ELSEVIER

Discrete Mathematics 141 (1995) 61–66

DISCRETE MATHEMATICS

The achromatic indices of the regular complete multipartite graphs ☆

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Received 14 April 1991; revised 31 August 1993

Abstract

In this paper, we study the achromatic indices of the regular complete multipartite graphs and obtain the following results:

(1) A good upper bound for the achromatic index of the regular complete multipartite graph which gives the exact values of an infinite family of graphs and solves a problem posed by Bouchet.

(2) An improved Bouchet coloring which gives the achromatic indices of another infinite family of regular complete multipartite graphs.

1. Introduction

An edge k-coloring of a simple graph $G=(V(G), E(G))$ is a surjection from $E(G)$ to the set $\{1,2,\ldots,k\}$ (which represents colors) so that any two incident edges of G receive different colors. Moreover, if for each pair of colors c_1 and c_2 there are incident edges e_1 and e_2 so that e_i is colored c_i, then the coloring is *complete*. The largest k so that there exists a complete edge k-coloring of G is the *achromatic index* $\Psi'(G)$ of G. The basic concepts related to graph colorings can be referred to [1, 3, 4, 8, 9].

The achromatic index of a complete graph, a regular complete multipartite graph, has been studied by Bouchet et al. in [2, 7, 11], respectively. Mainly, partial results are obtained. In particular, on regular complete multipartite graphs, Bouchet proved the following theorem.

Theorem 1.1. *Suppose that q is an odd integer and equal to the order of a projective plane. If n and m are integers such that $n \mid q+1$ and $m=q(q+1)/n$, then $\Psi'(K_{n[m]})\geqslant$*

☆ Research supported by the National Science Council of the Republic of China (NSC79-0208-M009-33).

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SSDI 0012-365X(93)E0207-K

$q(n-1)m$ where $K_{n[m]}$ denotes the regular complete n-partite graph with each partite set consisting of m vertices.

Furthermore, Bouchet posed a problem asking whether the above inequality could actually be an equality. In this paper, we find an upper bound for the achromatic index of the regular complete multipartite graph which forces the inequality to be an equality and a complete edge coloring for some kind of regular complete multipartite graphs which gives the achromatic indices of another class of infinitely many graphs.

2. The upper bound

In this section, we mainly give an upper bound for $\Psi'(K_{n[m]})$. We start with some definitions.

Let $g(x,y,z)=z(2y(x-1)-z-1)$ and $h(x,y,z)=x(x-1)y^2/2z$. Define $\beta_t(m,n)$ to be the maximum of $g(n,m,t)+1$ and $\lfloor h(n,m,t+1)\rfloor$. Moreover, let $B(n,m)=\min\{\beta_t(m,n)$: $t=1,2,\dots,m(n-1)-1\}$. Then we have the following lemma.

Lemma 2.1. $\Psi'(K_{n[m]})\leqslant B(n,m)$.

Proof. For a fixed $t\leqslant m(n-1)-1$, first suppose that there is a color class Γ which contains $s\leqslant t$ edges. Let the number of end-vertices of these edges which come from the ith partite set be s_i. Then it is not difficult to check that the number of edges which are incident with the edges of Γ is $2sm(n-1)-\frac{1}{2}(\sum_{i=1}^{n}s_i(s_1+\cdots+s_{i-1}+s_{i+1}+\cdots+s_n))-s=2sm(n-1)-2s^2+\frac{1}{2}\sum_{i=1}^{n}s_i^2-s\leqslant 2sm(n-1)-s^2-s$. Thus, the number of color classes is at most $2sm(n-1)-s^2-s+1$ which is $g(n,m,s)+1$. Since $g(n,m,x)$ is increasing on $[1,m(n-1)-\frac{1}{2}]$ when considering n and m as constants, it follows that in this case, $\Psi'(K_{n[m]})\leqslant g(n,m,t)+1$.

Secondly, if each color class has more than t edges, then the number of color classes is obviously less than or equal to $(n-1)m^2n/2(t+1)$, whence $\Psi'(K_{n[m]})\leqslant \lfloor h(n,m,t+1)\rfloor$ in this case.

