Transactions Briefs

L_p Norm Back Propagation Algorithm for Adaptive Equalization

Sammy Siu, Ching-Haur Chang, and Che-Ho Wei

Abstract— **The l_p norm back propagation algorithm for perceptron based adaptive equalization is analyzed taking account of possible numerical problem encountered when $p < 1$. Two methods are proposed to overcome the numerical problem. Computer simulations indicate that simultaneous improvement in convergence rate and bit-error-rate (BER) performance can be achieved by using $p < 2$.**

I. Introduction

In recent years, neural network has emerged as a powerful tool for nonlinear adaptive filtering [1]–[3]. The algorithm based on stochastic gradient with l_2 norm error criterion is often used in training the neural network. It is known that the l_2 norm gives equal weights for all errors during the weight updating process and may result in a slow convergence. The use of l_p norm with $p < 2$ gives small weights to large errors and hence reduces the influence of aberrant noise [4]–[9], while it gives large weights to small errors. This property can be used to improve the tracking capability of the network. Taking account of possible numerical problem for $p < 1$, the l_p norm back propagation algorithm with application to adaptive equalization is analyzed in this paper. Two methods are proposed to overcome the numerical problem.

II. Analysis of the L_p Norm Back Propagation Algorithm

Consider a back propagation model of an m-layer perceptron network with $m \in \{1, 2, \ldots, M\}$, $t_i(n)$ being the ith desired signal, and $v_i^{(M)}(n)$ being the ith estimated signal at the output layer [10], [11]. Then the l_p norm error function [9] is given as $\bar{E}(n) = p^{-1} \sum_j |t_j(n) - v_j^{(M)}(n)|^p$ where the factor p^{-1} is padded for mathematical convenience. Hereafter, $|e_i(n)|$ is used to stand for $|t_i(n) - v_i^{(M)}(n)|$ for simplicity in our later expressions. The ith error signal for the output layer is then given as

$$\delta_i^{(M)}(n) = \operatorname{sgn}(e_i(n))|e_i(n)|^{p-1} v_i'^{(M)}(n) \tag{1}$$

where $v_i'^{(M)}(n)$ denotes the derivative of $v_i^{(M)}(n)$. If the sigmoid function used is $f(x) = (1 - e^{-x})/(1 + e^{-x})$, then $v_i'^{(M)}(n) = (1 - v_i^{2(M)}(n))/2$. Eq. (1) indicates that changing the power metric p rescales $|e_i(n)|$ in $\delta_i^{(M)}(n)$. Fig. 1 depicts $|e_i(n)|^{p-1}$ versus $|e_i(n)|$ for different p. It indicates that $|e_i(n)|^{p-1}$ gives small weights for large $|e_i(n)|$ and large weights for small $|e_i(n)|$ for $p < 2$, and vice versa for $p > 2$. However, for $p = 2$, $|e_i(n)|^{p-1} = |e_i(n)|$. The algorithm then corresponds to the standard back propagation algorithm. For $p \geq 1$, $|e_i(n)|^{p-1}$ is bounded for all $|e_i(n)|$. However, for $p < 1$, $|e_i(n)|^{p-1}$ becomes very large as $|e_i(n)|$ is approaching zero. This may result in numerical problem during the weight updating process. The following two methods can be used to solve the problem. In Method I, $|e_i(n)|$ is replaced by θ whenever $|e_i(n)| < \theta$, where θ is a small positive value. In Method II, p is switched from $p_1 < 1$ to $1 \leq p_2 \leq 2$ whenever $|e_i(n)| < \theta$. Defining $\bar{\delta}_i^{(M)}(n) = \delta_i^{(M)}(n)|_{p=2}$, then $\delta_i^{(M)}(n) = |e_i(n)|^{p-2} \bar{\delta}_i^{(M)}(n)$. The increments of the weights for the output and hidden layers are then written as

$$\Delta w_{ij}^{(M)}(n+1) = \frac{\bar{\eta}}{|e_j(n)|^{2-p}} \bar{\delta}_j^{(M)}(n) v_i^{(M-1)}(n), \tag{2}$$

and

$$\Delta w_{ij}^{(m-1)}(n+1) = \frac{(1 - v_j^{2(m-1)}(n))}{2} \sum_l \frac{\bar{\eta}}{|e_l(n)|^{2-p}} \times \bar{\delta}_l^{(m)}(n) w_{lj}^{(m)}(n) v_i^{(m-2)}(n), \tag{3}$$

where $\bar{\eta}$ is the learning gain at $p = 2$. Similar results can be applied to update the threshold levels. It is seen from (2) or (3) that the effective learning gain can be expressed as $\eta_i(n) = \bar{\eta}/|e_i(n)|^{2-p}$. Assuming that $v_i^{(M)}(n)$ is uniformly distributed in [-1, 1] [12] and $t_i(n)$ belongs to the signal set {-1, 1}, the effect of p on the learning gain for both training mode and decision-directed mode is given below.

