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On positive solutions of some nonlinear differential equations – A probabilistic approach

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Abstract

By using connections between superdiffusions and partial differential equations (established recently by Dynkin, 1991), we study the structure of the set of all positive (bounded or unbounded) solutions for a class of nonlinear elliptic equations. We obtain a complete classification of all bounded solutions. Under more restrictive assumptions, we prove the uniqueness property of unbounded solutions, which was observed earlier by Cheng and Ni (1992).

Keywords: Branching particle systems; Measure-valued processes; Nonlinear elliptic equation; Range; Superdiffusions.

1. Introduction

Throughout this paper we consider positive solutions of the following nonlinear differential equation :

$$Lu(x) = k(x)u^{\alpha}(x), \quad x \in \mathbb{R}^d, \tag{1}$$

where $1 < \alpha \leqslant 2$, k is a bounded strictly positive continuous function on $\mathbb{R}^d$ satisfying condition (6) below, and L is a differential operator in $\mathbb{R}^d$ of the form

$$L = \sum_{i,j=1}^{d} a_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i} \tag{2}$$

such that it satisfies the following:

(1.a) The functions $a_{ij} = a_{ji}$ and b_i are bounded smooth functions in $\mathbb{R}^d$.

(1.b) There exists a constant $c > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x) u_i u_j \geqslant c \sum u_i^2$$

for all $x \in \mathbb{R}^n$ and all $u_1, u_2, ..., u_d$.

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If $k \equiv 0$, then Eq. (1) becomes a linear equation and it can be studied probabilistically by using paths of a diffusion $\xi = (\xi_t, \Pi_x)$ with the generator L. By using the superdiffusion $X = (X_t, X_\tau, P_\mu)$ with parameters (L, ψ), where $\psi(x,z) = k(x)z^\alpha$, we shall investigate the structure of the set of all positive solutions of Eq. (1).

If L corresponds to a recurrent diffusion, then there is no nontrivial bounded position solution to (1). (The following argument is provided by an anonymous referee. Assume u is such a solution and choose $x_0 \in \mathbb{R}^d$ and a ball B such that $u(x_0) > \sup_{y \in B} u(y)$. By Ito's formula, $\Pi_{x_0} u(\xi_{t \wedge \tau_B}) \geqslant u(x_0)$, where $\tau_B = \inf\{t \geqslant 0,\ \xi_t \in B\}$. By recurrence, $\Pi_{x_0}[\tau_B < \infty] = 1$. Thus letting $t \to \infty$ gives $u(x_0) > \Pi_{x_0} u(\xi_{\tau_B}) \geqslant u(x_0)$, a contradiction.) Therefore, we assume further that L corresponds to a transient diffusion.

The superdiffusion $X = (X_t, X_\tau, P_\mu)$ is a branching measure-valued Markov process describing the evolution of a random cloud. It can be obtained as a limit of branching particle systems by speeding up the branching rate, decreasing the mass of particles, and increasing the number of particles. For every $t > 0$, the random measure X_t is a limit of mass distribution of branching particle systems X^β at time t, as $\beta \to 0$. For every τ, the first exit time of ξ from a domain $D \subset \mathbb{R}^d$, the corresponding random measure X_τ can be obtained as a limit, as $\beta \to 0$, of the mass distribution of the particle systems X^β at the first exit time from D (see Section 2 for more detail). Let τ_n be the first exit time of ξ from the Euclidean ball of radius n, centered at 0 and let H stand for the set of all bounded positive functions h on $\mathbb{R}^d$ with $Lh(x) = 0$ for all $x \in \mathbb{R}^d$. Denote by $\langle f, \mu \rangle$ the integral of f with respect to μ. We write PY for the expected value of a random variable Y on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$. We obtain the following in Section 3.

Theorem I. (a) *If* $h \in H$, *then* $Z_h = \lim_{n \to \infty} \langle h, X_{\tau_n} \rangle$ *exists a.s. (which means* P_μ*-a.s. for all* μ*) and the function*

$$v_h(x) = -\log P_{\delta_x} \exp\{-Z_h\}$$

is the unique positive solution of (1) *with* $u = h$ *at* ∞. *(If* u *and* v *are two functions on* $\mathbb{R}^d$, *we write* $u = v$ *at* ∞ *if* $u(x) - v(x) \to 0$ *as* $\|x\| \to \infty$.)

