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ON COUPLED INTEGRAL H-LIKE EQUATIONS OF CHANDRASEKHAR*

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Abstract. A recently proposed "simple transport model" equation with an "angular shift" α $(0 \le \alpha \le 1)$ leads to a coupled integral H-like equation of Chandrasekhar. Such coupled H-like equations can be treated in terms of a one-parameter (k_1, $0 < k_1 < 1$) family. From there an a priori bound can be obtained, which is independent of k_1, α, and c $(0 \le c \le 1)$. Here c denotes the average total number of particles emerging from a collision. Consequently, we conclude that positive solutions of such coupled integral H-like equations exist. Moreover, we show that such equations have a unique positive solution pair for $c = 0$ or $c = 1$ and $\alpha = 0$ or $\alpha = 1$, and that the equations have exactly two positive solution pairs for $0 < c < 1$ and $0 \le \alpha < 1$ or $c = 1$ and α sufficiently close to 1.

Key words. integral equation, H-like functions of Chandrasekhar, a priori bound, existence and multiplicity

AMS subject classifications. 45G10, 82C70, 85A25

1. Introduction. In this work we study the coupled integral H-like equations of the form

$$H_1(\mu) = 1 + \frac{c}{2} H_1(\mu)(\mu + \alpha) \int_{\alpha}^{1} \frac{H_2(\mu'')}{\mu + \mu''} \, d\mu'', \; -\alpha \le \mu \le 1, \tag{1a}$$

and

$$H_2(\mu') = 1 + \frac{c}{2} H_2(\mu')(\mu' - \alpha) \int_{-\alpha}^{1} \frac{H_1(\mu'')}{\mu' + \mu''} \, d\mu'', \; \alpha \le \mu' \le 1. \tag{1b}$$

Here c denotes the average total number of particles emerging from a collision, which is assumed to be conservative, i.e., $c \le 1$, and α denotes an "angular shift" with $0 \le \alpha \le 1$. Equation (1) first appeared in [8], where it was derived from a "simple transport model" (see, e.g., [5], [8]) using Chandrasekhar's method of solution. For $\alpha = 0$, equation (1) reduces to Chandrasekhar's well-known integral equation. Various methods (see, e.g., [1]–[4], [6], [7], [10], [11]) have been applied to such equations. In summary, they have shown that Chandrasekhar's integral equation has one solution if $c = 1$ and at most two solutions if $c < 1$.

In this article, we first show that an a priori bound, which is independent of c and α, can be obtained by introducing a one-parameter $(k_1, 0 < k_1 < 1)$ family. Therefore, the degree theory is applied to show the existence of positive solutions. Second, the techniques used in [6], [10] are generalized to show that equation (1) has a unique positive solution pair for $c = 0$ or $c = 1$ and $\alpha = 0$ or $\alpha = 1$, and that equation (1) has exactly two positive solution pairs for $0 < c < 1$ and $0 \le \alpha < 1$ or $c = 1$ and α sufficiently close to 1. The above results are contained in §2.

We conclude this introductory section by noting that using the solutions obtained by equation (1), the simple transport model can then be treated as a "pure" initial

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value problem. More precisely, consider the following simple transport model:

$$(\mu+\alpha)\frac{\partial\phi(x,\mu)}{\partial x}+\phi(x,\mu)=\frac{c}{2}\int_{-1}^{1}\phi(x,\mu')\,d\mu',\ 0\le x<\infty,\ |\mu|\le 1,$$
$$\phi(0,\mu)=f(\mu),\ 1\ge\mu>-\alpha.$$

Then, for $-1\le\mu\le-\alpha$, we have (see equations (3) and (12) of [8]) that

$$\phi(0,\mu)=\frac{c}{2}\int_{-\alpha}^{1}\frac{\mu'+\alpha}{\mu'-\mu}H_1(\mu')H_2(-\mu)f(\mu')\,d\mu'.$$

Such an approach provides an interesting and effective alternative for solving the simple transport model theorectically as well as numerically.

2. Main results.

Notation. Set

$$x=\frac{c}{2}\int_{-\alpha}^{1}H_1(\mu)\,d\mu,\quad y=\frac{c}{2}\int_{\alpha}^{1}H_2(\mu')\,d\mu',$$

$$a=\frac{c^2}{4}\int_{-\alpha}^{1}\int_{\alpha}^{1}H_1(\mu)H_2(\mu'')\frac{\mu''-\alpha}{\mu+\mu''}\,d\mu''\,d\mu,$$

and

$$b=\frac{c^2}{4}\int_{-\alpha}^{1}\int_{\alpha}^{1}H_1(\mu)H_2(\mu'')\frac{\mu+\alpha}{\mu+\mu''}\,d\mu''\,d\mu.$$

Note that $a+b=xy$. We begin by deriving some integral properties which a solution of (1) must satisfy.

LEMMA 1. *If H_1 and H_2 are solutions of* (1), *then the following holds:*

$$(1-x)(1-y)=1-c. \tag{2}$$

Proof. Multiplying equation (1) by $\frac{c}{2}$ and integrating equations (1a) and (1b) over the ranges of μ' and μ, respectively, we obtain

$$x=\frac{c}{2}(1+\alpha)+xy-a \tag{3a}$$

and

$$y=\frac{c}{2}(1-\alpha)+xy-b. \tag{3b}$$

Adding up (3a) and (3b) would yield the assertion of Lemma 1.

