Pricing European Asian Options with Skewness and Kurtosis in the Underlying Distribution

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Numerical valuation model is extended for European Asian options while considering the higher moments of the underlying asset return distribution. The Edgeworth binomial lattice is applied and the lower and upper bounds of the option value are calculated. That the error bound in pricing Asian options from the Edgeworth binomial model is smaller than the error bound model by Chalasani et al. is shown. The approach is used to price the average rate currency option with different skewness and kurtosis. The numerical results show that this approach can effectively deal with the higher moments of the underlying distribution and provide better estimates of option value compared with various studies in literature. © 2008 Wiley Periodicals, Inc. Jrl Fut Mark 28:598–616, 2008

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INTRODUCTION

The payoff of an Asian option depends on the arithmetic or geometric price average of the underlying asset during the life of the option. Its value is path dependent, normally without the closed-form solution, and therefore more difficult to calculate than that of a standard option. However, the hedging effect of an Asian option, which is specifically widely used in the foreign exchange market, is better than that of a standard option and offers convenience and lower cost (Hu & Yu, 2000). Nielsen and Sandmann (2003) reported that the open interest of Asian options is in the range of 5–10 billion U.S. dollars on the over-the-counter market.

Many researchers have applied various methods to analyze Asian options: Haykov (1993), Boyle (1977), Grant, Vora, and Weeks (1997), and Kemna and Vorst (1990) have employed Monte Carlo (MC) simulations with variance reduction techniques; Hull and White (1993), Neave and Ye (2003), Chalasani, Jha, and Varikooty (1998), Chalasani, Jha, Egriboyun, and Varikooty (1999), and Reynaerts, Vanmaele, Dhaene, and Deelstra (2006) developed binomial trees, or lattices, with different efficiency enhancements; Dewynne and Wilmott (1995), Rogers and Shi (1995), and Alziary Decamps, and Koehl (1997) applied the partial differential equation approaches; Curran (1994) and Nielsen and Sandmann (1996, 2002, 2003) used general numerical methods; Geman and Yor (1993), Kramkov and Mordecky (1994), and Chacko and Das (1997) applied pseudo-analytic characterizations; and Turnbull and Wakeman (1991) (TW), Levy (1992), Vorst (1992), and Bouaziz, Briys, and Crouhy (1994) all have employed analytic approximations that produce closed-form expressions.

Most of the valuation methods for Asian options assume that the return distribution of the underlying asset is lognormal. However, practitioners and academics are well aware that the finite sum of the correlated lognormal random variables is not lognormal. It is for this reason that some researchers have tried to investigate other alternatives by considering the number of moments.

TW and Levy (1992) had applied the first two moments to price the average rate currency options and obtained reasonable approximations under low-volatility conditions. They had suggested using higher moments when volatility is high. Milevsky and Posner (1998) had used the fundamental method to derive the probability density function of the infinite sum of the correlated lognormal random variables and proved that it is a reciprocal gamma distribution under certain parameter restrictions. Fusai and Tagliani (2002) had also used moments to evaluate fixed exercise Asian options and showed that the density of the logarithm of the arithmetic average was uniquely determined. They had verified that entropy decreases significantly when the fourth moments are used, and their approximation is good at low-volatility levels. However, error increases for higher volatility and more moments may be required.

As return distributions in the currency market are usually not normal (Kearns & Pagan, 1997; Tucker & Pond, 1988), incorporating higher moments in the valuation of an Asian currency option should provide better results. In this study, the model developed by Chalasani et al. (1998) is extended for the valuation of European Asian options while considering the higher moments of the underlying asset return distribution. The Edgeworth binomial lattice (Rubinstein, 1998) is applied and the lower and upper bounds of the option value are calculated. The approach is used to price the average rate currency option with different skewness and kurtosis.

When the first two moments are used, the authors' model obtains a better value for an Asian option with low volatility than those of Levy (1992), Rogers and Shi (1995), and Chalasani et al. (1998). If four moments are used, the authors' model can provide satisfactory estimates for high-volatility Asian options comparing the results from the discrete Wilkinson approximation, the four-moment approximation, and the MC method.

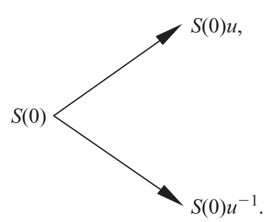
DEFINITIONS AND THE BASIC BINOMIAL MODEL

An underlying variable, $S(t)$, of an option at time t is generally assumed to satisfy the stochastic differential equation in a risk-neutral world:

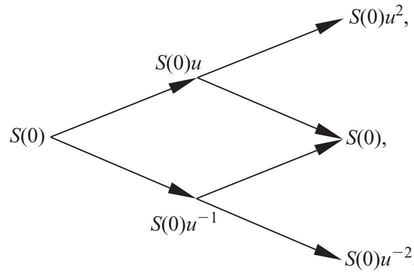
$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t)$$

where the drift μ and volatility σ are constant, and $\{B(t)\}$ denotes a Brownian motion process. Assume that the risk-free interest rate r is a constant, and that the option expires at time T.

A binomial tree (Cox, Ross, & Rubinstein, 1979) can approximate the continuous-time function $S(t)$, where one divides the life of the option into n time steps of length $\Delta t = T/n$. In each time step, the underlying asset may move up by a factor u with probability p_c, or down by a factor $d = u^{-1}$ with probability $q_c = 1 - p_c$, with $0 < d < 1 < u$. Firstly, the one-period case is considered, i.e. time step $k = 1$. The stock price at the end of the period will have two possible values, either up to a value $S(0)u$ with probability p_c or down to a value $S(0)u^{-1}$ with probability $1 - p_c$. These price movements can be represented in the following diagram:



Now consider a call option with two periods ($k = 2$) before its expiry date. The price process of the stock will show three possible values after two periods:



This price process of the stock can be extended to n time steps.

The stochastic differential equation describing this price process, i.e. $dS(t) = \mu S(t)\lambda dt + \sigma S(t)\lambda dB(t)$, has the following solution:

$$S(t) = S(0)e^{(\mu-(1/2)\sigma^2)t+\sigma\Phi\sqrt{t}}$$

where Φ is a standardized normal random variable.