Combining the above two cases, $\Psi'(K_{n[m]})\leqslant\max\{g(n,m,t)+1,\lfloor h(n,m,t+1)\rfloor\}=\beta_t$ for each fixed t. This implies that $\Psi'(K_{n[m]})\leqslant B(n,m)$. □

Hence, if we can find $B(n,m)$ explicitly, then we obtain an explicit upper bound of $\Psi'(K_{n[m]})$. So far, we have no answer for the general form. But we do have a very nice result for special values of n and m.

Let

$$P(x,y,z)=x(x-1)y^2-4(x-1)(z+1)zy+2(z+1)(z^2+z-1) \tag{1}$$

and

$$Q(x,y,z)=x(x-1)y^2-4(x-1)z^2y+2(z^3+z^2-z). \tag{2}$$

It is easy to see that $P(x,y,z)\geqslant 0$ if and only if $\lfloor h(x,y,z+1)\rfloor \geqslant g(x,y,z)+1$, and $Q(x,y,z)\geqslant 0$ if and only if $\lfloor h(x,y,z)\rfloor \geqslant g(x,y,z)+1$. Consider (1) and (2) as polynomials of y, then they are quadratic. Let D_1 and D_2 be the discriminants of (1) and (2), respectively. By a direct calculation, D_1 and D_2 are positive provided that $z\geqslant 2$ or $z=1$ and $n\geqslant 3$. Furthermore, let γ, δ and ε be the larger roots of $P(n,y,t)=0$, $Q(n,y,t)=0$ and $Q(n,y,t+1)=0$, respectively, and let γ' and ε' be the smaller roots of $P(n,y,t)=0$ and $Q(n,y,t+1)=0$, respectively. Also, by solving equations, we have $\gamma'<\delta<\gamma<\varepsilon$ and $\varepsilon'<\gamma$. With the above observation, we have the following theorem.

Theorem 2.2. *If $t\geqslant 2$ or $t=1$ and $n\geqslant 3$, then*
(i) $\Psi'(K_{n[m]})\leqslant g(n,m,t)+1$ *if $m\in[\delta,\gamma]$ and m is an integer; and*
(ii) $\Psi'(K_{n[m]})\leqslant \lfloor h(n,m,t+1)\rfloor$ *if $m\in[\gamma,\varepsilon]$ and m is an integer.*

Proof. (i) Since $m\in[\delta,\gamma]$, $P(n,m,t)\leqslant 0$ and $Q(n,m,t)\geqslant 0$, thus $\beta_t(m,n)=g(n,m,t)+1$. Now if u is an integer such that $t\leqslant u\leqslant m(n-1)-1$, then $\beta_u(n,m)\geqslant g(n,m,u)+1\geqslant g(n,m,t)+1=\beta_t(n,m)$. On the other situation, if s is an integer such that $s<t$, then since $Q(n,m,t)\geqslant 0$, i.e., $h(n,m,t)\geqslant g(n,m,t)+1$, hence $\beta_s(n,m)\geqslant h(n,m,s+1)\geqslant h(n,m,t)\geqslant g(n,m,t)+1=\beta_t(n,m)$. This concludes that $B(n,m)=\beta_t(n,m)=g(n,m,t)+1$. By Lemma 2.1, we have proved (i). Similarly, we can show that in (ii), $B(n,m)=\lfloor h(n,m,t+1)\rfloor$. The proof follows. □

With Theorem 2.2, we are able to obtain $B(n,m)$ for some special n and m.

Corollary 2.3. *Let k be an odd integer $\geqslant 3$ such that $n\mid k+1$ and $m=k(k+1)/n$. Then $B(n,m)=(n-1)km$.*

Proof. Let

$$t=\frac{k-1}{2} \qquad \text{then } m>\frac{(k+1)(k-1)}{2n}=\frac{2t(t+1)}{n}$$

and $P(n,m,t)>0$. Also,

$$m>\frac{(k+1)^2}{2n}=\frac{2(t+1)^2}{n} \quad \text{and } Q(n,m,t+1)<0.$$

Thus, $m\in[\gamma,\varepsilon]$. By Theorem 2.2, $B(n,m)=\lfloor h(n,m,t+1)\rfloor=\lfloor (n-1)km\rfloor=(n-1)km$ if $t\geqslant 2$, or $t=1$ and $n=3$. Hence, the case left is $k=3$ and $n=2$. Since $m=6$, $\beta_i(2,6)$ can be obtained directly, $i=1,2,\ldots,5$, and $B(2,6)=18$ follows easily.

