A Pragmatic Labeling Design of MIMO BICM-ID Systems Based on EXIT Chart

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***Abstract*—We propose a pragmatic labeling design based on the extrinsic information transfer (EXIT) chart for bit-interleaved coded modulation with iterative decoding on multiple-input-multiple-output channels. In our design, the EXIT chart is used to determine a candidate set of labelings which have representative demapper transfer curves. Then, a procedure to generate these labelings is provided based on the genetic algorithm. Given fixed channel code and signal-to-noise ratio (SNR), we can search within this candidate set for a most-suitable labeling to minimize the bit-error-rate (BER) with low complexity. Simulation results show that the labeling chosen from the candidate set exhibits a controllable BER performance gap compared to the optimal labeling found through exhaustive search. Besides, even though the channel code or the SNR is changed, the same candidate set can still be directly adopted to reduce the time on re-searching a new labeling.**

I. Introduction

Bit-interleaved coded modulation with iterative decoding (BICM-ID) [1][2] has been shown as an efficient and powerful transmission scheme over fading channels. It was pointed out in [1][2] that different BER performance can be obtained by employing different labelings. Recently, various researches have been done to find out good labelings of the BICM-ID systems for single-input-single-output (SISO) channels [3][4]. For multiple-input-multiple-output (MIMO) channels, single-dimensional labelings are investigated in [5][6], while [7][8] consider more general cases, called multi-dimensional labelings. Part of these works adopt the assumption of error-free feedback from the decoder and the labelings obtained turn out to have good BER performance at high signal-to-noise ratios (SNRs) but experience unacceptable performance loss at low or moderate SNRs.

In fact, it was mentioned in [4] that for any SNR the suitability of a labeling can always be determined with the help of the extrinsic information transfer (EXIT) chart [9], which has been known to be a powerful tool to understand the convergence behavior of iterative decoding schemes. Given a channel code and a fixed SNR, a good labeling should have the information transfer curve of demapper that makes a high first intersection with the decoder curve to allow the trajectory to go as far as possible. However, since any changes of the channel code and SNR may result in variations on the decoder and demapper curves respectively, the desired labeling should also be different. An intuitive way to keep the optimality is to exhaustively plot the demapper curves of all possible labelings on the EXIT chart to find a new one whenever the channel code or SNR is changed. Obviously, the complexity of such an exhaustive search turns out to be infeasible, especially for large number of transmit antennas and large constellations. To reduce the search complexity, the binary switch algorithms (BSA) are employed in [4] for labeling searching.

In this paper, a pragmatic methodology for labeling design is presented to accommodate the impact of different channel codes and SNRs. Based on the EXIT chart, an observation shows that for the demapper transfer curves that have similar shapes and similar values on their left and right end points, their corresponding behavior on the BER performance, including the occurrence of the waterfall region and the BER of the error floor region, are usually similar [10]. Therefore, we can first partition all possible labelings into groups according to their values of two end points and then choose a representative labeling from each group to form a set, called the candidate set. As long as the channel code and SNR are given, we can search within the candidate set for a most-suitable labeling with low search complexity. To practically generate the candidate set, a procedure is proposed based on the genetic algorithm (GA) [11]. Simulation results shows that the labeling we chosen from the candidate set can exhibit a controllable BER performance gap compared to the optimal labeling found through exhaustive search. Besides, even though the channel code or the SNR is changed, the same candidate set still can be directly adopted to reduce the time on re-searching a new labeling.

The rest of this paper is organized as follows. In Section II the system model of MIMO BICM-ID is given as well as the basic guideline of labeling design based on the EXIT chart. The idea of the candidate set and the procedure for generating the set are presented in Section III. In Section IV, simulation results are provided to verify the validity of our design for different channel codes and SNRs. Finally, a conclusion is drawn in Section V.

II. System Model and EXIT Chart Based Analysis

Consider the system model of MIMO BICM-ID with N_T transmit antennas and N_R receive antennas as shown in Fig.