(b) *If* u *is a bounded solution of* (1), *then* $u = v_h$ *for some* $h \in H$.

Therefore we characterize all bounded solutions of (1). (A similar result was established earlier by Cheng and Ni (1992) under more restrictive assumptions.)

Denote by E the set of all unbounded solutions $u(x)$ of (1) with $u(x) \to \infty$ as $\|x\| \to \infty$. The following theorem implies that E is not empty.

Theorem II. (a) *The function*

$$I(x) = -\log P_{\delta_x}[X_{\tau_n} = 0 \textit{ for } n \textit{ sufficiently large }], \quad x \in \mathbb{R}^d$$

is the largest element of E.

(b) *The function*

$$J(x) = -\log P_{\delta_x}[X_{\tau_n} \to 0 \textit{ as } n \to \infty\ ], \quad x \in \mathbb{R}^d$$

is the smallest element of E.

We will prove Theorem II in Section 4, and give equivalence conditions in Theorem 4.2, for the uniqueness of unbounded solutions u of (1) with $u(x) \to \infty$ as $\|x\| \to \infty$. As an application of Theorem 4.2, we give an alternative probabilistic proof of Cheng and Ni's result (cf. Cheng and Ni, 1992, Theorem II):

Theorem III. *If* $L = \Delta$, *where* $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$, *and* $k(x) \sim \|x\|^{-l}$, $l > 2$, *as* $\|x\| \to \infty$, *then there is only one unbounded solution* u *of* (1) *with* $u(x) \to \infty$, *as* $\|x\| \to \infty$.

(*Writing* $u \sim v$ *as* $\|x\| \to \infty$ *means there exist two constants* $c_1, c_2 > 0$ *such that* $c_1 v(x) \leqslant u(x) \leqslant c_2 v(x)$ *for* x *sufficiently large.*)

In Section 5, as an application of Theorem III, we evaluate the probability for the range $\mathscr{R}$ of X to be compact.

In this paper c always denotes a constant and it may have different values in different lines. The notation B_n stands for the open ball with radius n, centered at 0.

2. Preliminaries

2.1 Let L be a differential operator in $\mathbb{R}^d$ of the form (2) satisfying conditions (1.a) and (1.b). Then there exists a Markov process $\xi = (\xi_t, \Pi_x)$ in $\mathbb{R}^d$ with continuous paths such that for every bounded continuous function f on $\mathbb{R}^d$,

$$u_t(x) = \Pi_x f(\xi_t)$$

is the unique solution of the equation

$$\frac{\partial u}{\partial t} = Lu$$

with the property $u_t(x) \to f(x)$ as $t \downarrow 0$ (see, e.g. Stroock and Varadhan, 1979).

We call ξ the diffusion with the generator L.

2.2 Denote by $\mathscr{B}$ the Borel σ-algebra in $\mathbb{R}^d$ and by M the set of all finite measures on $\mathscr{B}$. Write $\mathscr{M}$ for the σ-algebra in M generated by the functions $f_B(\mu) = \mu(B), B \in \mathscr{B}$. For every positive bounded Borel function $k(x)$ in $\mathbb{R}^d$ and $1 < \alpha \leqslant 2$, there exists a Markov process $X = (X_t, P_\mu)$ in $(M, \mathscr{M})$ such that the following conditions are satisfied.

(2.2.a) If f is a bounded continuous function, then $\langle f, X_t \rangle$ is right continuous in t on $\mathbb{R}^+$.