For a binomial random walk to have the correct drift over a time period of Δt, the following is needed:

$$p_c Su + (1 - p_c)Sd = SE[e^{(\mu-(1/2)\sigma^2)\Delta t+\sigma\Phi\sqrt{\Delta t}}] = Se^{\mu\Delta t}$$

namely, $p_c u + (1 - p_c)d = e^{\mu\Delta t}$. Rearranging this equation the following can be obtained:

$$p_c = \frac{e^{\mu\Delta t} - d}{u - d}$$

with $u = e^{\sigma\sqrt{\Delta t}}$.

Here, let Ω_n be a sample space of an experiment including all possible sequences of n upticks and downticks. A typical element of Ω_n is presented as $\omega = \omega_1, \omega_2 \ldots \omega_n$, where ω_i denotes the ith uptick or downtick. Let $\{H_k(\omega)\}$ be an associated family of random variables, where $H_k(\omega)$ denotes the number of upticks at time k and $H_0(\omega) = 0$ for all ω. A symmetric random walk X_k can be defined, such that for each $k \geq 1$, $X_k = H_k - (k - H_k) = 2H_k - k$, which represents the number of upticks minus the number of downticks up to time k. It is used to define the nodes in a binomial lattice corresponding to the possible positions of the underlying random walk at different times. Specifically, a tree path ω is displayed to pass through or reach node (k, h) if and only if $H_k(\omega) = h$

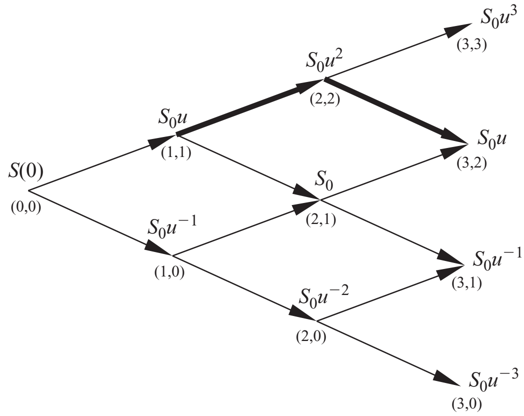


FIGURE 1
A binomial tree. Node (3, 2) means there are two upticks in any path reaching this node at time 3.

for times $k = 0,1, \ldots ,n$ and the number of possible upticks $h = 0,1, \ldots ,k$. Consequently, the underlying asset price at time k is S_k ($k = 0,1,\ldots,n$), where

$$S_k = S_0u^{X_k} = S_0u^{2H_k-k}.$$

For example, $S_3 = S_0u^{(2\times2-3)} = S_0u$ at node (3, 2) in the lattice diagram of Figure 1. The underlying asset price at node (k, h) is given by S_0u^{2h-k}, whose average at time k is defined as $A_k = (S_0 + S_1 + \uparrow + S_k)/(k + 1)$, $k \geq 0$. Therefore, the payoff of an Asian call with strike price L at time n is $V_n^+ = (A_n - L)^+ = \max\{A_n - L, 0\}$. The price of this option is the expected present value discounted to time 0, $C = E[V_n^+]/(1 + r)^n$. Note that $E[V_n^+]$ is a probability-weighted average given by $\Sigma_k P_k(A_k - L)^+$, where P_k denotes the risk-neutral probability associated with A_k at the expiration date.

EDGEWORTH BINOMIAL MODEL FOR ASIAN OPTION VALUATION

To consider the higher moments, the Edgeworth binomial tree model is first applied (Rubinstein, 1998). Assume that the tree has n time steps and $n + 1$ nodes ($h = 0,1, \ldots, n$) at step n. At each node h, there is a random variable $y_h = [2h - n]/n^{1/2}$ with a standardized binomial density $b(y_h) = [n!/h!(n - h)!](1/2)^n$. Giving predetermined skewness and kurtosis, the binomial density is transformed by the Edgeworth expansion up to the fourth moment. The result is

$$F(y_h) = f(y_h) \times b(y_h)$$

$$= \left[1 + \frac{1}{6}\gamma_1(\gamma_h^3 - 3y_h) + \frac{1}{24}(\gamma_2 - 3)(y_h^4 - 6y_h^2 + 3) \right.$$

$$+\frac{1}{72}\gamma_1(y_h^6 - 15y_h^4 + 45y_h^2 - 15)\Big] \times \left(\frac{1}{2}\right)^n\left[\frac{n!}{h!(n-h)!}\right] \tag{1}$$

with $f(y_h) = [1 + (1/6)\gamma_1(y_h^3 - 3y_h) + (1/24)(\gamma_2 - 3)(y_h^4 - 6y_h^2 + 3) + (1/72)\gamma_1(y_h^6 - 15y_h^4 + 45y_h^2 - 15)]$, where $\gamma_1 = E^Q[y_h^3]$ is the skewness and $\gamma_2 = E^Q[y_h^4]$ is the kurtosis of the underlying distribution under risk-neutral measure. Although the sum of $F(y_h)$ is not one, $F(y_h)$ is normalized by $F(y_h)/\Sigma_j F(y_j)$ and denoted as P_h.

The variable y_h, which has probability P_h, can be standardized as $x_h = (y_h - M)/V$ with $M = \Sigma_h P_h y_h$ and $V^2 = \Sigma_h P_h(y_h - M)^2$. The variable x_h is used later in Equation (2) to obtain the asset price and the corresponding risk-neutral probability, P_h, for a path to node h.