Remark. For $\alpha=0$, (2) reduces to some well-known expressions concerning the properties of H equations (see, e.g., [3, pp. 106–107]).

For $\alpha\ne 1$, we see immediately that if H_1 and H_2 are positive solutions of (1), then there must exist two positive numbers k_1 and k_2, where $0<k_1,\,k_2<1$ and $k_1+k_2=1$, so that

$$a=k_1xy \tag{4a}$$

and

$$b=k_2xy. \tag{4b}$$

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It then follows from (2), (3), and (4) that the following holds:

$$\text{(5a)}\quad x=\frac{1-\frac{c}{2}(1-\alpha)+k_1c\pm\sqrt{[1-\frac{c}{2}(1-\alpha)+k_1c]^2-2k_1(1+\alpha)c}}{2k_1}:=a_1\pm b_1,$$

$$\text{(5b)}\quad y=\frac{1-\frac{c}{2}(1+\alpha)+k_2c\pm\sqrt{[1-\frac{c}{2}(1+\alpha)+k_2c]^2-2k_2(1-\alpha)c}}{2k_2}:=a_2\pm b_2.$$

Since k_1 and k_2 are to be treated as real parameters, necessary conditions for (5) to be meaningful are that both $[1-\frac{c}{2}(1-\alpha)+k_1c]^2-2k_1c(1+\alpha)$ and $[1-\frac{c}{2}(1+\alpha)+k_2c]^2-2k_2c(1-\alpha)$ are nonnegative. However, these are so if $0\le\alpha\le 1$ and $0\le c\le 1$. To see this, we note that, for $c\neq 0$, $f_1(k_1):=[1-\frac{c}{2}(1-\alpha)+k_1c]^2-2k_1c(1+\alpha)$ has a minimim $(1+\alpha)(1-\alpha)(1-c)$, which is nonnegative whenever $0\le\alpha\le 1$ and $0\le c\le 1$.

We denote by S the feasible region $\{(k,c,\alpha): 0<k<1, 0\le c\le 1 \text{ and } 0\le\alpha\le 1\}$ for the solution of (1). The cross-section $\{(k,c,\alpha): 0<k<1, 0\le c\le 1 \text{ and } \alpha \text{ is fixed}\}$ of S will be denoted by S_α. The properties and signs of $1-x$ and $1-y$ will be examined in the next lemmas.

LEMMA 2. (i) $1-a_1+b_1\ge 0$ *and* $1-a_1-b_1\le 0$ *for all* $(k_1,c,\alpha)\in S$.

(ii) $1-a_2+b_2\ge 0$ *and* $1-a_2-b_2\le 0$ *for all* $(k_2,c,\alpha)\in S$.

(iii) *For each fixed* α, *where* $0\le\alpha<1$, *we have that* $1-a_1+b_1$ *and* $1-a_2+b_2$, *considered as functions from* $S_\alpha\to R$, *can be continuously extended to* $\bar{S}_\alpha$.

(iv) *Let* c *be sufficiently small, say* $0\le c\le\frac{1}{8}$. *Then* $1-a_1+b_1\ge\frac{1}{2}$ *and* $1-a_2+b_2\ge\frac{6}{7}$ *for all* k_1 *and* k_2, $0<k_1,k_2<1$, *and all* α, $0\le\alpha\le 1$.

Proof. Since the computation leading to (i) and (ii) is similar, we shall only illustrate (i). To see (i), it suffices to show that $b_1^2\ge(1-a_1)^2$, or equivalently

$$\left[1-\frac{c}{2}(1-\alpha)+k_1c\right]^2-2k_1c(1+\alpha)-\left[(2k_1-1)\left(1-\frac{c}{2}\right)-\frac{c\alpha}{2}\right]^2\ge 0.$$

Since the left-hand side of the inequality is equal to $4(1-k_1)(k_1)(1-c)$, the assertion of Lemma 2(i) thus follows. To prove (iii), we note that

$$a_1-b_1=\frac{(1+\alpha)c}{1-\frac{c}{2}(1-\alpha)+k_1c+\sqrt{[1-\frac{c}{2}(1-\alpha)+k_1c]^2-2k_1c(1+\alpha)}}:=\frac{(1+\alpha)c}{g_1(k_1,c,\alpha)}$$

and

$$a_2-b_2=\frac{(1-\alpha)c}{1-\frac{c}{2}(1+\alpha)+k_2c+\sqrt{[1-\frac{c}{2}(1+\alpha)+k_2c]^2-2k_2c(1-\alpha)}}:=\frac{(1-\alpha)c}{g_2(k_2,c,\alpha)}.$$

Since $g_1(k_1,c,\alpha)\ge\frac{1}{2}$ for all $(k_1,c,\alpha)\in\bar{S}$, we conclude that a_1-b_1, and hence $1-a_1+b_1$, can be continuously extended to $\bar{S}$. Now, if α is fixed as assumed, then $g_2(k_2,c,\alpha)\ge\frac{1}{2}(1-\alpha)>0$ for all (k_2,c). Therefore, for each fixed α, $1-a_2+b_2$ can be continuously extended to $\bar{S}_\alpha$.

