

Measures of similarity between vague sets

Shyi-Ming Chen

Department of Computer and Information Science, National Chiao Tung University, Hsinchu, Taiwan, ROC

Received May 1994; revised September 1994

Abstract

In this paper, we propose two similarity measures for measuring the degree of similarity between vague sets. The proposed measures can provide a useful way for measuring the degree of similarity between vague sets.

Keywords: Fuzzy set; Similarity measure; Vague set

1. Introduction

In 1965, Zadeh proposed the theory of fuzzy sets [10]. In recent years, many measures of similarity between fuzzy sets have been proposed [4, 6, 9, 12]. In [9], Pappis et al. made an assessment of measures of similarity of fuzzy values. In [12], Zwick et al. reviewed 19 similarity measures of fuzzy sets and compared their performance in an experiment. In [4], we proposed a similarity function F to measure the degree of similarity between fuzzy sets. In [6], Hyung et al. presented two similarity measures between fuzzy sets and between elements.

Roughly speaking, a fuzzy set is a class with fuzzy boundaries. A fuzzy set A of the universe of discourse U , $U = \{u_1, u_2, \dots, u_n\}$, is a set of ordered pairs $\{(u_1, \mu_A(u_1)), (u_2, \mu_A(u_2)), \dots, (u_n, \mu_A(u_n))\}$, where μ_A is the membership function of the fuzzy set A , $\mu_A: U \rightarrow [0, 1]$, and $\mu_A(u_i)$ indicates the grade of membership of u_i in A . When the universe of discourse U is a finite set, then the fuzzy set A can also be represented by

$$A = \mu_A(u_1)/u_1 + \mu_A(u_2)/u_2 + \dots + \mu_A(u_n)/u_n = \sum_{i=1}^n \mu_A(u_i)/u_i. \quad (1)$$

When the universe of discourse U is an interval of real numbers between a and b , then a fuzzy set A is often written in the form

$$A = \int_a^b \mu_A(u_i)/u_i, \quad (2)$$

where $u_i \in [a, b]$. For more details, please refer to [1–3, 7, 8, 10, 11]. It is obvious that $\forall u_i \in U$, the membership value $\mu_A(u_i)$ is a single value between zero and one. In [5], Gau et al. pointed out that this single value combines the evidence for $u_i \in U$ and the evidence against $u_i \in U$, without indicating how much there is

of each. They also pointed out that the single number tells us nothing about its accuracy. Therefore, in [5], Gau et al. presented the concepts of vague sets. They used a truth-membership function t_A and false-membership function f_A to characterize the lower bound on μ_A . These lower bounds are used to create a subinterval on $[0, 1]$, namely $[t_A(u_i), 1 - f_A(u_i)]$, to generalize the $\mu_A(u_i)$ of fuzzy sets, where $t_A(u_i) \leq \mu_A(u_i) \leq 1 - f_A(u_i)$. For example, let A be a vague set with truth-membership function t_A and false-membership function f_A , respectively. If $[t_A(u_i), 1 - f_A(u_i)] = [0.5, 0.8]$, then we can see that $t_A(u_i) = 0.5$, $1 - f_A(u_i) = 0.8$; $f_A(u_i) = 0.2$. It can be interpreted as “the degree that object u_i belongs to the vague set A is 0.5; the degree that object u_i does not belong to the vague set A is 0.2”. As another example, in a voting model, the vague value $[0.5, 0.8]$ can be interpreted as “the vote for resolution is 5 in favor, 2 against, and 3 abstentions”.

In this paper, we propose two similarity measures for measuring the degree of similarity between vague sets. The proposed measures differ from the previous ones in that the proposed measures deal with the similarity measures between vague sets rather than fuzzy sets. The proposed measures can provide a useful way for measuring the degree of similarity between vague sets.

The rest of this paper is organized as follows. In Section 2, we briefly review the theory of vague sets from [5]. In Section 3, we propose two similarity measures for measuring the degree of similarity between vague sets. The conclusions are discussed in Section 4.

