible option ($P_{5-7}^{LR} = 0$), one can get $P_{5-7} = 0$. After simplifying a position, one needs to check whether the position can be represented as a sum of simpler game. For example, $P_{4-3} = \{0|\uparrow\} = \uparrow\uparrow*$. The research in [20] provided an algorithm to simplify switches of up sums into up sums whenever possible. The game values in Table 1 have also been verified in CgSuite [21], a useful tool for deriving game values.

Fig. 4 shows the corresponding XT Domineering games values of positions in Fig. 2. The derivations for C, E, and $E+*$ are illustrated in Appendix A.

The sum in Fig. 4 is $\uparrow/2 + * + \downarrow + \downarrow\downarrow* + \uparrow\uparrow* = \downarrow/2 + * = \{\uparrow|\downarrow\downarrow\}$. Hence the first player can win the game.
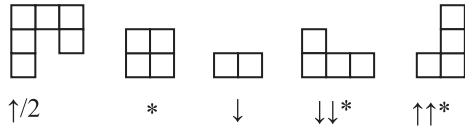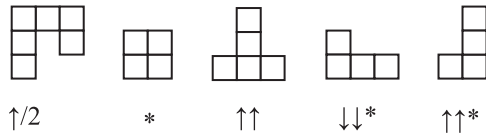
Fig. 4. Some game values in XT Domineering.

Fig. 5. Some game values in XT Domineering.

Assume that sub-position C is changed as shown in Fig. 5. Then, the sum in Fig. 5 becomes $\uparrow/2 + {}^* + \uparrow\uparrow + \downarrow\downarrow^* + \uparrow\uparrow^* = \uparrow\uparrow + \uparrow/2 + {}^*$. Since the atomic weight of the above sum is $2 + 1/2$, over 2, Left wins the game. From above examples, Table 1 becomes an important knowledge base for playing the game of XT Domineering.

5. Outcome of 3×3 XT Domineering

In the previous section, we derive the values of positions in Table 1. Then, we can easily determine the outcome of sums, if the atomic weights are at least 2 or at most -2. However, there are no simple rules when the atomic weights are between -2 and 2.

This section discusses the approach to determine the outcome of sums of 3×3 XT Domineering, even when the atomic weights are between -2 and 2. Since the game $\diamond$ will always be played before any other games in Table 1, we may only focus on the analysis of sums of the other 7 elementary games. Without loss of generality, a sum S of any positions in Table 1 can be written as:

Table 2
Minimum ups U required for $U + S_B + S_C > 0$.

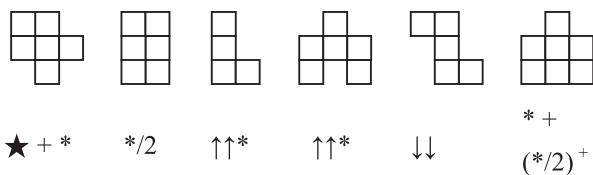| $S_C \backslash S_B$ ($n > 0$)	1	2	3	4
	0	$\star$	*	$\star + ^*$
1 $(n+1).\triangle_*$	0	$\uparrow^2$	0	$\uparrow^2$
2 $\triangle_*$	0	$\uparrow^2$	$\uparrow + 2.\uparrow^2$	$\uparrow + \uparrow^2$
3 0	$\uparrow^2, \uparrow/2 - \uparrow^2$	$\uparrow^2$	$\uparrow + 2.\uparrow^2$	$\uparrow + \uparrow^2$
4 $-n.\triangle_*$	$\uparrow + 2.\uparrow^2$	$\uparrow + \uparrow^2$	$\uparrow + 2.\uparrow^2$	$\uparrow + \uparrow^2$
5 $^*/2 + (n+2).\triangle_*$	$\uparrow/2$	$\uparrow/2 + \uparrow^2$	$\uparrow/2$	$\uparrow/2 + \uparrow^2$
6 $^*/2 + 2.\triangle_*$	$\uparrow/2$	$\uparrow/2 + \uparrow^2$	$\uparrow - \uparrow^2, \uparrow/2 + 2.\uparrow^2$	$\uparrow/2 + \uparrow^2$
7 $^*/2 + \triangle_*$	$\uparrow/2$	$\uparrow/2 + \uparrow^2$	$\uparrow/2 + \uparrow + 2.\uparrow^2$	$\uparrow/2 + \uparrow + \uparrow^2$
8 $^*/2$	$\uparrow - \uparrow^2, \uparrow/2 + 2.\uparrow^2$	$\uparrow/2 + \uparrow^2$	$\uparrow/2 + \uparrow + 2.\uparrow^2$	$\uparrow/2 + \uparrow + \uparrow^2$
9 $^*/2 - n.\triangle_*$	$\uparrow/2 + \uparrow + 2.\uparrow^2$	$\uparrow/2 + \uparrow + \uparrow^2$	$\uparrow/2 + \uparrow + 2.\uparrow^2$	$\uparrow/2 + \uparrow + \uparrow^2$

