NORTH-HOLLAND

Convergence Rates of Iterative Solutions of Algebraic Matrix Riccati Equations

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ABSTRACT

We consider the iterative solutions of a certain class of algebraic matrix Riccati equations with two parameters, $c(0 \leq c \leq 1)$ and $\alpha(0 \leq \alpha \leq 1)$. Here c denotes the fraction of scattering per collision and α is an angular shift. Equations of this class are induced via invariant imbedding and the shifted Gauss–Lengendre quadrature formula from a "simple transport model."

The purpose of this paper is to describe the effects of the parameters c, α, and N (the dimension of the matrix) on the convergence rates of the iterative solutions. We also compare the convergence rates of those iterative methods.

1. INTRODUCTION

Consider the algebraic matrix Riccati equation of the form

$$B - AS - SD + SCS = 0. \tag{1}$$

Here A, B, C, and D are matrices of approximate dimensions having the

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following structure:

$$
\begin{aligned}
A_{N^- \times N^-} &= \operatorname{diag}\left[\frac{1}{c(w_1^- + \alpha)}, \frac{1}{c(w_2^- + \alpha)}, \ldots, \frac{1}{c(w_{N^-}^- + \alpha)}\right] \\
&\quad - \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \left[\frac{c_1^-}{2(w_1^- + \alpha)}, \frac{c_2^-}{2(w_2^- + \alpha)}, \ldots, \frac{c_{N^-}^-}{2(w_{N^-}^- + \alpha)}\right] \\
&:= D_A - ia^T,
\end{aligned}
$$

where

$$
i = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix};
$$

$$
\begin{aligned}
D_{N^+ \times N^+} &= \operatorname{diag}\left[\frac{1}{c(w_1^+ - \alpha)}, \frac{1}{c(w_2^+ - \alpha)}, \ldots, \frac{1}{c(w_{N^+}^+ - \alpha)}\right] \\
&\quad - \left[\frac{c_1^+}{2(w_1^+ - \alpha)}, \frac{c_2^+}{2(w_2^+ - \alpha)}, \ldots, \frac{c_{N^+}^+}{2(w_{N^+}^+ - \alpha)}\right]^{T} i^T \\
&:= D_D - di^T; \\
B &= ii^T;
\end{aligned}
$$

and

$$
C = da^T.
$$

Equation (1) contains two parameters c and α. Here c denotes the average total number of particles emerging from a collision, which is assumed to be conservative, i.e., $0 \leq c \leq 1$, and $\alpha (0 \leq \alpha \leq 1)$ is an angular shift. The dimensionally dependent quantities w_i^- and w_i^+ denote the Gauss–Legendre sets on $[-\alpha, 1]$ and $[\alpha, 1]$, respectively; and c_i^- and c_i^+ are, respectively, their corresponding weights. Such equation is induced via invariant imbedding (see, e.g., [1, 2]), and the shifted Gauss–Legendre quadrature formula (see, e.g., [3]), from a "simple transport model" [4, 5].

For $\alpha = 0$, two iterative procedures, one corresponding to a nonlinear version of the Gauss–Jocobi method (GJ) and the other associated with a nonlinear version of the Gauss–Seidel method (GS) were proposed, respectively, by Shimizu and Aoki [6], and Juang and Lin [7]. Sufficient conditions for the convergence of the GJ and GS methods were given in [8] and [7], respectively. It was noted (see [9, Table 2]) that those sufficient conditions would fail if c is not far away from 1. And it was also observed (see [9, Theorem 1]) that both iterations converge as long as (1) with $\alpha = 0$ has a nonnegative solution (in the componentwise sense). Such observation can be easily extended to the case that $\alpha \neq 0$. Physically, one would expect that (1) has a nonnegative solution for all $0 \leq c \leq 1$ and $0 \leq \alpha \leq 1$. This is recently proved in [10]. Therefore, we shall not be worried about the problem of convergence in this article.

The purpose of this work is to analyze the behavior of the convergence rates of GJ and GS as parameters c, α and $N^{\pm}$ vary. In particular, we show that for fixed α and $N^{\pm}$ two methods GJ and GS converge slower as c increases, and that for fixed c and $N^{\pm}$, GJ and GS converge faster as α increases on $[\alpha^*, 1]$, for some $0 < \alpha^* < 1$. Some estimates for the convergence rates of both methods are obtained. Furthermore, we show that the GS method indeed converges no slower than the GJ method. Finally, some numerical results and concluding remarks are presented.

Since for $\alpha \approx 0$ and $c \approx 1$, such iterative procedures are extremely slow, the relaxation methods, such as the Jacobi overrelaxation (JOR) and successive overrelaxation (SOR) methods, are more desirable. However, the convergence of the relaxation methods is difficult to prove. Moreover, the search for the optimal w could be too expensive to be practical. Our analysis shall be a step forward toward understanding the phenomenon of the slow convergence as $c \approx 1$ and $\alpha \approx 0$, and can be hopefully put to use in developing better algorithms such as multilevel methods.

2. FORMULATION

We first rewrite (1) as

$$D_A S + S D_D = B + ia^T S + Sid^T + SCS. \tag{2}$$

In component form, (2) is

$$S_{ij} = \frac{c(w_i^- + \alpha)(w_j^+ - \alpha)}{(w_i^- + w_j^+)} \left[1 + \frac{1}{2}\sum_{k=1}^{N^-} \frac{c_k^- S_{kj}}{(w_k^- + \alpha)}\right] \left[1 + \frac{1}{2}\sum_{k=1}^{N^+} \frac{c_k^+ S_{ik}}{(w_k^+ - \alpha)}\right]$$

$$:= cr_{ij} \left[1 + \frac{1}{2}\sum_{k=1}^{N^-} \frac{c_k^- S_{kj}}{(w_k^- + \alpha)}\right] \left[1 + \frac{1}{2}\sum_{k=1}^{N^+} \frac{c_k^+ S_{ik}}{(w_k^+ - \alpha)}\right] \tag{3}$$

$$:= W_J S_{ij}.$$

The Gauss–Jacobi iteration is then defined as follows:

$$S_{ij}^{(p+1)} = W_J S_{ij}^{(p)}. \tag{4}$$

The Gauss-Seidel iteration can be formulated as

$$S_{ij}^{(p+1)} = cr_{ij} \left[1 + \frac{1}{2}\sum_{k=1}^{i-1} \frac{c_k^- S_{kj}^{(p+1)}}{(w_k^- + \alpha)} + \frac{1}{2}\sum_{k=i}^{N^-} \frac{c_k^- S_{kj}^{(p)}}{(w_k^- + \alpha)}\right]$$

$$\times \left[1 + \frac{1}{2}\sum_{k=1}^{j-1} \frac{c_k^+ S_{ik}^{(p+1)}}{(w_k^+ - \alpha)} + \frac{1}{2}\sum_{k=j}^{N^+} \frac{c_k^+ S_{ik}^{(p)}}{(w_k^+ - \alpha)}\right]$$

$$:= W_S(i,j,p). \tag{5}$$

Consequently, the JOR and SOR are, respectively,

$$T_{ij}^{(p+1)} = W_J S_{ij}^{(p)}, \tag{6a}$$

$$S_{ij}^{(p+1)} = wT_{ij}^{(p+1)} + (1-w)S_{ij}^{(p)}; \tag{6b}$$

and

$$T_{ij}^{(p+1)} = W_S(i,j,p), \tag{7a}$$

$$S_{ij}^{(p+1)} = wT_{ij}^{(p+1)} + (1-w)S_{ij}^{(p)}. \tag{7b}$$

Since we are only interested in positive solutions, the initial iteration for the procedures described above is defined to be

$$S_{ij}^{(0)} = 0 \qquad \text{for all } i,j. \tag{8}$$

To effectively analyze the convergence rate of both methods, the following transformation of S_{ij} into R_{ij} is essential, for reasons we shall detail later. In the case of the Gauss–Jacobi, consider the following iteration $\{R_{ij}^{(p)}\}_{p=0}^{\infty}$:

$$
\begin{aligned}
R_{ij}^{(p+1)} &= \left[1 + \frac{1}{2}\sum_{k=1}^{N^-} \frac{cc_k^-(w_j^+ - \alpha)R_{k_j}^{(p)}}{w_j^+ + w_k^-}\right] \\
&\quad \times \left[1 + \frac{1}{2}\sum_{k=1}^{N^+} \frac{cc_k^+(w_i^- + \alpha)R_{ik}^{(p)}}{w_i^- + w_k^+}\right] \qquad \text{(9a)} \\
&:= U_J(i,j,p) \qquad R_{ij}^{(0)} = 1 \qquad \text{for all } i,j. \qquad \text{(9b)}
\end{aligned}
$$

The JOR version of the iteration $\{R_{ij}^{(p)}\}_{p=0}^{\infty}$ is then defined as

$$
\begin{aligned}
T_{ij}^{(p+1)} &= U_J(i,j,p) && \text{(10a)} \\
R_{ij}^{(p+1)} &= wT_{ij}^{(p+1)} + (1-w)R_{ij}^{(p)} && \text{(10b)} \\
R_{ij}^{(0)} &= 1 \qquad \text{for all } i,j. && \text{(10c)}
\end{aligned}
$$

Similarly, the Gauss–Seidel and the SOR forms of the iteration $\{R_{ij}^{(p)}\}_{p=0}^{\infty}$ can be formulated, respectively, as follows:

$$
\begin{aligned}
R_{ij}^{(p+1)} &= \left[1 + \frac{1}{2}\sum_{k=1}^{i-1} \frac{cc_k^-(w_j^+ - \alpha)R_{kj}^{(p+1)}}{w_j^+ + w_k^-} + \frac{1}{2}\sum_{k=i}^{N^-} \frac{cc_k^-(w_j^+ - \alpha)R_{kj}^{(p)}}{w_j^+ + w_k^-}\right] \\
&\quad \times \left[1 + \frac{1}{2}\sum_{k=1}^{j-1} \frac{cc_k^+(w_i^- + \alpha)R_{ik}^{(p+1)}}{w_i^- + w_k^+} + \frac{1}{2}\sum_{k=j}^{N^+} \frac{cc_k^+(w_i^- + \alpha)R_{ik}^{(p)}}{w_i^- + w_k^+}\right] \qquad \text{(11a)} \\
&:= U_S(i,j,p) \qquad R_{ij}^{(0)} = 1 \qquad \text{for all } i,j; \qquad \text{(11b)}
\end{aligned}
$$

and

$$
\begin{aligned}
T_{ij}^{(p+1)} &= U_S(i,j,p) && \text{(12a)} \\
R_{ij}^{(p+1)} &= wT_{ij}^{(p+1)} + (1-w)R_{ij}^{(p)} && \text{(12b)} \\
R_{ij}^{(0)} &= 1 \qquad \text{for all } i,j. && \text{(12c)}
\end{aligned}
$$

REMARKS

(1) The existence and multiplicity of the positive solutions of (1) have been addressed in [10]. Since the iteration procedures defined by (4), (5), and (8) are monotonically increasing, any positive solution of (1) is an upper bound for both iterations. Therefore, the convergence

of such iterations as well as the iterations defined by (9) and (11) are assured. We shall denote the limits of the iterations $\{S_{ij}^{(p)}\}_{p=0}^{\infty}$ and $\{R_{ij}^{(p)}\}_{p=0}^{\infty}$ by $S_{ij}^{(\infty)}$ and $R_{ij}^{(\infty)}$, respectively.
Here, the matrix $R^{(\infty)} = (R_{ij}^{(\infty)})$ is a positive solution of the following equation:

$$R_{ij} = \left[1 + \frac{1}{2}\sum_{k=1}^{N^-} \frac{cc_k^-(w_j^+ - \alpha)R_{kj}}{w_j^+ + w_k^-}\right]\left[1 + \frac{1}{2}\sum_{k=1}^{N^+} \frac{cc_k^+(w_i^- + \alpha)R_{ik}}{w_i^- + w_k^+}\right]. \tag{13}$$

(2) Noting, via induction, that $S_{ij} \geq S_{ij}^{(p)}$ for all i, j, and p, we conclude that $S^{(\infty)} = (S_{ij}^{(\infty)})$ is the minimal positive solution of (1) in the sense that if S is any positive solution of (1), then $S_{ij} \geq S_{ij}^{(\infty)}$ for all i, j. Similarly, $R^{(\infty)}$ is the minimal positive solution of (13).

