Lyapunov-based Quantitative Analysis of Robust Stability

by HORNG-GIOU CHEN
Institute of Control, National Chiao Tung University, Hsinchu, Taiwan, Republic of China

and KUANG-WEI HAN
Yuan-ze Institute of Technology, Taoyuan, Taiwan, Republic of China

ABSTRACT: *The robust stability of control systems with perturbation is considered. Using Lyapunov functions, quantitative bounds on the perturbation are obtained such that the systems remain stable. Four classes of perturbations are treated and four measures of robust stability are proposed: the nondelay unstructured, delayed unstructured, nondelay structured, and delayed structured measures. Examples are given, and comparisons with the results given in current literature are made.*

Nomenclature

$\mathbb{R}^n$	real n-dimensional vector space
$\mathbb{R}^{n\times n}$	linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$
$[\cdot]^{-1}$	inverse matrix of an invertible matrix $[\cdot]$
$[\cdot]'$	transposed matrix of $[\cdot]$
$[\cdot]^{1/2}$	square-root of positive-definite matrix $[\cdot]$
$[\cdot]_s$	symmetric portion of square matrix $[\cdot]$
$[\cdot]_{ps}$	positive-semidefinite matrix formed by replacing each eigenvalue of the symmetric matrix $[\cdot]$ by its modulus value
$\sigma_{\max}([\cdot])$	maximum singular value of matrix $[\cdot]$
$\sigma_{\min}([\cdot])$	minimum singular value of matrix $[\cdot]$
$\|v\|$	Euclidean norm of vector v
$P > 0$	square symmetric matrix P being positive-definite
$P > Q$	square symmetric matrices P and Q that satisfy $P-Q > 0$

I. Introduction

The inclusion of plant uncertainty and parameter variations to the analysis and design of control systems has been an important problem. The uncertainty, described as unstructured perturbation, often arises from an imperfect knowledge of a system working at the presence of noise and disturbance. The parameter variations, described as structured perturbations, arise if a system is operated in

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widely different regimes. Such a system is termed a *robust system* if stability is preserved under a class of perturbations influencing the system. Presently, there are two approaches that quantitatively define the measures of robust stability: the *frequency domain* approach and the *state-space* approach. In frequency domain approach, the main direction of research is to extend the well-known SISO treatments to MIMO systems. A Bode-type criterion is generalized by use of the singular value decomposition method (**1**, **2**), and a Nyquist-type criterion is generalized by use of the E-contours method (**3**, **4**). The perturbations are mainly viewed in terms of the tolerable gain and phase changes of unstructured perturbations. On the other hand, the state-space approach can be categorized into three perspectives: the rootlocus-based analysis (**5–7**), the integral-inequality analysis (**8**), and the Lyapunov-based analysis (**9–12**). The state-space approach is more amenable to the consideration of the structured perturbations in the form of parameter variations and nonlinearities. This paper is an extension of the Lyapunov-based analysis.

While nondelay perturbations are considered, it has been shown that the rootlocus-based analysis produces better measures of robust stability than the Lyapunov-based analysis (**13**, **14**). Nevertheless, the Lyapunov-based analysis possesses the exclusive feature of accompanying the measures of robust stability with a quadratic Lyapunov function. In addition, this analysis takes an important part in the design of robust stabilizing controllers (**15–17**), and it is extendable to include delayed perturbations. In (**12**), the Lyapunov-based analysis has been extended to include a delay-independent analysis of robust stability for systems with delayed unstructured perturbations. However, a delay-independent criterion that does not include the information on delays is conservative especially when the delays are small. The motivation of making an extension to include delay-dependent analysis of robust stability for control systems with delayed perturbations guides us to do the research. It should be noted that, while delayed perturbations are considered, the rootlocus-based analysis fails and the integral-inequality analysis produces very conservative result.

In this paper, six theorems are presented to provide measures of robust stability for linear control systems with state-space models. Four classes of perturbations are considered: nondelay unstructured, delayed unstructured, nondelay structured, and delayed structured perturbations. Examples are given and comparisons with the results given in current literature are made.

II. Problem Formulations

In order to study the robust stability of linear control systems, we consider four measures of robust stability defined by the following definitions.

Definition I (measure of allowable nondelay unstructured perturbation)

Consider the perturbed closed-loop system described by

$$\mathrm{d}x(t)/\mathrm{d}t = Ax(t)+f(x(t),t), \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is the stable system matrix and $f(x(t), t) \in \mathbb{R}^n$ is a bounded perturbing function with $f(0, t) = 0$ for all time t. If stability of the perturbed system (1) is guaranteed for all bounded perturbations with perturbing magnitude given by

$$\max_{\forall(x \neq 0, t)} \left\{ \frac{\|f(x(t), t)\|}{\|x(t)\|} \right\} < \mu_n, \tag{2}$$

then μ_n is a nondelay unstructured robustness measure of the system with matrix A.