Consider a tree model of n steps. The asset price at the hth node ($h = 0, 1, \ldots, n$) during the final step, $\hat{S}_{n,h}$, is

$$\hat{S}_{n,h} = S_0 e^{\mu T + \sigma\sqrt{T}x_h} \tag{2}$$

with $\mu = r - (1/T)\ln\sum_{h=0}^{n} P_h e^{\sigma\sqrt{T}x_h}$, where S_0 is the initial asset price, r is the continuously compounded annual risk-free rate, T is the time for expiration of the option (in years), σ is the annualized volatility rate for the cumulative asset return, and x_h is a random variable from probability distribution P_h with mean 0 and variance 1. P_h is determined by modifying the binomial distribution using the Edgeworth expansion up to the fourth moment of $\ln(\hat{S}_{n,h}/S_0)$. Finally, μ is used to ensure that the expected risk-neutral asset return equals r. Solving backward recursively from the end of the tree, the nodal value, $S_{n-1,h}$, is

$$S_{n-1,h} = [p_e\hat{S}_{n,h+1} + q_e\hat{S}_{n,h}]\exp\left(-\frac{rT}{n}\right) \tag{3}$$

with $p_e = p_{n,h+1}/(p_{n,h+1} + p_{n,h})$ and $q_e = (1 - p_e)$, where $p_{n,h}$ is $P_h/[n!/h!(n-h)!]$.

The path dependence of Asian options is analyzed using the approach by Chalasani et al. (1999). To represent the refined binomial lattice, a new random variable $W_{k,h}$ denoting an area at time k is assigned. Its initial value W_0 is zero. For any node (k,h) in the tree, a lowest path reaching (k, h) is defined as the path with $k - h$ downticks followed by h upticks, and a highest path reaching (k, h) means the one with h upticks followed by $k - h$ downticks. The area $W_{k,h}(\omega)$ of a path ω reaching (k, h) can be defined as the number of diamond-shaped boxes enclosed between this path ω and the lowest path reaching this node. For example, the node (5, 2) means that the paths reaching it have two upticks at time 5. As demonstrated in Figure 2, a path passing through (5, 2) and reaching node (6, 2) is shown by the thick line segments. The area $W_{6,2}(\omega)$

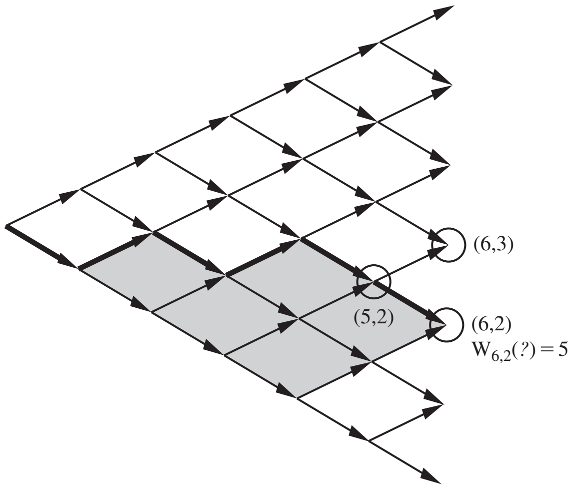


FIGURE 2
A binomial lattice. An example of diamond-shaped boxes for node (6, 2) is shown as the shaded area.

of this path is the number of diamond-shaped boxes, contained between this path and the lowest path reaching node (6, 2), as shown by the shaded area in the graph. The maximum area of any path reaching (k, h) is the number of boxes between the highest and the lowest paths reaching (k, h), that is, $h(k - h)$. The minimum area of any path reaching (k, h) is zero. The set of possible areas of paths reaching node (k, h) is therefore $\{0,1, \ldots ,h(k - h)\}$. Each node of the binomial lattice can be partitioned into "nodelets" based on the areas of the paths reaching this node. Therefore, any path reaching a given nodelet (k, h, a) has an area $W_{k,h}(\omega) = a$ with h upticks at time k. For instance, Figure 3 shows the nodelets in the nodes (5, 2), (6, 3), and (6, 2). As noted in Chalasani et al. (1999), there is a one–one correspondence between the possible areas and the possible geometric averages of underlying asset prices for paths reaching (k, h). Therefore, (k, h, a) represents all the paths in the binomial tree that reach node (k, h) and has the same geometric average asset price from time 0 to k.

Suppose the area of a path A reaching (k, h) is $W_{k,h}(A) = a$. If A has an uptick after this point, it reaches node $(k + 1, h + 1)$ at the next time step. The path A and the lowest path B reaching $(k + 1, h + 1)$ share the same edge linking (k, h) and $(k + 1, h + 1)$ in the lattice. Hence, the number of boxes between A and B at time $k + 1$ is the same as the number at time k. In this way the path A reaches nodelet $(k + 1, h + 1, a)$. On the other hand, if A has a downtick after time k, it will reach node $(k + 1, h)$. In this case, the number of boxes at time $k + 1$ between A and the lowest path reaching $(k + 1, h)$ will be increased by h to get $a + h$. The path A then reaches nodelet $(k + 1, h, a + h)$.

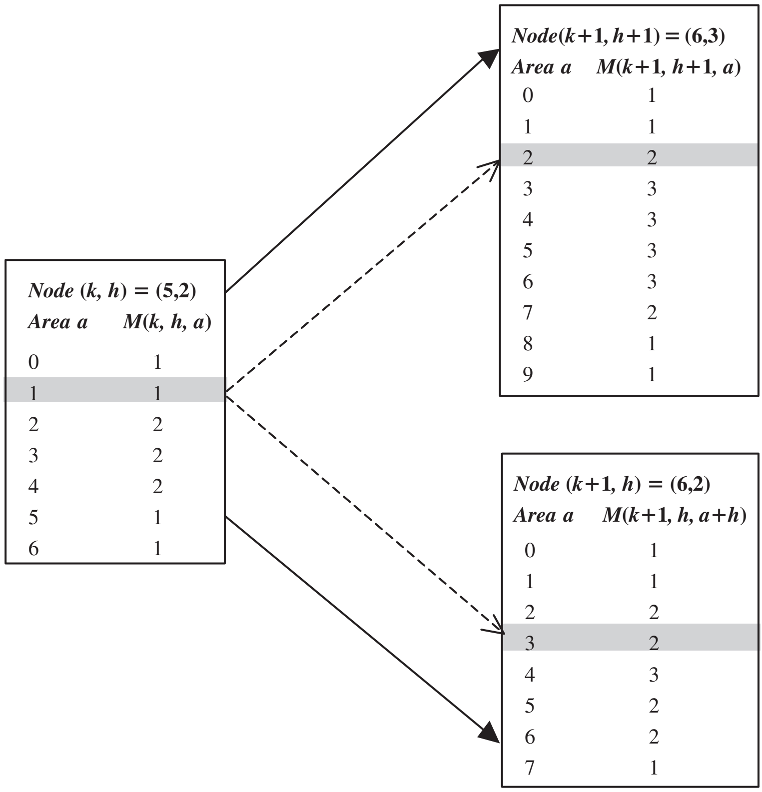


FIGURE 3
Nodelets in the nodes (5, 2), (6, 3), and (6, 2) as circled in Figure 2. This figure exhibits the number of paths $M(k, h, a)$ reaching each nodelet (k, h, a). An example is shown for nodelet (5, 2, 2), which is updated as in the nodelets (6, 3, 2) and (6, 2, 4).

