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Asymptotic Behavior of Positive Solutions to Semilinear Elliptic Equations on Expanding Annuli

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We study the asymptotic behavior of positive solutions of the semilinear elliptic equation $\Delta u + f(u) = 0$ in Ω_a, $u = 0$ on $\partial\Omega_a$, where $\Omega_a = \{x \in R^N : a < |x| < a+1\}$ are expanding annuli as $a \to \infty$, and f is positive and superlinear at both 0 and ∞. We first show that there are a priori bounds for some positive solutions $u_a(x)$ as $a \to \infty$. Then, if we fix any direction, after a suitable translation of u_a the limiting solutions are non-negative solutions on the infinite strip. We can obtain more detailed descriptions of these limits if u_a is radially symmetric, least-energy, or least-energy with a particular symmetry.

1. INTRODUCTION

In this paper we are interested in the asymptotic behavior of positive solutions of the semilinear elliptic equation

$$\Delta u + f(u) = 0 \qquad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{1.2}$$

where $\Omega = \Omega_a = \{x \in R^N : a < |x| < a+1\}$ are expanding annuli in R^N as $a \to +\infty$, $N \geqslant 2$, and f satisfies the following conditions:

(H-0) $f \in C^1(R^1)$ and $f(u) > 0$ for large u,

(H-1) $f(0) = 0$ and $f'(0) \leqslant 0$,

(H-2) there exists $\sigma > 0$ such that $uf'(u) \geqslant (1+\sigma) f(u)$ for all $u \geqslant 0$,

(H-3) for large u,

$$f(u) \leqslant \begin{cases} Cu^p & \text{for some } p < (N+2)/(N-2) \text{ and } C > 0 \textit{ if } N \geqslant 3, \\ \exp A(u) & \text{with } A(u) = o(u^2) \textit{ as } u \to \infty \textit{ if } N = 2. \end{cases}$$

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The existence of positive radially symmetric solutions on annular domains has been studied by many authors [1–3, 9, 16, 18, 22, 23]. The existence of positive non-radially symmetric solution on a given fixed annulus was first observed by Brezis and Nirenberg [4] for $f(u)=u^p$, p less than and close to $(N+2)/(N-2)$. Regarding the two dimensional case, Coffman [5] considered $f(u)=-u+u^p$, where $p=2m+1$ and m is a positive integer. He showed that the number of rotationally non-equativalent non-radial positive solutions is unbounded as $a\to\infty$. Coffman's method was to minimize the associated Rayleigh quotients on the class of all radial functions and the class of functions which are invariant under the rotating $2\pi/k$ angles with $k\geqslant 2$. By choosing appropriate test functions, he was able to show that the minima become different as soon as a is large enough. In [25, 26], Suzuki and Nagasaki gave a simpler proof of Coffman's results. Later, using the same idea, Li [12] extended these results to $N\geqslant 4$ and $p\in(1,(N+2)/(N-2))$. He also treated problems in which the nonlinearity is non-homogeneous.

In [19, 20], the present author took a different approach in studying the existence of many non-radial solutions. He studied (1.1) and (1.2) on $A_b=\{x\in R^N: b<|x|<1\}$, $N\geqslant 2$, and took $b\in(0,1)$ as a parameter. He also studied the linearized eigenvalue problems of (1.1) and (1.2) at radial solution $u_b(|x|)$. When the domain is thin enough, i.e., b is close to 1, then $u_b(|x|)$ is unstable with respect to certain non-radial modes w and the associated energy decreases along the direction w. Therefore, some non-radial solutions with the same symmetry as w can be obtained. In general, as the domains become thinner, more non-radial solutions with less symmetry can be generated. A similar result also holds for expanding domains as $a\to+\infty$.

Note that problems on expanding annuli Ω_a are related to those on shrinking annuli A_b by suitable transformations. Indeed, they are equivalent when $f(u)=u^p$ for the following reason: $\tilde{u}(x)=(a+1)^{(p-1)/2}\,u((a+1)\,x)$ is a solution on A_b with $b=a/(a+1)$ if and only if u is a solution on Ω_a. The problems on thin domains have been studied extensively by Hale and Raugel in [12–14].

In [17], the author also studied the asymptotic behavior of positive solutions $u_b(x)$ of (1.1) and (1.2) on A_b as $b\to 0^+$. After obtaining a priori bounds for u_b, he was able to prove that all positive solutions are necessarily radially symmetric when b is small enough, and the limits of u_b are positive solutions of (1.1) and (1.2) on a unit ball as $b\to 0^+$. Therefore, it is of interest to investigate the other asymptotic problem: what are the limits of solutions $u_b(x)$ as $b\to 1^-$?

Since there are many non-radial solutions with different symmetries as $b\to 1^-$, it is reasonable to expect that the limiting behavior of solutions is relatively complicated.

In this paper, we study the limits of solutions as $b \to 1^-$ by considering (1.1) and (1.2) on expending annuli, which are easier to work with. We first show that there are a priori bounds for some positive solutions $u_a(x)$ as $a \to \infty$. Then, if we fix any direction $\xi \in S^{N-1}$, the unit sphere, after a suitable translation of u_a, the limiting solutions are non-negative solutions of (1.1) and (1.2) on the infinite strip $S_\xi = \{t\xi + \eta : t \in (0, 1), \eta \in R^N$ and $\xi \cdot \eta = 0\}$. Furthermore, we can obtain more detailed descriptions of these limits if u_a is (i) radially symmetric, (ii) least-energy, or (iii) least-energy with a particular symmetry.

We also remark that Ni and Takagi [24] studied the asymptotic behavior of least-energy solutions to a semilinear Neumann problem on a fixed domain when the diffusion coefficient tends to 0.

The paper is organized as follows. In Section 2, we briefly discuss the existence of positive non-radially symmetric solutions with a particular partial symmetry by using a Nehari-type variational method and spectral analysis. In Section 3, we derive some a priori bounds for several classes of positive solutions, which including radially symmetric solutions and least-energy solutions. In Section 4, we study the limits of positive radially symmetric solutions as $a \to \infty$. In Section 5, we study the asymptotic behavior of least-energy solutions u_a^* as $a \to \infty$. We prove that after a suitable rotation and translation the energy of the u_a^*'s will concentrate on one infinite strip and their translations will converge to the positive, least-energy solution on that infinite strip. In Section 6, we study the asymptotic behavior of least-energy solutions with a particular partial symmetry. The limiting solutions are least-energy solutions on infinite strips of lower dimensions. Finally, in the Appendix, we review some results concerning the Bessel functions and Green's functions of $-\Delta$ on Ω_a as $a \to \infty$.

2. Symmetry-Breakings

In this section we shall briefly discuss the existence of positive non-radial solutions of (1.1) and (1.2) on expanding annuli. These problems are parallel to the problems on shrinking annuli A_b which have been studied in [19, 20].

We first introduce a Nehari-type [22] variational method and then use spectral analysis to study (1.1) and (1.2) at positive radial solutions. We consider the functionals

$$E(u) = \int_{\Omega_a} \{\tfrac{1}{2} |\nabla u|^2 - F(u^+)\},$$

and

$$J(u)=\int_{\Omega_a}\{|\nabla u|^2-u^+f(u^+)\}$$

on $H_0^1(\Omega_a)$, where $F(u)=\int_0^u f(t)\,dt$ and $u^+=\max\{u,0\}$, and manifold

$$M=M_a=\{u\in H_0^1(\Omega_a)\colon J(u)=0 \text{ and } u\neq 0\}.$$

For $N\geqslant 2$, let u_a be a positive radial solution of (1.1) and (1.2) on Ω_a. The linearized eigenvalue problem of (1.1) and (1.2) at u_a is

$$\Delta w+f'(u_a)\,w=-\mu w \qquad \text{in } \Omega_a, \tag{2.1}$$

$$w=0 \qquad \text{on } \partial\Omega_a. \tag{2.2}$$

In spherical coordinates, (2.1) and (2.2) are equivalent to

$$\varphi''(r)+\frac{N-1}{r}\varphi'(r)+\left\{f'(u_a(r))-\frac{\alpha_k}{r^2}\right\}\varphi(r)$$

$$=-\mu_{k,l}(u_a)\,\varphi(r), \qquad a<r<a+1, \tag{2.3}$$

$$\varphi(a)=0=\varphi(a+1) \tag{2.4}$$

where $\alpha_k=k(k+N-2)$, $k=0,1,2,...$, and $l=1,2,....$ Note that α_k are eigenvalues of Laplacian $-\Delta$ on S^{N-1}, the unit sphere. For $k\geqslant 1$, the space $S_{N,k}$ of associated eigenfunctions of $-\Delta$ on S^{N-1} is given by $S_{N,k}=\{\psi_k\colon S^{N-1}\to R^1\,|\,\psi_k(x)=P_k(x)$ for $|x|=1$, where $P_k(x)$ is a harmonic homogeneous polynomial of degree k on $R^N\}$. The associated eigenfunctions $w_{k,l}$ of (2.1) and (2.2) are given by $w_{k,l}=\varphi_{k,l}\psi_k$. For $l=1$, we shall denote $\varphi_{k,l}$ by φ_k.