$$S = S_A + S_B + S_C, \tag{37}$$

Fig. 6. Some game values in XT Domineering.

where S_A is a linear combination of $\uparrow^+$, $\uparrow$ and $\uparrow/2$,

S_B is a linear combination of * and $\star$, and
S_C is a linear combination of $^*/2$ and $\triangle_*$.

S_A measures the *up-ness* (or advantage for Left) of S; S_B is a sum that neither player has advantage; S_C consists of games with atomic weight *. There are only 4 possible cases of S_B, as shown in the column subhead of Table 2, and 9 possible cases of S_C, as shown in the row subhead of Table 2. Note that the atomic weight of S_C is 0 in row 1, 2, 3 and 4, and * in row 5, 6, 7, 8 and 9.

Table 2 is a set of 39 inequalities (note that there are two values in each of $grid(3,1)$, $grid(8,1)$ and $grid(6,3)$), $1 \le i \le 9$, $1 \le j \le 4$,

$$grid(i,j) + row(i) + col(j) > 0. \tag{38}$$

The proof for these inequalities is given in Appendix B. Let us illustrate by some example. The ups in $grid(9,2)$ is $\uparrow/2 + \uparrow + \uparrow^2$, it corresponds to the inequality:

$$\uparrow/2 + \uparrow + \uparrow^2 + {}^*/2 - n.\triangle_* + \star > 0, \quad \text{for } n > 0.$$

$Grid(3,1)$ represents 2 inequalities: $\uparrow^2 > 0$ and $\uparrow/2 - \uparrow^2 > 0$; $grid(8,1)$ represents 2 inequalities: $\uparrow - \uparrow^2 + {}^*/2 > 0$ and $\uparrow/2 + 2.\uparrow^2 + {}^*/2 > 0$. These inequalities are sufficient to determine the outcome of any sum of the 8 elementary games. The general steps to determine the outcome of a sum S of 3×3 XT Domineering is described as follows:

1. Check the game value of each of S's position from Table 1.
2. If there is any $\diamond$ in the sum, play it out first.
3. Denote the sum $S_A + S_B + S_C$, (37) by S, and determine the value of S_A, S_B and S_C.
4. Use S_B and S_C to lookup Table 2 for the minimum ups U required.
5. Determine whether $S_A \ge U$ or not. Inequalities (29)–(31) can help the determining process.
6. $S > 0$ if and only if $S_A \ge U$.
7. To determine whether $S < 0$ or not, it is equivalent to determining whether $-S > 0$ or not. Apply the above steps to $-S$.
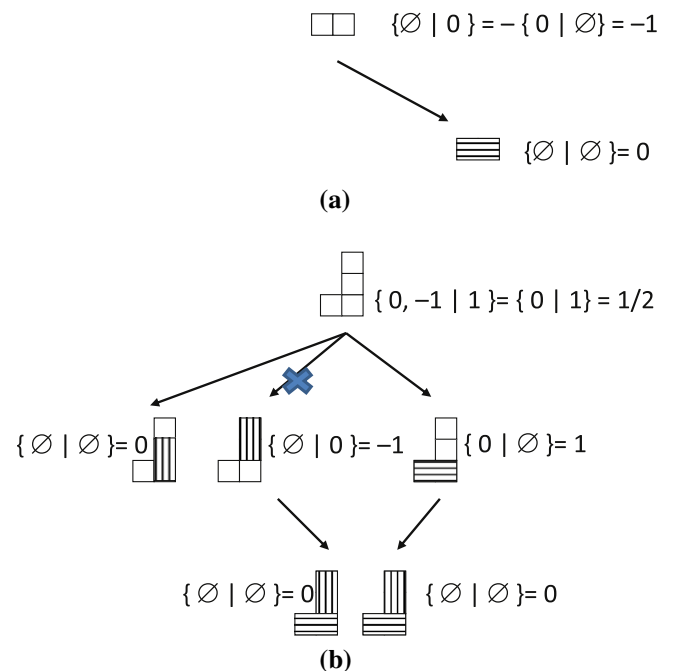