Definition II (measure of allowable nondelay structured perturbation)
Consider the perturbed closed-loop system described by

$$dx(t)/dt = \left[A + \eta_n \sum_{i=1}^{p} r_i(t) \Delta E_i \right] x(t) \tag{3}$$

where $A \in \mathbb{R}^{n \times n}$ is the stable system matrix, $\Delta E_i \in \mathbb{R}^{n \times n}$ are the perturbing matrices, and $r_i(t)$ are bounded scalar functions with $|r_i(t)| < 1$ for all time t. If the stability of the perturbed system (3) is guaranteed, then η_n is a nondelay structured robustness measure of the system with matrix A.

Definition III (measure of allowable delayed unstructured perturbation)
Consider the perturbed closed-loop system described by

$$dx(t)/dt = Ax(t) + f(x(t-\tau), t), \tag{4}$$

where $A \in \mathbb{R}^{n \times n}$ is the stable system matrix, $\tau > 0$ is any time-varying time delay, and $f(x(t-\tau), t) \in \mathbb{R}^n$ is a bounded perturbing function with $f(0, t) = 0$ for all time t. If stability of the perturbed system (4) is guaranteed for all bounded perturbations with perturbing magnitude given by

$$\max_{\forall(x \neq 0, t)} \left\{ \frac{\|f(x(t-\tau), t)\|}{\|x(t-\tau)\|} \right\} < \mu_d, \tag{5}$$

then μ_d is a delayed unstructured robustness measure of the system with matrix A.

Definition IV (measure of allowable delayed structured perturbation)
Consider the perturbed closed-loop system described by

$$dx(t)/dt = Ax(t) + \eta_d \sum_{i=1}^{p} r_i(t) \Delta E_i x(t-\tau), \tag{6}$$

where $A \in \mathbb{R}^{n \times n}$ is the stable system matrix, $\tau > 0$ is any time-varying time delay, $\Delta E_i \in \mathbb{R}^{n \times n}$ are the perturbing matrices, and $r_i(t)$ are bounded scalar functions with $|r_i(t)| < 1$ for all time t. If the stability of the perturbed system (6) is guaranteed, then η_d is a delayed structured robustness measure of the system with matrix A.

Remark I
Using the Lyapunov-based analysis of robust stability, two delayed unstructured robustness measures will be obtained. The measure which does not include infor-

mation on the delay time (τ) is called the *delay-independent measure* and is denoted by $\mu_d(\infty)$. On the other hand, the measure which carries information on the delay time is called the *delay-dependent measure* and is denoted by $\mu_d(\tau)$. Since the delay-independent measure is conservative especially when the delay time is small, the delay-dependent measure is used to complement the delay-independent measure such that

$$\mu_d = \max\{\mu_d(\infty), \mu_d(\tau)\}. \tag{7}$$

Obviously, we have

$$\mu_d(\tau = 0) = \mu_n, \tag{8}$$

where μ_n is the nondelay unstructured measure defined in Definition I.

Remark II

Similarly, two delayed structured robustness measures will be obtained: the delay-independent measure denoted by $\eta_d(\infty)$, and the delay-dependent measure denoted by $\eta_d(\tau)$. The delay-dependent measure is used to complement the delay-independent measure such that

$$\eta_d = \max\{\eta_d(\infty), \eta_d(\tau)\}. \tag{9}$$

We also have

$$\eta_d(\tau = 0) = \eta_n, \tag{10}$$

where η_n is the nondelay structured measure defined in Definition II.

III. Main Results

Suppose that a matrix $Q > 0$ has been selected, and a matrix $P > 0$ is determined by solving the Lyapunov equation, i.e.

$$A'P + PA = -2Q. \tag{11}$$

Given matrices P and Q that fulfil the Lyapunov equation (11), the measures of robust stability are proposed in the following six theorems.

Theorem I

The nondelay unstructured robustness measure of the system with matrix A is given by

$$\mu_n \equiv \frac{1}{\max\limits_{\forall(\|y\| = 1)} \{\|PQ^{-1/2}y\| \|Q^{-1/2}y\|\}}. \tag{12}$$

Proof: Select the Lyapunov function as

$$V(x(t)) = x(t)'Px(t), \tag{13}$$

where $P > 0$ is the Lyapunov matrix given in Eq. (11). A sufficient condition for the stability of the perturbed system (1) is

$$(Ax(t)+f(x(t),t))'Px(t)+x(t)'P(Ax(t)+f(x(t),t)) < 0. \tag{14}$$

Followed from the Lyapunov Eq. (11), we have

$$f(x(t),t)'Px(t) < x(t)'Qx(t). \tag{15}$$

This relation is sufficiently justified by

$$\|f(x(t),t)\|\,\|Px(t)\| < x(t)'Qx(t), \tag{16}$$

and by

$$\max_{\forall(x\neq 0,t)}\left\{\frac{\|f(x(t),t)\|}{\|x(t)\|}\right\} < \min_{\forall(x\neq 0,t)}\left\{\frac{x(t)'Qx(t)}{\|Px(t)\|\,\|x(t)\|}\right\} \equiv \mu_n. \tag{17}$$