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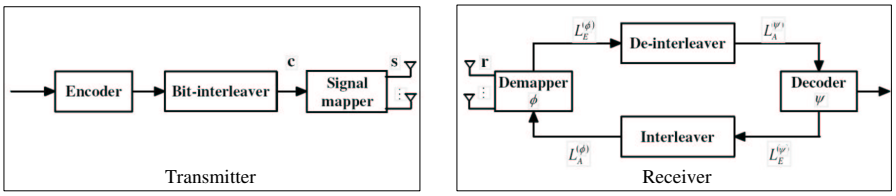


Fig. 1. Block diagram of BICM-ID systems.

1. A binary sequence $\boldsymbol{c} = [c_0, ..., c_{mN_T-1}]$ generated by an encoder C and an interleaver is first mapped to a vector $\boldsymbol{s} = [s_1, ..., s_{N_T}]^T$ by a labeling μ, where s_n is drawn from a regular constellation χ with 2^m signal points, $\forall\ 1 \le n \le N_T$. For single-dimensional labelings, $\boldsymbol{c}$ is first partitioned into N_T groups and then each group is mapped to a scalar signal point s_n before de-multiplexed to transmit antennas. Here, we consider multi-dimensional labelings [7] which directly map $\boldsymbol{c}$ to $\boldsymbol{s}$. The signal vector $\boldsymbol{s}$ is then transmitted through a frequency-nonselective MIMO channel, and the received signal $\boldsymbol{r} = [r_1, ..., r_{N_R}]^T$ can be expressed as

$$\boldsymbol{r} = \mathbf{H}\boldsymbol{s} + \boldsymbol{w}, \tag{1}$$

where the channel gain $\mathbf{H}$ is an $N_R \times N_T$ matrix with each element being identical independent complex Gaussian distribution of zero mean and unit variance and the noise $\boldsymbol{w}$ is an $N_R \times 1$ vector in which the elements are zero-mean complex Gaussian random variables with variance $N_0/2$ per dimension.

The receiver is divided into two main blocks: the demapper and the decoder. Upon receiving $\boldsymbol{r}$, the demapper ϕ takes $\boldsymbol{r}$ as well as $L_A^{(\phi)}(c_i)$'s, the a priori inputs of coded bits feedback from the decoder ψ to computes the extrinsic log-likelihood ratios (LLRs) $L_E^{(\phi)}(c_i)$'s according to

$$L_E^{(\phi)}(c_i) = \ln \frac{\sum\limits_{\boldsymbol{s}\in\kappa_i^1} \exp\left\{ -\frac{\|\boldsymbol{r}-\mathbf{H}\boldsymbol{s}\|^2}{N_0} + \sum\limits_{j\neq i} c_j L_A^{(\phi)}(c_j) \right\}}{\sum\limits_{\boldsymbol{s}\in\kappa_i^0} \exp\left\{ -\frac{\|\boldsymbol{r}-\mathbf{H}\boldsymbol{s}\|^2}{N_0} + \sum\limits_{j\neq i} c_j L_A^{(\phi)}(c_j) \right\}}, \tag{2}$$

where κ_i^0 and κ_i^1 denote the sets consisting of all the $\boldsymbol{s}$'s for which the corresponding c_i is of binary value 0 and 1, respectively. These extrinsic outputs of demapper are then de-interleaved and taken by the decoder as the *a priori* inputs $L_A^{(\psi)}(c_i)$'s. By an appropriate soft-input soft-output decoding algorithm, e.g. the BCJR algorithm [12], the decoder generates the extrinsic LLRs $L_E^{(\psi)}(c_i)$'s and passes them back to the demapper for further iterations of decoding.