To prove (iv), we see that if $0\le c\le\frac{1}{8}$, then $a_1-b_1=\frac{(1+\alpha)c}{g_1(k_1,c,\alpha)}\le 2c(1+\alpha)\le\frac{1}{2}$, for all k_1 and α. Thus, $1-a_1+b_1\ge\frac{1}{2}$ as asserted. Similarly, we have

$$a_2-b_2=\frac{(1-\alpha)c}{g_2(k_2,c,\alpha)}\le\frac{c}{1-\frac{c}{2}(1+\alpha)}\le\frac{c}{1-c}\le\frac{1}{7}.$$

Therefore, $1-a_2+b_2\ge\frac{6}{7}$ as asserted.

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Remarks. The function $1-a_1+b_1$, as indicated in the proof, can be continuously extended to $\bar{S}$. However, the same assertion fails for $1-a_2+b_2$. To see this, we note that if $\alpha=1$, then $a_2-b_2=0$ for all k_2 and c. However, if $c=1$ then $a_2-b_2=1$ for any $\alpha(\neq 1)$ and $k_2 \leq \frac{1}{2}(1-\alpha)$.

In view of (2), the case $c=1$ shall be further studied.

LEMMA 3. (i) $1-a_1+b_1=0$ *if and only if* $\frac{1}{2}(1+\alpha) \geq k_1$ *and* $c=1$. *Moreover,* $1-a_1-b_1=0$ *if and only if* $\frac{1}{2}(1+\alpha) \leq k_1$ *and* $c=1$.

(ii) $1-a_2+b_2=0$ *if and only if* $k_1 \geq \frac{1}{2}(1+\alpha)$ *and* $c=1$. *Furthermore,* $1-a_2-b_2=0$ *if and only if* $k_1 \leq \frac{1}{2}(1+\alpha)$ *and* $c=1$.

(iii) *If* $\frac{1}{2}(1+\alpha) < k_1$ *and* $c=1$, *then* $1-a_1+b_1=\frac{2k_1-1-\alpha}{2k_1}$. *Moreover, if* $\frac{1}{2}(1+\alpha) > k_1$ *and* $c=1$, *then* $1-a_1-b_1=\frac{2k_1-1-\alpha}{2k_1}$.

(iv) *If* $k_1 < \frac{1}{2}(1+\alpha)$ *and* $c=1$, *then* $1-a_2+b_2=\frac{1+\alpha-2k_1}{2(1-k_1)}$. *Furthermore, if* $k_1 > \frac{1}{2}(1+\alpha)$ *and* $c=1$, *then* $1-a_2-b_2=\frac{1+\alpha-2k_1}{2(1-k_1)}$.

Proof. The necessary parts of Lemma 3(i) follow from (2) and some simple algebra. The remainder of the proof is trivial and thus omitted.

Some simple algebra would yield the following equivalent formulation of (1).

LEMMA 4. *The functions* H_1 *and* H_2 *satisfy, respectively,*

$$[H_1(\mu)]^{-1} = (1-y) + \frac{c}{2}\int_{\alpha}^{1} \frac{\mu''-\alpha}{\mu+\mu''} H_2(\mu'')\, d\mu'' \tag{6a}$$

and

$$[H_2(\mu')]^{-1} = (1-x) + \frac{c}{2}\int_{-\alpha}^{1} \frac{\mu''+\alpha}{\mu'+\mu''} H_1(\mu'')\, d\mu'' \tag{6b}$$

if and only if H_1 *and* H_2 *satisfy* (1a) *and* (1b), *respectively.*

In view of (2), we see that if H_1 and H_2 are solutions of (1), then either

$$1-x \geq 0 \text{ and } 1-y \geq 0 \tag{7a}$$

or

$$1-x \leq 0 \text{ and } 1-y \leq 0. \tag{7b}$$

Let $C[-\alpha,1] \times C[\alpha,1]$ be the Banach space of pairs of bounded real-valued continuous functions with sup norm. That is, if $(h_1, h_2) \in C[-\alpha,1] \times C[\alpha,1]$, then

$$\| (h_1, h_2) \|_\infty := \max\left\{ \max_{-\alpha\leq\mu\leq 1} |\, h_1(\mu)\, | := \| h_1 \|_\infty,\ \max_{\alpha\leq\mu\leq 1} |\, h_2(\mu')\, | := \| h_2 \|_\infty \right\}.$$

In preparation for the use of a homotopy invariance argument define, for $(K_1, K_2) \in C[-\alpha,1] \times C[\alpha,1]$,

$$\psi_{1,c}(K_2(\mu)) = (1-y) + \frac{c}{2}\int_{\alpha}^{1} \frac{\mu''-\alpha}{\mu+\mu''} \frac{1}{K_2(\mu'')}\, d\mu'', \tag{8a}$$

$$\psi_{2,c}(K_1(\mu')) = (1-x) + \frac{c}{2}\int_{-\alpha}^{1} \frac{\mu''+\alpha}{\mu'+\mu''} \frac{1}{K_1(\mu'')}\, d\mu'', \tag{8b}$$

$$\psi_c(K_1(\mu), K_2(\mu')) = (\psi_{1,c}(K_2(\mu))), \psi_{2,c}(K_1(\mu')). \tag{8c}$$

An a priori bound, which is independent of k_1 and c, is obtained in the following lemma.