We now have the following result concerning the non-radial instability of positive radial solutions.

LEMMA 2.1. *Assume conditions* (H-0) ~ (H-2) *are satisfied. Then, for each* $k\geqslant 1$, *there exists an* $a_k=a_k(N,\sigma)\in(0,\infty)$ *such that for any* $a\in(a_k,\infty)$ *and any positive radial solution* u_a, *we have* $u_{k,1}(u_a)<0$.

Proof. $\mu_{k,1}=\mu_{k,1}(u_a)$ can be characterized as

$$\mu_{k,1}=\inf\left\{Q_k(v)\Big/\int_a^{a+1} r^{N-1}v^2\,dr\colon v\in H_0^1((a,a+1)) \text{ and } v\neq 0\right\}, \tag{2.5}$$

where

$$Q_k(v)=\int_a^{a+1} r^{N-1}\left\{v'^2-f'(u_a)\,v^2+\frac{\alpha_k}{r^2}v^2\right\}dr.$$

From (2.5), it is clear that $\mu_{k,1}$ is strictly increasing in k.

Since u_a is a solution of (1.1) and (1.2), we have

$$\int_{\Omega_a} |\nabla u_a|^2 = \int_{\Omega_a} u_a f(u_a).$$

By (H-2) and (2.6), we have

$$\begin{aligned}\omega_N Q_k(u_a) &= \int_{\Omega_a} \{u_a f(u_a) - f'(u_a)\, u_a^2\} + \alpha_k \int_{\Omega_a} u_a^2 r^{-2} \\ &\leqslant -\sigma \int_{\Omega_a} |\nabla u_a|^2 + \alpha_k a^{-2} \int_{\Omega_a} u_a^2, \\ &\leqslant (-\sigma\lambda_1(a) + \alpha_k a^{-2}) \int_{\Omega_a} u_a^2, \end{aligned} \tag{2.7}$$

where w_N is the area of S^{N-1} and $\lambda_1(a)$ is the least eigenvalue of $-\Delta$ on Ω_a with the Dirichlet boundary condition. Note that in deriving (2.7), the Poincaré inequality

$$\lambda_1(a) \int_{\Omega_a} v^2 \leqslant \int_{\Omega_a} |\nabla v|^2 \tag{2.8}$$

for all $v \in H_0^1(\Omega_a)$ has been used. Now

$$\lim_{a \to \infty} \lambda_1(a) = \pi^2. \tag{2.9}$$

(For details see Lemma A.1 in the Appendix.) Hence, there is an increasing sequence $a_k \uparrow \infty$ such that

$$\lambda_1(a)\, a^2 \geqslant \alpha_k / \sigma \tag{2.10}$$

for all $a \in (a_k, \infty)$. The lemma follows immediately. The proof is complete.

Next, we recall our earlier results in [19] concerning the change of $E(u)$ along the direction of non-radial mode w_k at positive radial solution u_a.

Lemma 2.2. *Assume conditions* (H-0) $\sim$ (H-2) *are satisfied. Let u_a be a positive radial solution of* (1.1) *and* (1.2), *w_0 and w_k be associated eigenfunctions with respect to $\mu_{0,1}$ and $\mu_{k,1}$, $k \geqslant 1$, respectively, and $\int_{\Omega_a} w_0^2 = \int_{\Omega_a} w_k^2 = 1$. Then there exist $\varepsilon > 0$ and a smooth function $\delta\colon (-\varepsilon, \varepsilon) \to R^1$ with $\delta(0) = \delta'(0) = 0$ such that for any $t \in (-\varepsilon, \varepsilon)$, we have*

$$J(u_a + \delta(t)\, w_0 + t w_k) = 0. \tag{2.11}$$

Moreover, we have

$$E(u_a+\delta(t)\,w_0+tw_k)=E(u_a)+\tfrac{1}{2}\mu_{0,1}\,\delta^2(t)+\tfrac{1}{2}\mu_{k,1}\,t^2+O(t^4), \qquad (2.12)$$

for $t\sim 0$.

For the proofs of (2.11) and (2.12), see Lemmas 6.1 and 6.2 in [19].

The subgroups of $O(N)$ that will be used to specify the partial symmetries later are $G_k\times O(N-2)$ and $O(l)\times O(N-l)$, which are defined as follows:

For $k\geqslant 2$, the rotational subgroup G_k is defined by

$$G_k=\Big\{g\in O(2): g(x_1,x_2)=\Big(x_1\cos\frac{2\pi l}{k}+x_2\sin\frac{2\pi l}{k},\ -x_1\sin\frac{2\pi l}{k}+x_2\cos\frac{2\pi l}{k}\Big),\ (x_1,x_2)\in R^2 \text{ and } l \text{ is an integer}\Big\}.$$

Furthermore, the following submanifolds of M with certain symmetry will be useful later.

$$V_k=\{u\in M: u\in G_k\times O(N-2)\}, \qquad (2.13)$$

and

$$\Sigma_l=\{u\in M: u\in O(l)\times O(N-l)\}. \qquad (2.14)$$

for $2\leqslant l\leqslant N-l$.

The following lemma shows that $S_{N,k}$ can provide eigenfunctions of $G_k\times O(N-2)$ symmetry for all $k\geqslant 1$, and more symmetric eigenfunctions when the dimension $N\geqslant 4$. For details see [20].

LEMMA 2.3. *Let* (ρ,θ) *be the polar coordinates in* R^2. *Then for* $N\geqslant 2$ *and for each* $k\geqslant 1$, *choosing* $\psi_k=\rho^k\cos k\theta$, *we have* $w_k=\varphi_k\psi_k\in G_k\times O(N-2)$. *Furthermore, for* $N\geqslant 4$, k *is even and* $2\leqslant l\leqslant N-l$. *Then there exist* $\psi_{k,l}$ *such that* $w_{k,l}=\varphi_k\psi_{k,l}\in O(l)\times O(N-l)$. *Moreover, for any decomposition* $L=(l_1,...,l_j)$ *of* N, $j\geqslant 2$, *i.e.,* l_i *satisfies* (i) $l_i\geqslant 2$ *for each* i *and* (ii) $\sum_{i=1}^{j} l_i=N$, *there exist* $w_{k,L}=\varphi_k\psi_{k,L}=\varphi\psi_{k,L}\in O(l_1)\times\cdots\times O(l_j)$.

On the basis of Lemmas 2.1–2.3, it is easy to obtain the following theorem; for details see [20].

THEOREM 2.4. *Assume conditions* (H-0) ~ (H-3) *are satisfied. Then there exists an increasing sequence* $a_k \to \infty$ *as* $k \to \infty$, *such that for any* $a \in (a_k, \infty)$, (1.1) *and* (1.2) *have a positive non-radial solution* $u_j \in V_j$, *and the* u_j*'s are non-equivalent for* $j = 1, 2, ..., k$. *Furthermore, if* $N \geqslant 6$, k *is even*, $3 \leqslant l \leqslant N - l$, *and* $a \in (a_k, \infty)$, *then there exist non-equivalent positive non-radial solutions* $u_{k,l} \in \Sigma_l$.

3. A PRIORI BOUNDS

In this section, we shall show that there are a priori bounds for some positive solutions of (1.1) and (1.2) when a is large. These results are essential in studying the asymptotic behavior of positive solutions as $a \to \infty$.

We first prove the following result for positive radial solutions of (1.1) and (1.2) under a very weak set of assumptions.

THEOREM 3.1. *For any* $a_0 \in (0, \infty)$, *let*

$$\wedge = \sup_{a_0 < a} \lambda_1(a) \tag{3.1}$$

If f *satisfies* (H-0) *and*

(H-2)′ *there are* $\varepsilon > 0$ *and* $U > 0$ *such that for any* $u \geqslant U$,

$$f(u) \geqslant (\wedge + \varepsilon)\, u, \tag{3.2}$$

then there is a positive constant $C = C(\wedge, \varepsilon, U, a_0)$ *such that for any* $a \in (a_0, \infty)$ *and any positive radial solution* u_a *of* (1.1) *and* (1.2), *we have*

$$\|u_a\|_\infty,\ \|u'_a\|_\infty \quad \text{and} \quad \|u''_a\|_\infty \leqslant C. \tag{3.3}$$

Proof. In the following, we shall use C as a generic constant that depends only on $\wedge$, ε, U and a_0. Since u_a is a radial solution on Ω_a, u_a satisfies

$$u'' + \frac{N-1}{r} u' + f(u(r)) = 0 \qquad \text{in } (a, a+1), \tag{3.4}$$

$$u(a) = 0 = u(a+1). \tag{3.5}$$

Let $\lambda_a = \lambda_1(a)$ and let $w_a(r) > 0$ be the associated eigenfunction of $-\Delta$ on Ω_a with $\|w_a\|_\infty = 1$. Thus w_a satisfies

$$w'' + \frac{N-1}{r} w' + \lambda_a w = 0 \qquad \text{in } (a, a+1), \tag{3.6}$$

$$w(a) = 0 = w(a+1). \tag{3.7}$$

Therefore, by (3.4) ~ (3.7), we have

$$\int_a^{a+1} r^{N-1} f(u_a)\, w_a = \lambda_a \int_a^{a+1} r^{N-1} u_a w_a. \tag{3.8}$$