Making the substitution

$$x(t) = Q^{-1/2}y, \tag{18}$$

the relation given in Eq. (17) becomes

$$\mu_n \equiv \min_{\forall(y\neq 0)}\left\{\frac{\|y\|^2}{\|PQ^{-1/2}y\|\,\|Q^{-1/2}y\|}\right\}. \tag{19}$$

Thus, the nondelay unstructured robustness measure of the system with matrix A is given by Eq. (12). ■

Theorem II

The nondelay structured robustness measure of the system with matrix A is given by

$$\eta_n \equiv \frac{1}{\sigma_{\max}\left(\sum_{i=1}^{p}[P_i]_{ps}\right)}, \tag{20}$$

where matrices P_i are defined by algebraic manipulations of the perturbation matrices ΔE_i given in Eq. (3) with the Lyapunov matrices P and Q given in the Lyapunov equation (11), i.e.

$$P_i \equiv Q^{-1/2}[P\Delta E_i]_s Q^{-1/2}, \quad i = 1,2,\ldots,p. \tag{21}$$

Proof: Select the Lyapunov function as $V(x(t)) = x(t)'Px(t)$, we will show that it is also a Lyapunov function of the structurally perturbed system (3) with η_n given in (20).

A simple computation shows that

$$\mathrm{d}V(x(t))/\mathrm{d}t = 2x(t)'Q^{1/2}\left(\eta_n \sum_{i=1}^{p} r_i(t)P_i - I\right)Q^{1/2}x(t). \tag{22}$$

It is clear that $\mathrm{d}V(t)/\mathrm{d}t < 0$ if

$$\eta_n \sigma_{\max}\left(\sum_{i=1}^{p} r_i(t)P_i\right) < 1. \tag{23}$$

Note that, for all $i = 1, 2, \ldots, p$ and for all $|r_i(t)| < 1$, we have

$$[P_i]_{ps} - r_i(t)P_i > 0. \tag{24}$$

Thus,

$$\sum_{i=1}^{p} [P_i]_{ps} > \sum_{i=1}^{p} r_i(t)P_i, \tag{25}$$

and

$$\sigma_{\max}\left(\sum_{i=1}^{p} [P_i]_{ps}\right) > \sigma_{\max}\left(\sum_{i=1}^{p} r_i(t)P_i\right). \tag{26}$$

Hence, Eq. (20) implies Eq. (23), and the nondelay structured robustness measure of the system with matrix A is given by Eq. (20). ■

Theorem III

The delay-independent delayed unstructured robustness measure of the system with matrix A is given by

$$\mu_d(\infty) \equiv \frac{\sigma_{\min}(P^{1/2})}{\max\limits_{\forall(\|y\|=1)} \{\|PQ^{-1/2}y\| \, \|P^{1/2}Q^{-1/2}y\|\}}. \tag{27}$$

Proof: Select the Lyapunov function as $V(x(t)) = x(t)'Px(t)$, where $P > 0$ is the Lyapunov matrix given in Eq. (11). The following inequality is assumed to hold for all time s and delay $h > 0$:

$$q^2 V(x(s)) > V(x(s-h)). \tag{28}$$

A sufficient condition for the stability of the perturbed system (4) is given by the Razumikhin theorem [(**18**), p. 127] as, for sufficiently small $q(q > 1)$,

$$(Ax(t) + f(x(t-\tau), t))'Px(t) + x(t)'P(Ax(t) + f(x(t-\tau), t)) < 0. \tag{29}$$

From the Lyapunov equation (11), we have

$$f(x(t-\tau), t)'Px(t) < x(t)'Qx(t). \tag{30}$$

This relation is sufficiently justified by

$$\|f(x(t-\tau), t)\| \, \|Px(t)\| < x(t)'Qx(t) \tag{31}$$

and by

$$\max_{\forall(x \neq 0, t)} \left\{\frac{\|f(x(t-\tau), t)\|}{\|x(t-\tau)\|}\right\} < \min_{\forall(x \neq 0, t)} \left\{\frac{x(t)'Qx(t)}{\|Px(t)\| \, \|x(t-\tau)\|}\right\}. \tag{32}$$