As for $n=2$, we can get a more clear form.

Corollary 2.4. *For a fixed $t \geqslant 2$, we have*

(i) $\Psi'(K_{2[m]}) \leqslant t(2m-t-1)+1$ *if* $\lfloor 2t^2 - t/2 \rfloor \leqslant m \leqslant \lceil 2t^2 + \frac{3}{2}t - 1 \rceil$; *and*

(ii) $\Psi'(K_{2[m]}) \leqslant m^2/(t+1)$ *if* $\lceil 2t^2 + \frac{3}{2}t \rceil \leqslant m \leqslant \lfloor 2(t+1)^2 - \frac{1}{2}(t+1) - 1 \rfloor$.

Now by Lemma 2.1, Corollary 2.3 and Theorem 1.1, we obtain our main theorem.

Theorem 2.5. *Let q be an odd order of a projective plane. If n and m are positive integers such that $n \mid q+1$ and $m = q(q+1)/n$, then $\Psi'(K_{n[m]}) = q(n-1)m$.*

3. The complete edge coloring

In this section, we use the property of finite projective plane and the construction technique of design theory [5, 10] to construct a complete edge colorings of a class of infinitely many regular complete partite graphs. Then, we obtain achromatic indices of more graphs.

In the edge coloring, the concept *overfull* is important. We say that a graph G is overfull if $|V(G)|$ is odd and $|E(G)|$ is greater than $\frac{1}{2}\Delta(G)(|V(G)|-1)$. Hoffman and Rodger [6] proved the following theorem.

Theorem 3.1. *Let G be a complete partite graph. Then*

$$\chi'(G) = \begin{cases} \Delta(G) & \text{if } G \text{ is not overfull; and} \\ \Delta(G)+1 & \text{otherwise.} \end{cases}$$

By the definition of overfull and Theorem 3.1, we know that every regular complete partite graph G of even order has a complete edge $\Delta(G)$-coloring.

Theorem 3.2. *Let q be an order of a projective plane. Suppose $q+1 = ns+r$ where $0 \leqslant r < n$ and $s \leqslant 1$.*

(i) *If q is odd, then*

(a) *if $r=0$, then for each l such that $0 \leqslant l \leqslant q-s-n+2$ and ln is even, we have $\Psi'(K_{n[m]}) \geqslant (n-1)mq$ where $m=(s+l)q$; and*

(b) *if $r \neq 0$, then for each l such that $1 \leqslant l \leqslant q-s-n+2$ and $nl-r$ is even, we have $\Psi'(K_{n[m]}) \geqslant (n-1)mq$ where $m=(s+l)q$.*

(ii) *If q is even, then*

(a) *if $r=0$, then for each l such that $1 \leqslant l \leqslant q-s-n+2$ and ln is odd, we have $\Psi'(K_{n[m]}) \geqslant (n-1)mq$ where $m=(s+l)q$; and*

(b) *if $r \neq 0$, then for each l such that $1 \leqslant l \leqslant q-s-n+2$ and $nl-r$ is odd, we have $\Psi'(K_{n[m]}) \geqslant (n-1)mq$ where $m=(s+l)q$.*

Proof. Let (P, p) be a projective plane of order q where P is the set of points and p is the collection of lines in the plane. Then $|p| = q^2+q+1$ and let ∞ be a point called infinity. Let $P'=P\setminus\{\infty\}$ and p' be obtained by replacing all the lines which contain ∞ with the lines with q points left. It is not difficult to see that (P',p') is a PBD with block sizes q and $q+1$. For convenience, let the collection of $q+1$ lines with q points be H and the collection of q^2 lines which contain $q+1$ points be V. Since $q+1\geqslant l+s+n-1$ and $q+1=ns+r$, we can distribute the lines in H into n parts $H_1,H_2,\ldots,H_n$ such that

(1) there are $l+s$ lines in H_1; and

(2) H_i contains at least one line and at most $l+s$ lines, $2\leqslant i\leqslant n$.