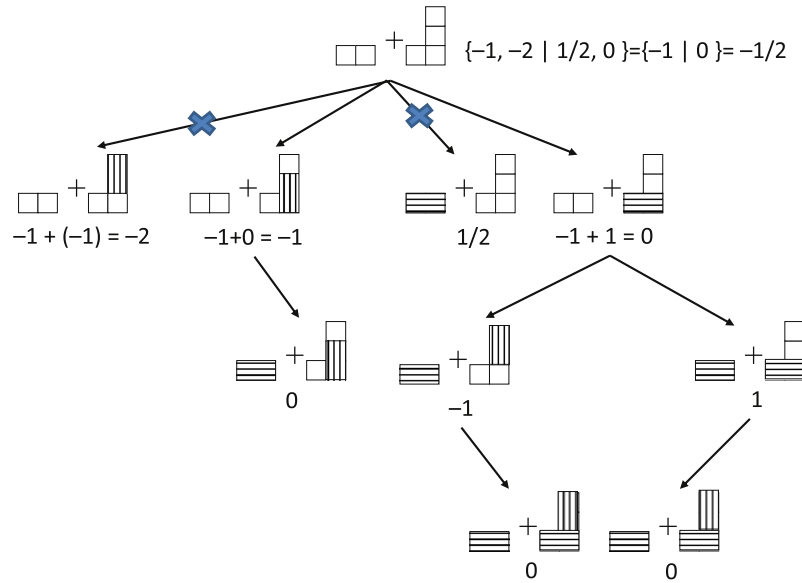**Fig. 7.** Deriving both game values of C and E of Fig. 2 in (a) and (b), respectively.

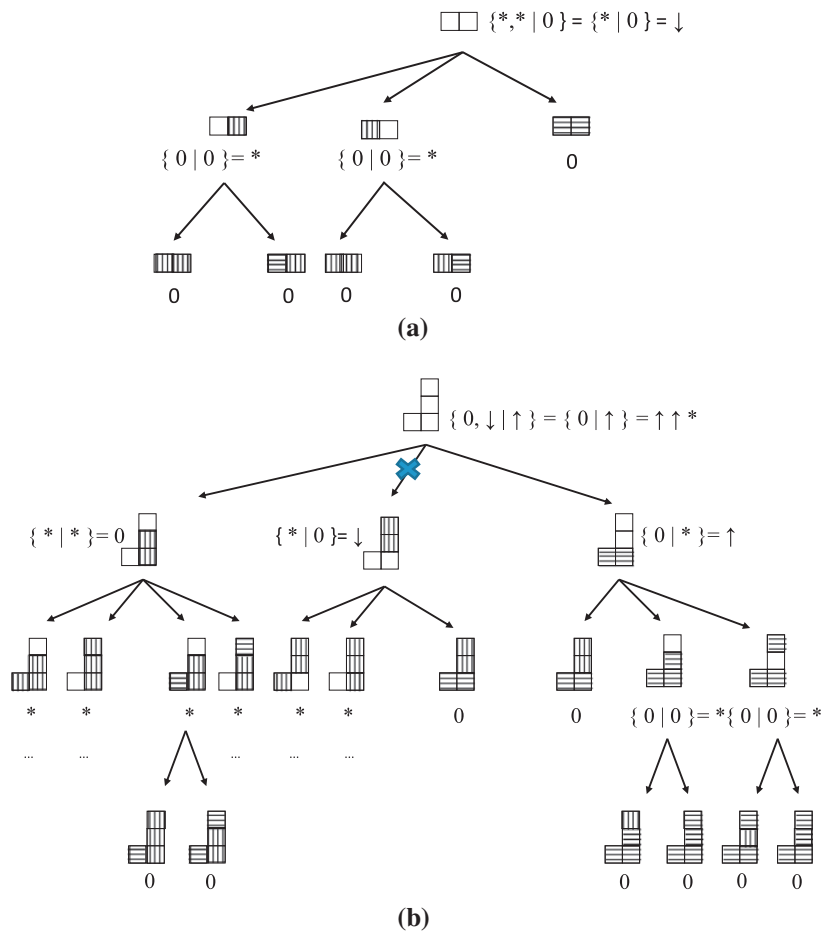**Fig. 8.** Deriving the game values of $C + E$.