Training Mode: Correct decisions make $|e_i(n)|$ to be uniformly distributed in [0, 1]. In this case, $E[|e_i(n)|^{2-p}] = (3 - p)^{-1}$ for $p \leq 2$ and $E[|e_i(n)|^{p-2}] = (p - 1)$ for $p \geq 2$. Thus the expected learning gain is given as

$$E[\eta_i(n)] = \eta_{cd} = \begin{cases} (3-p)\bar{\eta}, & p \leq 2 \\ (p-1)^{-1}\bar{\eta}, & p \geq 2 \end{cases} \tag{4}$$

where the subscript cd denotes "correct decision." It is seen that the expected learning gain increases by a factor of $(3-p)$ for $p \leq 2$, while decreases by $(p-1)^{-1}$ for $p \geq 2$. Incorrect decisions make $|e_i(n)|$ to be uniformly distributed in [1, 2]. Therefore, $E[|e_i(n)|^{2-p}] = (3-p)^{-1}(2^{3-p} - 1)$ for $p \leq 2$, $E[|e_i(n)|^{p-2}] = (p-1)^{-1}(2^{p-1} - 1)$

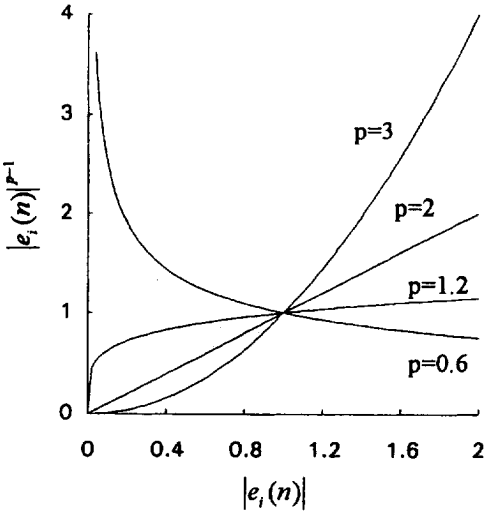

Fig. 1. $|e_i(n)|^{p-1}$ versus $|e_i(n)|$ for different p.

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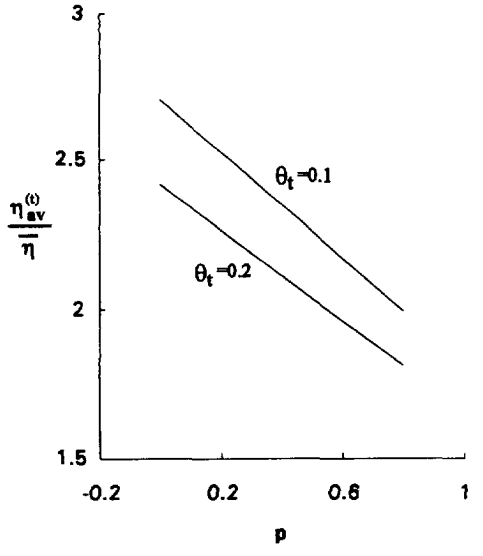
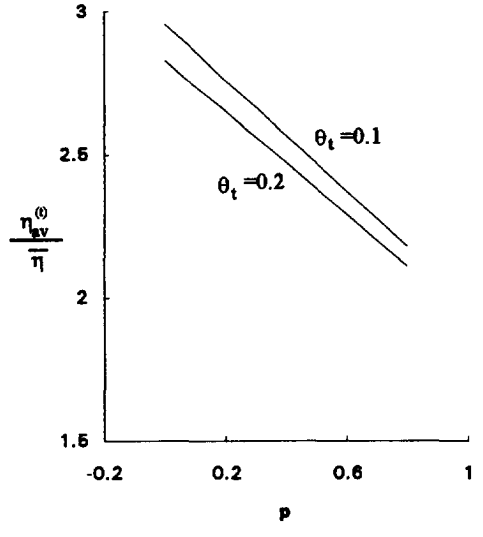
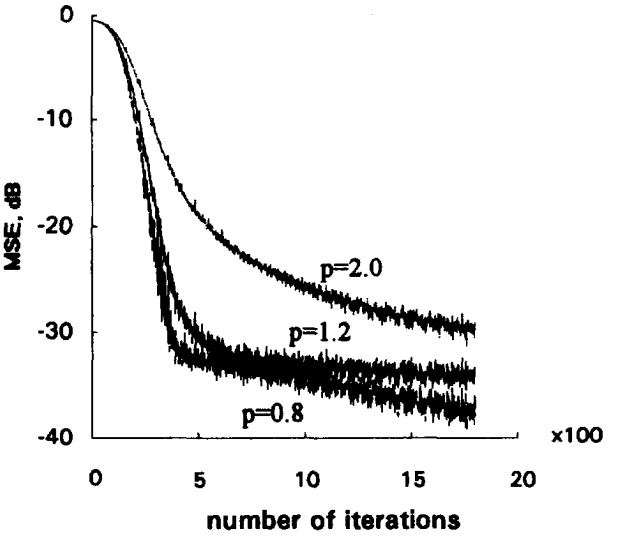