How the arithmetic average of underlying asset prices over all paths reaching (k, h, a) is computed is shown and denoted by $\overline{A}(k, h, a) = E[A_k | H_k = h, W_{k,h}(\omega) = a]$, $k = 0,1,\ldots,n$, $h \leq k$. It is simply the average of A_k over these paths. So the arithmetic average of stock prices over all paths reaching nodelet (k, h, a) can be expressed as

$$\overline{A}(k, h, a) = \frac{S''(k, h, a)}{(k + 1)M(k, h, a)} \tag{4}$$

where $S''(k, h, a) = \sum_{m=1}^{M} S'_m(k, h, a)$, $S'_m(k, h, a) = \sum_{i=0}^{K} S_{i,h}$, with $k = 0,1,\ldots,n$, $h = 0,1,\ldots,k$, $a = 0,1,\ldots,h(k - h)$, $m = 1,2,\ldots,M(k, h, a)$, and $M(k, h, a)$ is the number of paths reaching (k, h, a) with $M(0,0,0) = 1$. Here, $S''(k, h, a)$ is the sum of $S'_m(k, h, a)$ over all paths passing through (k, h, a) with $S''(0, 0, 0)$

$= S_0$, whereas $S'_m(k, h, a)$ is the sum of the asset prices along any possible path passing through (k, h, a) from time 0 to k.

Any path passing through nodelet (k, h, a) and having an uptick will get to nodelet $(k + 1, h + 1, a)$ at time $k + 1$. Thus, the number of paths reaching nodelet $(k + 1, h + 1, a)$, namely, $M(k + 1, h + 1, a)$, should include $M(k, h, a)$ paths through (k, h, a). The sum of the prices from all the paths reaching $(k + 1, h + 1, a)$, namely, $S''(k + 1, h + 1, a)$, would be $S''(k, h, a) + M(k, h, a)S_{k+1,h+1}$ for paths passing through (k, h, a). Likewise, all paths passing through nodelet (k, h, a) with a downtick will reach nodelet $(k + 1, h, a + h)$ at time $k + 1$. Similarly, $M(k + 1, h, a + h)$ should also include $M(k, h, a)$ and $S''(k + 1, h, a + h)$ would be $S''(k, h, a) + M(k, h, a)S_{k+1,h}$ in the forward induction process.

Next, the authors present how the value of an Asian option after obtaining the arithmetic average of the stock prices from Equation (4) is estimated. The approach used by Rogers and Shi (1995) is applied, where the lower bound and the error bound are calculated for the price of an Asian option. This lower bound on the price of an Asian call option with strike L can be expressed as

$$\begin{aligned} E[(A_n - L)]^+ &= E[E[(A_n - L)^+|Z]] \geq E[E[A_n - L|Z]^+] \\ &= E[E[(A_n|Z) - L]^+] \end{aligned} \tag{5}$$

where $Z = (W_{n,h}, S_{n,h})$ in which the random variable $W_{n,h}$ denotes the area at node (n, h), and $S_{n,h}$ represents the stock price reaching (n, h). The composition in the lower bound, $E[A_n|W_{n,h}, S_{n,h}]$, can be expressed as $\overline{A}(n, h, a)$, as in Equation (4). $\overline{A}(n, h, a)$ is the expectation of the average stock price A_n at node (n, h), where $A_n = (S_0 + S_{1,h} + \uparrow + S_{n,h})/(n + 1)$ on a tree path passing through (n, h, a). All paths through this nodelet have the same probability $P(W_{n,h}, S_{n,h})$, which is $M(n, h, a)p_e^h q_e^{n-h}$. Thus, the lower bound can be calculated as

$$C_0^d = E[[E(A_n|W_{n,h}, S_{n,h}) - L]^+] = \sum_{h=0}^{n} \sum_{a=0}^{h(n-h)} P(W_{n,h}, S_{n,h})[\overline{A}(n, h, a) - L]^+$$

$$= \sum_{h=0}^{n} \sum_{a=0}^{h(n-h)} M(n, h, a)p_e^h q_e^{n-h}[\overline{A}(n, h, a) - L]^+.$$

As a result, the error bound is

$$\begin{aligned} &E[E[(A_n - L)^+|W_{n,h}, S_{n,h}]] - E[E[(A_n - L)|W_{n,h}, S_{n,h}]^+] \\ &\quad = E[E[(A_n - L)^+|W_{n,h}, S_{n,h}] - E[(A_n - L)|W_{n,h}, S_{n,h}]^+] \\ &\quad \leq \frac{1}{2} E[[var(A_n - L|W_{n,h}, S_{n,h})]^{1/2}] \end{aligned} \tag{6}$$

assuming $V_n^{min}(Z) < 0$ and $V_n^{max}(Z) > 0$, where $V_n = A_n - L$.[1] Note that $var(A_n - L) = var\lambda A_n = EA^2{}_n - (EA_n)^2$ and $\overline{A}^2(n, h, a) = E[A_n^2 | W_{n,h}, S_{n,h}]$. Let $A^{min}(k, h, a)$ denote the minimal value of A_k and $A^{max}(k, h, a)$ its maximum over all paths passing through the nodelet (k, h, a). Thus, the error bound by Equation (6) equals

$$\frac{1}{2}\sum_{h=0}^{n} \sum_{\substack{a=0 \\ A^{min}(n,h,a)<L, A^{max}(n,h,a)>L}}^{h(n-h)} P(W_{n,h}, S_{n,h})(\overline{A}^2(n, h, a) - \overline{A}(n, h, a)^2)^{1/2}$$

$$=\frac{1}{2}\sum_{h=0}^{n} \sum_{\substack{a=0 \\ A^{min}(n,h,a)<L, A^{max}(n,h,a)>L}}^{h(n-h)} M(n, h, a)p_e^h q_e^{n-h}(\overline{A}^2(n, h, a) - \overline{A}(n, h, a)^2)^{1/2}. \quad (7)$$