(a)

(b)

Fig. 9. Deriving both game values of C and E of Fig. 2 in (a) and (b), respectively.

For example, consider the sum S of the sub-positions as shown in Fig. 6 and who wins the game.

The sum S can be simplified as:

$$S = \bigstar + {}^* + {}^*/2 + \uparrow\uparrow{}^* + \uparrow\uparrow{}^* + \downarrow\downarrow + {}^* + ({}^*/2)^+ = \bigstar + {}^* + {}^*/2 + \uparrow$$

$$\uparrow + {}^* + ({}^*/2)^+ = \bigstar - {}^*/2 + \uparrow\uparrow + {}^* + ({}^*/2)^+ = \bigstar + \uparrow\uparrow{}^* + \triangle_*$$

and,

$$S_A = \uparrow\uparrow,$$

$$S_B = \bigstar + {}^*,$$
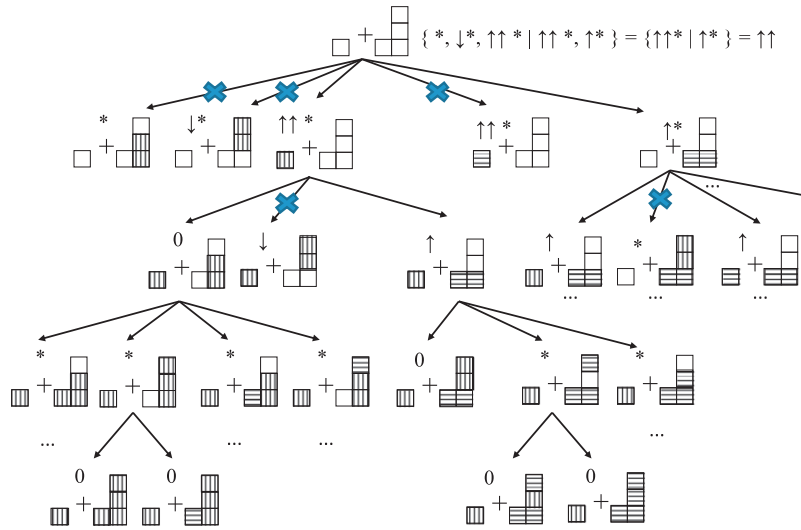
$$S_C = \triangle_*.$$

Fig. 10. Deriving the game values of $C + E$.

Using S_B and S_C to lookup Table 2, we get $U = \uparrow + \uparrow^2$. Since $S_A = \uparrow\uparrow > \uparrow + \uparrow^2 = U$, we conclude $S > 0$. Hence the game is a win for Left, no matter who moves first.

6. Conclusion and further consideration

This paper has the following three major contributions. First, we present a new game, XT Domineering, which has higher game-tree complexity [15] than Domineering.

Second, we also have presented a mathematical approach to solve sums of 3×3 XT Domineering. Again, this success demonstrates the potential of applying combinatorial game theory to solving more of other intelligent games.

After solving 3×3 XT Domineering, it is natural to think of 3×4, 4×4, or even larger size XT Domineering. According to our preliminary study, there seems to be no simple close form equation that can relate a given position to its game value. Thus a lookup table is required to store the values of all the positions. CgSuite [21] is a useful tool to derive the values. After deriving the canonical form of the game values, one still needs to check whether a game can be decomposed as a sum of simpler elementary games. Unfortunately, there are too many sub-positions in 3×4, 4×4, or even larger size XT Domineering, we cannot afford to examine all the positions and check whether they can be decomposed as simpler elementary games. An automated game decomposition procedure is in need and deserves further research in the future.

Third, we find several infinitesimal games with interesting properties, including $\star$, $^*/2$, $(^*/2)^+$ and $\uparrow/2$. It is worth further research to find more other interesting infinitesimal games. The game of XT Domineering is a rich source of infinitesimal games.