(a)

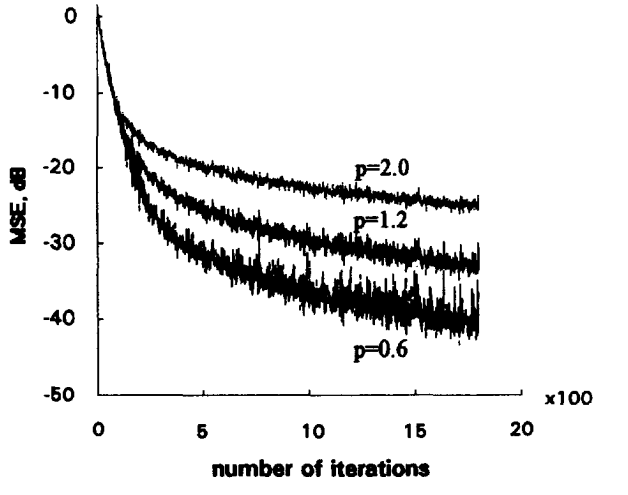


(b)

Fig. 2. Learning gain enhancement $(\eta_{\text{av}}^{(t)}/\bar{\eta})$ as a function of p for (a) Method I and (b) Method II under $P(e) \approx 0$. For Method II, the value of p is switched to $p = 1.0$ whenever $|e_i(n)| \leq \theta_t$.



(a)



(b)

Fig. 3. Learning curves for the two DFE's using different p and SNR= 20 dB. (a) MLP: $\bar{\eta} = 0.1$, $\beta = 0.05$, and $\theta_t = 0.2$ for $p = 0.8$ (b) PPS: $\bar{\eta} = 0.1$, $\beta = 0.05$, and $\theta_t = 0.2$ for $p = 0.6$.

for $p \geq 2$, and

$$E[\eta_i(n)] = \eta_{id} = \begin{cases} (3-p)(2^{3-p}-1)^{-1}\bar{\eta}, & p \leq 2 \\ (p-1)^{-1}(2^{p-1}-1)\bar{\eta}, & p \geq 2 \end{cases} \quad (5)$$

where the subscript id denotes "incorrect decision." Let $P(e)$ be the error probability. Then the average value of $E[|e_i(n)|^{2-p}]$ or $E[|e_i(n)|^{p-2}]$ can be found by assigning a weight of $P(e)$ to that when incorrect decisions are made and a weight of $(1 - P(e))$ to that when correct decisions are made. Thus $E[|e_i(n)|^{2-p}] = (3-p)^{-1}[1 + P(e)(2^{3-p} - 2)]$ for $p \leq 2$ and $E[|e_i(n)|^{p-2}] = (p-1)[1+P(e)(2^{p-1}-2)]^{-1}$ for $p \geq 2$. The average learning gain is then obtained as

$$\eta_{\text{av}}^{(t)} = \begin{cases} (3-p)[(1+P(e)2^{3-p}-2)]^{-1}\bar{\eta}, & p \leq 2 \\ (p-1)^{-1}[1+P(e)(2^{p-1}-2)]\bar{\eta}, & p \geq 2 \end{cases} \quad (6)$$

where the superscript t denotes "training mode." For small $P(e)$, $\eta_{\text{av}}^{(t)}$ approximates $(3-p)\bar{\eta}$ for $p \leq 2$ and approximates $(p-1)^{-1}\bar{\eta}$ for $p \geq 2$.