2. Vague sets

Let U be the universe of discourse, $U = \{u_1, u_2, \dots, u_n\}$, with a generic element of U denoted by u_i . A vague set A in U is characterized by a truth-membership function t_A and a false-membership function f_A ,

$$t_A : U \rightarrow [0, 1], \tag{3}$$

$$f_A : U \rightarrow [0, 1], \tag{4}$$

where $t_A(u_i)$ is a lower bound on the grade of membership of u_i derived from the evidence for u_i , $f_A(u_i)$ is a lower bound on the negation of u_i derived from the evidence against u_i , and $t_A(u_i) + f_A(u_i) \leq 1$. The grade of membership of u_i in the vague set A is bounded to a subinterval $[t_A(u_i), 1 - f_A(u_i)]$ of $[0, 1]$. The vague value $[t_A(u_i), 1 - f_A(u_i)]$ indicates that the exact grade of membership $\mu_A(u_i)$ of u_i may be unknown, but is bounded by $t_A(u_i) \leq \mu_A(u_i) \leq 1 - f_A(u_i)$, where $t_A(u_i) + f_A(u_i) \leq 1$. For example, Fig. 1 shows a vague set in the universe of discourse U .

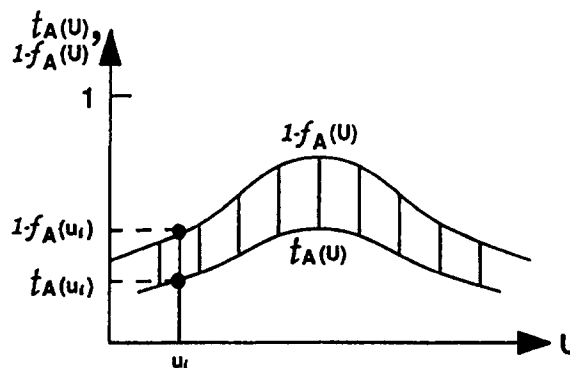


Fig. 1. A vague set.

When the universe of discourse U is continuous, a vague set A can be written as

$$A = \int_U [t_A(u_i), 1 - f_A(u_i)]/u_i. \quad (5)$$

When the universe of discourse U is discrete, a vague set A can be written as

$$A = \sum_{i=1}^n [t_A(u_i), 1 - f_A(u_i)]/u_i. \quad (6)$$

For example, let U be the universe of discourse, $U = \{6, 7, 8, 9, 10\}$. A vague set “LARGE” of U may be defined by

$$\text{LARGE} = [0.1, 0.2]/6 + [0.3, 0.5]/7 + [0.6, 0.8]/8 + [0.9, 1]/9 + [1, 1]/10.$$

3. Similarity measures

Let $x = [t_x, 1 - f_x]$ be a vague value, where $t_x \in [0, 1]$, $f_x \in [0, 1]$, and $t_x + f_x \leq 1$. Then, the score of x can be evaluated by the score function S shown as follows:

$$S(x) = t_x - f_x, \quad (7)$$

where $S(x) \in [-1, 1]$. Let $f_x^* = 1 - f_x$, then we can see that $x = [t_x, 1 - f_x] = [t_x, f_x^*]$. In this case, we can see that

$$S(x) = t_x - f_x = t_x - f_x^* - 1. \quad (8)$$

Let X and Y be two vague values, $X = [t_X, 1 - f_X] = [t_X, f_X^*]$ and $Y = [t_Y, 1 - f_Y] = [t_Y, f_Y^*]$. The degree of similarity between the vague values X and Y can be evaluated by the function M ,

$$M(X, Y) = 1 - \left| \frac{S(X) - S(Y)}{2} \right|, \quad (9)$$

where $S(X) = t_X - f_X = t_X + f_X^* - 1$ and $S(Y) = t_Y - f_Y = t_Y + f_Y^* - 1$. Let us consider the following cases:

Case 1: If the vague values $X = [1, 1]$ and $Y = [0, 0]$, then we can see that $S(X) = 1$ and $S(Y) = -1$. By applying Eq. (9), the degree of similarity between the vague values X and Y can be evaluated and is equal to

$$1 - \left| \frac{1 - (-1)}{2} \right| = 0. \quad (10)$$

Case 2: If the vague values $X = [1, 1]$ and $Y = [1, 0]$, then we can see that $S(X) = 1$ and $S(Y) = 0$. By applying Eq. (9), the degree of similarity between the vague values X and Y can be evaluated and is equal to