From the inequality given in Eq. (28), we have

$$q\,\|P^{1/2}x(t)\| > \sigma_{\min}(P^{1/2})\,\|x(t-\tau)\|. \tag{33}$$

Thus, as the parameter q approaches 1, relation (32) is sufficiently justified by

$$\max_{\forall(x\neq 0,t)}\left\{\frac{\|f(x(t-\tau),t)\|}{\|x(t-\tau)\|}\right\} < \sigma_{\min}(P^{1/2})\min_{\forall(x\neq 0,t)}\left\{\frac{x(t)'Qx(t)}{\|Px(t)\|\,\|P^{1/2}x(t)\|}\right\} \equiv \mu_d(\infty). \tag{34}$$

Making the substitution

$$y = Q^{1/2}x(t), \tag{35}$$

relation (34) becomes

$$\mu_d(\infty) \equiv \sigma_{\min}(P^{1/2})\min_{\forall(y\neq 0)}\left\{\frac{\|y\|^2}{\|PQ^{-1/2}y\|\,\|P^{1/2}Q^{-1/2}y\|}\right\}. \tag{36}$$

Thus, the delayed unstructured robustness measure of the system with matrix A is given by Eq. (27). ■

Theorem IV

The delay-dependent delayed unstructured robustness measure of the system with matrix A is given by $\mu_d(\tau) > 0$ that fulfills

$$\frac{1}{\mu_n} + \tau\sigma_{\min}(P^{1/2})\sigma_{\max}(AP^{-1/2})\frac{1}{\mu_d(\infty)} + \tau\mu_d(\tau)\frac{1}{\mu_d(\infty)} = \frac{1}{\mu_d(\tau)}, \tag{37}$$

where μ_n is the nondelay unstructured robustness measure given in Eq. (12), and $\mu_d(\infty)$ is the delay-independent delayed unstructured robustness measure given in Eq. (27).

Proof: Knowing that

$$x(t-\tau) = x(t) - \int_{t-\tau}^{t} \dot{x}(s)\,\mathrm{d}s = x(t) - \int_{t-\tau}^{t} \{Ax(s) + f(x(s-\tau),s)\}\,\mathrm{d}s, \tag{38}$$

we have

$$\begin{aligned}\|x(t-\tau)\| &< \|x(t)\| + \int_{t-\tau}^{t}\|Ax(s)\|\,\mathrm{d}s + \int_{t-\tau}^{t}\|f(x(s-\tau),s)\|\,\mathrm{d}s\\ &\leqslant \|x(t)\| + \sigma_{\max}(AP^{-1/2})\int_{t-\tau}^{t}\|P^{1/2}x(s)\|\,\mathrm{d}s\\ &\quad + \int_{t-\tau}^{t}\frac{\|f(x(s-\tau),s)\|}{\|x(s-\tau)\|}\,\|x(s-\tau)\|\,\mathrm{d}s, \end{aligned} \tag{39}$$

where $P > 0$ is the Lyapunov matrix given in Eq. (11). Select the Lyapunov function as $V(x(t)) = x(t)'Px(t)$; the following inequality is assumed to hold for all time s and delay $h > 0$:

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$$q^2 V(x(s)) > V(x(s-h)). \tag{40}$$

For all $s < t$, we have

$$q\|P^{1/2}x(t)\| > \|P^{1/2}x(s)\|, \tag{41}$$

and

$$q\|P^{1/2}x(t)\| > \sigma_{\min}(P^{1/2})\|x(s-\tau)\|. \tag{42}$$

Thus, relation (39) becomes

$$\|x(t-\tau)\| < \|x(t)\| + q\tau\sigma_{\max}(AP^{-1/2})\|P^{1/2}x(t)\| + q\tau\frac{\|P^{1/2}x(t)\|}{\sigma_{\min}(P^{1/2})}\max_{\forall(x\neq 0,t)}\left\{\frac{\|f(x(t-\tau),t)\|}{\|x(t-\tau)\|}\right\}. \tag{43}$$

A sufficient condition for the stability of the perturbed system (4) is given by the Razumikhin theorem as, for sufficiently small q ($q > 1$),

$$(Ax(t)+f(x(t-\tau),t))'Px(t)+x(t)'P(Ax(t)+f(x(t-\tau),t)) < 0. \tag{44}$$

From the Lyapunov Eq. (11), we have

$$f(x(t-\tau),t)'Px(t) < x(t)'Qx(t). \tag{45}$$

This relation is sufficiently justified by

$$\|f(x(t-\tau),t)\|\,\|Px(t)\| < x(t)'Qx(t). \tag{46}$$

Thus, for all $x \neq 0$ and for all time t, we need to fulfil

$$\frac{\|f(x(t-\tau),t)\|}{\|x(t-\tau)\|} < \frac{x(t)'Qx(t)}{\|Px(t)\|\,\|x(t-\tau)\|}. \tag{47}$$