Decision-Directed Mode: $|e_i(n)|$ is uniformly distributed in [0, 1] and hence the average learning gain is obtained as

$$\eta_{\text{av}}^{(d)} = \begin{cases} (3-p)\bar{\eta}, & p \leq 2 \\ (p-1)^{-1}\bar{\eta}, & p \geq 2 \end{cases} \quad (7)$$

where the superscript d denotes "decision-directed mode."

The effect of θ on the learning gain for the two methods used to overcome the numerical problem is analyzed below.

For Method I, $|e_i(n)|$ is uniformly distributed in $[\theta, 1]$ when correct decisions are made in training mode and when the equalizer is operated in decision-directed mode. The average learning gains for training mode and decision-directed mode are, respectively, found as

$$\eta_{\text{av}}^{(t)} = (3-p)(1-\theta_t)\{(1-\theta_t^{3-p}) - P(e) \times [(1-\theta_t^{3-p}) - (1-\theta_t)(2^{3-p}-1)]\}^{-1}\bar{\eta}, \quad (8)$$

[1]Hereafter p_1/p_2 is used to denote that p is switched from p_1 to p_2 whenever $|e_i(n)| \leq \theta$.

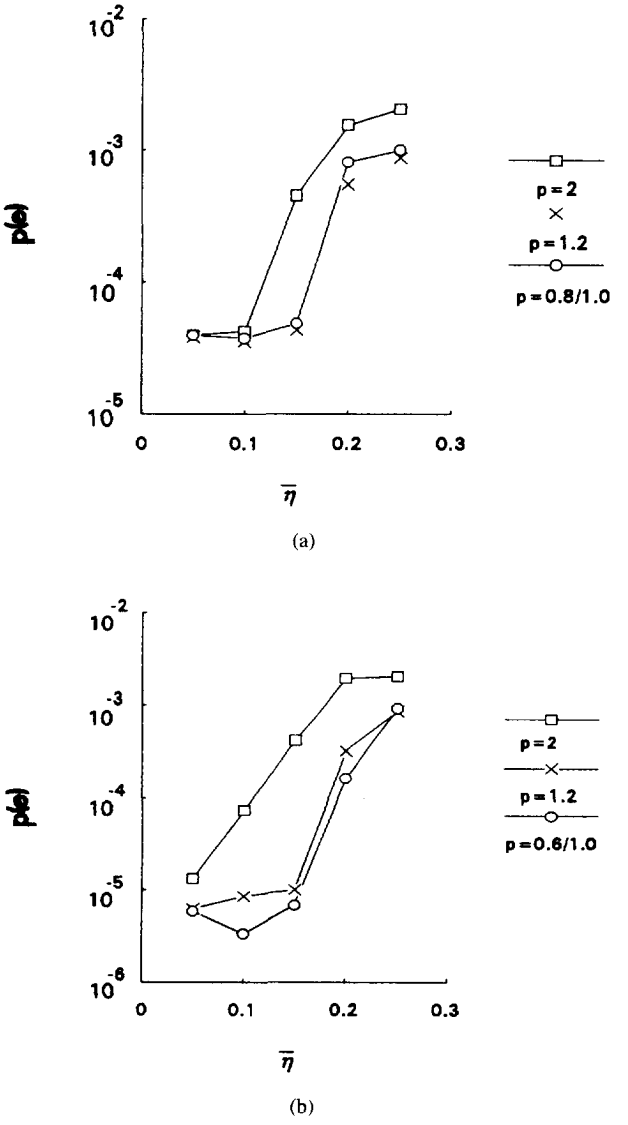


Fig. 4. BER performances for the two DFE's using different $\bar{\eta}$. (a)MLP: $\theta_t = 0.3$, $\theta_d = 1.0$ for $p = 0.8/1.0^1$, SNR = 16 dB (b) PPS: $\theta_t = 0.2$, $\theta_d = 1.0$ for $p = 0.6/1.0$, SNR = 18 dB.