The $\overline{A}(n, h, a)$ can be derived from Equation (4). Meanwhile, $A^{min}(k, h, a) = S^{min}(k, h, a)/(k + 1)$ and $A^{max}(k, h, a) = S^{max}(k, h, a)/(k + 1)$, where $S^{min}(k, h, a)$ and $S^{max}(k, h, a)$ are, respectively, the minimum value and maximum value of $S_{k,h}$ over these paths reaching (k, h, a). $\overline{A}^2(n, h, a)$ can also be calculated from the following:

$$\overline{A}^2(k, h, a) = \frac{\varphi(k, h, a) + 2\psi(k, h, a)}{(k + 1)^2 M(k, h, a)}$$

where $\varphi(k, h, a)$ is the sum of $\sum_{i=0}^{k} S_{i,h}^2$ and $\psi(k, h, a)$ is the sum of $\sum_{0\le i\le j\le k} S_{i,h} S_{ij,h}$.[2] With the lower bound and the error bound, the upper bound can be obtained.

Suppose one upward probability p_e, denoting the probability of the stock price moving up for the next step in the Edgeworth binomial tree, is lower than the other upward probability p_c, the probability of the stock price moving up in the binomial tree of Chalasani et al. (1999). The average stock price in a path with upward drift causes higher probability of $A^{min}(k, h, a) > L$, i.e. higher probability of zero variance. So the total variance of the average stock price will

[1]The minimum and maximum values of V_n over paths ω with $Z(\omega) = z$ is set to be $V_n^{max}(z) = \max_{\omega\in\Omega}\{V_n(\omega)|Z(\omega) = z\}$ and $V_n^{min}(z) = \min_{\omega\in\Omega}\{V_n(\omega)|Z(\omega) = z\}$. If $V_n^{max}(z_i) \le 0$, then for all paths ω with $Z(\omega) = z_i$, $V_n^+(\omega) = 0$ can be deduced, which implies $E(V_n^+|z_i) = 0$, and also $E(V_n|z_i) \le 0$, which implies $E(V_n|z_i)^+ = 0$. Hence, the error bound is zero. Similarly, if $V_n^{min}(z_i) \ge 0$, then for all paths ω with $Z(\omega) = z_i$, $V_n^+(\omega) = V_n(\omega)$ can be deduced, which implies $E(V_n^+|z_i) = E(V_n|z_i)$, and also $E(V_n|z_i) \ge 0$, which implies $E(V_n|z_i)^+ = E(V_n|z_i)$. Therefore, the error bound is again zero.

[2]To show how $\overline{A}^2(k, h, a)$ is derived, $(k + 1)^2 A_k^2 = \left(\sum_{i=0}^{k} S_{i,h}\right)^2 = \sum_{i=0}^{k} S_{i,h}^2 + 2\sum_{0\le i\le j\le k} S_{i,h}S_{j,h}$ can be written. Because all paths reaching (k, h, a) have the same probability, $\overline{A}^2(k, h, a)$ is the average of $A_k^2 = \left(\sum_{i=0}^{k} S_{i,h}^2 + 2\sum_{0\le i\le j\le k} S_{i,h}S_{j,h}\right) \div (k + 1)^2$ over these paths.

be smaller. According to Equation (7), the error bound of the option price with upward probability p_e will be smaller than the error bound with probability p_c. It can be shown in the following proposition that this can lead to tighter bounds on the error from approximating $E[V_n^+]$ if its upward probability is lower. The details are explained in the Appendix.

Proposition: The error bound in pricing a European Asian option from the modified Edgeworth binomial model is tighter than the error bound from the model by Chalasani et al. (1999).

NUMERICAL RESULTS

Valuation of European Asian Options Under Normal Skewness and Kurtosis

Microsoft Visual C++ is used to program the authors' algorithm. Considering first the normal skewness and kurtosis, the results are tested and compared with those in the literature. The call option to be valued has the initial stock price $S_0 = 100$, the maturity $T = 1$ year, and the strike prices $L = 95, 100, 105$, and 110, respectively. The underlying distribution has volatility $\sigma = 0.05, 0.1$, and 0.3, respectively, with normal skewness $\gamma_1 = 0$ and kurtosis $\gamma_2 = 3$. The risk-free rate r is set to be 0.05, 0.09, or 0.15. The time step N equals 30, and the computing time and memory space needed in the authors' algorithm are similar to those of Chalasani et al. (1998). The authors' simulation results are presented in Tables I and II.

In Table I, the authors' results are compared with those of Rogers and Shi (1995) and of Chalasani et al. (1998). When the call is in the money, the authors' valuation in general is smaller than those of Chalasani et al. (1998). However, the range of the authors' lower and upper bounds is narrower than theirs. For at-the-money and out-of-the-money calls, the authors' estimates are greater than theirs and closer to those of Rogers and Shi's, but the distance between the authors' lower and upper bounds is almost the same as that of Chalasani et al. The difference between the authors' calculations and those of Rogers and Shi's is owing to the authors' numerical approximation comparing with their continuous-time integrals.

In Table II, the authors' results are compared with those of MC simulations from Levy and Turnbull (1992) (LT). Because Chalasani et al. (1998) claim that their results are closer to MC estimations than those of Roger and Shi (1995), the authors also list their bounds. As depicted in the table, the authors' estimates are much closer to the results of MC simulations; hence, the authors' algorithm in pricing Asian options performs better than that of Chalasani et al. (1998).