To trace the effect of μ on MIMO BICM-ID, we use the EXIT chart for performance analysis on account of its iterative processing nature. By evaluating the mutual information between LLRs and the coded bits, two information transfer curves $\Phi_\mu(\cdot)$ and $\Psi_C(\cdot)$ are plotted in the EXIT chart to describe how the input information $I_A^{(\phi)}$ and $I_A^{(\psi)}$ are transferred to the output information $I_E^{(\phi)}$ and $I_E^{(\psi)}$ by the demapper ϕ and decoder ψ respectively, i.e.

$$I_E^{(\phi)} = \Phi_\mu(I_A^{(\phi)}) \quad \text{and} \quad I_E^{(\psi)} = \Psi_C(I_A^{(\psi)}). \tag{3}$$

Note that $\Phi_\mu(\cdot)$ is also a function of the SNR, though we omit the corresponding parameter of SNR for simplicity. We also define the two end points of the demapper curve by

$$\alpha_\mu = \Phi_\mu(0) \quad \text{and} \quad \beta_\mu = \Phi_\mu(1). \tag{4}$$

Let $p_{\mu,C}$ denote the first intersection of the demapper and decoder curves, i.e. the smallest mutual information that makes $\Phi_\mu(I) = \Psi_C^{-1}(I)$. According to the EXIT chart, the BER of BICM-ID systems after a sufficient iterations is determined by $p_{\mu,C}$. Since a labeling that achieves a higher value of $p_{\mu,C}$ is able to provide a lower BER, intuitively, one can always obtain a labeling which can achieve the lowest BER by exhaustive search such that

$$\mu_C^* = \arg\max_{\mu\in U} p_{\mu,C}, \tag{5}$$

where U denotes the set of all possible labelings.

III. Pragmatic Design Methodology of Labeling with Low Search Complexity

The exhaustive search in (5) can always guarantee the lowest BER at a fixed channel code and a fixed SNR. However, since the decoder curve depends on the channel code and the demapper curve varies with the SNR, to the best of our knowledge there does not exist an universal labeling of regular modulation that is optimal for all channel codes and SNRs. An intuitive way to keep the optimality is to perform (5) whenever the channel code or the SNR is changed. Obviously, the corresponding search complexity becomes infeasible as m or N_T increases. Although algorithms, such as BSA, can be used to speed up the search time, the resulting complexity is still high if the channel code or the SNR is changed frequently. Besides, given fixed channel code and SNR, we observe that the labelings with similar values of α_μ's and β_μ's, will usually have similar BER performance. (Here, the BER performance includes the SNRs where the waterfall regions occur and the BERs of the error floor regions.) Therefore, we may be able to perform a more efficient search if we can avoid trying the demapper curves with similar shapes.

Motivated by above two reasons, an intuitive idea is to first partition all possible labelings into N groups according to the similarity of their α_μ's and β_μ's. Then one representative labeling is picked from each group to form a set of labelings, call the candidate set Γ. Once the channel code and the SNR are known, the labeling can then be selected within Γ by

$$\mu_C^\Gamma = \arg\max_{\mu\in\Gamma} p_{\mu,C}. \tag{6}$$

In this way, we can both reduce the search complexity by controlling N, the number of labelings in Γ, and avoid trying demapper curves with similar shapes. However, it is infeasible to plot the demapper curves of all possible labelings on the EXIT chart, especially for large m and N_T. Therefore, to obtain Γ in practice, we would like to first determine the desired values of α_μ's and β_μ's we need in Γ, and then find corresponding labelings to approach these values. Let μ_0 denotes the Gray labeling, which is known as the one with the largest α_μ and smallest β_μ among all labelings. Now the procedure for generating Γ is listed as following:

Procedure for generating Γ:

Step 1: Use GA to find the labeling with the largest $\beta_\mu - \alpha_\mu$. Denote this labeling by μ_{N-1} and let $n = 0$.