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LEMMA 5. *Let K_1 and K_2 be any positive continuous solutions of* $(K_1, K_2) = \psi_c(K_1, K_2)$ *satisfying* (7a). *Then there is an* $m > 0$ (*independent of* c *and* α) *such that* $K_1(\mu) \geq m$ *and* $K_2(\mu') \geq m$ *for all* $(\mu, \mu') \in [-\alpha, 1] \times [\alpha, 1]$, *all* $0 \leq \alpha \leq 1$, *and all* $0 \leq c \leq 1$.

Proof. Clearly, $H_1 = \frac{1}{K_1}$ and $H_2 = \frac{1}{K_2}$ are positive solutions of (1). Consequently, $1 \geq K_1(\mu)$ and $1 \geq K_2(\mu')$ for all $[\mu, \mu') \in [-\alpha, 1] \times [\alpha, 1]$. Therefore,

$$K_1(\mu) \geq 1 - y + \frac{c}{2}\int_\alpha^1 \frac{\mu'' - \alpha}{\mu'' + 1}\, d\mu'' := 1 - y + g_1(c, \alpha)$$

and

$$K_2(\mu') \geq 1 - x + \frac{c}{2}\int_{-\alpha}^1 \frac{\mu'' + \alpha}{\mu'' + 1}\, d\mu'' := 1 - x + g_2(c, \alpha).$$

Since $1-y \geq 0$ and $1-x \geq 0$, there must exist positive constants k_1 and k_2, $k_1+k_2 = 1$, such that $1 - x = 1 - a_1 + b_1$ and $1 - y = 1 - a_2 + b_2$, where $a_1 - b_1$ and $a_2 - b_2$ are defined as in (5). Now, via Lemma 2(iv), $1 - x \geq \frac{1}{2}$ for $0 \leq c \leq \frac{1}{8}$. Since, for fixed c, $g_2(c, \alpha)$ is an increasing function (in α), we have that $\int_{-\alpha}^1 \frac{\mu''+\alpha}{\mu''+1}\, d\mu'' \geq 1 - \ell n2$. Consequently,

$$K_2 \geq \min\left\{\frac{1}{2}, \frac{1}{16}(1 - \ell n2)\right\} = \frac{1}{16}(1 - \ell n2) := m_2$$

for all $(\mu, \mu') \in [-\alpha, 1] \times [\alpha, 1]$, all $0 \leq c \leq 1$, and all $0 \leq \alpha \leq 1$. On the other hand,

$$y = \frac{c}{2}\int_\alpha^1 \frac{1}{K_2(\mu')}\, d\mu' \leq \frac{c(1-\alpha)}{2m_2},$$

and so $1 - y \geq 1 - \frac{c(1-\alpha)}{2m_2}$. Hence, if $0 \leq c \leq m_2$ or $\alpha \geq 1 - m_2$, then $1 - y \geq \frac{1}{2}$. However, if $1 \geq c \geq m_2$ and $0 \leq \alpha \leq 1 - m_2$ then

$$g_1(c, \alpha) \geq \frac{m_2}{2}\int_{1-m_2}^1 \frac{\mu'' - (1 - m_2)}{\mu'' + 1}\, d\mu'' := \bar{m}_1 > 0.$$

Consequently, $K_1(\mu) \geq \min\{\frac{1}{2}, \bar{m}_1\} := m_1$ as asserted. The assertion of the lemma now follows by choosing $m = \min\{m_1, m_2\}$.

Remark. The lower bound for K_2 is not sharp. A better bound can be obtained. To see this, let c be such that $0 \leq c \leq \frac{2}{9-\ell n2}$, then $a_1 - b_1 \leq 2c(1 + \alpha) \leq \frac{8}{9\ell n2}$. Thus, $1 - a_1 + b_1 \geq \frac{1-\ell n2}{9-\ell n2}$ for $0 \leq c \leq \frac{2}{9-\ell n2}$. Hence,

$$K_2(\mu') \geq 1 - x + \frac{c}{2}(1 - \ell n2) \geq \min\left\{\frac{1 - \ell n2}{9 - \ell n2}, \frac{1 - \ell n2}{9 - \ell n2}\right\} = \frac{1 - \ell n2}{9 - \ell n2}.$$

THEOREM 1. *For each* α *and* c, *where* $0 \leq \alpha < 1$ *and* $0 \leq c \leq 1$, ψ_c *has a fixed point satisfying* (7a).