The relationship between the iterations $\{R_{ij}^{(p)}\}_{p=0}^{\infty}$ and $\{S_{ij}^{(p)}\}_{p=0}^{\infty}$ are provided by the following lemma:

LEMMA 1. *The Gauss–Jacobi, Gauss–Seidel, JOR, and SOR versions of $\{S_{ij}^{(p)}\}_{p=0}^{\infty}$ are, respectively, related to those of $\{R_{ij}^{(p)}\}_{p=0}^{\infty}$ by the following formula.*

$$S_{ij}^{(p+1)} = cr_{ij}R_{ij}^{(p)} \quad \textit{for all } i, j \textit{ and all } p = 0, 1, 2, \ldots, \infty. \tag{14}$$

Consequently, for each i, j the iteration $\{S_{ij}^{(p)}\}_{p=0}^{\infty}$ enjoys the same convergence rate as its corresponding iteration $\{R_{ij}^{(p)}\}_{p=0}^{\infty}$.

PROOF. Since the proof leading to the assertion of the lemma for each iterative procedure is the same, we shall only illustrate the SOR method.

For $p = 0$, and i, j, clearly,

$$S_{ij}^{(1)} = cr_{ij} = cr_{ij}R_{ij}^{(0)}.$$

Suppose (14) is true up to $p = n + 1$, and $i \leq k - 1$ and $j \leq \ell - 1$. Then

$$\begin{aligned} S_{k\ell}^{(n+1)} &= wW_S(k, \ell, n) + (1 - w)S_{k\ell}^{(n)} \\ &= cr_{k\ell}wU_S(k, \ell, n - 1) + (1 - w)cr_{k\ell}R_{k\ell}^{(n-1)} \\ &= cr_{k\ell}\big(wT_{k\ell}^{(n)} + (1 - w)R_{kl}^{(n-1)}\big) \\ &= cr_{k\ell}R_{k,l}^{(n)}. \end{aligned}$$

The second equality above is justified by the induction hypothesis. This proves the first assertion of the lemma. The last assertion of the lemma now follows from (14). ∎

To further simplify the notation, the quadrature sets $\{w_i^-\}_{i=1}^{N^-}$ and $\{w_i^+\}_{i=1}^{N^+}$ and their corresponding weights $\{c_i^-\}_{i=1}^{N^-}$ and $\{c_i^+\}_{i=1}^{N^+}$ over the intervals $[-\alpha, 1]$ and $[\alpha, 1]$, respectively, are to be normalized into the standard interval $[-1, 1]$. In particular, suppose $\{x_i^-\}_{i=1}^{N^-}$ and $\{x_i^+\}_{i=1}^{N^+}$ are the quadrature sets and $\{d_i^-\}_{i=1}^{N^-}$ and $\{d_i^+\}_{i=1}^{N^+}$ are their associate quadrature weights over the interval $[-1, 1]$. Then, for all i,

$$w_i^- = \frac{1 - \alpha + x_i^-(1+\alpha)}{2}, \qquad w_i^+ = \frac{1 + \alpha + x_i^+(1-\alpha)}{2} \tag{15a}$$

$$c_i^- = \frac{(1+\alpha)d_i^-}{2}, \qquad c_i^+ = \frac{(1-\alpha)d_i^+}{2}. \tag{15b}$$

Without loss of generality, we shall assume henceforth that

$$\begin{aligned} -1 < x_1^+ < x_2^+ < \cdots < x_{N^+}^+ < 1 \qquad &\text{and} \\ -1 < x_1^- < x_2^- < \cdots < x_{N^-}^- < 1. & \end{aligned} \tag{16}$$

Substituting (15) into (9), (10), (11), and (12), respectively, we obtain, respectively,

$$\begin{aligned} R_{ij}^{(p+1)} &= \left[1 + \frac{c}{4}\sum_{k=1}^{N^-} \frac{d_k^-(1-\alpha^2)(1+x_j^+)R_{kj}^{(p)}}{2 + x_j^+(1-\alpha) + x_k^-(1+\alpha)}\right] \\ &\quad \times \left[1 + \frac{c}{4}\sum_{k=1}^{N^+} \frac{d_k^+(1-\alpha^2)(1+x_i^-)R_{ik}^{(p)}}{2 + x_i^-(1+\alpha) + x_k^+(1-\alpha)}\right] \\ &:= \left[1 + \frac{c}{4}\sum_{k=1}^{N^-} d_k^- g_j(x_k^-) R_{kj}^{(p)}\right]\left[1 + \frac{c}{4}\sum_{k=1}^{N^+} d_k^+ f_i(x_k^+) R_{ik}^{(p)}\right] && \text{(17a)} \\ &:= \overline{U}_J(i,j,p) \qquad R_{ij}^{(0)} = 1 \qquad \text{for all } i, j; && \text{(17b)} \end{aligned}$$

and

$$\begin{aligned} T_{ij}^{(p+1)} &= \overline{U}_J(i,j,p) && \text{(18a)} \\ R_{ij}^{(p+1)} &= wT_{ij}^{(p+1)} + (1-w)R_{ij}^{(p)} && \text{(18b)} \\ R_{ij}^{(0)} &= 1 \qquad \text{for all } i, j; && \text{(18c)} \end{aligned}$$

$$\begin{aligned} R_{ij}^{(p+1)} &= \left[1 + \frac{c}{4}\sum_{k=1}^{i-1} d_k^+ f_i(x_k^+) R_{ik}^{(p+1)} + \frac{c}{4}\sum_{k=i}^{N^+} d_k^+ f_i(x_k^+) R_{ik}^{(p)}\right] \\ &\quad \times \left[1 + \frac{c}{4}\sum_{k=1}^{j-1} d_k^- g_j(x_k^-) R_{kj}^{(p+1)} + \frac{c}{4}\sum_{k=j}^{N^-} d_k^- g_j(x_k^-) R_{kj}^{(p)}\right] && \text{(19a)} \\ &:= \overline{U}_S(i,j,p) \qquad R_{ij}^{(0)} = 1 \qquad \text{for all } i, j; && \text{(19b)} \end{aligned}$$

and

$$T_{ij}^{(p+1)} = \overline{U}_S(i,j,p) \tag{20a}$$

$$R_{ij}^{(p+1)} = wT_{ij}^{(p+1)} + (1-w)R_{ij}^{(p)} \tag{20b}$$

$$R_{ij}^{(0)} = 1 \qquad \text{for all } i,j. \tag{20c}$$

We shall henceforth work on the equations (17)–(20).