(2.2.b) For every $\mu \in M$ and for every positive bounded Borel function f,

$$P_\mu \exp\{-\langle f, X_t \rangle\} = \exp\{-\langle v_t, \mu \rangle\},$$

where v is the unique solution of the integral equation

$$v_t(x) + \Pi_x \left[\int_0^t k(\xi_s) v_{t-s}^{\alpha}(\xi_s) \, \mathrm{d}s \right] = \Pi_x f(\xi_t).$$

Moreover, to every $D \in \mathscr{B}$, there corresponds a random measure X_τ on $(\mathbb{R}^d, \mathscr{B})$ associated with the first exit time $\tau = \inf\{t, \xi_t \notin D\}$ from D by the formula

$$P_\mu \exp\{-\langle f, X_\tau \rangle\} = \exp\{-\langle v, \mu \rangle,\} \tag{3}$$

where v satisfies the integral equation

$$v(x) + \Pi_x \left[\int_0^\tau k(\xi_s) v^\alpha(\xi_s)\, \mathrm{d}s \right] = \Pi_x f(\xi_\tau). \tag{4}$$

(See Dawson (1993) or Dynkin (1994).) Note that for every $\mu \in M$ with $\operatorname{supp}(\mu) \subset D$, X_τ concentrates on $\partial D, P_\mu$-a.s.

We call $X = (X_t, X_\tau, P_\mu)$ the superdiffusion with parameters (L, ψ), where $\psi(x,z) = k(x)z^\alpha$. We explain the heuristic meaning of X_t and X_τ in terms of branching particle systems. Consider a system of particles which undergo random motion and branching on $\mathbb{R}^d$ according to the following rules.

a. Particles are distributed at time 0 according to the Poisson point process with intensity $\mu \in M$.

b. Each particle survives with probability $\exp\{-\int_0^t k(\xi_s)\,\mathrm{d}s\}$ at time t.

c. At the end of its lifetime, a dying particle gives birth to n offsprings at its own site, with probability p_n, where if $1 < \alpha < 2$,

$$p_n = \begin{cases} 0 & \text{if } n = 1\ , \\ \dfrac{1}{\alpha}(-1)^n \dbinom{\alpha}{n} & \text{if } n \neq 1, \end{cases}$$

and if $\alpha = 2$, $p_2 = p_0 = \frac{1}{2}$.

d. During its lifetime, the motion of each particle is governed by the process ξ.

e. All particle lifetimes, motions, and branching are independent of one another.

The historical path H_t^a of a particle consists of its own trajectory and the trajectories of all its ancestors. If each particle has mass β, then

$$X_t^\beta(B) = \beta \sum_a 1_B(H_t^a)$$

is the mass distribution at time t. (The sum is taken over all particles which are alive at time t.) Set

$$X_\tau^\beta(B) = \beta \sum_a 1_B(H_{\tau_a}^a),$$

where $\tau_a = \inf\{t, H_t^a \notin D\}$. (Here we identify particles a and b if $\tau_a = \tau_b$ and $H_s^a = H_s^b$ for all $s \leqslant \tau_a$.) If $k_\beta = \frac{k}{\beta^{\alpha-1}}$ and $\mu_\beta = \frac{\mu}{\beta}$, then X_t^β and X_τ^β converge weakly to X_t and X_τ as $\beta \to 0$. Let τ be the first exit time of ξ from a domain $D \subset \mathbb{R}^d$. A boundary point y of D is called regular if $\Pi_y[\tau = 0] = 1$. We quote two theorems from Dynkin (1991).

Theorem 2.1. *Let D be a bounded domain in $\mathbb{R}^d$. For every positive bounded Borel function f on ∂D, the function*

$$v(x) = -\log P_{\delta_x} \exp\{-\langle f, X_\tau \rangle\}$$

satisfies the equation

$$Lv(x) = k(x)v^\alpha(x) \tag{5}$$

for $x \in D$. *Moreover if* D *is regular and if* f *is continuous, then* $v = f$ *on* ∂D. (*We write* $v = f$ *on* $K \subset \partial D$ *if for every* $y \in K$, $\lim_{x\in D, x\to y} v(x) = f(y)$.)