On the other hand, by (3.2), we have

$$\int_a^{a+1} r^{N-1} f(u_a)\, w_a \geqslant (\wedge + \varepsilon) \int_{S_a^+} r^{N-1} u_a w_a + \int_{S_a^-} r^{N-1} f(u_a)\, w_a,$$

where

$$S_a^+ = \{r \in (a, a+1) : u_a(r) \geqslant U\},$$

and

$$S_a^- = \{r \in (a, a+1) : u_a(r) < U\}.$$

Hence,

$$\int_a^{a+1} r^{N-1} f(u_a)\, w_a \geqslant (\wedge + \varepsilon) \int_a^{a+1} r^{N-1} u_a w_a$$

$$- (\wedge + \varepsilon)\, U \int_a^{a+1} r^{N-1} w_a + \int_{S_a^-} r^{N-1} f(u_a)\, w_a.$$

If we combine the last inequality with (3.8), we have shown that there is a positive constant C such that

$$\int_a^{a+1} u_a w_a \leqslant C \qquad \text{and} \qquad \int_a^{a+1} f(u_a)\, w_a \leqslant C. \tag{3.9}$$

Now, it is known that u_a has a unique maximum in $(a, a+1)$, which is denoted by η_a (see [9]). Furthermore, by the symmetry results of Gidas *et al.* [10], we have

$$\eta_a \leqslant a + \tfrac{1}{2} \tag{3.10}$$

and

$$u_a'(r) \leqslant 0 \qquad \text{in } (\eta_a, a+1). \tag{3.11}$$

Let $\delta \in (0, \frac{1}{2})$. Then by Lemma A.1 there is an $m = m(\delta, a_0) > 0$ such that

$$w_a(a + 1 - \delta) \geqslant m, \tag{3.12}$$

and

$$w_a'(r) < 0 \quad \text{in } (a + \tfrac{1}{2}, a + 1) \tag{3.13}$$

for all $a \in (a_0, \infty)$.

By (3.9), (3.10) and (3.11), we have

$$\int_{a+1/2}^{a+1-\delta} u_a w_a \geqslant (\tfrac{1}{2} - \delta)\, u_a(a + 1 - \delta)\, w_a(a + 1 - \delta),$$

which entails that

$$u_a(a + 1 - \delta) \leqslant (\tfrac{1}{2} - \delta)^{-1} m^{-1} C. \tag{3.14}$$

Therefore, by (3.9), (3.10), (3.11), and (3.14), there is a positive constant C such that

$$\int_{\eta_a}^{a+1} f(u_a) \leqslant C. \tag{3.15}$$

Now, for any $r \in (\eta_a, a + 1)$, (3.4) entails

$$-r^{N-1} u_a'(r) = \int_{\eta_a}^{r} s^{N-1} f(u_a(s))\, ds.$$

Hence, (3.15) implies

$$-u_a'(r) \leqslant C \tag{3.16}$$

for $r \in (\eta_a, a + 1)$ and some positive constant C. Therefore, there is a positive constant C such that

$$\|u_a\|_\infty \leqslant C. \tag{3.17}$$

Furthermore,

$$\int_a^{a+1} f(u_a) \leqslant C \tag{3.18}$$

for some positive constant C.

Next, we want to prove the u_a''s are uniformly bounded for $a \in [a_0, \infty)$. By (3.16), it suffices to show $u_a'(r) \leqslant C$ also holds for $r \in (a, \eta_a)$. To do so, let $a < b < \eta_a < b_1 < a + 1$ such that

$$u_a(b) = u_a(b_1). \tag{3.19}$$

Integrating (3.4) from b to b_1, we have

$$u_a'(b) - u_a'(b_1) + (N-1)\frac{u_a(b)}{b} - (N-1)\frac{u_a(b_1)}{b_1}$$

$$= \int_b^{b_1} \frac{N-1}{r^2} u_a + \int_b^{b_1} f(u_a).$$

By (3.16) $\sim$ (3.19) and the last equality, we have

$$\|u_a'\|_\infty \leqslant C \tag{3.20}$$

for some positive constant C.

Finally, the uniform bound of u_a'' follows by (3.4), (3.17) and (3.20). The proof is complete.

Next, we shall show that the L^∞-norm of positive solutions can be bounded by their H^1-norm when (H-3) is satisfied. We begin by proving the following lemma:

LEMMA 3.2. *Assume condition* (H-3) *is satisfied. Then there exist positive constants C and τ such that for any positive solution u_a of* (1.1) *and* (1.2) *on Ω_a, $a \in [0, \infty)$, we have*

$$\|u_a\|_\infty \leqslant C(1 + \|\nabla u_a\|_2^\tau). \tag{3.21}$$

Proof. The proof of this lemma is similar to that of Lemma 2.3 of [17], in which $\|u\|_\infty$ is bounded by $\|\nabla u\|_2$ and the volume of the domains. Since the volume $|\Omega_a|$ of Ω_a tends to ∞ as $a \to \infty$, to prove (3.21) we need to modify of the argument in [17] slightly.

Thus instead of bounding $\int u^\alpha$ by

$$\int_\Omega u^\alpha \leqslant |\Omega|^{p/(\alpha+p)} \|u\|_{\alpha+p}^\alpha, \tag{3.22}$$

as in line 5 of p. 622 in [17], we shall bound it by

$$\|u\|_\alpha^\alpha \leqslant \|u\|_2^{2p/(\alpha+p-2)} \|u\|_{\alpha+p}^{\alpha+p}. \tag{3.23}$$

Here it is assumed that $\|u\|_{\alpha+p} \geqslant 1$. The rest of the argument is the same as in [17]. We therefore know that there exist constants $C_1 > 0$ and $\alpha, \beta \in (1, \infty)$ such that

$$\|u\|_\infty \leqslant C_1(1 + \|u\|_2^\alpha)(1 + \|\nabla u\|_2^\beta). \tag{3.24}$$

Since there is a positive constant $\tilde{\lambda}$ such that

$$\lambda_1(a) \geqslant \tilde{\lambda} \tag{3.25}$$

for all $a \in [0, \infty)$, where Ω_0 is the unit ball, therefore, by (3.24), (3.25) and the Poincaré inequality, (3.21) follows. The proof is complete.

An immediate consequence of Lemma 3.2 is the existence of a priori bounds for least-energy solutions u_a^* of (1.1) and (1.2) on Ω_a; here u_a^* satisfies

$$E(u_a^*) = \inf\{E(u) : u \in M_a\}. \tag{3.26}$$

THEOREM 3.3. *Assume* (H-0) ~ (H-3) *are satisfied. Then there is a positive constant C such that for any least-energy solution u_a^* of* (1.1) *and* (1.2) *on Ω_a, $a \in [0, \infty)$, we have*

$$\|u_a^*\|_\infty \leqslant C. \tag{3.27}$$

Proof. Let $R_a = (a + \frac{1}{2}, 0, ..., 0)$ and $B_{1/2}(R_a)$ be the ball of radius $\frac{1}{2}$ centered at R_a. Let $\tilde{u}$ be a positive radial solution of (1.1) and (1.2) on $B_{1/2}(O)$, where O is the origin. Then $\tilde{u}_a(x) = \tilde{u}(x - R_a)$ solves (1.1) and (1.2) on $B_{1/2}(R_a)$. Since $B_{1/2}(R_a) \subset \Omega_a$, we may extend $\tilde{u}_a$ by 0 in $\Omega_a - B_{1/2}(R_a)$ and still denote it by $\tilde{u}_a$. Therefore $\tilde{u}_a \in M_a$ and $E(\tilde{u}_a) = E(\tilde{u})$, which implies

$$E(u_a^*) \leqslant E(\tilde{u}), \tag{3.28}$$

a constant independent of $a \in [0, \infty)$.

To show (3.27), it suffices to prove there is a positive constant C_1 such that

$$\|\nabla u\|_2^2 \leqslant C_1 E(u) \tag{3.29}$$

for all $u \in M_a$ and $a \in [0, \infty)$. Indeed, it is easy to verify (H-2) entails

(H-2)* $uf(u) \geqslant (2 + \sigma) F(u)$ for all $u \geqslant 0$.

Therefore, for any $u \in M_a$,

$$E(u) = \int_{\Omega_a} \{\tfrac{1}{2}uf(u) - F(u)\} \geqslant \gamma \int_{\Omega_a} uf(u) = \gamma \int_{\Omega_a} |\nabla u|^2, \tag{3.30}$$

where $\gamma = 1/2 - 1/(2 + \sigma) > 0$. Hence, (3.29) follows. Now (3.27) follows by (3.28), (3.29) and Lemma 3.2. The proof is complete.

It is not clear whether all positive solutions can be uniformly bounded by some positive constant as $a \to \infty$ under the assumptions (H-0) ~ (H-3).

However, we can obtain an a priori bound for positive solutions of (1.1) and (1.2) on Ω_a, $a \in [0, \infty)$, by assuming f behaves like u^p as $u \to \infty$, as Gidas and Spruck [11] did for a fixed domain.