Insert the result given in relation (43), the condition given in Eq. (47) is sufficiently justified by

$$\max_{\forall(x\neq 0,t)}\left\{\frac{\|f(x(t-\tau),t)\|}{\|x(t-\tau)\|}\right\} < \frac{1}{\max_{\forall(x\neq 0,t)}\{M(q,\tau,x(t))\}}, \tag{48}$$

where

$$M(q,\tau,x(t)) \equiv \frac{\|Px(t)\|\,\|x(t)\|}{x(t)'Qx(t)} + q\tau\sigma_{\max}(AP^{-1/2})\frac{\|Px(t)\|\,\|P^{1/2}x(t)\|}{x(t)'Qx(t)} + \frac{q\tau}{\sigma_{\min}(P^{1/2})}\max_{\forall(x\neq 0,t)}\left\{\frac{\|f(x(t-\tau),t)\|}{\|x(t-\tau)\|}\right\}\frac{\|Px(t)\|\,\|P^{1/2}x(t)\|}{x(t)'Qx(t)}. \tag{49}$$

Given the definitions of μ_n in Eq. (17) and $\mu_d(\infty)$ in Eq. (34), and as the parameter q approaches 1, we have

$$\max_{\forall(x\neq 0,t)}\{M(q,\tau,x(t))\}\leqslant\frac{1}{\mu_n}+\tau\sigma_{\min}(P^{1/2})\sigma_{\max}(AP^{-1/2})\frac{1}{\mu_d(\infty)}+\tau\max_{\forall(x\neq 0,t)}\left\{\frac{\|f(x(t-\tau),t)\|}{\|x(t-\tau)\|}\right\}\frac{1}{\mu_d(\infty)}. \tag{50}$$

From Eq. (48), a sufficient condition for the stability of the perturbed system (4) is given by

$$\max_{\forall(x\neq 0,t)}\left\{\frac{\|f(x(t-\tau),t)\|}{\|x(t-\tau)\|}\right\}<\mu_d(\tau), \tag{51}$$

where $\mu_d(\tau)>0$ and fulfills

$$\frac{1}{\mu_n}+\tau\sigma_{\min}(P^{1/2})\sigma_{\max}(AP^{-1/2})\frac{1}{\mu_d(\infty)}+\tau\mu_d(\tau)\frac{1}{\mu_d(\infty)}=\frac{1}{\mu_d(\tau)}. \tag{52}$$

Thus, the delayed unstructured robustness measure of the system with matrix A is given by Eq. (37). ■

Theorem V

The delay-independent delayed structured robustness measure of the system with matrix A is given by

$$\eta_d(\infty)\equiv\frac{1}{\sum_{i=1}^{p}\max_{\forall(\|y\|=1)}\{\|P^{1/2}Q^{-1/2}y\|\|P^{-1/2}\Delta E_i'PQ^{-1/2}y\|\}}. \tag{53}$$

Proof: Select the Lyapunov function as $V(x(t))=x(t)'Px(t)$, where $P>0$ is the Lyapunov matrix given in Eq. (11). The following inequality is assumed to hold for all time s and delay $h>0$:

$$q^2V(x(s))>V(x(s-h)). \tag{54}$$

A sufficient condition for the stability of the perturbed system (6) is given by the Razumikhin theorem as, for sufficiently small q ($q>1$),

$$x(t)'\eta_d(\infty)P\sum_{i=1}^{p}r_i(t)\Delta E_ix(t-\tau)<x(t)'Qx(t). \tag{55}$$

From the inequality given in Eq. (54), we have

$$q\|P^{1/2}x(t)\|>\|P^{1/2}x(t-\tau)\|. \tag{56}$$

Since

$$x(t)'\eta_d(\infty)P\sum_{i=1}^{p} r_i(t)\Delta E_i x(t-\tau) \leqslant \eta_d(\infty)\sum_{i=1}^{p} |x(t)'P\Delta E_i x(t-\tau)|$$

$$\leqslant \eta_d(\infty)\sum_{i=1}^{p} \|P^{-1/2}\Delta E_i' P x(t)\|\,\|P^{1/2}x(t-\tau)\|, \quad (57)$$

the relation (55) is sufficiently justified by

$$x(t)'Qx(t) > q\eta_d(\infty)\|P^{1/2}x(t)\|\sum_{i=1}^{p}\|P^{-1/2}\Delta E_i' P x(t)\|. \quad (58)$$

As the parameter q approaches 1, we need to fulfil

$$\frac{1}{\eta_d(\infty)} > \sum_{i=1}^{p}\frac{\|P^{1/2}x(t)\|\,\|P^{-1/2}\Delta E_i' P x(t)\|}{x(t)'Qx(t)}. \quad (59)$$

Making the replacement of

$$y = Q^{1/2}x(t), \quad (60)$$

relation (59) becomes

$$\frac{1}{\eta_d(\infty)} > \sum_{i=1}^{p}\frac{\|P^{1/2}Q^{-1/2}y\|\,\|P^{-1/2}\Delta E_i' P Q^{-1/2}y\|}{\|y\|^2}. \quad (61)$$