Theorem 2.2. *Let* D *be an arbitrary domain in* $\mathbb{R}^d$. *Choose a sequence of bounded regular domains* $\{D_n\}_n$ *with* $D_n \uparrow D$ *and let* τ_n *be the first exit time of* ξ *from* D_n. *If* u *is a solution of* (5) *in* D, *then* $Z_u = \lim_{n\to\infty}\langle u, X_{\tau_n}\rangle$ *exists a.s. and* $u(x) = -\log P_{\delta_x}\exp\{-Z_u\}$ *for all* $x \in D$.

3. Bounded solutions of $Lu = ku^\alpha$

Throughout this paper we consider a strictly positive bounded continuous function $k(x)$ on $\mathbb{R}^d$ satisfying the condition :

$$\lim_{r\to\infty}\sup_{x\in\mathbb{R}^d}\int_{\|y\|>r} g(x,y)k(y)\,\mathrm{d}y = 0, \tag{6}$$

where $g(x,y)$ is the Green function of the operator L on $\mathbb{R}^d$.

Example 1. If $k(x) \leqslant h(\|x\|)$ for x sufficiently large, and $\int_a^\infty sh(s)\mathrm{d}s < \infty$ for some $a > 0$, then k satisfies condition (6) for $L = \Delta$ and $d = 3$.

Proof. It follows from Eq. (13.74) in Dynkin (1965) that $g(x,y) \leqslant c\|x-y\|^{2-d}$ for all $x, y \in \mathbb{R}^d$. Our conclusion follows from Zhao (1993, Propositions 1 and 2). □

As before, τ_n stands for the first exit time of the diffusion ξ from the domain B_n and $X = (X_t, X_\tau, P_\mu)$ is the superdiffusion with parameters (L, ψ), where $\psi(x,z) = k(x)z^\alpha$. Put $Z_{h,n} = \langle h, X_{\tau_n}\rangle$ if $h \in H$.

Lemma 3.1. *For every* $\mu \in M$ *and* $h \in H$, P_μ-*a.s.*, $Z_h = \lim_{n\to\infty} Z_{h,n}$ *exists and* $Z_h < \infty$.

Proof. Let $\mathscr{F}_n = \sigma\{X_{\tau_k}, 1 \leqslant k \leqslant n\}$. If $m < n$, then, by the strong Markov property and Eqs. (3) and (4),

$$P_\mu[\exp\{-Z_{h,n}\}|\mathscr{F}_m] = P_{X_{\tau_m}}\exp\{-Z_{h,n}\} = \exp\{-\langle w_n, X_{\tau_m}\rangle\}, \tag{7}$$

where w_n satisfies the equation

$$w_n(x) + \Pi_x\left[\int_0^{\tau_n} w_n^\alpha(\xi_s)k(\xi_s)\,\mathrm{d}s\right] = h(x), \qquad x \in B_n.$$

Clearly $w_n(x) \leqslant h(x)$ and we have, by (7),

$$P_\mu[\exp\{-Z_{h,n}\}|\mathscr{F}_m] \geqslant \exp\{-Z_{h,m}\}.$$

Therefore, $(\exp\{-Z_{h,n}\},\mathscr{F}_n,P_\mu)$ is a bounded submartingale and $Z_h = \lim_{n\to\infty} Z_{h,n}$ exists P_μ-a.s.

It follows from (3) and (4) that $P_\mu Z_{h,n} = \Pi_\mu h(\xi_{\tau_n}) = \langle h,\mu\rangle$. By Fatou's lemma,

$$P_\mu Z \leqslant \liminf_{n\to\infty} P_\mu Z_{h,n} = \langle h,\mu\rangle < \infty,$$

which implies that $Z_h < \infty, P_\mu$-a.s. □

We will write Z for Z_h if $h(x) = 1$ for all $x \in \mathbb{R}^d$.

Theorem 3.2. *For every $h \in H$, the function*

$$v_h(x) = -\log P_{\delta_x} \exp\{-Z_h\}, \quad x \in \mathbb{R}^d,$$

is the unique solution of (1) with $v_h = h$ at ∞. Moreover v_h satisfies the integral equation

$$v(x) + \int_{\mathbb{R}^d} g(x,y)k(y)v^\alpha(y)\,\mathrm{d}y = h(x) \tag{8}$$

in $\mathbb{R}^d$.