THEOREM 3.4. *Assume f satisfies* (H-0) *and*

(H-4) *there exists $p \in (1, (N+2)/(N-2))$ such that*

$$\lim_{u \to \infty} f(u)/u^p = l > 0. \tag{3.31}$$

Then there exists a positive constant C such that for any positive solution u_a of (1.1) *and* (1.2) *on Ω_a, $a \in [0, \infty)$, we have*

$$\|u_a\|_\infty \leqslant C.$$

Proof. The proof of this theorem is similar to a proof used by Gidas and Spruck [11]. In their proof, they used a blow-up technique and the following uniqueness result:

If V is a non-negative solution of

$$\Delta V + V^p = 0 \qquad \text{in } R^N,$$

where $p \in (1, (N+2)/(N-2))$, then

$$V \equiv 0. \tag{3.32}$$

Suppose the theorem is false. Then there exist a sequence $\{a_n\} \subset [0, \infty)$, a sequence of points $P_n \in \Omega_n = \Omega_{a_n}$, and functions $u_n \equiv u_{a_n}$ such that

$$L_n = \max_{x \in \Omega_n} u_n(x) = u_n(P_n) \to \infty \tag{3.33}$$

as $n \to \infty$. If $P_n \to \tilde{P} \in R^N$, then $\{a_n\}$ is bounded. Therefore, by an argument like that in [11], we obtain a contradiction to (3.32). Therefore, (3.33) is impossible.

On the other hand, if $\{P_n\}$ is unbounded, then we may assume that $P_n = (b_n, 0..., 0)$ and $b_n \to \infty$. Now, let

$$y = (x - P_n)\,\alpha_n^{-1},$$

and

$$V_n(y) = \alpha_n^{2/(p-1)}\, u_n(x),$$

where α_n is chosen such that

$$\alpha_n^{2/(p-1)} L_n = 1.$$

The remaining argument is similar to that in [11] and implies a contradiction to (3.32). Hence, (3.33) is again impossible. The proof is complete.

4. Limiting Behaviour: Radial Case

In this section we shall briefly discuss the asymptotic behavior of positive radial solutions of (1.1) and (1.2) on Ω_a as $a \to \infty$. It is natural to study the equations

$$v''(s) + f(v(s)) = 0 \qquad \text{in } (0, 1), \tag{4.1}$$

$$v(0) = 0 = v(1), \tag{4.2}$$

which are the limiting equations of (3.4) and (3.5) as $a \to \infty$. Indeed, for any radial solution u_a of (3.4) and (3.5), let

$$v_a(s) = u_a(a+s). \tag{4.3}$$

Then v_a satisfies

$$v'' + \frac{N-1}{a+s} v' + f(v) = 0 \qquad \text{in } (0, 1), \tag{4.4}$$

$$v(0) = 0 = v(1). \tag{4.5}$$

The following theorem is an easy consequence of the uniqueness result of Ni and Nussbaum [23] and Theorem 3.2.

Theorem 4.1. *Assume f satisfies* (H-0) *and*

(H-2)″ $uf'(u) > f(u) > 0$ *for all* $u > 0$.

Let

$$a_0 = \begin{cases} \{(N-1)^{1/(N-2)} - 1\}^{-1} & \text{if } N \geqslant 3 \\ \{1 - e^{-1}\}^{-1} & \text{if } N = 2. \end{cases} \tag{4.6}$$

Then, for any $a \in (a_0, \infty)$, there is a unique positive radial solution u_a of (1.1) *and* (1.2) *on Ω_a and the associated solutions v_a of* (4.4) *and* (4.5) *tend to V uniformly as $a \to \infty$, where V is the unique positive solution of* (4.1) *and* (4.2).

Proof. The uniqueness of the positive radial solution of (1.1) and (1.2) on Ω_a, $a \in (a_0, \infty)$, follows by [23]. The uniqueness of the positive solution of (4.1) and (4.2) can be proved either by computing the associated time map or by studying the linearized eigenvalue problem with the help of the Sturm Comparison Theorem. The details of the proof are omitted.

Since v_a satisfies (4.4) and (4.5), by Theorem 3.1, it is easy to verify that v_a tends to V uniformly on $[0,1]$ as $a \to \infty$.

The proof is complete.

We shall need the following notion in order to state more general results later.

DEFINITION 4.2. Let V be a solution of (4.1) and (4.2); then V is called *non-degenerate* if 0 is not an eigenvalue of the linearized eigenvalue problem

$$\varphi'' + f'(V)\,\varphi = -\mu\varphi \qquad \text{in } (0, 1), \tag{4.7}$$

$$\varphi(0) = 0 = \varphi(1). \tag{4.8}$$

THEOREM 4.3. *Assume f satisfies* (H-0), (H-1) *and* (H-2)$'$. *If $a_n \to \infty$ as $n \to \infty$ and $v_n = v_{a_n}$ is a solution of* (4.4) *and* (4.5), *then there is a subsequence $\tilde{v}_n$ of v_n such that $\tilde{v}_n$ converges to a positive solution $\tilde{V}$ of* (4.1) *and* (4.2) *in $C^2([0, 1])$ as $n \to \infty$.*

Conversely, if V is a positive non-degenerate solution of (4.1) *and* (4.2) *then there is $a^* > 0$ such that for any $a \in (a^*, \infty)$,* (4.4) *and* (4.5) *have a positive solution v_a and v_a converges uniformly to V as $a \to \infty$.*

Proof. The first part of the theorem follows easily by Theorem 3.1 and by applying the Arzela–Ascoli theorem. We need only to verify that the limiting function V is positive in (0, 1). However, for any positive solution u_a of (1.1) and (1.2), we have

$$\int_{\Omega_a} (f(u_a) - \lambda_1(a)\, u_a)\, w_a = 0, \tag{4.9}$$

which implies

$$\sup_{x \in \Omega_a} f(u_a(x))/u_a(x) \geqslant \lambda_1(a). \tag{4.10}$$

Therefore, there is a positive constant m such that

$$\|u_a\|_\infty \geqslant m. \tag{4.11}$$

Now, by (3.3) and (4.11), we have

$$\|V\|_\infty \geqslant m.$$

To prove the second part of the theorem, we rewrite Eq. (4.4) as

$$v''(s) + \varepsilon \frac{N-1}{1+\varepsilon s} v'(s) + f(v(s)) = 0 \qquad \text{in } (0, 1), \tag{4.12}$$

where $\varepsilon = 1/a$. Therefore, (4.12) and (4.5) are equivalent to the integral equation

$$v(s) = \int_0^1 G_\varepsilon(s, t) f(v(t))\, dt \tag{4.13}$$

where G_ε is the Green's function of $-(d^2/ds^2 + \varepsilon \cdot (N-1)/(1+\varepsilon s)(d/ds))$ with Dirichlet boundary condition (G_ε is given explicitly in Lemma A.2 in the Appendix). Now consider the nonlinear operator

$$H(\varepsilon, v) = v - \int_0^1 G_\varepsilon(\cdot, t) f(v(t))\, dt,$$

which is defined on $(-\varepsilon_0, \varepsilon_0) \times C_0([0, 1])$, where $\varepsilon_0 > 0$ and small. Then H is C^1. If V is a positive solution of (4.1) and (4.2), then

$$H(0, V) = 0.$$

Furthermore, if V is non-degenerate, then $(\partial H/\partial v)(0, V)$ is invertible on $C_0([0, 1])$. Therefore, by the Implicit Function Theorem, there is an $\varepsilon_1 \in (0, \varepsilon_0)$ and a continuous function $w(\cdot): (-\varepsilon_1, \varepsilon_1) \to C_0([0, 1])$ such that $H(\varepsilon, w(\varepsilon)) = 0$ for all $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$ and $w(0) = V$. Hence, the theorem holds.

The proof is complete.

Remark 4.4. If v_a converges to V in $C^2([0, 1])$ as $a \to \infty$, then

$$E(v_a) = \{C\tilde{E}(V) + o(1)\}\, a^{N-1} \tag{4.14}$$

as $a \to \infty$, where C is a positive constant and

$$\tilde{E}(V) = \int_0^1 \{\tfrac{1}{2}(V')^2 - F(V)\}.$$

Note that $\tilde{E}(V) > 0$ if (H-2)″ is satisfied. From (3.28), (4.14) and some energy estimates in Section 6, it can be shown that the greater the symmetry of solutions the larger their energies.

5. Limiting Behaviour of Least-Energy Solutions

In this section we shall study the asymptotic behavior of least-energy solutions of (1.1) and (1.2) as $a \to \infty$.

We shall need the following notations.