$$1 - \left| \frac{1 - 0}{2} \right| = \frac{1}{2}. \quad (11)$$

Case 3: If the vague values $X = [1, 0]$ and $Y = [1, 1]$, then we can see that $S(X) = 0$ and $S(Y) = 1$. By applying Eq. (9), the degree of similarity between the vague values X and Y can be evaluated and is equal to

$$1 - \left| \frac{0 - 1}{2} \right| = \frac{1}{2}. \quad (12)$$

Case 4: If the vague values $X = [0, 1]$ and $Y = [0, 1]$, then we can see that $S(X) = -1$ and $S(Y) = 1$. By applying Eq. (9), the degree of similarity between the vague values X and Y can be evaluated and is equal to

$$1 - \left| \frac{-1 - 1}{2} \right| = 0. \quad (13)$$

It is obvious that if X and Y are identical vague values (i.e., $X = Y$), then $S(X) = S(Y)$. By applying Eq. (9), we can see that $M(X, Y) = 1$, i.e., the degree of similarity between the vague values X and Y is equal to 1.

Let A and B be two vague sets in the universe of discourse U , $U = \{u_1, u_2, \dots, u_n\}$, where

$$\begin{aligned} A &= [t_A(u_1), 1 - f_A(u_1)]/u_1 + [t_A(u_2), 1 - f_A(u_2)]/u_2 + \dots + [t_A(u_n), 1 - f_A(u_n)]/u_n \\ &= [t_A(u_1), f_A^*(u_1)]/u_1 + [t_A(u_2), f_A^*(u_2)]/u_2 + \dots + [t_A(u_n), f_A^*(u_n)]/u_n \\ &= \sum_{i=1}^n [t_A(u_i), f_A^*(u_i)]/u_i, \end{aligned} \quad (14)$$

$$\begin{aligned} B &= [t_B(u_1), 1 - f_B(u_1)]/u_1 + [t_B(u_2), 1 - f_B(u_2)]/u_2 + \dots + [t_B(u_n), 1 - f_B(u_n)]/u_n \\ &= [t_B(u_1), f_B^*(u_1)]/u_1 + [t_B(u_2), f_B^*(u_2)]/u_2 + \dots + [t_B(u_n), f_B^*(u_n)]/u_n \\ &= \sum_{i=1}^n [t_B(u_i), f_B^*(u_i)]/u_i, \end{aligned} \quad (15)$$

where $f_A^*(u_i) = 1 - f_A(u_i)$, $f_B^*(u_i) = 1 - f_B(u_i)$, and $1 \leq i \leq n$. Let $V_A(u_i) = [t_A(u_i), f_A^*(u_i)]$ be the vague membership value of u_i in the vague set A , and let $V_B(u_i) = [t_B(u_i), f_B^*(u_i)]$ be the vague membership value of u_i in the vague set B . By applying Eq. (8), we can see that

$$S(V_A(u_i)) = t_A(u_i) + f_A^*(u_i) - 1, \quad (16)$$

$$S(V_B(u_i)) = t_B(u_i) + f_B^*(u_i) - 1, \quad (17)$$

where $1 \leq i \leq n$. The degree of similarity between the vague sets A and B can be evaluated by the function T ,

$$\begin{aligned} T(A, B) &= \frac{1}{n} \sum_{i=1}^n M(V_A(u_i), V_B(u_i)) \\ &= \frac{1}{n} \sum_{i=1}^n \left(1 - \left| \frac{S(V_A(u_i)) - S(V_B(u_i))}{2} \right| \right), \end{aligned} \quad (18)$$

where $T(A, B) \in [0, 1]$. The larger the value of $T(A, B)$, the more the similarity between the vague sets A and B .

Example 3.1. Let A and B be two vague sets of the universe of discourse U ,

$$U = \{u_1, u_2, u_3, u_4, u_5\},$$

$$A = [0.2, 0.4]/u_1 + [0.3, 0.5]/u_2 + [0.5, 0.7]/u_3 + [0.7, 0.9]/u_4 + [0.8, 1]/u_5$$

$$B = [0.3, 0.5]/u_1 + [0.4, 0.6]/u_2 + [0.6, 0.8]/u_3 + [0.7, 0.9]/u_4 + [0.9, 1]/u_5,$$

where

$$\begin{aligned} V_A(u_1) &= [0.2, 0.4], & V_B(u_1) &= [0.3, 0.5], \\ V_A(u_2) &= [0.3, 0.5], & V_B(u_2) &= [0.4, 0.6], \\ V_A(u_3) &= [0.5, 0.7], & V_B(u_3) &= [0.6, 0.8], \\ V_A(u_4) &= [0.7, 0.9], & V_B(u_4) &= [0.7, 0.9], \\ V_A(u_5) &= [0.8, 1], & V_B(u_5) &= [0.9, 1]. \end{aligned}$$