3. THE MAIN RESULTS

Our objective in this section is to study the effect of parameters c, α and that of the dimension on the iterative procedures defined by (17)–(18).

LEMMA 2. *Let $\{R_{ij}^{(p)}\}_{p=0}^{\infty}$ be the sequence defined by* (17), *and let $r_{\max}^{(p)} := \max_{i,j} R_{ij}^{(p)}$, and $r_{\min}^{(p)} := \min_{i,j} R_{ij}^{(p)}$. Then the following holds:*

(i) *For fixed i,j,α, and $N^{\pm}$, then $R_{ij}^{(p)}$ is increasing with respect to c for $p = 1,2,\ldots,\infty$.*
(ii) *For fixed c, α, and j, $R_{ij}^{(p)}$ is increasing with respect to i for all $p = 1,2,\ldots,\infty$. Likewise, $R_{ij}^{(p)}$ is increasing with respect to j for all fixed c,α,j, and p. Here $p = 1,2,\ldots,\infty$.*
(iii) *For fixed c and $N^{\pm}$, there exists an α^*, where $0 \le \alpha^* < 1$ such that $R_{ij}^{(p)}$ is decreasing with respect to α on $[\alpha^*,1]$ for all i,j and $p = 1,2,\ldots,\infty$.*
(iv) *For $\alpha = 0$, $N^- = N^+, R^{(\infty)}$ is a symmetric matrix.*

PROOF. The assertion of (i) and (ii) follows from (17a) and an induction on p. To see (iii), differentiating $f_i(x_j^+)$ with respect to α, we obtain that

$$\begin{aligned}
&\left.\frac{\partial f_i(x_j^+)}{\partial \alpha}\right|_{\alpha=1} \\
&\quad = \left.\frac{(1+x_i^-)\left[(x_j^+ - x_i^-)\alpha^2 - 2\alpha(2+x_i^- + x_j^+) + (x_j^+ - x_i^-)\right]}{\left(2 + x_i^-(1+\alpha) + x_j^+(1-\alpha)\right)^2}\right|_{\alpha=1} \\
&\quad = \frac{-4(1+x_i)^2}{\left(2 + x_i^-(1+\alpha) + x_j^+(1-\alpha)\right)^2} < 0.
\end{aligned}$$

Similarly, $\partial g_i(x_j^-)/\partial\alpha|_{\alpha=1} < 0$.

An induction on p from (17a) will give (iii) as asserted. To prove (iv), we see, again, that an induction will give $R_{ij}^{(p)} = R_{ji}^{(p)}$ for all i,j and p, and hence $R_{ij}^{(\infty)} = R_{ji}^{(\infty)}$. ∎

REMARKS. (1) From computational data, it is expected that for $i \le j$, $R_{ij}^{(p)}$ is decreasing in α on $[0,1]$, and that for $i > j$, $R_{ij}^{(p)}$ is increasing in α on $[0,\overline{\alpha}]$ and decreasing in α on $[\overline{\alpha},1]$. Here $\overline{\alpha}$ depends on i,j and is in between 0 and 1. (2) Similar assertions to (i)–(iii) of Lemma 2 hold for the iteration $\{R_{ij}^{(p)}\}_{p=0}^{\infty}$ defined by (19).

Let $\{R_{ij}^{(p)}\}_{p=0}^{\infty}$ be the sequence defined by (17), and recall that $\{R_{ij}^{(p)}\}_{p=0}^{\infty}$ converges monotonically upward to the minimal positive solution $R_{ij}^{(\infty)}$ of (13). Set

$$R_{ij}^{(\infty)} - R_{ij}^{(p)} := e_{ij}^{(p)} \tag{21}$$

so that

$$\begin{aligned}
e_{ij}^{(p+1)} &= \frac{c}{4}\sum_{k=1}^{N^-} d_k^- g_j(x_k^-) e_{kj}^{(p)} + \frac{c}{4}\sum_{k=1}^{N^+} d_k^+ f_i(x_k^+) e_{ik}^{(p)} \\
&\quad + \frac{c^2}{16}\left(\sum_{k=1}^{N^-} d_k^- g_j(x_k^-) e_{kj}^{(p)}\right)\left(\sum_{k=1}^{N^+} d_k^+ f_i(x_k^+) R_{ik}^{(p)}\right) \\
&\quad + \frac{c^2}{16}\left(\sum_{k=1}^{N^-} d_k^- g_j(x_k^-) R_{kj}^{(p)}\right)\left(\sum_{k=1}^{N^+} d_k^+ f_i(x_k^+) e_{ik}^{(p)}\right) \\
&\le \frac{c}{4}\sum_{k=1}^{N^-} d_k^- g_j(x_k^-)\left[1 + \frac{c}{4}\sum_{k=1}^{N^+} d_k^+ f_i(x_k^+) R_{ik}^{(\infty)}\right] e_{kj}^{(p)} \\
&\quad + \frac{c}{4}\sum_{k=1}^{N^+} d_k^+ f_i(x_k^+)\left[1 + \frac{c}{4}\sum_{k=1}^{N^-} d_k^- g_j(x_k^-) R_{kj}\right] e_{ik}^{(p)} \\
&:= \frac{c}{4}\sum_{k=1}^{N^-} (\tilde{g}_{kj}\overline{f}_i) e_{kj}^{(p)} + \frac{c}{4}\sum_{k=1}^{N^+} (\tilde{f}_{ik}\overline{g}_j) e_{ik}^{(p)}.
\end{aligned} \tag{22}$$