Definition 5.1. For any $\xi \in S^{N-1}$, the unit sphere, the infinite strip with unit width that is perpendicular to ξ is denoted by

$$S_\xi = \{t\xi + \eta : t \in (0, 1), \eta \in R^N \text{ and } \xi \cdot \eta = 0\}. \tag{5.1}$$

The two parallel planes P_ξ^0 and P_ξ^1 that cover S_ξ are given by

$$P_\xi^0 = \{\eta \in R^N : \xi \cdot \eta = 0\} \tag{5.2}$$

and

$$P_\xi^1 = \{\xi + \eta : \eta \in P_\xi^0\}. \tag{5.3}$$

The translation of Ω_a along the ξ-direction to the origin is denoted by

$$\tilde{\Omega}_a(\xi) = \{x - a\xi : x \in \Omega_a\}. \tag{5.4}$$

The domain with a cap shape is defined by

$$K_{a,\xi} \equiv K_a(\xi) \equiv S_\xi \cap \tilde{\Omega}_{a,\xi}. \tag{5.5}$$

For any function u_a defined on Ω_a, its translation along the ξ-direction,

$$v_{a,\xi}(x) = u_a(x - a\xi), \tag{5.6}$$

is defined on $\tilde{\Omega}_a(\xi)$ and in particular it is well-defined on $K_{a,\xi}$. For these $v_{a,\xi}$, we may extend them to

$$K'_{a,\xi} = S_\xi \backslash K_{a,\xi} \tag{5.7}$$

by 0, and still denote these extensions by $v_{a,\xi}$. In particular, if $\xi = \xi_1 = (1, 0, ..., 0)$, then the above notations are simplified as follows:

$$S = S_{\xi_1} = (0, 1) \times R^{N-1}, \tag{5.8}$$

$$\tilde{\Omega}_a = \tilde{\Omega}_a(\xi_1), \tag{5.9}$$

$$K_a = K_{a,\xi_1}, \tag{5.10}$$

and

$$v_a = v_{a,\xi_1}. \tag{5.11}$$

After this preparation, we can prove a simple generalization of Theorem 4.3 on each direction $\xi \in S^{N-1}$.

THEOREM 5.2. *Assume conditions* (H-0) $\sim$ (H-3) *are satisfied. If* $a_n \to \infty$ *and* $\{u_n\}$ *is a sequence of positive solutions of* (1.1) *and* (1.2) *on* Ω_{a_n}, *with*

$$\|u_n\|_\infty \leqslant C \tag{5.12}$$

for some positive constant C, then for each $\xi \in S^{N-1}$, *there is a subsequence* $\{a_{n'}\}$ *of* $\{a_n\}$ *such that the translated solutions* $v_{n',\xi} \equiv v_{a_{n'},\xi}$ *satisfy*

$$v_{n',\xi} \to V_\xi \qquad \text{pointwise in } S_\xi, \tag{5.13}$$

and the convergence is in $C^2(K)$ *for any compact subset K of* S_ξ, *where* $V_\xi \geqslant 0$ *is a solution of* (1.1) *on* S_ξ *and*

$$V_\xi = 0 \qquad \text{on} \quad P_\xi^0 \cup P_\xi^1. \tag{5.14}$$

Furthermore, if

$$\max_{t \in [0,1]} v_{n',\xi}(t\xi) \geqslant m > 0 \tag{5.15}$$

for all n′, then

$$V_\xi > 0 \qquad \text{in } S_\xi. \tag{5.16}$$

Proof. It is clear that

$$K_{a_1,\xi} \subset K_{a_2,\xi} \tag{5.17}$$

if $a_1 \leqslant a_2$ and

$$\bigcup_{a \geqslant a_0} K_{a,\xi} = S_\xi \tag{5.18}$$

for any $a_0 \geqslant 0$. Therefore, for any compact subset K of S_ξ, there is an n_0 such that

$$K_{a_n,\xi} \supset K \tag{5.19}$$

for all $n \geqslant n_0$. Then, using (5.19) and L^p-estimates, Schauder estimates, and the Arzela–Ascoli Theorem, there is a subsequence $\{v_{n',\xi}\}$ of $\{v_{n,\xi}\}$ such that $v_{n',\xi}$ converges pointwise to V_ξ in S_ξ and uniformly in $C^2(K)$ for any compact subset K of S_ξ, where $V_\xi \geqslant 0$ is a solution of (1.1) on S_ξ. Finally, by an argument similar to that used in proving Theorem 4.3, it is easy to see that (5.16) holds when (5.15) is satisfied.

The proof is complete.

Remark 5.3. Let

$$C_{a,\xi} = a\xi + K_{a,\xi} = \{a\xi + \eta : \eta \in K_{a,\xi}\}. \tag{5.20}$$

Then, for any $\xi \neq \xi'$ we have

$$C_{a,\xi} \cap C_{a,\xi'} = \phi \tag{5.21}$$

if a is large enough, although their translations $K_{a,\xi}$ and $K_{a,\xi'}$ may have non-empty intersections.

Since the limiting behavior of positive solutions can be characterized by solutions on an infinite strip, it is necessary to review some of that results concerning (1.1) and (1.2) on infinite strips which have been studied by several previous authors [8, 21].

PROPOSITION 5.4. *Assume f satisfies* (H-0) ~ (H-3).

(i) *For any $u \in H_0^1(S)$, we have*

$$\pi^2 \int_S u^2 \leqslant \int_S |\nabla u|^2. \tag{5.22}$$

(ii) *Let Ω be a domain in R^N. Then for any $u \in H_0^1(\Omega)$, $u \geqslant 0$, there is a unique $t \in (0, \infty)$ such that*

$$J(tu) = 0, \tag{5.23}$$

and

$$J(t', u) > 0 \quad \text{if} \quad t' \in (0, t), \tag{5.24}$$

$$J(t'', u) < 0 \quad \text{if} \quad t'' \in (t, \infty). \tag{5.25}$$

(iii) *For each $\alpha > 0$, let*

$$I_{-\alpha} = \inf\{E(u) : u \in H_0^1(S), u \neq 0 \text{ and } J(u) = -\alpha\} \tag{5.26}$$

and

$$I \equiv I(S) = \inf\{E(u) : u \in H_0^1(S), u \neq 0 \text{ and } J(u) = 0\} \tag{5.27}$$

Then

$$0 < I < I_{-\alpha}. \tag{5.28}$$

(iv) *I is achieved by some $V^* \in H_0^1(S)$, which is a positive solution of* (1.1) *and* (1.2) *on S.*

(v) *Let*

$$H_{0,s}^1(S) = \{u \in H_0^1(S) : u(x_1, y) = u(x_1, |y|), \text{ where } y \in R^{N-1}\}.$$

When $N = 2$, we also assume $u(x_1, |y|)$ is non-increasing along $|y|$. Let

$$I_s = \inf\{E(u) : u \in H_{0,s}^1(S), u \neq 0 \text{ and } J(u) = 0\}. \tag{5.29}$$

Then

$$I_s = I \tag{5.30}$$

and is achieved by some $V_s \in H_{0,s}^1(S)$, which is also a positive solution of (1.1) *and* (1.2) *on S.*

Proof. (i) (5.22) was proved by Esteban [8]. Note that π^2 is the first eigenvalue of $-\Delta$ on $(0, 1)$ with the Dirichlet boundary condition. In [8], it was shown that a similar result holds for a general strip-like domain $\omega \times R^l$, where ω is a bounded smooth domain in R^m.

(ii) This result was proved by Ding and Ni [7]. Indeed, for $s \in (0, \infty)$

$$\frac{d}{ds} E(su) = \int \{s |\nabla u|^2 - uf(su)\}$$

$$= \frac{1}{s} J(su) \tag{5.31}$$

and

$$\frac{d^2}{ds^2} E(su) = \int \{|\nabla u|^2 - u^2 f'(su)\}$$

$$= s^{-2} \int \{|\nabla(su)|^2 - (su)^2 f'(su)\}. \tag{5.32}$$

Therefore, $(d/ds) E(su)|_{s=t} = 0$ if and only if $J(tu) = 0$, ad in this case,

$$\frac{d^2}{ds^2} E(su)|_{s=t} = t^{-2} \int \{tuf(tu) - (tu)^2 f'(tu)\} < 0.$$

Therefore, there is a unique global maximum of $E(su)$ in $(0, \infty)$ at the zero of $J(su)$.

(iii) This was proved by P. Lions in Theorem III.1 (p. 269) of [21], and the dichotomy case in applying the concentration compactness principle is ruled out.

(iv) By our assumptions on f, Lemma I.1 (p. 231) in [21] is applicable and the vanishing case is ruled out. When we combine (iv) with (iii), I is achieved by some $V^* \in H_0^1(S)$.

(v) That result that I_s is achieved was proven by Esteban in [8]. To show (5.30), let $\tilde{V}$ be the Steiner-symmetrization of V^* with respect to $x_1 = 0$, i.e.,

$$\tilde{V}(x_1, y) = \tilde{V}(x_1, |y|) \tag{5.33}$$

for all $y \in R^{N-1}$ and $x_1 \in (0, 1)$. Then, it is known that

$$\int \tilde{U} f(\tilde{U}) = \int U^* f(U^*), \tag{5.34}$$

$$\int F(\tilde{U}) = \int F(U^*), \tag{5.35}$$

and

$$\int |\nabla \tilde{U}|^2 \leqslant \int |\nabla U^*|^2. \tag{5.36}$$

See the reference in [8].

We claim that

$$\int |\nabla \tilde{U}|^2 = \int |\nabla U^*|^2. \tag{5.37}$$

Otherwise, (5.34) ~ (5.36) imply

$$J(\tilde{U}) < J(U^*) = 0$$

and

$$E(\tilde{U}) < E(U^*) = I,$$

which also imply $I_{-\alpha} \leqslant I$, where $\alpha = -J(\tilde{U})$, a contradiction to (5.27). Since $I \leqslant I_s$, and (5.34) ~ (5.37) imply $I_s \leqslant I$, (5.30) follows. The proof is complete.