TABLE I

Model Comparisons for Asian Options Valuations Under Normal Skewness and Kurtosis

Strike L	*Vol. σ*	*r*	*E-LB*	*E-UB*	*RS-LB*	*RS-UB*	*C-LB*	*C-UB*
95	0.05	0.05	7.177	7.177	7.178	7.183	7.178	7.178
100	0.05	0.05	2.712	2.712	2.716	2.722	2.708	2.708
105	0.05	0.05	0.332	0.332	0.337	0.343	0.309	0.309
95	0.05	0.09	8.811	8.811	8.809	8.821	8.811	8.811
100	0.05	0.09	4.306	4.306	4.308	4.318	4.301	4.301
105	0.05	0.09	0.957	0.957	0.958	0.968	0.892	0.892
95	0.05	0.15	11.100	11.100	11.094	11.114	11.100	11.100
100	0.05	0.15	6.799	6.799	6.794	6.810	6.798	6.798
105	0.05	0.15	2.745	2.745	2.744	2.761	2.667	2.667
90	0.10	0.05	11.947	11.947	11.951	11.973	11.949	11.949
100	0.10	0.05	3.635	3.635	3.641	3.663	3.632	3.632
110	0.10	0.05	0.319	0.320	0.331	0.353	0.306	0.306
90	0.10	0.09	13.385	13.385	13.385	13.410	13.386	13.386
100	0.10	0.09	4.909	4.909	4.915	4.942	4.902	4.902
110	0.10	0.09	0.621	0.621	0.630	0.657	0.582	0.583
90	0.10	0.15	15.404	15.404	15.399	15.445	15.404	15.404
100	0.10	0.15	7.024	7.024	7.028	7.066	7.015	7.015
110	0.10	0.15	1.411	1.412	1.413	1.451	1.316	1.317
90	0.30	0.05	13.928	13.936	13.952	14.161	13.929	13.938
100	0.30	0.05	7.924	7.932	7.944	8.153	7.924	7.932
110	0.30	0.05	4.041	4.051	4.070	4.279	4.040	4.049
90	0.30	0.09	14.961	14.968	14.983	15.194	14.964	14.972
100	0.30	0.09	8.811	8.818	8.827	9.039	8.807	8.815
110	0.30	0.09	4.672	4.682	4.695	4.906	4.661	4.671
90	0.30	0.15	16.494	16.500	16.512	16.732	16.499	16.506
100	0.30	0.15	10.197	10.205	10.208	10.429	10.187	10.195
110	0.30	0.15	5.715	5.725	5.728	5.948	5.685	5.696

Note. The European Asian option to be valued has initial stock price $S_0 = 100$ dollars and option life $T = 1.0$ year. Using time steps $N = 30$, the lower and upper bounds from the authors' algorithm are indicated by E-LB and E-UB, respectively, whereas those from Rogers and Shi (1995) are indicated by RS-LB and RS-UB, and those from Chalasani et al. (1998) by C-LB and C-UB. The authors used normal skewness $\gamma_1 = 0$ and kurtosis $\gamma_2 = 3$ in their algorithm.

TABLE II

Comparisons with Monte Carlo Simulations Under Normal Skewness and Kurtosis

Strike L	*Vol. σ*	*r*	*Monte Carlo*	*E-LB*	*E-UB*	*C-LB*	*C-UB*
95	0.10	0.09	8.91	8.91	8.91	8.91	8.91
100	0.10	0.09	4.91	4.91	4.91	4.90	4.90
105	0.10	0.09	2.06	2.07	2.07	2.03	2.03
90	0.30	0.09	14.96	14.96	14.97	14.96	14.97
100	0.30	0.09	8.81	8.81	8.82	8.81	8.82
110	0.30	0.09	4.68	4.67	4.68	4.66	4.67
90	0.50	0.09	18.14	18.14	18.18	18.15	18.19
100	0.50	0.09	12.98	12.98	13.02	12.99	13.03
110	0.50	0.09	9.10	9.07	9.11	9.08	9.12

Note. The European Asian option to be valued has initial stock price $S_0 = 100$ dollars and option life $T = 1.0$ year. Using time steps $N = 30$, the lower and upper bounds from the authors' algorithm are indicated by E-LB and E-UB, respectively, whereas Monte Carlo estimates from Levy and Turnbull (1992) are indicated by Monte Carlo, and those from Chalasani et al. (1998) are indicated by C-LB and C-UB. The authors used normal skewness $\gamma_1 = 0$ and kurtosis $\gamma_2 = 3$ in their algorithm.

Tables I and II demonstrate the performance of the authors' algorithm under normal skewness and kurtosis. In the next section, the Asian options under various non-normal skewness and kurtosis are priced and the results are compared with those in the literature.

Valuation of European Asian Options Under Various Skewness and Kurtosis

The valuation performance of the European Asian option in Table III is based on the initial stock price $S_0 = 100$, the risk-free rate $r = 0.09$, and the maturity $T = 1$ year with varying skewness and kurtosis. The results of the numerical analysis are compared with those of the Edgeworth expansion model by TW, the modified Edgeworth expansion method by LT, and the four-moment approximation model by Posner and Milevsky (1998) (PM). All these models are considered up to the fourth moments in their valuations.

For low-volatility cases ($\sigma = 0.05$ and 0.1) in Table III, the authors' results from in-the-money or at-the-money calls are very close to MC estimates, which is the benchmark used by LT under lognormal distribution. This is similar to those of LT and TW. For out-of-the-money calls, the authors' outcomes are the same as theirs under right-skewed conditions. For high-volatility cases ($\sigma = 0.3$ and 0.5), the authors' outcomes for at-the-money or deep-in-the-money calls approach the results from MC simulation with positive skewness and a slight

TABLE III

Model Comparisons for Asian Option Valuations Under Various Skewness and Kurtosis

Strike L	*Vol. σ*	γ_1	γ_2	*MC*	*E-LB*	*E-UB*	*LT*	*TW*	*PM*
95	0.05	0	3	8.81 (0.00)	8.81	8.81	8.81	8.81	NAN
100	0.05	0	3	4.31 (0.00)	4.31	4.31	4.31	4.31	NAN
105	0.05	0.03	3	0.95 (0.00)	0.95	0.95	0.95	0.95	NAN
95	0.1	0	3	8.91 (0.00)	8.91	8.91	8.91	8.91	NAN
100	0.1	0	3	4.91 (0.00)	4.91	4.91	4.91	4.91	NAN
105	0.1	0.02	3	2.06 (0.00)	2.06	2.06	2.06	2.06	NAN
90	0.3	0	3	14.96 (0.01)	14.97	14.98	15.00	14.91	14.96
100	0.3	0.01	3	8.81 (0.01)	8.80	8.82	8.84	8.78	8.80
110	0.3	0	3	4.68 (0.01)	4.68	4.70	4.69	4.69	4.67
90	0.5	0.01	3	18.14 (0.03)	18.14	18.21	18.13	17.66	18.14
100	0.5	0	3.02	12.98 (0.03)	12.97	13.03	13.00	12.86	12.97
110	0.5	0	3	9.10 (0.03)	9.09	9.16	9.12	9.22	9.07