Proof. Set $v_{h,n}(x) = -\log P_{\delta_x}\exp\{-Z_{h,n}\}, \quad x \in B_n$. It follows from Eqs. (3) and (4) that $v_{h,n}$ satisfies the equation

$$v_{h,n}(x) + \int_{B_n} g_n(x,y)k(y)v_{h,n}^\alpha(y)\,\mathrm{d}y = h(x), \qquad x \in B_n, \tag{9}$$

where $g_n(x,y)$ is the Green's function of L on B_n. Set $g_n(x,y) = 0$ if $x \notin B_n$ or $y \notin B_n$, and put $v_{h,n}(x) = h(x)$ if $x \notin B_n$. Then equation (9) becomes

$$v_{h,n}(x) + \int_{\mathbb{R}^d} g_n(x,y)k(y)v_{h,n}^\alpha(y)\,\mathrm{d}y = h(x), \quad \forall x \in \mathbb{R}^d. \tag{10}$$

Note that $g_n(x,y) \uparrow g(x,y)$ as $n \to \infty$, and, by Lemma 3.1, $v_h(x) = \lim_{n\to\infty} v_{h,n}(x)$ for all $x \in \mathbb{R}^d$. Therefore, for every $x \in \mathbb{R}^d$,

$$g_n(x,y)v_{h,n}^\alpha(y) \to g(x,y)v_h^\alpha(y) \quad \text{as } n \to \infty.$$

Since $v_{h,n}(y) \leqslant \|h\|$, we have $v_h(y) \leqslant \|h\|$ and so

$$g_n(x,y)v_{h,n}^\alpha(y)k(y) \leqslant c g(x,y)k(y).$$

For every $x \in \mathbb{R}^d$, condition (6) implies that $g(x,y)k(y)$ is integrable. Letting $n \to \infty$ in (10), the dominated convergence theorem implies that v_h satisfies Eq. (8).

For $n > m$, both the functions $v_{h,n}$ and $-\log P_{\delta_x}\exp\{-\langle v_{h,n}, X_{\tau_m}\rangle\}$ are solutions of (5) in B_m, and they have the same boundary values on ∂B_m. The maximum principle (see, e.g. Dynkin (1991, Theorem 0.5)) implies that

$$v_{h,n}(x) = -\log P_{\delta_x}\exp\{-\langle v_{h,n}, X_{\tau_m}\rangle\}, \quad x \in B_m.$$

Letting $n \to \infty$, we get $v_h(x) = -\log P_{\delta_x} \exp\{-\langle v_h, X_{\tau_m} \rangle\}$ in B_m. Theorem 2.1 implies that v_h is a solution of (5). To check $v_h = h$ at ∞, it suffices, by (8), to show that $\int_{\mathbb{R}^d} g(x,y)k(y)\,\mathrm{d}y \to 0$ as $\|x\| \to \infty$. Write

$$\int_{\mathbb{R}^d} g(x,y)k(y)\,\mathrm{d}y = A(x,r) + B(x,r), \tag{11}$$

where

$$A(x,r) = \int_{\|y\| \leqslant r} g(x,y)k(y)\,\mathrm{d}y$$

and

$$B(x,r) = \int_{\|y\| \geqslant r} g(x,y)k(y)\,\mathrm{d}y.$$

Since k is bounded and $g(x,y) \to 0$ as $\|x-y\| \to \infty$, for every $r > 0$, $A(x,r)$ goes to 0 as $\|x\| \to \infty$. Clearly condition (6) implies that $\sup_{x \in \mathbb{R}^d} B(x,r) \to 0$ as $r \to \infty$. Letting $\|x\| \to \infty$ and then $r \to \infty$ in (11), we observe that $\int g(x,y)k(y)\,\mathrm{d}y \to 0$ as $\|x\| \to \infty$.