Before giving a detailed description of the limiting behavior of least-energy solutions, we need the following lemma, which states that a least-energy solution on S can be approximated by one on K_a as $a \to \infty$.

LEMMA 5.5. *Assume conditions* (H-0) ~ (H-3) *are satisfied. Let* w_a *be a least-energy solution of* (1.1) *and* (1.2) *on* K_a. *Then*

$$w_a \to V_s \quad \text{pointwise in } S$$

and the convergence is in $C^2(K)$ *for any compact subset* K *of* S, *where* $V_s > 0$ *is a minimizer of* I_s. *Furthermore,*

$$\lim_{a \to \infty} E(w_a) = I_s. \tag{5.38}$$

Proof. It is clear that the minimization problem

$$\inf\{E(u) \colon u \in H_0^1(K_a), u \neq 0 \text{ and } J(u) = 0\}, \tag{5.39}$$

can be solved by some $w_a \in H_0^1(K_a)$ that is also a positive solution of (1.1) and (1.2) on K_a. Furthermore, by the symmetry principle of Gidas *et al.* [10], $w_a(x_1, y) = w_a(x_1, |y|)$. Since $K_a \neq K_{a'}$ if $a < a'$, we have $E(w_a) > E(w_{a'})$ if $a < a'$. By the result of Esteban [8], there is a $W \in H_{0,s}^1(S)$ such that

$$w_a \to W > 0 \qquad \text{in } S$$

and

$$\lim_{a \to \infty} E(w_a) = E(W) \geqslant I_s. \tag{5.40}$$

Next, we claim that

$$I_s \geqslant E(W). \tag{5.41}$$

Let V_s be a minimizer of I_s and $\varphi_a \in H_{0,s}^1(K_a)$ such that $0 \leqslant \varphi_a \leqslant 1$ and

$$\varphi_a \to 1 \qquad \text{in } S \text{ as } \quad a \to \infty \tag{5.42}$$

and the convergence is uniform on any compact set in S. Then, by Proposition 5.4(ii) there is a $t_a \in (0, \infty)$ such that

$$J(t_a \varphi_a V_s) = 0. \tag{5.43}$$

Then we have

$$E(w_a) \leqslant E(t_a \varphi_a V_s). \tag{5.44}$$

We claim that

$$\lim_{a \to \infty} E(t_a \varphi_a V_s) = E(V_s). \tag{5.45}$$

To prove (5.45), it suffices to show that

$$\lim_{a \to \infty} t_a = 1. \tag{5.46}$$

Since f is superlinear as $u \to \infty$, it is easy to verify t_a is bounded above. On the other hand, $I_s > 0$ implies $t_a \geqslant m > 0$ for large a. Finally, if $t_{a_j} \to \tilde{t}$ as $a_j \to \infty$, then (5.43) implies $J(\tilde{t} V_s) = 0$. Hence $\tilde{t} = 1$ by Proposition 5.3(ii). Therefore, (5.46) follows, and (5.41) does as well. The proof is complete.

THEOREM 5.6. *Assume f satisfies* (H-0) $\sim$ (H-3). *Let u_a^* be a least-energy solution of* (1.1) *and* (1.2) *on Ω_a, and*

$$\max_{t \in [a, a+1]} u_a^*(t\xi_1) = \|u_a^*\|_\infty. \tag{5.47}$$

Let v_a be the translation of u_a^* according to (5.6). *Then*

$$v_a \to V^* \qquad \text{in } S \tag{5.48}$$

and

$$u_a^* \to 0 \qquad \text{uniformly in} \quad \Omega_a - C_a. \tag{5.49}$$

Proof. Since $K_a \subset \tilde{\Omega}_a$, we have

$$E(u_a^*) = E(v_a) < E(w_a), \tag{5.50}$$

where w_a is a minimizer of E on K_a obtained in Lemma 5.5. Hence by (5.50) and (5.38) we have

$$\lim_{a \to \infty} E(v_a) = I \tag{5.51}$$

and then

$$v_a \to V^* \qquad \text{in } S,$$

where $E(V^*) = I$.

Next, we claim that

$$u_a^* \to 0 \qquad \text{uniformly on} \quad \Omega_a - C_a.$$

Otherwise, there exists $\delta > 0$ and a sequence $a_n \to \infty$ and $x_n \in \Omega_a - C_a$ such that

$$u_{a_n}^*(x_n) \geqslant \delta. \tag{5.52}$$

Since $x_n/|x_n| \in S^{N-1}$, we may assume that $x_n/|x_n| \to \xi \in S^{N-1}$ as $n \to \infty$. If $\xi \neq \xi_1$, then $C_a \cap C_a(\xi) = \phi$ for large a, and by Theorem 5.2, we have

$$\begin{aligned} E(u^*_{a_n}) &\geq E(u^*_{a_n} \mid C_a) + E(u^*_{a_n} \mid C_a(\xi)) \\ &\geq \tfrac{3}{2} I \end{aligned} \tag{5.53}$$

for large n, where

$$E(u \mid A) = \int_A \{\tfrac{1}{2} |\nabla u|^2 - F(u)\},$$

for any subset A of R^N. However, (5.53) contradicts (5.51), hence $\xi = \xi_1$. Now, if $x_n/|x_n| \to \xi_1$, then $x_n \in \Omega_a - C_a$ implies

$$x_n - a_n \xi_1 \to 0 = (0, ..., 0)$$

as $n \to \infty$. Therefore,

$$v_{a_n}(x_n - a_n \xi_1) \to V^*(0) = 0 \qquad \text{as} \quad n \to \infty,$$

a contradiction to (5.52). Hence $u^*_a \to 0$ uniformly on $\Omega_a - C_a$ as $a \to \infty$. The proof is complete.

A similar result holds for any sequence of positive solutions with bounded energy; the details of this proof are omitted.

THEOREM 5.7. *Assume f satisfies* (H-0) $\sim$ (H-3). *If $\{u_n\}$ is a sequence of positive solutions of* (1.1), (1.2) *on Ω_{a_n} such that*

$$E(u_n) \leq C \tag{5.54}$$

for $a_n \to \infty$ and some positive constant C, then there are finite many directions $\eta_1, \eta_2, ..., \eta_l$ such that

$$u_{a_n} \to 0 \qquad \text{uniformly on} \quad \Omega_{a_n} - \bigcup_{j=1}^{l} C_a(\eta_j), \tag{5.55}$$

and the translated solutions $v_{n,j} = v_{a_n, \eta_j}$ converge to V_j, which is a non-negative solution of (1.1), (1.2) *on S_{η_j}, $j = 1, ..., l$.*

In particular, we have the following result.

COROLLARY 5.8. *Assume conditions* (H-0) $\sim$ (H-3) *are satisfied. If $N = 2$ and $k \geq 2$, $a \in (a_k, \infty)$ and $u_{a,j}$ are least-energy solutions with G_j-symmetry, $j = 1, ..., k$. Then, for each j, there are $\eta_1, \eta_2 = \eta_1 + 2\pi/j, ..., \eta_j = \eta_1 + (j-1)\, 2\pi/j$, directions such that their translated solutions $v_{a,i}$ converge to V_i as $a \to \infty$, for $i = 1, ..., j$ and $u_{a,j} \to 0$ uniformly on $\Omega_a - \bigcup_{i=1}^{j} C_a(\eta_i)$.*

6. Least-Energy Solutions with Partial Symmetry

In this section, we shall study the asymptotic behavior of some least-energy solutions with partial symmetry. We shall concentrate on positive solutions with $O(2)$-symmetry when $N=3$ and with $O(l)\times O(N-l)$-symmetry when $N\geqslant 4$ and $2\leqslant l\leqslant N-l$.

We begin with the case where $N=3$. Suppose $u\in H_0^1(\Omega_a)$ is $O(2)$-symmetric. We may assume, without loss of generality, that u_a is rotationally symmetric with respect to the z-axis. Hence, in spherical coordinates $x=r\sin\theta\cos\varphi$, $y=r\sin\theta\cos\varphi$, $z=r\cos\theta$, $\theta\in[0,\pi)$ and $\varphi\in[0,2\pi)$. Then

$$u=u(r,\theta), \tag{6.1}$$

or equivalently, in cylindrical coordinates $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, z, $\varphi\in[0,2\pi]$,

$$u=u(\rho,z). \tag{6.2}$$

Let

$$D_a=\{(x,z)\in R^2\colon a^2<x^2+z^2<(a+1)^2\}.$$

Then, with each $u\in H_0^1(\Omega_a)$ with $O(2)$-symmetry, we can associate a unique $w\in H_0^1(D_a)$ such that

$$w(x,z)=u(x,z) \tag{6.3}$$

and

$$w(-x,z)=u(x,z) \tag{6.4}$$

for all $(x,z)\in D_a$, and vice versa. From (6.4), it is clear that

$$\frac{\partial w}{\partial x}(0,z)=0 \tag{6.5}$$

for all $z\in(a,a+1)$. Now the energy of u can be expressed in terms of w by

$$E(u)=E^s(w)\equiv\pi\int_{D_a}|x|\,\{\tfrac{1}{2}|\nabla w|^2-F(w)\}, \tag{6.6}$$

and

$$J(u)=J^s(w)\equiv\pi\int_{D_a}|x|\,\{|\nabla w|^2-wf(w)\}. \tag{6.7}$$

Define

$$I_a^s = \inf\{E(u) \colon u \in M_a \text{ and } u \text{ is } O(2)-\text{symmetric}\}; \tag{6.8}$$

then we have

$$I_a^s = \inf\{E^s(w) \colon w \in H_0^1(D_a),\ w \neq 0,\ J^s(w) = 0 \text{ and } w \text{ satisfies (6.4)}\}. \tag{6.9}$$