Acknowledgements

Appendix A

The power of using combinatorial theory is to derive the game value (or result) without tree search as many board games do. This is well described in many articles such as [6,7]. In this appendix, a simple Domineering example with C and E in Fig. 2 as well as a XT Domineering example is illustrated to demonstrate the power of using combinatorial theory.

First, let us investigate the game of Domineering. The game value of C, -1, is derived in Fig. 7(a). The negative game value indicates that Right wins the game. The game value of E, $1/2$, is derived in Fig. 7(b). The positive value indicates that Left wins the game. In the derivation, a cross is used to indicate that Left does not choose -1 since choosing 0 is better to Left.

If both C and E are left in a game, we can derive the game value, $-1 + 1/2 = -1/2$, by using the combinatorial theory, and easily conclude that Right wins the game due to the negative game value. However, in case of using tree search, we need to derive the same game value as shown in Fig. 8, whose computational complexity grows exponentially as more are added.

Now, let us investigate the game of XT Domineering. As described in Section 3, the game becomes more complex since 1×1 dominos are also allowed to be placed. For both games C and E, the derivations for both are shown in Fig. 9(a) and (b), respectively. The game value of C, $\downarrow$ (a negative infinitesimal), indicates that Right still wins the game, while the game value of E, $\uparrow\uparrow^*$, indicates that Left wins the game. The derivations for both are clearly much more complex, when compared with Fig. 7.

For simplicity, we choose the game $^* + E$, as shown in Fig. 10. Its game value is $\uparrow\uparrow$ with atomic weight 2, which indicates that Left wins the game, as described in Section 3.

Appendix B

Proposition 1. *The ups in the grids of Table 2 are the sufficient and necessary conditions for*

$$grid(i,j) + row(i) + col(j) > 0.$$

Proof. Let $(G_{i,j})$ denote the inequality

$$grid(i,j) + row(i) + col(j) > 0.$$

We first show the sufficiency of the conditions.

- Since $\uparrow > \uparrow/2 > \uparrow^2 > 0$, we have $(G_{3,1})$, $\uparrow^2 > 0$ and $\uparrow/2 - \uparrow^2 > 0$.
- Since $^*/2 + \triangle_* + \uparrow/2 > 0$, we have $(G_{7,1})$.
- Since $^*/2 + \uparrow - \uparrow^2 > 0$ and $^*/2 + \uparrow/2 + 2.\uparrow^2 > 0$, we have $(G_{8,1})$.

- Since $*/2 + \uparrow/2 + \uparrow^2 > \star$, we have $(G_{8,2})$.
- Since $\triangle_* + \triangle_* > *$, we have $(G_{1,3})$, and

$$(G_{7,1}) \Rightarrow (G_{5,3}),$$
$$(G_{8,1}) \Rightarrow (G_{6,3}), \text{ and}$$
$$(G_{8,2}) \Rightarrow (G_{6,4}).$$

- Since $*/2 + \uparrow/2 + \uparrow + \uparrow^2 > \star + *$, we have $(G_{8,4})$. $(G_{8,2})$ and $(G_{8,4}) \Rightarrow (G_{9,2})$ and $(G_{9,4})$.
- Since $\uparrow^2 > \star$ and $\uparrow + \uparrow^2 > \star + *$, we have $(G_{3,2})$ and $(G_{3,4})$. $(G_{3,2})$ and $(G_{3,4}) \Rightarrow (G_{4,2})$ and $(G_{4,4})$.
- Since $\uparrow^2 > \star$, we have $(G_{3,2})$, and

$$(G_{4,2}) \Rightarrow (G_{4,1}),$$
$$(G_{9,2}) \Rightarrow (G_{9,1}),$$
$$(G_{1,3}) \Rightarrow (G_{1,4}),$$
$$(G_{4,4}) \Rightarrow (G_{4,3}), \text{ and}$$
$$(G_{9,4}) \Rightarrow (G_{9,3}).$$