By applying Eq. (8), we can get

$$\begin{aligned} S(V_A(u_1)) &= 0.2 + 0.4 - 1 = -0.4, \\ S(V_A(u_2)) &= 0.3 + 0.5 - 1 = -0.2, \\ S(V_A(u_3)) &= 0.5 + 0.7 - 1 = 0.2, \\ S(V_A(u_4)) &= 0.7 + 0.9 - 1 = 0.6, \\ S(V_A(u_5)) &= 0.8 + 1.0 - 1 = 0.8, \\ S(V_B(u_1)) &= 0.3 + 0.5 - 1 = -0.2, \\ S(V_B(u_2)) &= 0.4 + 0.6 - 1 = 0, \\ S(V_B(u_3)) &= 0.6 + 0.8 - 1 = 0.4, \\ S(V_B(u_4)) &= 0.7 + 0.9 - 1 = 0.6, \\ S(V_B(u_5)) &= 0.9 + 1.0 - 1 = 0.9. \end{aligned}$$

By applying Eq. (18), the degree of similarity between the vague sets A and B can be evaluated by the function T ,

$$T(A, B) = \frac{1}{5} \sum_{i=1}^5 \left(1 - \left| \frac{S(V_A(u_i)) - S(V_B(u_i))}{2} \right| \right) = 0.93. \quad (19)$$

It indicates that the degree of similarity between the vague sets A and B is equal to 0.93.

In the following, we present a weighted similarity measure between vague sets. Let A and B be two vague sets in the universe of discourse U , where

$$U = \{u_1, u_2, \dots, u_n\},$$

$$A = \sum_{i=1}^n [t_A(u_i), 1 - f_A(u_i)]/u_i = \sum_{i=1}^n [t_A(u_i), f_A^*(u_i)]/u_i,$$

$$B = \sum_{i=1}^n [t_B(u_i), 1 - f_B(u_i)]/u_i = \sum_{i=1}^n [t_B(u_i), f_B^*(u_i)]/u_i,$$

where $f_A^*(u_i) = 1 - f_A(u_i)$, $f_B^*(u_i) = 1 - f_B(u_i)$, and $1 \leq i \leq n$.

Assume that the weight of the element u_i in the universe of discourse U is w_i , respectively, where $w_i \in [0, 1]$ and $1 \leq i \leq n$, then the degree of similarity between the vague sets A and B can be evaluated by the weighting function W ,

$$\begin{aligned} W(A, B) &= \frac{\sum_{i=1}^n w_i^* M(V_A(u_i), V_B(u_i))}{\sum_{i=1}^n w_i} \\ &= \frac{\sum_{i=1}^n w_i^* \left(1 - \left| \frac{S(V_A(u_i)) - S(V_B(u_i))}{2} \right| \right)}{\sum_{i=1}^n w_i}, \end{aligned} \quad (20)$$

where $W(A, B) \in [0, 1]$. The larger the value of $W(A, B)$, the more the similarity between the vague sets A and B .

Example 3.2. Let A and B be two vague sets of the universe of discourse U , where

$$U = \{u_1, u_2, u_3, u_4, u_5\},$$

$$A = [0.1, 0.3]/u_1 + [0.2, 0.6]/u_2 + [0.4, 0.8]/u_3 + [0.6, 0.8]/u_4 + [0.8, 1]/u_5$$

$$B = [0.2, 0.5]/u_1 + [0.3, 0.7]/u_2 + [0.5, 0.8]/u_3 + [0.7, 0.9]/u_4 + [0.9, 1]/u_5.$$