Proof. Note, via Lemma 2(iii), that there exists a positive constant $\tilde{m}$ such that

$$\max\left\{\max_{(k_1,c,\alpha)\in\bar{S}_\alpha}(1 - a_1 + b_1), \max_{(k_2,c,\alpha)\in\bar{S}_\alpha}(1 - a_2 + b_2)\right\} \leq \tilde{m}.$$

Choose $a = \min(\frac{1}{2}, \frac{m}{2})$ and $b = \frac{1}{m} + \tilde{m} + 1$, where m is chosen as in Lemma 5. Set $D = \{(K_1, K_2) \in C[-\alpha, 1] \times C[\alpha, 1] : a < K_1(\mu), K_2(\mu') < b$ for all $(\mu, \mu') \in [-\alpha, 1] \times [\alpha, 1]\}$. Clearly, D is a nonempty bounded open subset of $C[-\alpha, 1] \times C[\alpha, 1]$, and $\psi_c : \bar{D} \to C[-\alpha, 1] \times C[\alpha, 1]$ is compact. Next, we show that if $(K_1, K_2) =$

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$\psi_c(K_1, K_2)$ for $(K_1, K_2) \in \bar{D}$, then $(K_1, K_2) \in D$. To prove this note first that from the a priori bound for (K_1, K_2), we see that $K_1(\mu), K_2(\mu') \geq m > a$ for all μ, μ'. Second,

$$\| (K_1, K_2) \|_\infty = \| \psi_c(K_1, K_2) \|_\infty \leq \tilde{m} + \frac{c(1+\alpha)}{2m} < b.$$

Thus $u \in D$ as asserted. The preparations for the use of degree are now complete. Consider the homotopy $I - \psi_c$. By homotopy invariance (see, e.g., Theorem 13.6 of [9]), since $(1,1) \in D$,

$$d(I - \psi_c, (0,0), D) = d(I - \psi_0, (0,0), D) = d(I, (1,1), D) = 1.$$

Therefore, the existence of equation (1) now follows from the Leray–Schauder fixed point theorem.

To show the uniqueness of equation (1) satisfying (7a), we need the following lemma.

LEMMA 6. *Equation* (1) *has minimal positive solutions* $H_{1,\min}(\mu)$ *and* $H_{2,\min}(\mu')$ *in the following sense : if* $H_1(\mu)$ *and* $H_2(\mu')$ *are positive solutions of* (1), *then* $H_{1,\min}(\mu) \leq H_1(\mu)$ *and* $H_{2,\min}(\mu') \leq H_2(\mu')$ *for all* μ, μ'.

Proof. Consider the two iterates $\{H_1^{(p)}\}$ and $\{H_2^{(p)}\}$ defined as follows:

$$H_1^{(1)}(\mu) = 1, \tag{9a}$$

$$H_2^{(1)}(\mu') = 1 \text{ for all } \mu, \mu', \tag{9b}$$

$$H_1^{(p+1)}(\mu) = 1 + \frac{c}{2} H_1^{(p)}(\mu)(\mu + \alpha) \int_\alpha^1 \frac{H_2^{(p)}(\mu'')}{\mu + \mu''} \, d\mu'', \tag{9c}$$

and

$$H_2^{(p+1)}(\mu') = 1 + \frac{c}{2} H_2^{(p)}(\mu')(\mu' - \alpha) \int_{-\alpha}^1 \frac{H_1^{(p)}(\mu'')}{\mu' + \mu''} \, d\mu''. \tag{9d}$$

Clearly, for each μ and μ', $\{H_1^{(p)}(\mu)\}$ and $\{H_2^{(p)}(\mu')\}$ are both increasing sequences. It follows from Theorem 1 that equation (1) has positive solutions, say $H_1(\mu)$ and $H_2(\mu')$. Since $H_1(\mu) \geq 1$ and $H_2(\mu') \geq 1$ for all μ, μ', an easy induction would yield $H_1^{(p)}(\mu) \leq H_1(\mu)$ and $H_2^{(p)}(\mu') \leq H_1(\mu')$ for all μ, μ' and all p. Hence, the sequences $\{H_1^{(p)}(\mu)\}$ and $\{H_2^{(p)}(\mu')\}$, respectively, converge upward to two limits, say $\bar{H}_1(\mu)$ and $\bar{H}_2(\mu')$. It then follows from the monotone convergence theorem that $\bar{H}_1$ and $\bar{H}_2$ solve equation (1), and that $\bar{H}_1(\mu) \leq H_1(\mu)$ and $\bar{H}_2(\mu') \leq H_2(\mu')$ for all μ, μ'. The proof of the lemma is thus complete.

THEOREM 2. *For* $c = 0$ *or* $\alpha = 1$, *equation* (1) *has unique solutions. Furthermore, for* $0 < c < 1$, *equation* (1) *has unique solutions* H_1 *and* H_2 *satisfying* (7a).