Note. E-LB and E-UB indicate the lower and the upper bounds from the authors' model with various skewness (γ_1) and kurtosis (γ_2). The approximations of Levy and Turnbull (1992) are represented by LT, and of Turnbull and Wakeman (1991) by TW; MC represents the Monte Carlo estimates in the Levy and Turnbull (1992), and PM represents the four-moment approximation by Posner and Milevsky (1998). The simulations assume the option life $T = 1$ year, the domestic interest rate $r = 0.09$, the time steps $N = 52$, and the initial spot price $S_0 = 100$.

TABLE IV
Model Comparisons for Asian Currency Option Valuations Under Various Skewness and Kurtosis

Strike L	*Vol. σ*	*N*	γ_1	γ_2	*MC*	*E-LB*	*E-UB*	*Levy*	*PM*
1.8	0.3	4	0.15	3	0.0235 (0.00017)	0.0235	0.0235	0.0231	0.0234
1.65	0.3	4	0.15	3.04	0.0517 (0.00017)	0.0517	0.0517	0.0515	0.0516
1.5	0.3	4	0.10	3.01	0.1034 (0.00017)	0.1034	0.1034	0.1038	0.1034
1.35	0.3	4	0.02	3.02	0.1858 (0.00017)	0.1858	0.1858	0.1864	0.1858
1.2	0.3	4	−0.03	3.08	0.2958 (0.00017)	0.2958	0.2958	0.2961	0.2958
1.8	0.3	12	0.03	3	0.0249 (0.00014)	0.0249	0.0250	0.0243	0.0249
1.65	0.3	12	0.03	3.03	0.0540 (0.00014)	0.0540	0.0540	0.0537	0.0538
1.5	0.3	12	0.04	3.01	0.1061 (0.00014)	0.1061	0.1061	0.1067	0.1061
1.35	0.3	12	0.01	3	0.1882 (0.00014)	0.1881	0.1882	0.1890	0.1880
1.2	0.3	12	0	3	0.2966 (0.00014)	0.2965	0.2965	0.2974	0.2967

Note. E-LB and E-UB indicate the lower and the upper bounds from the modified Edgeworth binomial model with various time steps (N), skewness (γ_1), and kurtosis (γ_2). Levy represents the discrete Wilkinson approximation. MC represents the Monte Carlo estimates. In addition, PM represents the four-moment approximation by Posner and Milevsky. The simulations assume the option life $T = 1$ year, the domestic interest rate $r_d = 0.15$, the foreign interest rate $r_f = 0.1$, and the initial spot price $S_0 = 1.5$ as units of domestic currency per unit of foreign currency.

leptokurtic. Under lognormal distribution, when the call is deep out of money, the authors' lower bounds are more accurate than the estimates from all the other methods. The results from MC method are consistent with the authors' lower and upper bounds. Overall, the authors' outcomes are better than those of LT and TW, and similar to PM.

The authors' modified model can be used to price European average rate currency options when $\rho = r_d - r_f$ substitutes for r in Equations (2) and (3). Valuation results are compared with those of Levy (1992), which applies the discrete Wilkinson approximation (Levy) and the MC method, and with those of the four-moment approximation by PM. The impact of the higher moments on the value of the option is explored. Based on Equations (2) and (3), the authors' simulations are constructed with maturity $T = 1$ year, the domestic interest rate $r_d = 0.15$, the foreign interest rate $r_f = 0.1$, and the initial spot price $S_0 = 1.5$, which is in domestic currency per unit of foreign currency. Various skewness and kurtosis are expressed in Table IV.

Using MC estimates as the authors' benchmark, it is found in Table IV that for quarterly averaged options, the authors' valuation results are almost the same under right-skewed and leptokurtic conditions, except in the case of $L = 1.2$.[3] The authors' performance is similar to that of the four-moment method, but superior to the discrete Wilkinson approximation. Meanwhile, for monthly

[3]MC estimates were calculated by averaging 10,000 replications of $\ln\lambda M(t)$. Under the null hypotheses of zero skewness, the asymptotic standard error of skewness with 10,000 replications was 0.0245. See Levy (1992, p. 484).

average Asian options, the authors' results are also similar to those from MC, just as in the quarterly average cases. However, the difference from Levy is rather huge. Thus, the authors' valuation method is more accurate than the discrete Wilkinson approximation. The pricing model of Asian options should emphasize the higher moments when underlying assets have higher volatility.

CONCLUSION

The modified Edgeworth binomial model to price European-style Asian options with higher moments in the underlying return distribution was developed. Specifically, the values of average rate currency options are simulated under various skewness and kurtosis. Combining the Edgeworth approximation and the averaging algorithm by Chalasani et al. (1998), the authors' method is faster and more accurate in the sense that the estimates have a smaller error bound. The numerical results show that this approach can effectively deal with the higher moments of the underlying distribution and provide better option value estimates than those found in various studies in the literature.

APPENDIX A: ANALYTICAL EXPLANATION FOR THE PROPOSITION

That the error bound in approximating $E[V_n^+]$ from a modified Edgeworth binomial tree model and that from a binomial tree model employed by Chalasani et al. (1999) are proportional to their upward probabilities, respectively, in the binomial paths is first shown. For this, a discrete approximation method similar to the lattice approach is used. Let T be the time for expiration of the option. At time T, let $Y_e(T)$ denote the variance of the arithmetic average of the stock prices in the authors' modified Edgeworth binomial tree with upward probabilities p_e, and $Y_c(T)$ denote the variance of the average price in a binomial tree from Chalasani et al. with upward probability p_c. From Equation (2), the asset price in the Edgeworth model is affected by the drift with upward trend, resulting in higher average than the case in Chalasani et al. From Equation (7), higher average price increases the probability of $A^{min}(k, h, a) > L$, i.e. higher probability of zero variance based on explanations in footnote 1. Thus $Y_c(T) \geq Y_e(T)$ is obtained.