Thus, delayed structured robustness measure of the system with matrix A is given by Eq. (53). ■

Theorem VI

The delay-dependent delayed structured robustness measure of the system with matrix A is given by $\eta_d(\tau) > 0$ that fulfills

$$\frac{1}{\eta_n} + \tau\frac{1}{\eta_{ea}} + \tau\eta_d(\tau)\frac{1}{\eta_{ee}} = \frac{1}{\eta_d(\tau)}, \quad (62)$$

where η_n is the nondelay structured robustness measure given in Eq. (20), and

$$\eta_{ea} \equiv \frac{1}{\sum_{i=1}^{p}\max_{\forall(\|y\|=1)}\{\|P^{1/2}Q^{-1/2}y\|\,\|P^{-1/2}A'\Delta E_i' P Q^{-1/2}y\|\}}, \quad (63)$$

$$\eta_{ee} \equiv \frac{1}{\sum_{j=1}^{p}\sum_{i=1}^{p}\max_{\forall(\|y\|=1)}\{\|P^{1/2}Q^{-1/2}y\|\,\|P^{-1/2}\Delta E_j'\Delta E_i' P Q^{-1/2}y\|\}}. \quad (64)$$

Proof: We know that

$$x(t-\tau) = x(t) - \int_{t-\tau}^{t} \dot{x}(s)\,\mathrm{d}s$$

$$= x(t) - \int_{t-\tau}^{t} \left\{ Ax(s) + \eta_d(\tau) \sum_{i=1}^{p} r_i(s)\Delta E_i x(s-\tau) \right\} \mathrm{d}s, \quad (65)$$

and that the following inequality is assumed to hold for all time s and delay $h > 0$:

$$q^2 V(x(s)) > V(x(s-h)), \quad (66)$$

where $P > 0$ is the Lyapunov matrix given in Eq. (11) and the Lyapunov function is selected as $V(x(t)) = x(t)'Px(t)$.

A sufficient condition for the stability of the perturbed system (6) is given by the Razumikhin theorem as, for sufficiently small q ($q > 1$),

$$x(t)'\eta_d(\tau)P \sum_{i=1}^{p} r_i(t)\Delta E_i x(t-\tau) < x(t)'Qx(t). \quad (67)$$

Inserting Eq. (65), and defining $P_i \equiv Q^{-1/2}[P\Delta E_i]_s Q^{-1/2}$ we have

$$\frac{1}{\eta_d(\tau)} x(t)'Qx(t) > \sum_{i=1}^{p} r_i(t)x(t)'Q^{1/2}P_iQ^{1/2}x(t) - \sum_{i=1}^{p} r_i(t)x(t)'P\Delta E_i \int_{t-\tau}^{t} Ax(s)\,\mathrm{d}s$$

$$- \sum_{i=1}^{p} r_i(t)x(t)'P\Delta E_i \int_{t-\tau}^{t} \left\{ \eta_d(\tau) \sum_{j=1}^{p} r_j(s)\Delta E_j x(s-\tau) \right\} \mathrm{d}s. \quad (68)$$

Note that, for all $i = 1, 2, \ldots, p$ and for all $|r_i(t)| < 1$,

$$[P_i]_{ps} - r_i(t)P_i > 0. \quad (69)$$

Thus, we have

$$\sum_{i=1}^{p} r_i(t)x(t)'Q^{1/2}P_iQ^{1/2}x(t) < \sum_{i=1}^{p} x(t)'Q^{1/2}[P_i]_{ps}Q^{1/2}x(t). \quad (70)$$

From the inequality given in Eq. (66), we get

$$q\|P^{1/2}x(t)\| > \|P^{1/2}x(s-\tau)\|, \quad (71)$$

and

$$q\|P^{1/2}x(t)\| > \|P^{1/2}x(s)\|. \quad (72)$$

Thus, we also have

$$-\sum_{i=1}^{p} r_i(t)x(t)'P\Delta E_i \int_{t-\tau}^{t} Ax(s)\,\mathrm{d}s \leqslant \sum_{i=1}^{p} \left| \int_{t-\tau}^{t} x(t)'P\Delta E_i Ax(s)\,\mathrm{d}s \right|$$

$$\leqslant \sum_{i=1}^{p} \int_{t-\tau}^{t} \|P^{-1/2}A'\Delta E_i'Px(t)\|\ \|P^{1/2}x(s)\|\,\mathrm{d}s$$

$$< q\tau \sum_{i=1}^{p} \|P^{-1/2}A'\Delta E_i'Px(t)\|\ \|P^{1/2}x(t)\|, \quad (73)$$