Let u be a solution of (1) with $u = h$ at ∞. Since $Z < \infty$ and $u = h$ at ∞, $\langle u, X_{\tau_n} \rangle \to Z_h$. By Theorem 2.2, $u(x) = -\log P_{\delta_x} \exp\{-Z_h\} = v_h(x)$. □

Remark. (a) If $L = \Delta$ and k is radial, then $v_{c,n}$ is radial for every constant function $h = c$ and so is v_c.

(b) Under the same assumptions on $k(x)$ as in Example 1, Kawano (1984) and Cheng and Ni (1992) obtained similar results for $L = \Delta$. By using Brownian path integration and potential theory, Zhao (1993) studied related problems for $L = \Delta$.

Theorem 3.3. *If u is a bounded solution of* (1), *then $v = h$ at ∞ for some $h \in H$ and $v(x) = v_h(x)$.*

Proof. Note that if u satisfies Eq. (8) for some $h \in H$, our conclusions follow from the same arguments as in the proof of Theorem 3.2. Since $u(x) = -\log P_{\delta_x} \exp\{-\langle u, X_{\tau_n} \rangle\}$, (3) and (4) imply that u satisfies the equation

$$u(x) + \int_{B_n} g_n(x,y)k(y)u^\alpha(y)\,\mathrm{d}y = h_n(x), \quad x \in B_n, \tag{12}$$

where $h_n(x) = \Pi_x[u(\xi_{\tau_n})]$. Clearly $Lh_n = 0$ and $h_n(x) = \Pi_x u(\xi_{\tau_n}) \leqslant \|u\|$ for all n. By passing n to the limit in (12), $h(x) = \lim_{n\to\infty} h_n(x)$ exists for all $x \in \mathbb{R}^d$. Therefore, h is bounded and $Lh = 0$ in $\mathbb{R}^d$. By passing to the limit in (12) again, u satisfies Eq. (8). □

Remark. The special case of Theorem 3.2, where $L = \Delta$ and k satisfies the conditions in Example 1, was observed earlier by Cheng and Ni (1992).

4. Unbounded solutions of $Lu = ku^\alpha$

Lemma 4.1. (a) *Let* $B = \{x;\ \|x - x^0\| < R\}$ *and*

$$u(x) = \lambda(R^2 - r^2)^{-\frac{2}{\alpha-1}}$$

where λ *is a positive constant and* $r = \|x - x_0\|$. *We have*

$$\lim_{x\to a, x\in B} u(x) = \infty$$

for all $a \in \partial B$, *and*

$$Lu - ku^\alpha \leqslant 0 \text{ in } B$$

for some λ *depending only on* α, *the dimension* d *and the upper bounds for* $\tilde{a}_{ij} = \frac{a_{ij}}{k}$ *and* $\tilde{b}_i = \frac{b_i}{k}$ *in* B.

(b) *If* $\bar{B} \subset D$ *for some open set* D, *and* v *is a solution of* (5) *in* D, *then* $v \leqslant u$ *in* B.

Proof. (a) is quoted from Dynkin (1991, Lemma 3.1) and (b) follows easily from the maximum principle. □

Proof of Theorem II. (a) For $x \in B_n$ and $m > 0$, set $I_{n,m}(x) = -\log P_{\delta_x} \exp\{-mZ_{1,n}\}$. Theorem 2.1 implies that $I_{n,m}$ satisfies (5) in B_n and $I_{n,m} = m$ on ∂B_n. By the maximum principle, $I_{n,m}$ is increasing in m. Therefore, for every $x \in B_n$,

$$I_n(x) = \lim_{m\to\infty} I_{n,m}(x)$$

exists. Clearly $I_n(x) = -\log P_{\delta_x}[Z_{1,n} = 0]$ and $I_n = \infty$ on ∂B_n. Let B be an arbitrary bounded open ball and let τ be the first exit time of ξ from B. If $\bar{B} \subset B_n$ for some n, Theorem 2.1 and the maximum principle imply that

$$I_{n,m}(x) = -\log P_{\delta_x} \exp\{-\langle I_{n,m}, X_\tau\rangle\},\ x \in B. \tag{13}$$

Note that, by Lemma 4.1, $|I_{n,m}| \leqslant c$ in $\bar{B}$, for all m. Letting m go to ∞ in (13), we have

$$I_n(x) = -\log P_{\delta_x} \exp\{-\langle I_n, X_\tau\rangle\}. \tag{14}$$

Clearly $I(x) = \lim_{n\to\infty} I_n(x)$ and, letting $n \to \infty$ in (14), we observe $I(x) = -\log P_{\delta_x} \exp\{-\langle I, X_\tau\rangle\}$. By Theorem 2.1, I satisfies Eq. (1).