Proof. The uniqueness for $c = 0$ or $\alpha = 1$ is trivial. For $0 < c < 1$, and $1 - x > 0$ and $1 - y > 0$, we have that

$$1 - x_{\min} := 1 - \frac{c}{2} \int_{-\alpha}^1 H_{1,\min}(\mu) d\mu \geq 1 - x > 0,$$

$$1 - y_{\min} := 1 - \frac{c}{2} \int_\alpha^1 H_{2,\min}(\mu') d\mu' \geq 1 - y > 0,$$

and

$$[H_2(\mu')]^{-1} = \frac{1-c}{1-y} + \frac{c}{2}\int_{-\alpha}^{1} \frac{\mu''+\alpha}{\mu'+\mu''} H_1(\mu'')\, d\mu''$$

$$\geq \frac{1-c}{1-y_{\min}} + \frac{c}{2}\int_{-\alpha}^{1} \frac{\mu''+\alpha}{\mu'+\mu''} H_{1,\min}(\mu'')\, d\mu''$$

$$\geq [H_{2,\min}(\mu')]^{-1}.$$

Therefore, $H_2 = H_{2,\min}$, and hence $H_1 = H_{1,\min}$, and the lemma is proved.

Our final result is concerned with the number of positive solutions for equation (1). The techniques for proving this result are motivated by those of Leggett [10]. To this end, we first prove the following lemma.

LEMMA 7. *Let $0 < c < 1$ and $0 \leq \alpha < 1$, and let (H_1, H_2) and $(\bar{H}_1, \bar{H}_2)$ be positive solutions pairs of equation* (1) *satisfying* (7a) *and* (7b), *respectively. Then the following holds:*

(i) *There exist, respectively, two positive constants k_1 and k_2, where $0 < k_1 < \frac{1}{1+\alpha}$ and $0 < k_2 < \frac{1}{1-\alpha}$, such that*

$$\text{(10a)} \qquad \frac{c}{2}\int_{-\alpha}^{1} \frac{H_1(\mu'')}{1-k_1(\mu''+\alpha)}\, d\mu'' = 1$$

and

$$\text{(10b)} \qquad \frac{c}{2}\int_{\alpha}^{1} \frac{H_2(\mu'')}{1-k_2(\mu''-\alpha)}\, d\mu'' = 1.$$

Furthermore, such choices of k_1 and k_2 are unique.

(ii) *There exist, respectively, two positive constants $\bar{k}_1$ and $\bar{k}_2$, where $0 < \bar{k}_1 < \frac{1}{1+\alpha}$ and $0 < \bar{k}_2 < \frac{1}{1-\alpha}$, such that*

$$\text{(10c)} \qquad \frac{c}{2}\int_{-\alpha}^{1} \frac{\bar{H}_1(\mu'')}{1+\bar{k}_2(\mu''+\alpha)}\, d\mu'' = 1$$

and

$$\text{(10d)} \qquad \frac{c}{2}\int_{\alpha}^{1} \frac{\bar{H}_2(\mu'')}{1+\bar{k}_1(\mu''-\alpha)}\, d\mu'' = 1.$$

Moreover, such choices of $\bar{k}_1$ and $\bar{k}_2$ are unique.

Proof. Since the analysis leading to (10a), (10b), (10c), and (10d) is similar, we illustrate only (10b) and (10c). Define the function $T : (0, \frac{1}{1-\alpha}) \to R$ by

$$T(k) = \frac{c}{2}\int_{\alpha}^{1} \frac{H_2(\mu'')}{1-k(\mu''-\alpha)}\, d\mu''.$$

Then

$$\text{(11)} \qquad \lim_{k \to (\frac{1}{1-\alpha})^-} T(k) = \frac{c}{2}\int_{\alpha}^{1} \frac{(1-\alpha)H_2(\mu'')}{1-\mu''}\, d\mu'',$$

since $(1-k(\mu''-\alpha))^{-1}$ increases monotonically with k, $0 < k < \frac{1}{1-\alpha}$. Note that the improper integral in (11) diverges to $+\infty$. Since $T(0) = \frac{c}{2}\int_{\alpha}^{1} H_2(\mu'')\, d\mu'' < 1$, and

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since $T(k)$ is strictly increasing with $T(\frac{1}{1-\alpha}) = +\infty$, there exists a unique $k_2 \in (0, \frac{1}{1-\alpha})$ for which (10b) holds. Now suppose that $\bar{H}_1$ and $\bar{H}_2$ satisfy (1) and (7b). Then $\frac{c}{2}\int_{-\alpha}^{1} \bar{H}_1(\mu'')\,d\mu'' > 1$, and

$$\frac{c}{2}\int_{-\alpha}^{1} \frac{\bar{H}_1(\mu'')}{1+\frac{1}{1-\alpha}(\mu''+\alpha)}\,d\mu'' = \frac{c}{2}\int_{-\alpha}^{1} \frac{1-\alpha}{1+\mu''}\bar{H}_1(\mu'')\,d\mu''$$

$$= 1 - [\bar{H}_2(1)]^{-1} < 1.$$

Therefore, there exists a unique $\bar{k}_2$, $0 < \bar{k}_2 < \frac{1}{1-\alpha}$, such that (10c) holds.

THEOREM 3. *Equation* (1) *has exactly two positive solutions if* $0 < c < 1$ *and* $0 \le \alpha < 1$.