Let $E = (e_{11}, e_{12}, \ldots, e_{1N^+}, e_{21}, \ldots, e_{2N^+}, e_{31}, \ldots, e_{N^-1}, \ldots, e_{N^-N^+})^T$. The inequality (22) can be written as

$$E^{(p+1)} \le G_J E^{(p)}.$$

Here G_J is a large and sparse matrix with the following structure;

$$G_J = \frac{c}{4}\begin{bmatrix} D_{11}+B_1 & D_{12} & \cdots & \cdots & D_{1N^-} \\ D_{21} & D_{22}+B_2 & \cdots & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ D_{N^-1} & D_{N^-2} & \cdots & \cdots & D_{N^-N^-}+B_{N^-} \end{bmatrix}_{N^+N^- \times N^+N^-},$$

where $D_{ij} = \operatorname{diag}[\widetilde{g}_{j1}\overline{f}_i, \widetilde{g}_{j2}\overline{f}_i, \ldots, \widetilde{g}_{jN^+}\overline{f}_i]$, and $B_k = (\overline{g}_i\widetilde{f}_{kj})_{N^+\times N^+}$. In the case of the Gauss–Seidel iteration, one can similarly obtain that

$$e_{ij}^{(p+1)} - \frac{c}{4}\sum_{k=1}^{i-1}(\widetilde{g}_{kj}\overline{f}_i)e_{kj}^{(p+1)} - \frac{c}{4}\sum_{k=1}^{j-1}(\widetilde{f}_{ik}\overline{g}_j)e_{ik}^{(p+1)} \leq \frac{c}{4}\sum_{k=i}^{N^-}(\widetilde{g}_{kj}\overline{f}_i)e_{kj}^{(p)} + \frac{c}{4}\sum_{k=j}^{N^+}(\widetilde{f}_{ik}\overline{g}_j)e_{ik}^{(p)}. \tag{23}$$

Let $G_J = D + L + U$, where L and U are strictly lower and strictly upper triangular matrices of G_J. Then (23) in the matrix form is

$$E^{(p+1)} \leq G_S E^{(p)},$$

where $G_S = (I - L)^{-1}(D + U)$.

DEFINITION 1. The convergence rates of the Gauss–Jacobi and the Gauss–Seidel methods are defined to be the spectral radii $\rho(G_J)$ and $\rho(G_S)$ of G_J and G_S, respectively. To emphasize the dependency of $\rho(G_J)$ on the parameters, we shall denote $\rho(G_J)$ by $\mathrm{CGJ}(c, \alpha, N^{\pm})$. Likewise, $\rho(G_S)$ by $\mathrm{CGS}(c, \alpha, N^{\pm})$.

REMARK. As noted earlier the iterations defined by the Gauss–Jacobi and the Gauss–Seidel methods converges to the minimal positive solution of (13). Therefore, it is reasonable to assume from hereafter that $\rho(G_J)$ and $\rho(G_S)$ are no greater than 1.

We are now ready to state our first main result.

THEOREM 1.

(i) *For fixed c, α and $N^{\pm}$, the Gauss–Seidel iteration always converges no slower than the Gauss–Jacobi method, i.e., $CGJ(c, \alpha, N^{\pm}) \geq CGS(c, \alpha, N^{\pm})$.*
(ii) *For fixed α and $N^{\pm}$, CGJ and CGS are increasing in c.*
(iii) *For fixed c and $N^{\pm}$, CGJ and CGS are decreasing in α for $\alpha \in [\alpha^*, 1]$, where α^* is as in Lemma* 2(iii).

PROOF. We first note that G_J and G_S are nonnegative matrices (in the componentwise sense). It follows from the well-known Perron–Frobenius theorem that $\rho(G_S)$ is an eigenvalue of G_S. Let $\rho(G_S) = \lambda \leq 1$, then there exists a vector $E \neq 0$ such that

$$(I - L)^{-1}(D + U)E = \lambda E.$$

Hence,

$$(\lambda L + D + U)E = \lambda E.$$

Since $G_J \geq \lambda L + D + U$, we conclude that

$$\rho(G_J) \geq \rho(\lambda L + D + U) \geq \lambda.$$

The second and the last assertions of the theorem are direct consequences of Lemma 2 and the Perron–Frobenius theorem. ■

REMARK. The numerical results suggest that $\alpha^* = 0$.

LEMMA 3.

(i) $\max_{1\leq j\leq N^+} j g_j(x) = g_{N^+}(x)$, $\max_{1\leq i\leq N^-} f_i(x) = f_{N^-}(x)$.

(ii) $\sum_{k=1}^{N^-} d_k^- g_j(x_k^-)$ *and* $\sum_{k=1}^{N^+} d_k^+ f_i(x_k^+)$ *converges up to* $\int_{-1}^1 g_j(x)\,dx$ *and* $\int_{-1}^1 f_i(x)\,dx$, *respectively, as* $N^\pm \to \infty$.

(iii) $\sup_{1\leq N^+<\infty} \max_{1\leq j\leq N^+} \int_{-1}^1 g_j(x)\,dx = \lim_{j\to N^+} \int_{-1}^1 g_j(x)\,dx = 2(1-\alpha)\ell n\, 2/(1-\alpha) := G(\alpha)$.

(iv) $\sup_{1\leq N^-<\infty} \max_{1\leq i\leq N^-} \int_{-1}^1 f_i(x)\,dx = \lim_{i\to N^-} \int_{-1}^1 f_i(x)\,dx = 2(1+\alpha)\ell n\, 2/(1+\alpha) := F(\alpha)$.

PROOF. A direct calculation will give the assertion of Lemma 3(i). To see (ii), we note, (e.g., see [(3, 5.3.29)] that

$$\begin{aligned} E_N(h) &:= \int_{-1}^1 h(x)\,dx - \sum_{k=1}^N d_k h(x_k) \\ &= \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n!)]^2} \cdot \frac{h^{(2n)}(\eta)}{(2n)!}. \end{aligned}$$

The assertion of the Lemma 3(ii) now follows from the above error formula. The proof of (iii) and (iv) is similar, we illustrate only (iv).