Assume u is a solution of (1). Since $I_n = \infty$ on ∂B_n, the maximum principle implies that $u \leqslant I_n$ in B_n and so $u \leqslant I$.

(b) Write v_c for v_h in Theorem 3.2 if h is a constant function c. Clearly $v_c(x) \uparrow J(x)$ as $c \uparrow \infty$ and $J(x) \to \infty$ as $\|x\| \to \infty$. Since $v_c(x) = -\log P_{\delta_x} \exp\{-\langle v_c, X_{\tau_n}\rangle\}$ in B_n, we have $J(x) = -\log P_{\delta_x} \exp\{-\langle J, X_{\tau_n}\rangle\}$ in B_n. By Lemma 4.1, J is bounded on ∂B_n, and so Theorem 2.1 implies that J is a solution of (1).

Assume $u \in E$. For every $c > 0$, $\langle u, X_{\tau_n}\rangle \geqslant cZ_{1,n}$ for sufficiently large n. For any $c > 0$,

$$u(x) = \lim_{n\to\infty} - \log P_{\delta_x} \exp\{-\langle u, X_{\tau_n}\rangle\}$$

$$\geqslant \lim_{n\to\infty} - \log P_{\delta_x} \exp\{-cZ_{1,n}\} = v_c(x).$$

Letting $c \uparrow \infty, u(x) \geqslant J(x)$ for all $x \in \mathbb{R}^d$. Therefore J is the minimal element in E. □

Remark. (a) Assume $L = \Delta$ and k is radial. Then both $I_{n,m}$ and I_n are radial. Therefore I is radial. Clearly J is radial.

(b) Let k_1, k_2 be two bounded strictly positive continuous functions on $\mathbb{R}^d$ which both satisfy condition (6). Assume further that $k_1(x) \leqslant k_2(x)$ for all x. For $s = 1, 2$, let $I_{s,n,m}, I_s$, and J_s denote $I_{n,m}, I$, and J respectively, with k replaced by k_s. For $x \in B_n$, we have

$$LI_{2,n,m} - k_2 I^{\alpha}_{2,n,m} = 0 = LI_{1,n,m} - k_1 I^{\alpha}_{1,n,m} \geqslant LI_{1,n,m} - k_2 I^{\alpha}_{1,n,m}.$$

The Maximum principle implies that $I_{2,n,m} \leqslant I_{1,n,m}$ on B_n. Therefore $I_2 \leqslant I_1$. Similar arguments imply that $J_2 \leqslant J_1$.

Denote by $|E|$ the cardinality of E. By Theorem II, $|E| \geqslant 1$.

Theorem 4.2. *The following three statements are equivalent.*

(a) $|E| = 1$.

(b) *For every measure $\mu \in M$ with compact support, we have*

$$P_\mu[Z_{1,n} \to 0] = P_\mu[Z_{1,n} = 0 \textit{ for sufficiently large } n\ ].$$

(c) *There exists a constant c such that*

$$I(x) \leqslant cJ(x) \quad \textit{for } x \textit{ sufficiently large,}$$

where I and J are functions in Theorem II.