We can see that

$$V_A(u_1) = [0.1, 0.3], \quad V_B(u_1) = [0.2, 0.5],$$

$$V_A(u_2) = [0.2, 0.6], \quad V_B(u_2) = [0.3, 0.7],$$

$$V_A(u_3) = [0.4, 0.8], \quad V_B(u_3) = [0.5, 0.8],$$

$$V_A(u_4) = [0.6, 0.8], \quad V_B(u_4) = [0.7, 0.9],$$

$$V_A(u_5) = [0.8, 1], \quad V_B(u_5) = [0.9, 1],$$

By applying Eq. (8), we can get

$$S(V_A(u_1)) = 0.1 + 0.3 - 1 = -0.6,$$

$$S(V_A(u_2)) = 0.2 + 0.6 - 1 = -0.2,$$

$$S(V_A(u_3)) = 0.4 + 0.8 - 1 = 0.2,$$

$$S(V_A(u_4)) = 0.6 + 0.8 - 1 = 0.4,$$

$$S(V_A(u_5)) = 0.8 + 1 - 1 = 0.8,$$

$$S(V_B(u_1)) = 0.2 + 0.5 - 1 = -0.3,$$

$$S(V_B(u_2)) = 0.3 + 0.7 - 1 = 0,$$

$$S(V_B(u_3)) = 0.5 + 0.8 - 1 = 0.3,$$

$$S(V_B(u_4)) = 0.7 + 0.9 - 1 = 0.6,$$

$$S(V_B(u_5)) = 0.9 + 1 - 1 = 0.9.$$

Assume that the weights of $u_1, u_2, u_3, u_4,$ and u_5 are 0.5, 0.8, 1.0, 0.7, and 1.0, respectively, then by applying Eq. (20), the degree of similarity between the vague sets A and B can be evaluated by the weighting function W ,

$$W(A, B) = \frac{\sum_{i=1}^5 w_i^* \left(1 - \left| \frac{S(V_A(u_i)) - S(V_B(u_i))}{2} \right| \right)}{\sum_{i=1}^5 w_i} = 0.90625. \quad (21)$$

It indicates that the degree of similarity between the vague sets A and B is equal to 0.90625.

4. Conclusions

Although many similarity measures have been proposed in the literature for measuring the degree of similarity between fuzzy sets, those measures cannot deal with the similarity measures between vague sets. In this paper, we propose two similarity measures for measuring the degree of similarity between vague sets. The proposed measures can provide a useful way to deal with the similarity measures of vague sets.

Acknowledgments

This work was supported in part by the National Science Council, Republic of China, under Grant NSC 83-0408-E-009-041.

References

- [1] S.M. Chen, A new approach to handling fuzzy decisionmaking problems, *IEEE Trans. Systems Man Cybernet.* **18** (1988) 1012–1016.
- [2] S.M. Chen, An improved algorithm for inexact reasoning based on extended fuzzy production rules, *Cybernet. Systems* **23** (1992) 463–481.
- [3] S.M. Chen, A new approach to inexact reasoning for rule-based systems, *Cybernet. Systems* **23** (1992) 561–562.
- [4] S.M. Chen, A weighted fuzzy reasoning algorithm for medical diagnosis, *Decision Support Systems* **11** (1994) 37–43.
- [5] W.L. Gau and D.J. Buehrer, Vague sets, *IEEE Trans. Systems Man Cybernet.* **23** (1993) 610–614.
- [6] L.K. Hyung, Y.S. Song and K.M. Lee, Similarity measure between fuzzy sets and between elements, *Fuzzy Sets and Systems* **62** (1994) 291–293.
- [7] A. Kaufmann and M.M. Gupta, *Introduction to Fuzzy Arithmetic* (Van Nostrand Reinhold, New York, 1985).
- [8] A. Kaufmann and M.M. Gupta, *Fuzzy Mathematical Models in Engineering and Management Science* (North-Holland, Amsterdam, 1988).
- [9] C.P. Pappis and N.I. Karacapilidis, A comparative assessment of measures of similarity of fuzzy values, *Fuzzy Sets and Systems* **56** (1993) 171–174.
- [10] L.A. Zadeh, Fuzzy sets, *Inform. and Control* **8** (1965) 338–356.
- [11] H.J. Zimmermann, *Fuzzy Sets Theory and Its Applications* (Kluwer-Nijhoff, Dordrecht, 1991).
- [12] R. Zwick, E. Carlstein and D. Budescu, Measures of similarity among fuzzy sets: a comparative analysis, *Internat. J. Approx. Reason.* **1** (1987) 221–242.