If H_i contains less than $l+s$ lines, then add extra copies of a line in H_i in order that each part H_i, $i=1,2,\ldots,n$, contains $l+s$ lines as a result. Let H_i' be the set of all points which belong to some line in H_i, $i=1,2,\ldots,n$. Now construct a regular complete n partite graph $K_{n[m]}$, $m=(l+s)q$, by using $H_i'(i=1,2,\ldots,n)$ as partite set and defining $wv\in E(K_{n[m]})$ if and only if w and v belong to different partite sets.

For each line L in V, let $H_i^L=\{v\in H_i'$: $v\in L$ or v is copied from $w\in H_i'$ and $w\in L\}$, $i=1,2,\ldots,n$. Since each line in V intersects each line in H in exactly one point, we have $|H_i^L|=l+s$ for every $i=1,2,\ldots,n$. Let $V^L=H_1^L\cup H_2^L\cup\cdots\cup H_n^L$. Then V^L induces a regular complete n-partite subgraph $K_{n[s+l]}^L=[V^L]_{K_{n[m]}}$. Each $K_{n[s+l]}^L$ is of order $(s+l)n$ with each vertex of degree $(n-1)(l+s)$. In each case of (i) and (ii), $(s+l)n$ is even. Hence, by Theorem 3.1, there is a complete edge $(n-1)(l+s)$-coloring for each $K_{n[s+l]}^L$ and each vertex in $K_{n[s+l]}^L$ is incident with every color of these $(n-1)(l+s)$ colors.

It is easy to check that for any pair of points from different partite sets appears at least once in $K_{n[l+s]}^L$ for some $L\in V$ and the total number of edges of these q^2 subgraphs is equal to the number of edges in $K_{n[m]}$. Hence, these q^2 induced subgraphs form an edge decomposition of $K_{n[m]}$. If we color each of these q^2 subgraphs with a distinct set of $(n-1)(l+s)$ colors, then we get an edge $q^2(n-1)(l+s)$-coloring of $K_{n[m]}$. Since any pair of these q^2 lines has at least one point in common, the colorings of all the subgraphs form a complete edge coloring of $K_{n[m]}$ using $q^2(s+l)(n-1)=(n-1)mq$ colors. Hence $\Psi'(K_{n[m]})\geqslant(n-1)mq$. $\square$

By Theorem 3.2 and Theorem 2.2, we obtain the following result.

Theorem 3.3. *Let q be an order of a projective plane. Suppose $q+1=ns+r$ where $0<r<n$ and $s\geqslant 1$. Let $m=(s+1)q$. Then*

(i) *If q is odd and $n-r$ is even, then $\Psi'(K_{n[m]})=(n-1)mq$; and*

(ii) *If q is even and $n-r$ is odd, then $\Psi'(K_{n[m]})=(n-1)mq$.*

Proof. By Theorem 3.2, it is clear that $\Psi'(K_{n[m]})\geqslant(n-1)mq$.

On the other hand, take $t=(s+1)n/2-1$. Then, it is easy to check that $2(t+1)t/n<m$; $2(t+1)^2/n<m$; $P(n,m,t)>0$ and $Q(n,mt+1)<0$. Hence, $m=(s+1)q\in[\gamma,\varepsilon]$ and then $B(n,m)=(n-1)m$ and $\Psi'(K_{n[m]})\leqslant(n-1)mq$. Therefore, $\Psi'(K_{n[m]})=(n-1)mq$. $\square$

It can be seen from the papers about achromatic index that to determine the exact value of $\Psi'(G)$ is very difficult. We expect to obtain more results on this topic in the future.

Acknowledgements

We would like to express our thanks to Dr. Song-Tyang Liu for his help in checking some of the results by using computer and also to the referees for their helpful comments.

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