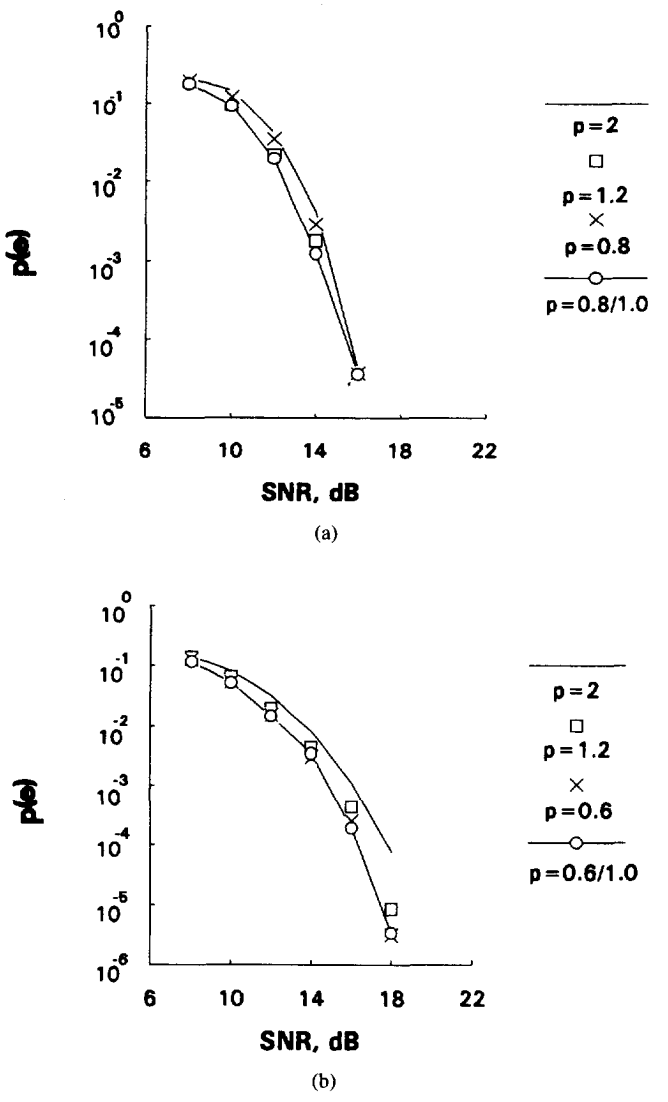


Fig. 5. BER performances for the two DFE's as a function of SNR. (a) MLP: $\bar{\eta} = 0.1$, $\beta = 0.05$, $p = 0.8/1.0$, and $\theta_t = 0.3$, $\theta_d = 1.0$ (b) PPS: $\bar{\eta} = 0.1$, $\beta = 0.05$, $p = 0.6/1.0$, and $\theta_t = 0.2$, $\theta_d = 1.0$.

and

$$\eta_{\text{av}}^{(d)} = (3-p)(1-\theta_d)(1-\theta_d^{3-p})^{-1}\bar{\eta} \tag{9}$$

where the subscripts t and d denote "training mode" and "decision-directed mode." For small θ_t or θ_d, both $\eta_{\text{av}}^{(t)}$ (when $P(e) \approx 0$) and $\eta_{\text{av}}^{(d)}$ approximate $(3-p)\bar{\eta}$.

For Method II, $1 \le p_2 \le 2$ is used when $|e_i(n)|$ is distributed in $[0, \theta]$, while $p_1 < 1$ is used when $|e_i(n)|$ distributed in $[\theta, 1]$ or [1,2]. Therefore, the average learning gains in training mode and decision-directed mode are respectively obtained as

$$\eta_{\text{av}}^{(t)} = \left\{[1-P(e)]\left(\frac{\theta_t^{3-p_2}}{3-p_2} + \frac{1-\theta_t^{3-p_1}}{3-p_1}\right) + P(e)\frac{2^{3-p_1}-1}{3-p_1}\right\}^{-1}\bar{\eta}, \tag{10}$$

and

$$\eta_{\text{av}}^{(d)} = \left\{(3-p_2)^{-1}\theta_d^{3-p_2} + (3-p_1)^{-1}(1-\theta_d^{3-p_1})\right\}^{-1}\bar{\eta}. \tag{11}$$

For $\theta_t^{3-p_1} \ll 1$ or $\theta_d^{3-p_1} \ll 1$, both $\eta_{\text{av}}^{(t)}$ (when $P(e) \approx 0$) and $\eta_{\text{av}}^{(d)}$ tend to $(3-p_1)\bar{\eta}$. For $\theta_d = 1$, $\eta_{\text{av}}^{(d)}$ becomes $(3-p_2)\bar{\eta}$. Defining $\eta_{\text{av}}^{(t)}/\bar{\eta}$ as the learning gain enhancement, Fig. 2 depicts $\eta_{\text{av}}^{(t)}/\bar{\eta}$ as a function of p for the two methods. Method II actually reflects small errors to $\delta_i^{(M)}(n)$, instead of limiting small errors by θ. It is therefore seen that $\eta_{\text{av}}^{(t)}/\bar{\eta}$ is larger for Method II than that for Method I. Also, both $\eta_{\text{av}}^{(t)}$ and $\eta_{\text{av}}^{(d)}$ increase with decreasing θ_t or θ_d for both methods. However, the value of θ can only be determined empirically.