Assume for the moment that p_e is less than p_c. At time $t = T/3$, the conditional expectations of the variances with upward probabilities p_e and p_c are given by $E_{3t}[Y_e(T)]$ and $E_{3t}[Y_c(T)]$, respectively. It can be seen (ignoring the discount factors) that

$$E_{3t}[Y_c(T)]^{1/2} \geq E_{3t}[Y_e(T)]^{1/2}$$

because

$$\sum_{h=0}^{3} \sum_{\substack{a=0 \\ A^{max}>L,\, A^{min}<L}}^{2} M(3, h, a) p_c^h (1 - p_c)^{3-h} (var(A_3|a, S_{3,h}))^{1/2}$$

$$\geq \sum_{h=0}^{3} \sum_{\substack{a=0 \\ A^{max}>L,\, A^{min}<L}}^{2} M(3, h, a) p_e^h (1 - p_e)^{3-h} (var(A_3|a, S_{3,h}))^{1/2}.$$

Similarly, at time $t = T/4$,

$$E_{4t}[Y_c(T)]^{1/2} \geq E_{4t}[Y_e(T)]^{1/2}$$

because

$$\sum_{h=0}^{4} \sum_{\substack{a=0 \\ A^{max}>L,\, A^{min}<L}}^{4} M(4, h, a) p_c^h (1 - p_c)^{4-h} (var(A_4|a, S_{4,h}))^{1/2}$$

$$\geq \sum_{h=0}^{4} \sum_{\substack{a=0 \\ A^{max}>L,\, A^{min}<L}}^{4} M(4, h, a) p_e^h (1 - p_e)^{4-h} (var(A_4|a, S_{4,h}))^{1/2}.$$

Therefore, as long as the above inequality continuously holds for all time $t \leq T/5$, the error bound for a tree model with upward probability p_e will be tighter than that for a tree with upward probability p_c, given that p_e is less than p_c.

Next, that the upward probability p_e in an Edgeworth binomial model is indeed less than the upward probability p_c in the model employed by Chalasani et al. (1998) is shown.

As noted in Equation (3), $p_{n,h} = P_h/[n!/h!(n - h)!]$ and $p_{n,h+1} = P_{h+1}/[n!/(h + 1)!(n - (h + 1))!]$, where $P_h = F(y_h)/\Sigma_j F(y_j)$ and the Edgeworth-corrected probability $F(y_h) = f(y_h) \times b(y_h)$, as discussed in Equation (1). The upward probability p_e and the downward probability q_e are defined as follows:

$$p_e = \frac{p_{n,h+1}}{p_{n,h+1} + p_{n,h}} = \frac{1}{1 + (p_{n,h}/p_{n,h+1})} = \frac{1}{1 + (f(x_h)/f(x_{h+1}))}$$

$$q_e = 1 - p_e$$

where $f(x_h)$ denotes an Edgeworth expansion function, and x_h represents the normalized random variable from y_h, i.e. x_h equals $(y_h - M)/V$ with $M = \Sigma_h P_h y_h$, $V^2 = \Sigma_h P_h (y_h - M)^2$, and P_h is the probability distribution. On the basis of p_e and q_e as defined above, two possible cases can be obtained:

(1) If $f(x_h) \geq f(x_{h+1})$, then $0 < p_e \leq 0.5$, and $0.5 \leq q_e < 1$.

(2) If $f(x_h) < f(x_{h+1})$, then $0.5 < p_e < 1$, and $0 < q_e < 0.5$.

In case (1), $P_h \geq P_{h+1}$ can be inferred because $f(x_h) \geq f(x_{h+1})$. Like the argument discussed in the first paragraph in this Appendix, the drift with upward trend in an Edgeworth model will affect the average price of the underlying asset. From the error bound in Equation (7), higher average price increases the probability for $A^{min}(k, h, a) > L$, i.e. higher probability of zero variance.

If $f(x_h) < f(x_{h+1})$ as in case (2), $P_h < P_{h+1}$ is obtained. The underlying asset price in an Edgeworth model is affected by the drift with downward trend. All average stock prices with non-lognormal distributions are smaller than those with lognormal distributions in an Edgeworth model, but higher than those in a binomial tree from Chalasani et al. when the drifts are greater than zero in Equation (2). So the higher average prices still increase the probability of $A^{min}(k, h, a) > L$ and also the probability of zero variance.

In both cases, if the upward probabilities in an Edgeworth model are lower than or equal to those in a binomial tree from Chalasani et al., then, according to Equation (7), the error bound of an Edgewroth model will be tighter than that of a binomial tree from Chalasani et al.

When the underlying asset return exhibits a lognormal distribution, $f(x_h) = f(x_{h+1})$ is obtained. The upward probability in the Edgeworth model is then equal to 0.5 ($p_e = q_e = 0.5$). Meanwhile, the corresponding upward probability, p_c, in the binomial tree model described by Chalasani et al. (1999) is more than 0.5 under $\sigma < (2r)^{0.5}$. Therefore, $p_e < p_c$ is obtained. If $\sigma > (2r)^{0.5}$, then the upward probability in the binomial model by Chalasani et al. is less than 0.5. The higher the volatility of the stock price, the greater the total variance of the average stock price. The error bounds of the binomial model by Chalasani et al. become larger when the upward probability is less than 0.5. Similar to the above cases (1) and (2), an upward probability in the Edgeworth model less than or equal to that of Chalasani et al can be set. The drifts with upward trend then will affect the average stock prices in the Edgeworth model; hence, its error bound is smaller.

As a result, the upward probability in the authors' Edgeworth binomial model is smaller than that in the model employed by Chalasani et al. Hence, the error bound in pricing an Asian option from the authors' modified Edgeworth binomial model should be smaller than that in Chalasani et al.

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