Furthermore, if u is a solution of (1.1) and (1.2) on Ω_a with $O(2)$-symmetry, then w satisfies

$$\Delta w + \frac{1}{x}\frac{\partial w}{\partial x} + f(w) = 0 \qquad \text{in } D_a, \tag{6.10}$$

$$w = 0 \qquad \text{on } \partial D_a. \tag{6.11}$$

Now, for each $a \in (0, \infty)$, let u_a^s be a positive least-energy solution of (1.1) and (1.2) on Ω_a with $O(2)$-symmetry, i.e.,

$$E(u_a^s) = I_a^s. \tag{6.12}$$

Then, we have the following a priori estimates for I_a^s and u_a^s.

LEMMA 6.1. *Assume conditions* (H-0) $\sim$ (H-3) *are satisfied and* $N = 3$. *Then there exists a positive constant* C *such that*

$$\|u_a^s\|_\infty \leqslant C \tag{6.13}$$

and

$$E(u_a^s) \leqslant C. \tag{6.14}$$

Proof. For $a > 0$, let $P_a = (0, 0, a + \frac{1}{2})$ and $B_a = B_{1/2}(P_a)$, that is, the ball with center at P_a and radius $\frac{1}{2}$. Let U_a be a positive least-energy solution of (1.1) and (1.2) on B_a. By the symmetry result of Gidas *et al.* [10],

$$U_a = U_a(t) \tag{6.15}$$

where

$$t = |(x, y, z) - P_a| = \{x^2 + y^2 + (z - a - \tfrac{1}{2})^2\}^{1/2}. \tag{6.16}$$

Hence, $U_a = U(\rho, z)$ is $O(2)$-symmetric. Furthermore, $B_a \subset \Omega_a$ implies

$$E(u_a^s) \leqslant E(U_a) = C_1. \tag{6.17}$$

Hence C_1 is a positive constant that is independent of a, since U_a can be chosen such that

$$U_a(\cdot) = U_{a'}(\cdot + P_{a'} - P_a).$$

This proves (6.14). Therefore, (6.13) follows by (6.14) and Lemma 3.2. The proof is complete.

On the other hand, if u_a does not concentrate its energy on the $\pm(0, 0, 1)$ directions, then we shall prove that the energy $E(u_a)$ will go to ∞ as $a \to \infty$.

LEMMA 6.2. *Assume conditions* (H-0) $\sim$ (H-3) *are satisfied and* $N = 3$. *For* $a \in (0, \infty)$, *let* u_a *be a positive solution of* (1.1) *and* (1.2) *on* Ω_a *with* $O(2)$*-symmetry. If*

$$\|u_a\|_\infty \leqslant C \tag{6.18}$$

for some positive constant C, *and*

$$\max_{t \in [0, 1]} u_a(a + t\xi) \geqslant m > 0 \tag{6.19}$$

for some positive constant m *and* $\xi \in S^2$ *with*

$$\xi \neq \pm(0, 0, 1), \tag{6.20}$$

then

$$E(u_a) \to \infty \qquad as \quad a \to \infty. \tag{6.21}$$

Proof. Let v_a be the translation of u_a along the ξ-direction, as in Section 5. Then (6.18) and (6.19) imply

$$v_a \to V > 0 \qquad \text{in } S_\xi \tag{6.22}$$

as $a \to \infty$, and the convergence is uniform in every compact subset of S_ξ. (Here we may choose a subsequence from $\{u_a\}$ if necessary.)

Now, by (3.30), we have

$$\begin{aligned} E(u_a) = E^s(w_a) &\geqslant \frac{\gamma}{2} \int_{D_a} |x|\, |\nabla w|^2 \\ &\geqslant \frac{\gamma}{2} \int_{C_{a,\eta}} |x|\, |\nabla w|^2, \end{aligned} \tag{6.23}$$

where η is rotated from ξ to the $x-z$ plane by keeping the θ-angle fixed. Then

$$\eta \neq \pm(0, 1). \tag{6.24}$$

Therefore, from (6.24), there is a constant $m_1 > 0$ such that

$$|x| \geqslant m_1 a \tag{6.25}$$

for all $(x, z) \in C_{a,\eta}$ and large a. Hence, by (6.23) and (6.25) we have

$$E(u_a) \geqslant \tfrac{1}{2} \gamma m_1 a \int_{C_{a,\eta}} |\nabla w|^2.$$

Now (6.22) implies

$$\int_{C_{a,\eta}} |\nabla w|^2 \geqslant m_2 > 0 \tag{6.26}$$

for some positive constant m_2 when a is large. Hence, (6.21) follows by the last two inequalities. The proof is complete.

From the last two lemmas, we can obtain the following theorem.

THEOREM 6.3. *Assume conditions* (H-0) ~ (H-3) *are satisfied and* $N = 3$. *Then*

$$I_a^s \to I^s \quad \textit{as} \quad a \to \infty \tag{6.27}$$

and the associated translated solutions v_a^s *satisfy*

$$v_a^s \to V > 0 \quad \textit{in } S_\xi, \tag{6.28}$$

where $\xi = (0, 0, 1)$ *or* $(0, 0, -1)$ *and* V *is a positive least-energy solution of* (6.10) *and* (6.11) *on* $R^1 \times (0, 1)$ *with*

$$\int_{R^1 \times (0,1)} |x| \left\{ \tfrac{1}{2} |\nabla V|^2 - F(V) \right\} = I^s.$$

Proof. By Lemmas 6.1 and 6.2, we may assume the energy of $E(u_a^s)$ concentrates in the $\xi_3 = (0, 0, 1)$ direction. Let

$$C_a = C_{a,\xi_3},$$

and u_a^c be a positive least-energy solution of (1.1) and (1.2) on C_a. Then, the symmetry result of Gidas et al. [10] implies u_a^c is $O(2)$-symmetric, and we have

$$E(u_a^s) \leqslant E(u_a^c).$$

By an argument like that used in Lemma 5.5, we can prove

$$\lim_{a \to \infty} E(u_a^c) = I^s.$$

Hence

$$\limsup_{a \to \infty} I_a^s \leqslant I^s. \tag{6.29}$$

On the other hand,

$$\max_{t \in [0, 1]} u_a^s(a + t\xi_3) \geqslant m > 0$$

for some positive constant m, which implies (6.28) holds and

$$\liminf_{a \to \infty} I_a^s \geqslant I^s. \tag{6.30}$$

Hence (6.27) follows by (6.29) and (6.30). The proof is complete.

COROLLARY 6.4. *Assume conditions* (H-0) ~ (H-3) *are satisfied and* $N=3$. *For any* $k \geqslant 2$, *let* $u_{a,k}$ *be a positive least-energy solution of* (1.1) *and* (1.2) *on* Ω_a *with* G_k*-symmetry, i.e.,*

$$E(u_{a,k}) = I_{a,k}, \tag{6.31}$$

where

$$I_{a,k} = \inf\{E(u) : u \in V_k\}.$$

V_k *is given in* (2.13).
Then

$$I_{a,k} \to I \qquad \text{as} \quad a \to \infty \tag{6.32}$$

for any $k \geqslant 2$.

Proof. For $k \geqslant 2$, it is clear that

$$E(u_a^*) \leqslant E(u_{a,k}) \leqslant E(u_a^s) \tag{6.33}$$

for all a. Then (6.32) follows by Theorem 6.3 and (6.33). The proof is complete.

Remark 6.5. The asymptotic results of G_k-symmetric least-energy solutions are very different for the cases $N=2$ and $N=3$. According to Corollary 6.4, in the case $N=3$ all energy of $u_{a,k}$ concentrates on the z-axis eventually as $a \to \infty$. However, in the case where $N=2$, there is no room for $u_{a,k}$ to move all its energy into a single direction. Thus (6.32) indicates the arguments in [5, 15] work only for $N=2$ or $G_k \times O(N-2)$-symmetry for $N \geqslant 4$ but not for $N=3$.

Finally, we shall discuss the asymptotic behavior of least-energy solutions with $O(l) \times O(N-l)$-symmetry, $2 \leqslant l \leqslant N-l$ and $N \geqslant 4$.