Step 2: Let $\delta_\alpha = \frac{\alpha_{\mu_0} - \alpha_{\mu_N}}{N-1}$ and $\delta_\beta = \frac{\beta_{\mu_N} - \beta_{\mu_0}}{N-1}$. Let $\alpha_k^* = \alpha_{\mu_0} - k * \delta_\alpha$ and $\beta_k^* = \beta_{\mu_0} + k * \delta_\beta$, $\forall k = 1, 2, ..., N-2$.

Step 3: Let $n = n + 1$ and use GA to find the labeling μ_n that minimizes the fitness function $\theta_n(\mu)$, where $\theta_n(\mu) = |\alpha_n^* - \alpha_\mu| + |\beta_n^* - \beta_\mu|$

Step 4: If $n = N - 1$, Let $\Gamma = \{\mu_0, \mu_1, ..., \mu_{N-1}\}$ and stop the procedure. Else, go back to Step. 3.

Following the procedure, we can successively obtain new labelings with decreasing α_μ's and increasing β_μ's. In Step. 1 and 3, the labeling is found according to the genetic algorithm [11] shown as Fig. 2. Firstly, a set of labelings are randomly generated as the initial population with P members. Secondly, the fitness function $\theta(\mu)$ of each labeling in the population is evaluated and the first $P/2$ labelings with smaller $\theta(\mu)$'s are kept, while the others are eliminated. Note that only α_μ and β_μ of each labeling are needed to be calculated here. Then, a pair of labelings are randomly selected to carry out the crossover, where the label on each signal point of one is randomly decided with probability 0.5 whether or not to adopt the label of its mate to create an offspring. Every offspring has a probability of q to suffer a mutation, where two randomly-selected signal points on the constellation exchanges their labels. Sufficient pairs are picked until the total number of the survivors and the offsprings equals P. After that, a new iteration will start from the re-evaluation of the fitness functions. The GA will stop until I iterations have been done. Instead of BSA, GA is chosen in our design because that the randomly-generated initial population can make the search efficiency less depend on the initial labelings and that the mutation step is more likely to get the solutions out of local optimums.

Examples of the candidate sets Γ_{I} and Γ_{II} for SISO 16QAM are given in Table I and Table II with $N = 6$ and 10 respectively, wherein the Gray labeling is used as the indexing of the signal points and the decimal representation is used for the binary labels. The labelings in the tables are listed in a decreasing order in terms of their α_μ's and their corresponding curves are plotted in Fig. 3(a) and Fig. 3(b) with $E_s/N_0 = 6$ dB at AWGN channels. Using large N is able to create a candidate set with smaller δ_α and δ_β. (Note that although Gray labeling is not unique, using different Gray labelings on indexing the signal points in our tables will lead to identical set of demapper curves.) Once the SNR and the channel code are known, the labeling can then be select within Γ_{I} or Γ_{II} according to (6). Note that the number of labelings needed to be compared is extremely reduced from 16! (in (5)) to 6 or 10. For the MIMO case, a candidate set Γ_{III} for BPSK with $N_T = N_R = 4$ on MIMO Rayleigh fading channels with $E_s/N_0 = -1$ dB is listed with the same order in Table III and the corresponding curves are plotted in Fig. 4. The number of labelings needed to be compared is extremely reduced from 16! to 10 in Γ_{III}. Note that the candidate sets are obtained by setting $I = 30$, $P = 30$, and $q = 0.05$. Further increasing I or P brings only trivial improvement on the minimum $\theta_n(\mu)$ among the populations, according to our experience from experiments.

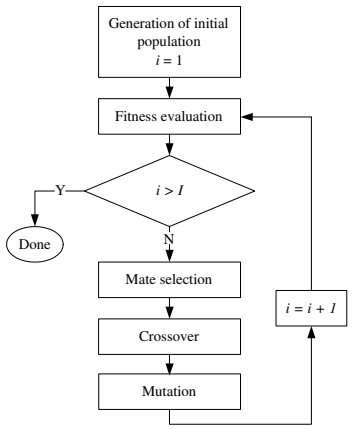


Fig. 2. A block diagram of GA.

