

MONTE CARLO SIMULATION FOR CORRELATED VARIABLES WITH MARGINAL DISTRIBUTIONS^a

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The authors present a practical multivariate Monte Carlo simulation that preserves the marginal distribution of random variables and their correlation structure without requiring that the complete joint distribution be known. This is achieved by transforming the original correlation coefficient into an equivalent correlated coefficient in standard normal space. The procedure was applied to the reliability analysis of a bridge pier against scouring.

There are other reliability methods that overcome the two major concerns in the practical application of Monte Carlo simulations mentioned by the authors. These concerns are: (1) the requirement of tremendous computation for generating random variables; and (2) the presence of correlation among stochastic system parameters.

The point estimate method (PEM) is one reliability technique that is more straightforward to implement and seems to be more efficient than the Monte Carlo simulation for problems involving correlated random variables.

PEM usually refer to those methods that require only the knowledge of the model response, $Y = f(\mathbf{X})$, at a specific set of values of the n random variables, $\mathbf{X} = (x_1, x_2, \dots, x_n)$, for calculating the statistical moments of the model response. Rosenblueth (1975) developed a second-order two-point PEM, which uses the mean, variance, and correlation coefficient of the random variables and only requires 2^n evaluations of $Y = f(\mathbf{X})$. Li (1992) developed a new PEM that is more accurate and efficient than Rosenblueth's method. The fourth-order three-point method only requires $(n^2 + 3n + 2)/2$ evaluations of $Y = f(\mathbf{X})$ and uses the mean, variance, skewness, kurtosis, and correlation coefficient of the random variables.

For Li's second-order three-point method, which requires the same number of evaluations of $Y = f(\mathbf{X})$ as the fourth-order method, the expected value of a function, with several correlated random variables, can be expressed in the form of

$$E(Y) = \left(1 - \frac{3n}{2} - \frac{\rho}{2}\right) \cdot f(\mathbf{u}) + \frac{1}{2} \sum_{i=1}^n [(3 - 2\rho_i) \cdot f_i^+ + f_i^-] + \sum_{i,j} f_{ij}^+ \cdot \rho_{ij} \quad (12)$$

where

$$\rho_i = \sum_{j=1}^n \rho_{ij}, \quad \rho = \sum_{i=1}^n \rho_i, \\ f_i^+ = f(\mu_1, \mu_2, \dots, \mu_i + \sigma_i, \dots, \mu_n), \\ f_i^- = f(\mu_1, \dots, \mu_i - \sigma_i, \dots, \mu_n) \quad (13)$$

σ = standard deviation; μ_i = mean of the random variable x_i ; ρ_{ij} = correlation coefficient between x_i and x_j ; $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_n)$ and $f(\boldsymbol{\mu}) = f(\mu_1, \mu_2, \dots, \mu_n)$.

No assumption was made about the distribution of the

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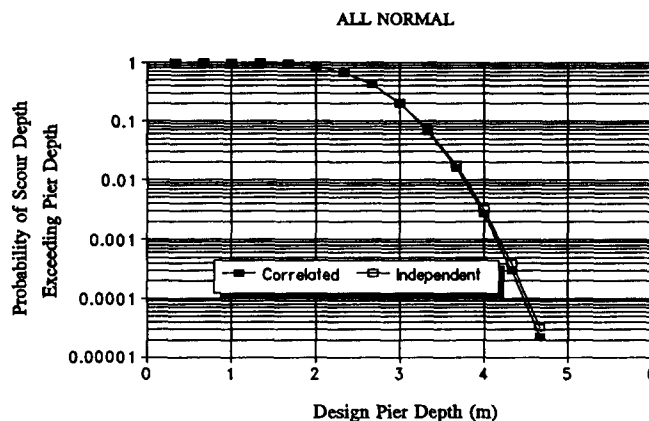


FIG. 10. Effect of Correlation on Failure Probability Curves Using All Normal Parameters and Point Estimate Method for $b = 2.0$ m

random variables in the derivation of (12). There is no restriction on the distribution type or the combination of distribution types for the random variables involved. Only the mean, variance, and correlation coefficient are required and it is applicable to both Gaussian and non-Gaussian correlated random variables.