For $x=(x_1, ..., x_l, x_{l+1}, ..., x_N) \in R^N$, let $y=(x_1, ..., x_l)$, $z=(x_{l+1}, ..., x_N)$, $s=|y|$, and $t=|z|$, and let

$$D_a^+ = \{(s,t) \in R^2 : s>0,\ t>0 \text{ and } a^2 < s^2+t^2 < (a+1)^2\}.$$

Then, if $u \in H_0^1(\Omega_a)$ with $O(l) \times O(N-l)$-symmetry, there is a $w \in H^1(D_a^+)$ such that

$$u(y,z) = w(|y|, |z|) \tag{6.34}$$

and

$$\frac{\partial w}{\partial s}(0,t) = 0 = \frac{\partial w}{\partial t}(s,0), \tag{6.35}$$

$$w=0 \quad \text{on} \quad s^2+t^2=a^2 \quad \text{and} \quad s^2+t^2=(a+1)^2. \tag{6.36}$$

By (6.35) and (6.36), w can be extended to D_a, and denoted by w again, so that $w \in H_0^1(D_a)$ with

$$w(-s,t) = w(s,-t) = w(-s,-t) = w(s,t) \tag{6.37}$$

for all $(s,t) \in D_a^+$. Then converse is also ture. The energy of u can now be expressed in terms of w by

$$E(u) = \tilde{E}(w) \equiv \tilde{C} \int_{D_a} s^{l-1} t^{N-l-1} \{\tfrac{1}{2} |\nabla w|^2 - wf(w)\}, \tag{6.38}$$

and

$$J(u) = \tilde{J}(w) \equiv \tilde{C} \int_{D_a} s^{l-1} t^{N-l-1} \{|\nabla w|^2 - wf(w)\}, \tag{6.39}$$

where $\tilde{C} = \tilde{C}(N, l)$ is a positive constant.

Furthermore, if u_a is a solution of (1.1) and (1.2) on Ω_a with $O(l)\times O(N-l)$-symmetry, then the associated solution w_a according to (6.34) will satisfy the equation

$$\frac{\partial^2 w}{\partial s^2}+\frac{l-1}{s}\frac{\partial w}{\partial s}+\frac{\partial^2 w}{\partial t^2}+\frac{N-l-1}{t}\frac{\partial w}{\partial t}+f(w)=0 \qquad \text{in } D_a, \tag{6.40}$$

$$w=0 \qquad \text{on } \partial D_a. \tag{6.41}$$

As before, for any $\eta\in S^1$, we can translate w_a to v_a on $K_{a,\eta}$ and study its limit V_η on an infinite strip as $a\to\infty$.

For $a>0$ and $2\leqslant l\leqslant N-l$, let

$$\tilde{I}_{a,l}=\inf\{E(u): u\in\Sigma_l\} \tag{6.42}$$

and let $\tilde{u}_{a,l}\in\Sigma_l$ be a minimizer of $\tilde{I}_{a,l}$, i.e.,

$$E(\tilde{u}_{a,l})=\tilde{I}_{a,l}, \tag{6.43}$$

where Σ_l is given in (2.14).

We can now obtain a lower bound of $E(u_a)$ if we know the direction in which w_a concentrates its energy.

LEMMA 6.6. *Assume conditions* (H-0) $\sim$ (H-3) *are satisfied and* $N\geqslant 4$, $2\leqslant l\leqslant N-l$. *Let* u_a *be a positive solution of* (1.1) *and* (1.2) *on* Ω_a *with* $O(l)\times O(N-l)$*-symmetry. Let* $\eta\in S^1$, *if the translated solutions* v_a *tend to a positive limit* V_η *in the* η*-direction as* $a\to\infty$. *Then*

$$E(u_a)\geqslant\begin{cases} ca^{l-1} & \text{if } \eta=\pm(1,0),\\ ca^{N-l-1} & \text{if } \eta=\pm(0,1),\\ ca^{N-2} & \text{otherwise,}\end{cases} \tag{6.44}$$

where c *is a positive number dependent on* V_η.

Proof. The proof is similar to that empolyed in proving Lemma 6.2 by using (6.38) and (6.39). The details are omitted.

From (6.44), we expect that the least-energy solutions will occur when they concentrate in the $\pm(1,0)$-direction.

THEOREM 6.7. *Assume conditions* (H-0) $\sim$ (H-4) *are satisfied*, $N\geqslant 4$ *and* $2\leqslant l\leqslant N-l$. *Then*

$$a^{1-l}\tilde{I}_{a,l}\to\tilde{I}_l \qquad as \quad a\to\infty \tag{6.45}$$

and the translated solutions $\tilde{v}_{a,l}$, *satisfy*

$$\tilde{v}_{a,l} \to V \qquad in \quad (0,1) \times R^{N-l} \equiv \tilde{S}_l,$$

where V is a positive least-energy solution satisfying

$$\Delta V + f(V) = 0 \qquad in\ \tilde{S}_l, \tag{6.46}$$

$$V = 0 \qquad on\ \partial\tilde{S}_l, \tag{6.47}$$

with

$$\int_{\tilde{S}_l} t^{N-l-1}\{\tfrac{1}{2}|\nabla V|^2 - F(V)\} = \tilde{I}_l. \tag{6.48}$$

Proof. We first claim that there is a positive constant C such that

$$\tilde{I}_{a,l} \leqslant Ca^{l-1}. \tag{6.49}$$

For large a, consider the rectangles

$$R_a = \{(s,t) \in D_a : a < s < a + \tfrac{1}{2} \text{ and } -1 < t < 1\}.$$

Let $\tilde{w}_a$ be a positive (least-energy) solution of (6.40) and (6.41) on R_a. Since $R_a \subset C_a$, where $C_a = C_{a,\eta_1}$ and $\eta_1 = (1,0)$, we have

$$\tilde{I}_{a,l} \leqslant \tilde{E}(\tilde{w}_a) \leqslant C(a+1)^{l-1} \int_{R_a} \{\tfrac{1}{2}|\nabla \tilde{w}_a|^2 - F(\tilde{w}_a)\}. \tag{6.50}$$

By an argument similar to the one used in the proof of Theorem 3.4, there is a positive constant C_1 such that

$$\|\tilde{w}_a\|_\infty \leqslant C_1. \tag{6.51}$$

If we combine (6.50) with (6.51), (6.49) follows.

An immediate consequence of (6.49) and Lemma 6.6 is that the translated solutions $\tilde{v}_{a,l}$ can only concentrate their energies on the s-axis. Finally, by an argument like that empolyed in proving Lemma 5.5 and Theorem 6.3, (6.45) ~ (6.48) follow. The details of the proofs are omitted here.

The proof is complete.

APPENDIX

We first recall some results for Bessel and modified Bessel functions that were used in this paper.

Lemma A.1. *Let $\lambda_{k,j}(a)$ be the jth eigenvalue of*

$$\varphi'' + \frac{N-1}{r}\varphi' - \frac{\alpha_k}{r^2}\varphi = -\lambda_{k,j}(a)\,\varphi \qquad \text{in } (a, a+1), \tag{A.1}$$

$$\varphi(a) = 0 = \varphi(a+1), \tag{A.2}$$

where $\alpha_k = k(k+N-2)$, and let $\lambda_j = j^2\pi^2$ be the jth eigenvalue of

$$\varphi'' = -\lambda_j \varphi \qquad \text{in } (0, 1), \tag{A.3}$$

$$\varphi(0) = 0 = \phi(1), \tag{A.4}$$

where $k = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots$. Then

$$\lim_{a \to \infty} \lambda_{k,j}(a) = \lambda_j. \tag{A.5}$$

Furthermore, let $\varphi_{k,j,a}(r)$ and $\varphi_j(r) = \sin j\pi r$ be the associated eigenfunctions with $\|\cdot\|_\infty = 1$. Then

$$\varphi_{k,j,a} \to \varphi_j$$

uniformly on $[0,1]$ as $a \to \infty$, for each $k = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots$.

Proof. See pp. 364–365 of [27].

Lemma A.2. *For any $\varepsilon \in (-1, 1)$, the Green's function $G_\varepsilon(s,t)$ of operator*

$$-\left\{v'' + \varepsilon \frac{N-1}{1+\varepsilon t} v'\right\} \tag{A.6}$$

with

$$v(0) = 0 = v(1) \tag{A.7}$$

is given by

$$G_\varepsilon(s,t) = G_o(s,t) + \varepsilon L(s,t) + \varepsilon^2 Q(s,t) + o(\varepsilon^2) \tag{A.8}$$

as $\varepsilon \to 0$, where

$$G_o(s,t) = \begin{cases} (1-s)t & \text{if } 0 < s < t < 1, \\ (1-t)s & \text{if } 0 < t < s < 1, \end{cases}$$

$$L(s,t) = \begin{cases} \dfrac{N-1}{2}(1-s)(t-s) & \text{if } 0 < s < t < 1, \\ L(t,s) & \text{if } 0 < t < s < 1, \end{cases}$$

and

$$Q(s,t)=\begin{cases}(1-s)\,t\left\{\dfrac{(N-1)(N-3)}{6}t^2-\left(\dfrac{N-1}{2}\right)^2 st\right.\\ \qquad\left.+\dfrac{N-1}{2}\left[-4N+(3-N)\,s+2Ns^2\right]\right\}, & \text{if } 0<s<t<1,\\ Q(s,t) & \text{if } 0<t<s<1.\end{cases}$$

Note that $G_o(s,t)$ is the Green's function of $-v''$ with the Dirichlet boundary conditions (A.7).

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