- Since $\triangle_* > 0$, we have $(G_{2,1})$, and

$$(G_{2,1}) \Rightarrow (G_{1,1}),$$
$$(G_{7,1}) \Rightarrow (G_{6,1}) \Rightarrow (G_{5,1}),$$
$$(G_{3,2}) \Rightarrow (G_{2,2}) \Rightarrow (G_{1,2}),$$
$$(G_{8,2}) \Rightarrow (G_{7,2}) \Rightarrow (G_{6,2}) \Rightarrow (G_{5,2}),$$
$$(G_{4,3}) \Rightarrow (G_{3,3}) \Rightarrow (G_{2,3}),$$
$$(G_{9,3}) \Rightarrow (G_{8,3}) \Rightarrow (G_{7,3}),$$
$$(G_{3,4}) \Rightarrow (G_{2,4}),$$
$$(G_{6,4}) \Rightarrow (G_{5,4}), \text{ and}$$
$$(G_{8,4}) \Rightarrow (G_{7,4}).$$

This completes proof for the sufficiency of the conditions. □

Next, we prove the necessary of the conditions. We need to show that any sums of ups less than or confused with the value in a corresponding grid will result in an insufficient condition. Note that the smallest increments of sums ups are $\uparrow^2$ and $\uparrow/2 - \uparrow^2$, and the only possible sums of ups confusing with 0 are $\uparrow/2 - (n+1).\uparrow^2$, $n > 0$.

For $(G_{3,1})$, $(G_{6,3})$ and $(G_{8,1})$, we only need to show that if the value in the corresponding grid reduced by $\uparrow^2$, then the inequality will not hold. For all the other grids, in order to prove the necessary conditions, we need to show that if the value in a grid reduced by $\uparrow^2$ or $\uparrow/2 - \uparrow^2$, or, if the value in a grid increased or reduced by $\uparrow/2 - (n+1).\uparrow^2$, $n > 0$, then the corresponding inequality will not hold. Since $\uparrow/2 - 2.\uparrow^2 \geq \uparrow/2 - (n+1).\uparrow^2 > -(\uparrow/2 - \uparrow^2) > -(\uparrow/2 - (n+1).\uparrow^2)$, it is sufficient to show: if the value in a grid reduced by $\uparrow^2$ or increase by $\uparrow/2 - 2.\uparrow^2$ then the corresponding inequality will not hold.

- Consider $(G_{3,1})$, $\uparrow^2 > 0$ and $\uparrow/2 - \uparrow^2 > 0$.
  But $0 \not> 0$ and $\uparrow/2 - 2.\uparrow^2 \not> 0$.
  Thus $\uparrow^2$ or $\uparrow/2 - \uparrow^2$ is a necessary condition.
- Consider $(G_{6,3})$,

$$*/2 + 2.\triangle_* + \uparrow/2 + 2.\uparrow^2 > * \quad \text{and} \quad */2 + 2.\triangle_* + \uparrow - \uparrow^2 > *.$$

But

$$*/2 + 2.\triangle_* + \uparrow/2 + \uparrow^2 \not> * \quad \text{and} \quad */2 + 2.\triangle_* + \uparrow - 2.\uparrow^2 \not> *.$$

Thus $\uparrow/2 + 2.\uparrow^2$ or $\uparrow - \uparrow^2$ is a necessary condition.

Note that, since $2.\triangle_* > *$, the necessary condition of $(G_{6,3})$ implies the necessary condition of $(G_{8,1})$.

- Consider $(G_{1,2})$, $(n+1).\triangle_* + \uparrow^2 > \star$.
  But $(n+1).\triangle_* \not> \star$ and $(n+1).\triangle_* + \uparrow/2 - \uparrow^2 \not> \star$.
  Thus $\uparrow^2$ is a necessary condition.
- Consider $(G_{1,4})$, $(n+1).\triangle_* + \uparrow^2 > \star + *$.
  But $(n+1).\triangle_* \not> \star + *$ and $(n+1).\triangle_* + \uparrow/2 - \uparrow^2 \not> \star + *$
  Thus $\uparrow^2$ is a necessary condition.