and

$$-\sum_{i=1}^{p} r_i(t)x(t)'P\Delta E_i \int_{t-\tau}^{t} \left\{ \eta_d(\tau) \sum_{j=1}^{p} r_j(s)\Delta E_j x(s-\tau) \right\} \mathrm{d}s$$

$$\leqslant \eta_d(\tau) \sum_{i=1}^{p} \sum_{j=1}^{p} \int_{t-\tau}^{t} |x(t)'P\Delta E_i \Delta E_j x(s-\tau)| \, \mathrm{d}s,$$

$$\leqslant \eta_d(\tau) \sum_{i=1}^{p} \sum_{j=1}^{p} \int_{t-\tau}^{t} \|P^{-1/2}\Delta E_j' \Delta E_i' P x(t)\| \, \|P^{1/2}x(s-\tau)\| \, \mathrm{d}s,$$

$$< q\tau\eta_d(\tau) \sum_{i=1}^{p} \sum_{j=1}^{p} \|P^{-1/2}\Delta E_j' \Delta E_i' P x(t)\| \, \|P^{1/2}x(t)\|. \tag{74}$$

Thus, relation (68) is sufficiently justified by

$$\frac{1}{\eta_d(\tau)} x(t)'Qx(t) > \sum_{i=1}^{p} x(t)'Q^{1/2}[P_i]_{ps}Q^{1/2}x(t)$$

$$+q\tau \sum_{i=1}^{p} \|P^{-1/2}A'\Delta E_i' P x(t)\| \, \|P^{1/2}x(t)\|$$

$$+q\tau\eta_d(\tau) \sum_{i=1}^{p} \sum_{j=1}^{p} \|P^{-1/2}\Delta E_j' \Delta E_i' P x(t)\| \, \|P^{1/2}x(t)\|. \tag{75}$$

Making the substitution

$$y = Q^{1/2}x(t), \tag{76}$$

and as the parameter q approaches 1, relation (75) is sufficiently justified by

$$\frac{\|y\|^2}{\eta_d(\tau)} > \sigma_{\max}\left(\sum_{i=1}^{p} [P_i]_{ps}\right)\|y\|^2 + \tau \sum_{i=1}^{p} \|P^{-1/2}A'E_i'PQ^{-1/2}y\| \, \|P^{1/2}Q^{-1/2}y\|$$

$$+\tau\eta_d(\tau) \sum_{i=1}^{p} \sum_{j=1}^{p} \|P^{-1/2}E_j'E_i'PQ^{-1/2}y\| \, \|P^{1/2}Q^{-1/2}y\|. \tag{77}$$

Given the definition of η_n in Eq. (20), the delayed structured robustness measure of the system with matrix A is given by Eq. (62). ■

IV. Examples

Given matrices P and Q that fulfil the Lyapunov equation (11), measures of robust stability can be determined. The Lyapunov-based robustness-measure problem demands a judicious choice of the matrix Q such that less conservative measures can be achieved. In comparison with the results given in current literature, the proposed measures are investigated in the following examples.

Remark III

If an upper bound on the delay time (τ) is available, the measures of robust stability for delayed perturbations given by applying Theorem III and Theorem V can be improved respectively by using Theorem IV and Theorem VI. In the sequel, only the delay-independent measures ($\mu_d(\infty)$, $\eta_d(\infty)$) are evaluated for delayed perturbations.

Example I

Consider a system with nominal system matrix given by

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}. \tag{78}$$

For nondelay unstructured perturbation, the rootlocus-based analysis given in (**14**) produces the exact bound of robust stability of $\mu_n = 0.5402$. In (**9**), while the matrix $Q = I_2$ is selected, Patel and Toda provides a Lyapunov-based measure of $\mu_n = 0.3820$. Here, we obtain a measure of $\mu_n = 0.4842$ using Theorem I with

$$Q = \begin{bmatrix} 5.2361 & 2.6180 \\ 2.6180 & 2.6180 \end{bmatrix}, \tag{79}$$

where matrix Q is obtained by use of the algorithm given in the Appendix (**19**). As far as Lyapunov-based methods are concerned, Theorem I provides a less conservative measure of robust stability for nondelay unstructured perturbation.

For delayed unstructured perturbation, since the matrix measure of A given in Eq. (78) is positive, the integral-inequality analysis fails to achieve a bound of robust stability. In (**12**), Cheres *et al.* produce a Lyapunov-based measure of $\mu_d(\infty) = 0.178$. Here, we obtain a measure of $\mu_d(\infty) = 0.3842$ using Theorem III with the matrix Q given in Eq. (79). Thus, Theorem III provides a less conservative measure of robust stability for delayed unstructured perturbation.