Proof. Note that for every $\mu \in M$ with compact support, we have

$$P_\mu[Z = 0] = \exp\{-\langle J, \mu\rangle\}$$

and

$$P_\mu[Z_{1,n} = 0 \text{ for sufficiently large } n\ ] = \exp\{-\langle I, \mu\rangle\}.$$

Therefore (a) and (b) are equivalent. Clearly (b) implies (c). Assume that (c) holds. To prove (a), it suffices to show that $I = J$. Fix $x \in \mathbb{R}^d$. By Theorem 2.2, both $Z_I = \lim_{n\to\infty}\langle I, X_{\tau_n}\rangle$ and $Z_J = \lim_{n\to\infty}\langle I, X_{\tau_n}\rangle$ exist P_x-a.s. Since $I(x) \to \infty$ and $J(x) \to \infty$, $Z_I = J_J = \infty$ on $\{Z > 0\}$. Combining with Theorem 2.2, we have

$$- \log P_{\delta_x}[Z = 0] = J(x) = - \log P_{\delta_x} \exp\{-Z_J\}$$

$$= - \log P_{\delta_x}[\exp\{-Z_J\}, Z = 0]$$

Therefore $Z_J = 0$ on $\{Z = 0\}P_x$-a.s. By assumption, we have $\langle I, X_{\tau_n}\rangle \leqslant c\langle J, X_{\tau_n}\rangle$ for n sufficiently large, which implies that $Z_I = 0$ on $\{Z = 0\}, P_x$-a.s. Therefore

$$\begin{aligned} I(x) &= -\log P_{\delta_x} \exp\{-Z_I\} \\ &= -\log P_{\delta_x}[\exp\{-Z_I\}, Z = 0] \\ &= -\log P_{\delta_x}[Z = 0] = J(x). \qquad \square \end{aligned}$$

If $L = \Delta$ and if k is radial with $k(x) = c|x|^{-l}, l > 2$, for large x, then the first part of Theorem 4.3 of Cheng and Ni (1992), implies that for every radial solution u of (1), we have $u(x) \sim |x|^{\frac{l-2}{\alpha-1}}$ as $\|x\| \to \infty$. By using this observation and Theorem 4.2, we prove Theorem III.

Proof of Theorem III. By assumption, there exist two constants c_1, c_2 and two radial functions k_1 and k_2 with $k_1(x) = |x|^{-l} = k_2(x)$ for x sufficiently large and

$$c_1 k_1(x) \leqslant k(x) \leqslant c_2 k_2(x) \quad \text{for all } x \in \mathbb{R}^d.$$

For $s = 1, 2$, let I_s, J_s denote I and J respectively with k replaced by $c_s k_s$. By Remark (b) following the proof of Theorem II, we have, for all x,

$$I_2(x) \leqslant I(x) \leqslant I_1(x)$$

and

$$J_2(x) \leqslant J(x) \leqslant J_1(x).$$

Since $I_s(x) \sim |x|^{\frac{l-2}{\alpha-1}}$ and $J_s(x) \sim |x|^{\frac{l-2}{\alpha-1}}, s = 1, 2$, as $\|x\| \to \infty$, we get

$$\frac{I(x)}{J(x)} \leqslant \frac{I_1(x)}{J_2(x)} \leqslant c$$

for x sufficiently large. Our result follows from Theorem 4.2. $\square$

Remark. In general we do not know if condition (6) is a sufficient condition for $|E| = 1$.

5. Application

The range $\mathcal{R}$ of the superdiffusion X is the smallest closed subset of $\mathbb{R}^d$ such that $\mathcal{R}$ contains supports of X_t for all $t \geqslant 0$. For constant k, Iscoe (1986) proved that $\mathcal{R}$ is compact a.s. for $L = \Delta$ and Dynkin (1991) observed this for general L.

Theorem 5.1. *If $L = \Delta$ and $k(x) \sim |x|^{-l}$, $l > 2$, as $\|x\| \to \infty$, then for every $\mu \in M$ with compact support,*

$$P_\mu[\ \mathcal{R} \text{ is compact}] = \exp\{-\langle I, \mu\rangle\},$$

where I is the unique unbounded solution of (1).

Proof. If $\mu \in M$ has compact support, we have, P_μ-a.s.,

$$\{\mathscr{R} \text{ is compact}\} = \bigcup_{n=1}^{\infty} \{Z_{1,n} = 0\}.$$

(See Dynkin, 1991). Our statement follows from Theorem II and Theorem III. □

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