- Consider $(G_{2,3})$, $\triangle_* + \uparrow + 2.\uparrow^2 > *$.
  But $\triangle_* + \uparrow + \uparrow^2 \not> *$ and $\triangle_* + \uparrow + \uparrow/2 \not> *$.
  Thus $\uparrow + 2.\uparrow^2$ is a necessary condition.
- Consider $(G_{4,1})$, $-n.\triangle_* + \uparrow + 2.\uparrow^2 > 0$.
  But $-n.\triangle_* + \uparrow + \uparrow^2 \not> 0$ and $-n.\triangle_* + \uparrow + \uparrow/2 \not> 0$.
  Thus $\uparrow + 2.\uparrow^2$ is a necessary condition.
- Consider $(G_{5,2})$, $*/2 + n.\triangle_* + \uparrow/2 + \uparrow^2 > \star$.
  But $*/2 + n.\triangle_* + \uparrow/2 \not> \star$ and $*/2 + n.\triangle_* + \uparrow - \uparrow^2 \not> \star$.
  Thus $\uparrow/2 + \uparrow^2$ is a necessary condition.
- Consider $(G_{5,4})$, $*/2 + (n+2).\triangle_* + \uparrow/2 + \uparrow^2 > \star + *$.
  But $*/2 + (n+2).\triangle_* + \uparrow/2 \not> \star + *$ and $*/2 + (n+2).\triangle_* + \uparrow - \uparrow^2 \not> \star + *$.
  Thus $\uparrow/2 + \uparrow^2$ is a necessary condition.
- Consider $(G_{7,3})$, $*/2 + \triangle_* + \uparrow/2 + \uparrow + 2.\uparrow^2 > *$.
  But $*/2 + \triangle_* + \uparrow/2 + \uparrow + \uparrow^2 \not> *$ and $*/2 + \triangle_* + 2.\uparrow \not> *$.
  Thus $\uparrow/2 + \uparrow + 2.\uparrow^2$ is a necessary condition.
- Consider $(G_{9,1})$, $*/2 - n.\triangle_* + \uparrow/2 + \uparrow + 2.\uparrow^2 > 0$.
  But $*/2 - n.\triangle_* + \uparrow/2 + \uparrow + \uparrow^2 \not> 0$ and $*/2 - n.\triangle_* + 2.\uparrow \not> 0$.
  Thus $\uparrow/2 + \uparrow + 2.\uparrow^2$ is a necessary condition.
  Let $(G_{i,j})^*$ denote the inequalities.
  $grid(i,j) + row(i) + col(j) - \uparrow^2 \not> 0$, and $grid(i,j) + row(i) + col(j) + \uparrow/2 - 2.\uparrow^2 \not> 0$.
- Since $\uparrow^2 > \star$, we have

$$(G_{1,2})^* \Rightarrow (G_{1,1})^*,$$
$$(G_{1,4})^* \Rightarrow (G_{1,3})^*,$$
$$(G_{2,3})^* \Rightarrow (G_{2,4})^*,$$
$$(G_{4,1})^* \Rightarrow (G_{4,2})^*,$$
$$(G_{5,2})^* \Rightarrow (G_{5,1})^*,$$
$$(G_{5,4})^* \Rightarrow (G_{5,3})^*,$$
$$(G_{7,3})^* \Rightarrow (G_{7,4})^*, \text{ and}$$
$$(G_{9,1})^* \Rightarrow (G_{9,2})^*.$$

- Since $\triangle_* > 0$, we have

$$(G_{1,1})^* \Rightarrow (G_{2,1})^*,$$
$$(G_{5,1})^* \Rightarrow (G_{6,1})^* \Rightarrow (G_{7,1})^*,$$
$$(G_{1,2})^* \Rightarrow (G_{2,2})^* \Rightarrow (G_{3,2})^*,$$
$$(G_{5,2})^* \Rightarrow (G_{6,2})^* \Rightarrow (G_{7,2})^* \Rightarrow (G_{8,2})^*,$$
$$(G_{2,3})^* \Rightarrow (G_{3,3})^* \Rightarrow (G_{4,3})^*,$$
$$(G_{7,3})^* \Rightarrow (G_{8,3})^* \Rightarrow (G_{9,3})^*,$$
$$(G_{2,4})^* \Rightarrow (G_{3,4})^* \Rightarrow (G_{4,4})^*,$$
$$(G_{5,4})^* \Rightarrow (G_{6,4})^*, \text{ and}$$
$$(G_{7,4})^* \Rightarrow (G_{8,4})^* \Rightarrow (G_{9,4})^*.$$

This completes the proof for the necessary of the conditions.

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