Example II

Consider a system with nominal system matrix given by

$$A = \begin{bmatrix} 79.0 & 20.0 & -30.0 & -20.0 \\ -41.0 & -12.0 & 17.0 & 13.0 \\ 167.0 & 40.0 & -60.0 & -38.0 \\ 33.5 & 9.0 & -14.5 & -11.0 \end{bmatrix}. \tag{80}$$

In (**7**), the rootlocus-based analysis produces the measure of robust stability of $\mu_n = 0.08234$ for "nonlinear" nondelay unstructured perturbation. Here, we obtain a measure of $\mu_n = 0.07007$ using Theorem I with

$$Q = \begin{bmatrix} 824.1 & 247.7 & -298.4 & -227.1 \\ 247.7 & 77.19 & -89.62 & -67.94 \\ -298.4 & -89.62 & 109.2 & 82.30 \\ -227.1 & -67.94 & 82.30 & 64.66 \end{bmatrix}, \tag{81}$$

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where matrix Q is obtained by use of the algorithm given in the Appendix. Note that, for nondelay unstructured perturbation, the result of applying Theorem I is fairly close to the result given by use of the rootlocus-based analysis. Note also that, the Lyapunov-based analysis can be applied to a wider class of unstructured perturbations. Using Theorem III with the matrix Q given in Eq. (81), we obtain a measure of $\mu_d(\infty) = 0.03220$ for delayed unstructured perturbation.

Example III

Consider the following matrices:

$$A = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix}, \tag{82}$$

$$\Delta E_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \tag{83}$$

and

$$\Delta E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{84}$$

For nondelay structured perturbation, the rootlocus-based analysis given in (**13**) produces the measure of robust stability of $\eta_n = 1.7500$. In (**11**), while the matrix $Q = I_3$ is selected, Zhou and Khargonekar provide a Lyapunov-based measure of $\eta_n = 1.5533$. Here, we obtain a better measure of $\eta_n = 1.6894$ using Theorem II with

$$Q = \begin{bmatrix} 1.2789 & 0.0567 & 0.6383 \\ 0.0567 & 1.9710 & 0.3509 \\ 0.6383 & 0.3509 & 2.5118 \end{bmatrix}, \tag{85}$$

where matrix Q is obtained by use of the algorithm given in the Appendix. As far as Lyapunov-based methods are concerned, Theorem II provides a less conservative measure of robust stability for nondelay structured perturbation. Note also that the result of applying Theorem II is fairly close to the result given by use of the rootlocus-based analysis.

For delayed structured perturbation, by use of the integral-inequality analysis proposed by Wang *et al.* (**8**), a measure of robust stability of $\eta_d(\infty) = 0.4566$ is obtained. Here, we obtain a measure of $\mu_d(\infty) = 0.8753$ using Theorem V with the matrix Q given in Eq. (85). Thus, Theorem V provides a less conservative measure of robust stability for delayed structured perturbation.

V. Conclusions

In this paper, Lyapunov-based analysis of robust stability has been extended to include four classes of perturbations: the nondelay unstructured, delayed unstructured, nondelay structured, and delayed structured perturbations. Both delay-independent analysis and delay-dependent analysis have been presented for systems with delayed perturbations. As illustrated in the examples, the allowable perturbation bounds obtained by use of the proposed theorems are less conservative than those given in current literature.

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Appendix

Algorithm [computes the matrix Q for Lyapunov equation (**19**)]
Step 1. Assign

$$Q_1 = I, \quad \text{and} \quad Q_2 = I. \tag{A1}$$

Step 2. Equate Lyapunov equations to acquire the matrices P_1 and P_2, i.e.

$$A'P_1 + P_1A + 2Q_1 = 0, \tag{A2}$$

and

$$P_2A' + AP_2 + 2Q_2 = 0. \tag{A3}$$

Step 3. Equate matrices Y_1 and Y_2, where

$$Y_1 = aQ_1 + bP_2^{-1}Q_2P_2^{-1}, \tag{A4}$$

and

$$Y_2 = cQ_2 + dP_1^{-1}Q_1P_1^{-1}, \tag{A5}$$

and the parameters a, b, c and d are chosen such that

$$\sigma_{\max}(aQ_1)\sigma_{\min}(aQ_1) = 1, \tag{A6}$$

$$\sigma_{\max}(bP_2^{-1}Q_2P_2^{-1})\sigma_{\min}(bP_2^{-1}Q_2P_2^{-1}) = 1, \tag{A7}$$

$$\sigma_{\max}(cQ_2)\sigma_{\min}(cQ_2) = 1, \tag{A8}$$

$$\sigma_{\max}(dP_1^{-1}Q_1P_1^{-1})\sigma_{\min}(dP_1^{-1}Q_1P_1^{-1}) = 1. \tag{A9}$$

Step 4. Make the replacement of

$$Q_1 = Y_1/\sigma_{\min}(Y_1), \quad \text{and} \quad Q_2 = Y_2/\sigma_{\min}(Y_2); \tag{A10}$$

and repeat from Step 2 until a convergent condition is detected. Then, we have $Q = Q_1$.