TABLE I
LABELINGS IN Γ_{I}

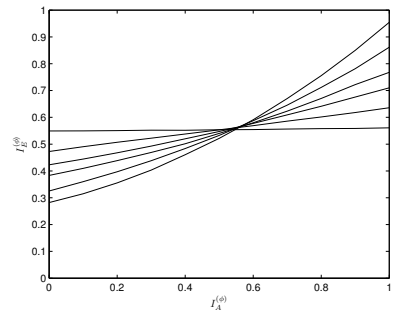
μ	$(\mu(0), \mu(1), \ldots, \mu(15))$
μ_0^{I}	(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)
μ_1^{I}	(0,1,14,2,4,5,7,6,12,13,15,10,8,9,11,3)
μ_2^{I}	(7,1,5,12,3,11,13,6,0,9,15,14,2,10,8,4)
μ_3^{I}	(9,3,1,2,5,15,13,7,4,6,14,11,8,0,12,10)
μ_4^{I}	(15,3,0,6,13,5,10,12,14,4,9,1,7,2,8,11)
μ_5^{I}	(12,14,4,8,2,6,7,13,11,9,0,3,10,15,1,5)

TABLE II
LABELINGS IN Γ_{II}

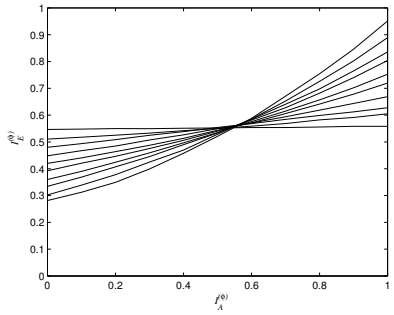
μ	$(\mu(0), \mu(1), \ldots, \mu(15))$
μ_0^{II}	(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)
μ_1^{II}	(0,1,2,3,4,5,6,7,8,9,15,11,12,13,10,14)
μ_2^{II}	(1,4,2,6,0,5,3,7,9,8,11,10,12,13,14,15)
μ_3^{II}	(1,0,2,3,4,5,14,7,8,11,10,9,12,13,6,15)
μ_4^{II}	(9,15,12,14,11,1,8,2,5,4,3,6,13,7,10,0)
μ_5^{II}	(12,14,4,8,2,6,7,13,11,9,0,3,10,15,1,5)
μ_6^{II}	(1,13,15,5,11,9,3,12,2,4,7,0,8,14,6,10)
μ_7^{II}	(5,9,15,3,0,2,6,12,8,1,4,7,10,11,14,13)
μ_8^{II}	(15,11,1,2,6,8,7,4,0,9,10,3,5,12,13,14)
μ_9^{II}	(4,14,9,3,7,13,10,0,2,8,15,5,1,11,12,6)

TABLE III
LABELINGS IN Γ_{III}

μ	$(\mu(0), \mu(1), \ldots, \mu(15))$
μ_0^{III}	(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)
μ_1^{III}	(8,0,2,10,4,12,6,14,1,9,3,11,5,13,7,15)
μ_2^{III}	(8,0,2,10,4,12,14,6,1,9,3,11,5,13,7,15)
μ_3^{III}	(1,0,2,3,4,5,7,6,9,8,11,10,12,13,14,15)
μ_4^{III}	(8,4,2,10,0,12,14,6,1,9,11,3,5,13,7,15)
μ_5^{III}	(1,13,6,15,5,8,2,10,4,9,3,14,0,12,7,11)
μ_6^{III}	(2,13,6,9,5,8,1,15,4,10,3,14,0,12,7,11)
μ_7^{III}	(0,15,7,9,5,8,10,13,4,1,3,14,2,12,6,11)
μ_8^{III}	(8,15,7,9,5,0,10,13,6,1,11,14,2,12,4,3)
μ_9^{III}	(15,4,0,11,2,9,13,6,1,10,14,5,12,7,3,8)



(a)



(b)

Fig. 3. The demapper transfer curves of (a) Γ_{I} in Table I and (b) Γ_{II} in Table II with $E_s/N_0 = 6$ dB at AWGN channels.