Let $y_i = (1+\alpha)(1+x_i)$. Then

$$\begin{aligned} \int_{-1}^1 f_i(x)\,dx &= (1+\alpha)(1+x_i)\,\ell n\,\frac{(1+\alpha)x_i + (3-\alpha)}{(1+\alpha)(1+x_i)} \\ &= y_i\,\ell n\,\frac{y_i + 2 - 2\alpha}{y_i} := H(y_i). \end{aligned}$$

Now,

$$\frac{dH}{dy_i} = \ell n(y_i + 2 - 2\alpha) - \ell n\, y_i - \frac{2-2\alpha}{y_i + 2 - 2\alpha}$$

and

$$\frac{d^2H}{dy_i^2} = (2-2\alpha)\left[\frac{1}{(y_i+2-2\alpha)^2} - \frac{1}{(y_i+2-2\alpha)y_i}\right] \le 0.$$

Since $0 \le x_i \le 1$, we have that $0 \le y_i \le 2(1+\alpha)$. Therefore, the minimum of dH/dy_i occurs at $y_i = 2(1+\alpha)$. Hence, $dH/dy_i \ge dH/dy_i|_{y_i=2(1+\alpha)} = \ell n\, 2 - \ell n(1+\alpha) - (1-\alpha)/2 \ge 0$. Therefore, H is increasing in x_i. Moreover, x_i are zeros of Legendre polynomial, so that $x_{N^-} \to 1$ as $N^- \to \infty$. Consequently,

$$\begin{aligned}\sup_{1<N^-<\infty}\max_{1\le N\le N^-}\int_{-1}^{1} f_i(x)\,dx &= \sup_{1\le N^-<\infty} H((1+\alpha)(1+x_1))\\ &= H(2(1+\alpha)) = 2(1+\alpha)\ell n\left(\frac{2}{1+\alpha}\right).\end{aligned}$$

■

In the following, we are to obtain upper and lower bounds for CGJ$(c,\alpha,\pm N)$. Using (22) and Lemma 3(i), we see immediately that

$$\begin{aligned} e_{ij}^{(p+1)} &\le \frac{c}{4}\sum_{k=1}^{N^-}(\widetilde{g}_{kN^+})(\overline{f}_i)e_{kj}^{(p)} + \frac{c}{4}\sum_{k=1}^{N^+} e_{ik}^{(p)}(\widetilde{f}_{N^-k})(\overline{g}_i)\\ &\le \frac{c}{4}\sum_{k=1}^{N^-}(\widetilde{g}_{kN^+})(\overline{f}_{N^-})e_{kj}^{(p)} + \frac{c}{4}\sum_{k=j}^{N^+} e_{ik}^{(p)}(\widetilde{f}_{N^-k})(\overline{g}_{N^+}). \end{aligned} \tag{24}$$

Let rank one matrices M_2 and N_2 be defined as follows: $M_2 = (\overline{f}_{N^-}\widetilde{g}_{jN^+})_{N^-\times N^-}$ and $N_2 = (\widetilde{f}_{N^-i}\overline{g}_{N^+})_{N^+\times N^+}$, then (24) gives the following form:

$$E^{(p+1)} \le GE^{(p)},$$

where $G = \frac{c}{4}(M_2 \times I + I \times N_2)$. The notation $\times$ denotes the Konecker product (see, e.g., [1, p. 235]). Noting that M_2 and N_2 are matrices of rank one, we conclude that

$$\rho(M_2) = \left[\sum_{k=1}^{N^-} d_k^- g_{N^+}(x_k^-)\right]\left[1 + \frac{c}{4}\sum_{k=1}^{N^+} d_k^+ f_{N^-}(x_k^+)R_{N^-k}^{(\infty)}\right] \tag{25a}$$

and

$$\rho(N_2) = \left[\sum_{k=1}^{N^+} d_k^+ f_{N^-}(x_k^+)\right]\left[1 + \frac{c}{4}\sum_{k=1}^{N^-} d_k^- g_{N^+}(x_k^-)R_{kN^+}^{(\infty)}\right]. \tag{25b}$$

It follows from the fact $0 \leq G_J \leq G$ that

$$\mathrm{CGJ}(c, \alpha, N^{\pm}) \leq \frac{c}{4}(\rho(M_2)) + \rho(N_2)).$$

As $N^{\pm} \to \infty$, we see that $c/4(\rho(M_2)) + \rho(N_2)$ approaches

$$\begin{aligned}
&\frac{c}{4}[F(\alpha) + G(\alpha)] + \frac{(1-\alpha^2)c^2}{8} \\
&\quad \times \left[\int_{-1}^{1} \frac{R(1, y')}{3 + \alpha + y'(1-\alpha)} \, dy' + \int_{-1}^{1} \frac{R(x', 1,)}{3 - \alpha + x'(1+\alpha)} \, dx' \right] \\
&:= U(c, \alpha, \infty).
\end{aligned} \tag{26}$$

Here $F(\alpha)$ and $G(\alpha)$ are defined as in the Lemma 3, and $R(x, y)$ satisfies the following nonlinear integral equation:

$$\begin{aligned}
R(x, y) = &\left(1 + \frac{c(1-\alpha^2)}{4} \int_{-1}^{1} \frac{(1+y)R(x', y)}{2 + y(1-\alpha) + x'(1+\alpha)} \, dx' \right) \\
&\times \left(1 + \frac{c(1-\alpha^2)}{4} \int_{-1}^{1} \frac{(1+x)R(x, y')}{2 + x(1+\alpha) + y'(1-\alpha)} \, dy' \right).
\end{aligned} \tag{27}$$

Noting that $\max_{-1 \leq x, y \leq 1} R(x, y) = R(1, 1)$, we see that (26) is bounded above by

$$\frac{c}{4} \left[G(\alpha) + F(\alpha) + \frac{cR(1, 1)}{2} G(\alpha) F(\alpha) \right]. \tag{28}$$

Similarly, replacing M_2 and N_2 by $M_1 = (\widetilde{f}_1 \widetilde{g}_{j1})$ and $N_1 = (\widetilde{f}_{1i} \widetilde{g}_1)$, respectively, one would conclude that

$$\rho(M_1) + \rho(N_1) \leq \mathrm{CGJ}(c, \alpha, n^{\pm}).$$

Here

$$\rho(M_1) = \left[\frac{c}{4} \sum_{k=1}^{N^-} d_k^- g_1(x_k^-) \right] \left[1 + \frac{c}{4} \sum_{k=1}^{N^+} d_k^+ f_1(x_k^+) R_{1k}^{(\infty)} \right] \tag{29a}$$

$$\rho(N_1) = \left[\frac{c}{4} \sum_{k=1}^{N^+} d_k^+ f_1(x_k^+) \right] \left[1 + \frac{c}{4} \sum_{k=1}^{N^-} d_k^+ g_1(x_k^-) R_{k1}^{(\infty)} \right]. \tag{29b}$$

We summarize the above results in the following theorem:

THEOREM 2.