Proof. Let H_1 and H_2 be positive solutions of (1) satisfying (7a). Define

$$\bar{H}_1(\mu) = \frac{1+k_2\mu+k_2\alpha}{1-k_1\mu-k_1\alpha}H_1(\mu) \tag{12a}$$

and

$$\bar{H}_2(\mu'') = \frac{1+k_1\mu''-k_1\alpha}{1-k_2\mu''+k_2\alpha}H_2(\mu''). \tag{12b}$$

Here k_1 and k_2 are chosen as in Lemma 7. Now, using (10b), we find

$$\frac{c}{2}\int_{\alpha}^{1} \frac{\mu+\alpha}{\mu+\mu''}\cdot \bar{H}_2(\mu'')\,d\mu'' = \frac{c}{2}\int_{\alpha}^{1} \frac{(\mu+\alpha)(1+k_1\mu''-k_1\alpha)}{(\mu+\mu'')(1-k_2\mu''+k_2\alpha)}H_2(\mu'')\,d\mu''$$

$$= \frac{1-k_1\mu-k_1\alpha}{1+k_2\mu+k_2\alpha}\left(\frac{c}{2}\right)\int_{\alpha}^{1} \frac{\mu+\alpha}{\mu+\mu''}H_2(\mu'')\,d\mu''$$

$$+ \frac{(k_1+k_2)(\mu+\alpha)}{1+k_2\mu+k_2\alpha}\left(\frac{c}{2}\right)\int_{\alpha}^{1} \frac{H_2(\mu'')}{1-k_2\mu''+k_2\alpha}\,d\mu''$$

$$= \frac{1-k_1\mu-k_1\alpha}{1+k_2\mu+k_2\alpha}\left[1-\frac{1}{H_1(\mu)}\right] + \frac{(k_1+k_2)(\mu+\alpha)}{1+k_2\mu+k_2\alpha}$$

$$= 1 - \frac{1}{\bar{H}_1(\mu)}.$$

A similar computation would yield that

$$\frac{c}{2}\int_{-\alpha}^{1} \frac{\mu'-\alpha}{\mu'+\mu''}\bar{H}_1(\mu'')\,d\mu'' = 1 - \frac{1}{\bar{H}_2(\mu')}.$$

That is, $\bar{H}_1$ and $\bar{H}_2$ satisfy equation (1). Hence, $\bar{H}_1$ and $\bar{H}_2$ must satisfy either (7a) or (7b). Since H_1 and H_2 are the unique positive solutions of (1) satisfying (7a), and since $\bar{H}_1(\mu) > H_1(\mu)$, $\bar{H}_2(\mu') > H_2(\mu')$ for almost all μ, μ', $\bar{H}_1$ and $\bar{H}_2$ must satisfy (7b). Thus, we have shown that equation (1) has at least two positive solutions when c and α are as assumed. It remains to show that such an equation has at most two

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solutions. To this end, we suppose that $\bar{H}_1$ and $\bar{H}_2$ are positive solutions satisfying (1) and (7b). Define

$$H_1(\mu) = \frac{1 - \bar{k}_1\mu - \bar{k}_1\alpha}{1 + \bar{k}_2\mu + \bar{k}_2\alpha}\bar{H}_1(\mu) \tag{13a}$$

and

$$H_2(\mu'') = \frac{1 - \bar{k}_2\mu'' + \bar{k}_2\alpha}{1 + \bar{k}_1\mu'' - \bar{k}_1\alpha}\bar{H}_2(\mu''). \tag{13b}$$

Here $\bar{k}_1$ and $\bar{k}_2$ are chosen as in Lemma 7. Now, using (10c), we obtain

$$\begin{aligned}
\frac{c}{2}\int_{-\alpha}^{1} \frac{\mu' - \alpha}{\mu' + \mu''} \cdot H_1(\mu'')\, d\mu'' &= \frac{c}{2}\int_{-\alpha}^{1} \frac{(\mu' - \alpha)(1 - \bar{k}_1\mu'' - \bar{k}_1\alpha)}{(\mu' + \mu'')(1 + \bar{k}_2\mu'' + \bar{k}_2\alpha)}\bar{H}_1(\mu'')\, d\mu'' \\
&= \frac{1 + \bar{k}_1\mu' - \bar{k}_1\alpha}{1 - \bar{k}_2\mu' + \bar{k}_2\alpha}\left(\frac{c}{2}\right)\int_{-\alpha}^{1} \frac{\mu' - \alpha}{\mu' + \mu''}\bar{H}_1(\mu'')\, d\mu'' \\
&\quad - \frac{(\bar{k}_1 + \bar{k}_2)(\mu' - \alpha)}{1 - \bar{k}_2\mu' + \bar{k}_2\alpha}\left(\frac{c}{2}\right)\int_{-\alpha}^{1} \frac{\bar{H}_1(\mu'')}{1 + \bar{k}_2\mu'' + \bar{k}_2\alpha}\, d\mu'' \\
&= \frac{1 + \bar{k}_1\mu' - \bar{k}_1\alpha}{1 - \bar{k}_2\mu' + \bar{k}_2\alpha}\left[1 - \frac{1}{\bar{H}_2(\mu')}\right] - \frac{(\bar{k}_1 + \bar{k}_2)(\mu' - \alpha)}{1 - \bar{k}_2\mu' + \bar{k}_2\alpha} \\
&= 1 - \frac{1}{H_2(\mu')}.
\end{aligned}$$