III. Computer Simulations

Two perceptron based decision feedback equalizers (DFE) are employed in the simulations. One is based on a multilayer perceptron (MLP) with 9 neurons in hidden layer 1, 3 neurons in hidden layer 2, and 1 neuron in the output layer [2], and the other based on a third-order polynomial-perceptron structure (PPS) [13], [14]. The channel chosen is of the form $H(z) = 0.3482 + 0.8704z^{-1} + 0.3482z^{-2}$. The input to the channel is random in {-1, 1}. The channel noise is white Gaussian with zero-mean. The number of taps in the feedforward section of the DFE is 4 and in the feedback section is 1. Fig. 3 shows the learning curves for the two structures using different p and SNR = 20 dB. It indicates that both structures converge faster for smaller p. Fig. 4 shows the BER performances for the two structures using different $\bar{\eta}$. It is seen that substantial performance improvement

is obtained with increasing $\bar{\eta}$. This is because the noise caused by larger $\bar{\eta}$ can be suppressed by using $p < 2$. Fig. 5 shows the BER performances for the two structures as a function of SNR. It can be seen that better BER performance can be achieved by using smaller p and this is especially true for the PPS structure.

IV. Conclusion

The l_p norm back propagation algorithm for adaptive equalization, taking account of possible numerical problem encountered when $p < 1$, is analyzed. Two methods are proposed to overcome the numerical problem. Simulation results indicate that simultaneous improvement in convergence rate and BER performance can be obtained by using $p < 2$.

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Closed-Form Impulse Responses of Discrete-Domain Multidimensional Filters

Dave Jin and L. T. Bruton

***Abstract*— It is known that useful two-dimensional (2-D) and three-dimensional (3-D) discrete-domain recursive transfer functions may be designed by applying the MD bilinear transformation to the continuous-domain transfer functions of prototype MD inductance-resistance networks. Closed-form expressions are derived for the impulse responses of these 2- and 3-D discrete-domain filters.**

I. Introduction

Prototype three-dimensional (3-D) inductance-resistance continuous-domain networks, having Laplace transform transfer functions of the form

$$T_1(s_1, s_2, s_3) = \frac{R}{R + s_1 L_1 + s_2 L_2 + s_3 L_3}, \tag{1}$$

have been shown [1] to be useful for the design of 3-D discrete-domain recursive filter transfer functions by applying the triple bilinear transformation to (1). In particular, such filters can be used to selectively enhance 3-D linear trajectory (LT) and 3-D plane wave (PW) space-time signals.

The demonstrated usefulness of such filters in image processing has motivated this work, in which closed-form expressions are derived for the impulse responses, $h(\mathbf{n})$ (where the boldface $\mathbf{n}$ represents the integer m-tuple $n_1, n_2, \ldots, n_m$), of both the 2-D and 3-D LT filters.

Closed-form expressions for $h(\mathbf{n})$ are not generally available for MD filters, primarily because of the lack of a Fundamental Theorem of Algebra for multivariate polynomials[1]. Therefore, MD transfer functions cannot be expanded by the method of partial fraction expansion as in the 1-D case. However, closed-form expressions are available [2] for *purely first-order* m-D z-transform transfer functions. This result has led to a deeper understanding of the 2-D stability of transfer functions obtained via the 2-D bilinear transformation [3].

The availability of algebraic expressions for $h(\mathbf{n})$ facilitates further research on the input-output properties of LT and PW 3-D recursive filters, including issues relating to MD convolution and MD stability. The transfer functions considered here are more general than in [2]. In this brief, we present the derivation for the impulse response of the 2-D LT filter using the method of residues. The extension to the 3-D case is straightforward but very lengthy, and is given in [4].

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[1] For single-variable polynomials, the Fundamental Theorem of Algebra allows any Nth degree single-variable polynomial to be factored into N first degree factors.