Higher-order statistical moments of the model response, $Y = f(\mathbf{X})$ can be estimated using the PEM. The statistical moments can be used to fit a distribution, such as a Pearson or Johnson family of frequency curves (Kendall et al. 1987) to the model response (e.g. Zoppou and Li 1992). The probability of failure of the model response can be inferred from the fitted distribution.

The authors' example, which estimates the probability that the scour depth exceeds the pier depth, will be used to illustrate the application of the PEM. Assuming that all the variables are normally distributed and that the predicted scour depth, D_s , is also normally distributed, then only the expected value and variance of D_s are required. Eq. (12) was used to estimate the statistical moments for D_s . The estimated mean and standard deviation of D_s for $b = 2.0$ m are: $\mu = 2.5603$ m and $\sigma = 0.5166$ m. Using these values and the assumed distribution for D_s , the probability curve for correlated parameters is shown in Fig. 10. The probability curve for independent parameters is also shown in Fig. 10.

Comparing the results shown in Fig. 10 with the results obtained by the authors using the Monte Carlo simulation, shown in Fig. 4, reveals that the second-order PEM has produced results similar to the Monte Carlo simulation results. Deviations have occurred at the tail of the distribution. Both the authors' method and the PEM are approximations and neither may be reliable in this region. However, the PEM could have been used to estimate higher statistical moments (skewness and kurtosis) for D_s , and either a Pearson or Johnson family of curves fitted to provide an improved fit to the tail of the distribution of D_s than a normal distribution.

In the example presented by the authors, where $n = 4$, 100,000 samples were generated. To reproduce the results in Fig. 10 only 15 simulations were required, a considerable savings indeed. In addition, the transformation described by the authors is unnecessary. However, for a large number of random variables, Monte Carlo simulation may be more efficient than the PEM.

The PEM for multivariate correlated functions is simple to implement, efficient and for many hydraulic and hydrological problems, could prove to be an alternative method to the Monte Carlo simulation.

APPENDIX I. REFERENCES

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APPENDIX II. NOTATION

The following symbols are used in this paper:

- $E[\cdot]$ = expected value;
 f_i^- = $f(\mu_1, \dots, \mu_{i-1}, \mu_i \pm \sigma_i, \mu_{i+1}, \dots, \mu_n)$;
 f_{ij}^{++} = $f(\mu_1, \dots, \mu_{i-1}, x_i \pm \sigma_i, \dots, \mu_{j-1}, x_j \pm \sigma_j, \dots, \mu_n)$;
 $f(\mathbf{X})$ = model response for a specific set of random vectors \mathbf{X} ;
 $f(\mu)$ = $f(\mu_1, \mu_2, \dots, \mu_n)$;
 n = number of random variables;
 \mathbf{X} = vector of random variables $\mathbf{X} = (x_1, x_2, \dots, x_n)$;
 x_i = random variable;
 Y = model response $Y = f(\mathbf{X})$;
 μ = $(\mu_1, \mu_2, \dots, \mu_n)$; and
 ρ_{ij} = correlation coefficient between x_i and x_j .

Closure by Che-Hao Chang,⁶ Yeou-Koung Tung,⁷ Member, ASCE, and Jinn-Chuang Yang,⁸ Member, ASCE

The writers appreciate the discussers for pointing out the computational advantages of their probabilistic point-estimate method for uncertainty and reliability analysis of hydrologic and hydraulic problems. Although (12) was derived with no assumption made about the distribution of the random variables, it is only a special case to (17) in Li (1992). In other words, (12) is appropriate when one only wishes to consider the first two moments of random variables in estimating $E[f(\mathbf{X})]$. This is not equivalent to saying that (12) is good for dealing with problems involving nonnormal random variables. To estimate the higher-order moments about $f(\mathbf{X})$, the higher-order moments of involved random variables would become important. Furthermore, information about the distribution of random variables is important in estimating tail probability. Recently, a point-estimation method that takes into account the marginal distribution information of multivariate random variables has been developed by the first author (Chang 1994). The method requires only $2n$ function evaluations for estimating the statistical moments of $f(\mathbf{X})$.

APPENDIX. REFERENCE

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