IV. SIMULATION RESULTS

For SISO 16QAM, the examples of labeling selection in Γ_{II} are given in Fig. 5(a) and Fig. 5(b) for BCJR decoder of convolutional code with with generator matrix $(1+D+D^2+D^3, 1+D+D^3)$, denoted by CC(17,15) for simplicity, and AWGN channels with $E_b/N_0 = 3$ dB and 4 dB, respectively. For $E_b/N_0 = 3$ dB, the labeling that achieves the highest first intersection is μ_6^{II}, while for $E_b/N_0 = 4$ dB, μ_9^{II} becomes a favorable choice. The optimality of μ_6^{II} and μ_9^{II} among Γ_{II} for the two different SNRs can be easily verified by the BER simulation in Fig. 6, wherein a bit-interleaver of block length 80000 bits, an MAP demapper in (2) and a BCJR decoder are employed for iterative decoding with 40 iterations. It can also be observed that on one hand the labelings $\mu_0, \mu_1, ..., \mu_9$ which have an order of regularly decreasing α_μ can result in increasing SNRs of the occurrence of the waterfall regions and on the other hand the order of regularly increasing β_μ also leads to a regularly decreasing BERs at the error floors.

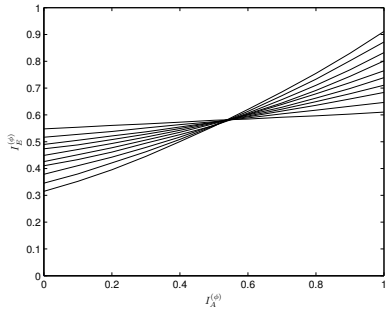
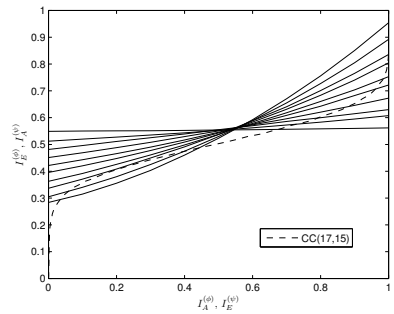
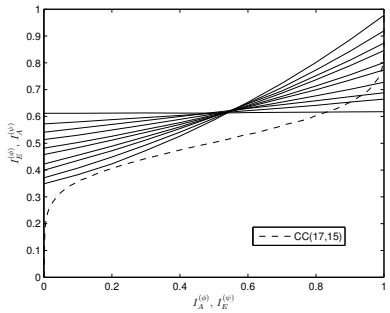


Fig. 4. The demapper transfer curves of Γ_{III} in Table. III with $E_s/N_0 = -1$ dB at Rayleigh fading channels.



(a)



(b)

Fig. 5. EXIT chart with the transfer curves of decoder CC(17,15) and the demapper curves in Γ_{II} for AWGN channels with (a) $E_b/N_0 = 3$ dB (b) $E_b/N_0 = 4$ dB.

Besides, each labeling is favorable at its own SNR interval and thus we can directly adopt this candidate set on different SNRs to reduce the time on re-searching a new labeling using BSA or exhaustive search. For the MIMO case, the BER simulations of labelings in Γ_{III} are shown in Fig. 7 for BCJR decoder of convolutional code with with generator matrix $(1+D+D^2, 1+D^2)$ and Rayleigh fading channels. It can also be observed that it is beneficial to select proper labelings from Γ_{III} on individual SNR intervals, though the EXIT charts to illustrate the corresponding selection are omitted here due to the length limitation.