(i) $\rho(M_1) + \rho(N_1) \leq CGJ(c, \alpha, N^{\pm}) \leq \rho(M_2) + \rho(N_2)$, *where* $\rho(M_2)$, $\rho(N_2)$, $\rho(M_1)$ *and* $\rho(N_1)$ *are defined in* (25a), (25b) *and* (29a), (29b) *respectively.*
(ii) $CGJ(c, \alpha, \infty) \leq U(c, \alpha, \infty) \leq c/4[G(\alpha) + F(\alpha) + R(1,1)F(\alpha)G(\alpha)/2]$, *where* $U(c, \alpha, \infty)$ *is defined as in* (26).

REMARKS

(1) For $N^{\pm} = 1, c = 1$ and $\alpha = 0$, (17) reduces to

$$R^{(p+1)} = \left(1 + \frac{R^{(p)}}{4}\right)^2 \tag{30a}$$

$$R^{(0)} = 0. \tag{30b}$$

A simple calculation would give that $\{R^{(p)}\}_{p=0}^{\infty}$ converges extremely slowly to the exact solution $R = 4$. In this special case $\rho(M_1) = \rho(N_1) = \rho(M_2) = \rho(N_2) = 1/2$, and hence CGJ$(1,0,1) = 1$; this indeed, suggests that the entire scheme almost stalls.
(2) When $c \approx 1$ and $\alpha \approx 0$, the upper bound for CGJ$(c, \alpha, N^{\pm}), N^{\pm} = 1, 2, \ldots, \infty$, is not as good as when $c \approx 0$ and $\alpha \approx 1$.
(3) If the convergence rate analysis were to proceed without using the transformation (14), then we shall accordingly obtain $f_i(w_k^+) = 1/(w_k^+ - \alpha)$ and $g_i(w_k^-) = 1/(w_k^- + \alpha)$, both of which are then not even integrable over $[\alpha, 1]$ and $[-\alpha, 1]$, respectively.

We shall next consider the asymptotic rates of convergence of the GJ method as $N^+ = N^- \to \infty$. Consider the following iteration:

$$\begin{aligned} R^{(p+1)}(x,y) &= \left(1 + \frac{c(1-\alpha^2)}{4} \int_{-1}^{1} \frac{(1+y)R^{(p)}(x',y)}{2 + y(1-\alpha) + x'(1+\alpha)}\, dx'\right) \\ &\quad \times \left(1 + \frac{c(1-\alpha^2)}{4} \int_{-1}^{1} \frac{(1+x)R^{(p)}(x,y')}{2 + x(1+\alpha) + y'(1-\alpha)}\, dy'\right), \\ &:= U_J(x,y,p), \qquad R^{(0)}(x,y) = 1. \end{aligned}$$

Suppose (27) has a positive solution $R(x,y)$. Then the monotonically increasing sequence $\{R^{(0)}(x,y)\}_{p=1}^{\infty}$ has an upper bound $R(x,y)$. Therefore, $R^{(p)}(x,y)$ converges to a limit, say $R^{(\infty)}(x,y)$, which is a solution of (27).

Let $e^{(p)}(x,y) := R^{(\infty)}(x,y) - R^{(p)}(x,y)$.

Then a similar procedure as done in (22) will give

$$\begin{aligned} e^{(p+1)}(x,y) &= \frac{c(1-\alpha^2)}{4}\left(\int_{-1}^{1} \frac{(1+x)e^{(p)}(x,y')}{2+x(1+\alpha)+y'(1-\alpha)}\,dy'\right) \\ &\quad \times \left[1+\frac{c(1-\alpha^2)}{4}\int_{-1}^{1}\frac{(1+y)R^{(\infty)}(x',y)}{2+y(1-\alpha)+x'(1+\alpha)}\,dx'\right] \\ &\quad + \frac{c(1-\alpha^2)}{4}\left(\int_{-1}^{1} \frac{(1+y)e^{(p)}(x',y)}{2+y(1-\alpha)+x'(1+\alpha)}\,dx'\right) \\ &\quad \times \left[1+\frac{c(1-\alpha^2)}{4}\int_{-1}^{1}\frac{(1+x)R^{(\infty)}(x,y')}{2+x(1+\alpha)+y'(1-\alpha)}\,dy'\right] \\ &:= \left(\overline{G}_J e^{(p)}\right)(x,y). \end{aligned}$$

Following the standard technique (see, e.g., [11, Theorem 8.7-5]), one will be able to show that $\overline{G}: X \to X$, where $X = C([0,1]\times[0,1])$, is a linear compact integral operator. Moreover, the calculations similar to those in Lemma 3 will give

$$\|\overline{G}_J\| \le U(c,\alpha,\infty).$$

REMARK. We would expect that the spectral radius of G_J as N gets larger would be a good approximation of that of $\overline{G}_J$.