Similarly, we obtain that

$$\frac{c}{2}\int_{\alpha}^{1} \frac{\mu + \alpha}{\mu + \mu''}H_2(\mu'')\, d\mu'' = 1 - \frac{1}{H_1(\mu)}.$$

Therefore, H_1 and H_2 satisfy equation (1). It follows from (13) and (10c), (10d) that $\frac{c}{2}\int_{-\alpha}^{1} H_1(\mu)\, d\mu < 1$ and $\frac{c}{2}\int_{\alpha}^{1} H_2(\mu')\, d\mu' < 1$; i.e., H_1 and H_2 satisfy (7a). Since the solutions of (1) satisfying (7a) are unique, we conclude that the solutions of (1) satisfying (7b) are also unique, and the theorem is proved.

THEOREM 4. *Let $c = 1$ and let α be sufficiently close to* 1. *Then equation* (1) *has exactly two positive solutions.*

Proof. Let H_1 and H_2 be solutions of equation (1) satisfying (7a). It follows from Lemma 5 that y must approach zero as α approaches 1 from the left. Hence if $c = 1$ and α is chosen to be sufficiently close to 1, then $x = 1$ and $y < 1$. Define $\bar{H}_1$ and $\bar{H}_2$ as follows:

$$\bar{H}_1(\mu) = (1 + k_2\mu + k_2\alpha)H_1(\mu)$$

and

$$\bar{H}_2(\mu'') = \frac{H_2(\mu'')}{1 - k_2\mu'' + k_2\alpha},$$

where k_2 is uniquely satisfied by (10b). Using a procedure similar to the proof of Theorem 3, it follows that $\bar{H}_1$ and $\bar{H}_2$ are positive solutions of (1) satisfying (7b). Since $H_1 \neq \bar{H}_1$ and $H_2 \neq \bar{H}_2$, it remains to show that such an equation has at most

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two positive solutions. Suppose $\bar{H}_1$ and $\bar{H}_2$ are positive solutions of (1) satisfying (7b). Then either

$$\bar{x} := \frac{1}{2}\int_{-\alpha}^{1} \bar{H}_1(\mu)d\mu > 1 \text{ and } \bar{y} := \frac{1}{2}\int_{\alpha}^{1} \bar{H}_2(\mu')d\mu' = 1, \tag{14a}$$

or

$$\bar{x} = 1 \text{ and } \bar{y} > 1, \tag{14b}$$

or

$$\bar{x} = 1 \text{ and } \bar{y} = 1. \tag{14c}$$

If (14c) held, then $\bar{H}_1$ and $\bar{H}_2$ would also satisfy (7a), and hence $\bar{y} \to 0$ as $\alpha \to 1$, a contradication. Thus, (14c) should be ruled out. If (14b) were the case, then H_1 and H_2, defined as in (13a) and (13b), respectively, with $\bar{k}_1 = 0$ and k_2 satisfying (10c), were positive solutions of (1) satisfying (7a). Since $H_1 \leq \bar{H}_1$ and $H_1 \neq \bar{H}_1$, we see immediately that $x < 1$ and $y = 1$. This is not possible. Therefore, (14a) must hold.

Define H_1 and H_2 as in (13a) and (13b), respectively, with $\bar{k}_2 = 0$ and $\bar{k}_1$ satisfying (10d). Then such H_1 and H_2 are the positive solutions of (1) satisfying (7a). Now, if we can show that the positive solutions of equation (1) satisfying (7a) are unique, then the proof of the theorem will be complete. To this end, we note, as observed in the first paragraph of the proof, that $x_{\min}$ must be equal to 1. Therefore, $\int_{-\alpha}^{1}(H_1(\mu) - H_{1,\min}(\mu))\,d\mu = 0$, and so $H_1 \equiv H_{1,\min}$ and $H_2 \equiv H_{2,\min}$. Thus, the theorem is proved.

We conclude this paper with the following remarks.

Remarks. 1. We may conclude, via the proofs of Theorems 3 and 4, that for $c = 1$, if x and y are not both equal to 1, then equation (1) admits exactly two positive solutions.

2. On the other hand, if $x = y = 1$, then equation (1) has unique positive solutions. To see this, we note that either $x_{\min} := \frac{1}{2}\int_{\alpha}^{1} H_{1,\min}(\mu)\,d\mu$ or $y_{\min} := \frac{1}{2}\int_{-\alpha}^{1} H_{2,\min}(\mu')\,d\mu'$ is equal to 1. We assume, without loss of generality, that $y_{\min} = 1$. Thus,

$$0 = \frac{1}{2}\int_{-\alpha}^{1} H_{2,\min}(\mu') - H_2(\mu')\,d\mu'.$$

Since $H_{2,\min} - H_2$ is a continuous nonpositive function, we find that $H_{2,\min} \equiv H_2$, and hence $H_{1,\min} \equiv H_1$. Note that for $\alpha = 0$ and $c = 1$, we have $x = y = 1$.

3. The case where c is not a constant can be easily generalized.

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