Fig. 8 shows the effect of N on the labeling selection. The labelings μ_3^{I} and μ_6^{II} are chosen from Γ_{I} ($N = 6$) and Γ_{II} ($N = 10$) respectively to achieve the highest first intersection with the decoder curve of BCJR decoder of CC(17,15), while the labeling μ_C^* is obtained according to (5) with the help of

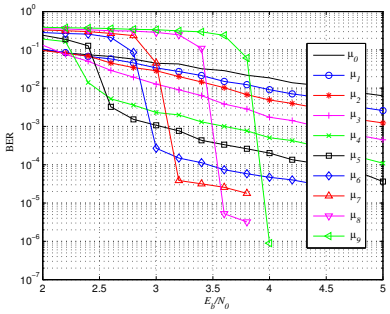


Fig. 6. The BER performance of the candidate set Γ_{II} for Table. II at AWGN channels.

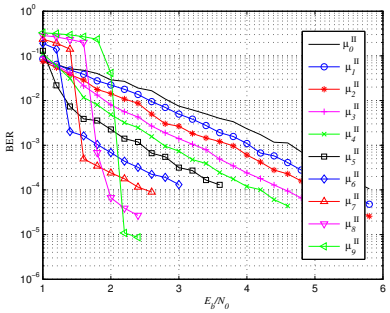


Fig. 7. The BER performance of the candidate set Γ_{III} for Table. III at fast Rayleigh fading channels.

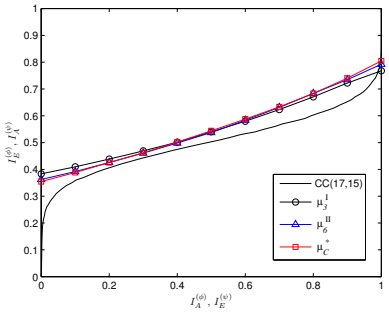


Fig. 8. EXIT chart with the transfer curves of decoder CC(17,15) and the demapper curves of labelings μ_3^{I}, μ_6^{II} and μ_C^* with $\mu_C^* = (14, 13, 0, 4, 7, 8, 5, 1, 9, 15, 3, 10, 11, 2, 6, 12)$.

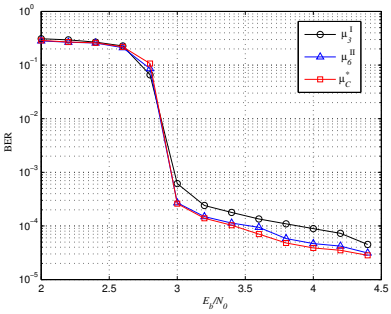


Fig. 9. The BER performance of the labelings μ_3^{I}, μ_6^{II} and μ_C^* at AWGN channels, note that the curve of μ_C^* at $E_b/N_0 = 3$dB is obtained by performing 80 iterations.

GA. It is shown in Fig. 8 that μ_C^* has the highest intersection, closely followed by μ_6^{II} and then μ_3^{I} at $E_b/N_0 = 3$ dB. The corresponding BER performance in Fig. 9 sustains the observations in Fig. 8. Usually, the labeling chosen from the candidate with large N is more likely to achieve close BER performance to the BER performance of μ_C^*. In our case, $N = 10$ is good enough to have the close-to-optimum BER performance, while the number of labelings needed to be compared is extremely reduced to 10.

V. Conclusion

A pragmatic design of labeling for MIMO BICM-ID is presented in this paper with the help of the EXIT-chart based analysis. To both reduce the search complexity and avoid trying labelings with similar shapes, the idea of the candidate set is first specified on the EXIT chart and realized by a procedure based on GA. Once the channel code and SNR are known, we can search within the candidate set for a most-suitable labeling to achieve the lowest BER with low complexity. Simulation results show that the labeling chosen from the candidate can achieve a very close BER performance to the labeling obtained by exhaustive search. Besides, even though the channel code or the SNR is changed, the same candidate set still can be directly adopted to reduce the time on re-searching new labelings using BSA or exhaustive search.

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