We shall conclude this section with the following example:

Let $N^- = N^+ = 2$, then

$$\begin{aligned} G_J &= \begin{pmatrix} D_{11}+B_1 & D_{12} \\ D_{21} & D_{22}+B_2 \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{g}_{11}\overline{f}_1 + \overline{g}_1\widetilde{f}_{11} & \overline{g}_1\widetilde{f}_{12} & \widetilde{g}_{21}\overline{f}_1 & 0 \\ \overline{g}_2\widetilde{f}_{11} & \widetilde{g}_{12}\overline{f}_1 + \overline{g}_2\widetilde{f}_{12} & 0 & \widetilde{g}_{22}\overline{f}_1 \\ \widetilde{g}_{11}\overline{f}_2 & 0 & \widetilde{g}_{21}\overline{f}_2 + \overline{g}_1\widetilde{f}_{21} & \overline{g}_1\widetilde{f}_{22} \\ 0 & \widetilde{g}_{12}\overline{f}_2 & \overline{g}_2\widetilde{f}_{21} & \widetilde{g}_{22}\overline{f}_2 + \overline{g}_2\widetilde{f}_{22} \end{pmatrix} \end{aligned}$$

By noting that

$$\begin{aligned} &\overline{f}_i = \overline{g}_i \qquad \text{for } i = 1,2, \\ &\widetilde{g}_{ii} = \widetilde{f}_{ii} = \tfrac{1}{2} \qquad \text{for } i = 1,2, \\ &\widetilde{g}_{ij} = \widetilde{f}_{ji} \qquad \text{for } i,j = 1,2 \qquad \text{and} \qquad i \neq j, \end{aligned}$$

and

$$\widetilde{g}_{ij} + \widetilde{g}_{ji} = \widetilde{f}_{ji} + \widetilde{f}_{ij} = \tfrac{1}{2} \qquad \text{for } i \neq j,$$

we conclude that each of the column sums of G_{J_2} is equal to $c/2(\overline{f}_1 + \overline{f}_2)$, and hence $\rho(G_{J_2}) = c/2(\overline{f}_1 + \overline{f}_2)$.

4. NUMERICAL EXAMPLES AND CONCLUDING REMARKS

We provide the following tables:

TABLE 1

$\alpha = 0.1$ AND $N^- = N^+ = 3$

c	GJ	GS	JOR/w	SOR/w	UB	LB	CGJ	CGS
$c = 0.1$	10	9	8/1.03	7/1.02	0.069	0.025	0.051	0.028
$c = 0.2$	13	11	10/1.07	9/1.03	0.143	0.051	0.105	0.059
$c = 0.3$	16	13	11/1.11	10/1.07	0.223	0.078	0.162	0.094
$c = 0.4$	19	15	13/1.15	11/1.09	0.312	0.106	0.224	0.138
$c = 0.5$	22	18	15/1.20	12/1.13	0.411	0.134	0.290	0.189
$c = 0.6$	27	21	18/1.24	14/1.17	0.525	0.164	0.364	0.251
$c = 0.7$	34	26	22/1.29	16/1.25	0.659	0.196	0.448	0.329
$c = 0.8$	44	33	27/1.40	20/1.32	0.826	0.229	0.546	0.428
$c = 0.9$	66	48	38/1.53	26/1.50	1.060	0.266	0.673	0.570
$c = 1.0$	263	186	140/1.84	88/1.99	1.584	0.312	0.913	0.877

TABLE 2

$c = 0.9$ AND $N = 3$

α	GJ	GS	JOR/w	SOR/w	UB	LB	CGJ	CGS
$\alpha = 0.0$	68	50	40/1.52	28/1.49	1.081	0.269	0.684	0.582
$\alpha = 0.1$	66	48	38/1.53	26/1.50	1.060	0.266	0.673	0.570
$\alpha = 0.2$	59	44	36/1.47	24/1.46	1.001	0.258	0.644	0.536
$\alpha = 0.3$	51	38	31/1.45	22/1.40	0.915	0.244	0.600	0.486
$\alpha = 0.4$	44	33	27/1.39	20/1.32	0.810	0.225	0.544	0.427
$\alpha = 0.5$	37	28	24/1.30	17/1.28	0.695	0.202	0.480	0.363
$\alpha = 0.6$	30	24	20/1.26	15/1.22	0.573	0.174	0.409	0.296
$\alpha = 0.7$	25	20	17/1.20	13/1.17	0.446	0.141	0.331	0.230
$\alpha = 0.8$	20	16	14/1.14	12/1.10	0.313	0.102	0.242	0.164
$\alpha = 0.9$	14	13	11/1.08	10/1.05	0.169	0.056	0.138	0.092

TABLE 3

$c = 0.9$ AND $\alpha = 0.1$

Dim.	GJ	GS	JOR/w	SOR/w	UB	LB	CGJ	CGS
$N = 2$	65	52	38/1.53	26/1.65	0.991	0.409	0.673	0.599
$N = 3$	66	48	39/1.53	26/1.50	1.060	0.266	0.673	0.570
$N = 4$	66	46	39/1.53	26/1.46	1.088	0.187	0.673	0.552
$N = 5$	66	44	39/1.53	26/1.42	1.102	0.140	0.673	0.539
$N = 6$	66	43	39/1.53	26/1.40	1.110	0.109	0.673	0.530

Here the column of GJ denotes the number iterates necessary to solve (13) within the prescribed error by the Gauss–Jacobi method. Likewise for GS, JOR, and SOR. The stopping criterion for all the iterative processes is $\max_{i,j} |R_{ij}^{(m+1)} - R_{ij}^{(m)}| < 10^{-11}$. The optimal w in the fourth and fifth columns are obtained by trial and error. The columns of UB and LB give the upper and lower bounds for the convergence rate of the GJ method, which are given in Theorem 2(i). The eighth and ninth columns of Table 1 gives the numerical convergence rates of the GJ and GS methods. It is estimated by using the quantity $\max_{i,j} = |R_{ij}^{(m+1)} - R_{ij}^{(m)}|/|R_{ij}^{(m)} - R_{ij}^{(m+1)}|$.

Based upon the results presented here, the following related matters would appear to warrant further investigation:

(1) It appears numerically that the smaller α is the slower the convergence is. However, we are unable to prove this at this point.

(2) From Table 3, it appears that the choice of the optimal w is very insensitive to the dimension of the matrix. The advantage of such observation, if confirmed, is apparent.

(3) Although our examples only deal with small dimension, (1) or (13) is indeed large, dense, and nonlinear. And the convergence rates of the GJ, GS, JOR, and SOR methods are not satisfactory for $c \approx 1$ and $\alpha \approx 0$. Therefore, how to accelerate the iteration, and/or how the multigrid methods can be brought in is of great interest.

(4) It would be also interesting to analyze the errors produced by the discretization of the continuous model, such as (27), as well as those created by iterative procedures together.

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