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# Numerical ranges of weighted shift matrices with periodic weights ☆

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ABSTRACT

Let $A$ be an $n$-by-$n$ ($n \geqslant 2$) matrix of the form

$$\begin{bmatrix} 0 & a_1 & & \\ & 0 & \ddots & \\ & & \ddots & a_{n-1} \\ a_n & & & 0 \end{bmatrix}.$$

We show that if the $a_j$'s are nonzero and their moduli are periodic, then the boundary of its numerical range contains a line segment. We also prove that $\partial W(A)$ contains a noncircular elliptic arc if and only if the $a_j$'s are nonzero, $n$ is even, $|a_1| = |a_3| = \cdots = |a_{n-1}|$, $|a_2| = |a_4| = \cdots = |a_n|$ and $|a_1| \neq |a_2|$. Finally, we give a criterion for $A$ to be reducible and completely characterize the numerical ranges of such matrices.



An $n$-by-$n$ ($n \geqslant 2$) *weighted shift matrix* $A$ is one of the form

$$\begin{bmatrix} 0 & a_1 & & \\ & 0 & \ddots & \\ & & \ddots & a_{n-1} \\ a_n & & & 0 \end{bmatrix},$$

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where the $a_j$'s, called the *weights* of $A$, are complex numbers. The purpose of this paper is to study the numerical ranges of such matrices with periodic weights.

Recall that the *numerical range* $W(A)$ of an $n$-by-$n$ complex matrix $A$ is by definition the subset $\{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}$ of the complex plane, where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the standard inner product and Euclidean norm in $\mathbb{C}^n$. It is known that $W(A)$ is a nonempty compact convex subset of $\mathbb{C}$. For other properties, the reader may consult [4, Chapter 1] or [2].

The study of the numerical ranges of the weighted shift matrices was started in [6]. It was proven in [6, Theorem 1] that the boundary of the numerical range of such a matrix $A$ has a line segment if and only if the weights are all nonzero and all its $(n-1)$-by-$(n-1)$ principal submatrices have identical numerical ranges (which are necessarily circular discs centered at the origin). In such discussions, we may assume that the weights are all nonnegative. In Theorem 1 below, we show that if the weights $a_1, \ldots, a_n$ of an $n$-by-$n$ weighted shift matrix $A$ are nonzero and periodic with period k, then $W(A) = W(B)$, where $B = \sum_{j=0}^{(n/k)-1} \oplus(e^{ij\theta}C)$ and $C$ is the $k$-by-$k$ weighted shift matrix with weights $a_1, \ldots, a_{k-1}, \alpha a_k$, $\alpha = (a_1 \cdots a_n)^{k/n}/(a_1 \cdots a_k)$ and $\theta = 2\pi/n$. In this case, the boundary of $W(A)$ has a line segment. In Theorem 3, we give a necessary and sufficient condition for the boundary of $W(A)$ to have a noncircular elliptic arc. More specifically, it is shown that this is the case if and only if the $a_j$'s are nonzero, $n$ is even, $|a_1| = |a_3| = \cdots = |a_{n-1}|$, $|a_2| = |a_4| = \cdots = |a_n|$ and $|a_1| \neq |a_2|$. For $n = 4$, this essentially generalizes [6, Proposition 12]. Finally, we give a criterion for $A$ to be reducible and characterize their numerical ranges in Theorem 4. In particular, it says that, for $n = 4$, $A$ is reducible if and only if either (1) $a_i = a_j = 0$ for some $i$ and $j$, $1 \leqslant i < j \leqslant n$, or (2) $|a_1| = |a_3| \neq 0$ and $|a_2| = |a_4| \neq 0$.

For an $n$-by-$n$ matrix $A$, let $A^T$ denote its transpose, $A^*$ its adjoint, Re $A$ its real part $(A + A^*)/2$ and Im $A$ its imaginary part $(A - A^*)/2i$. For $1 \leqslant i_1 < \cdots < i_m \leqslant n$, let $A[i_1, \ldots, i_m]$ denote the $(n-m)$-by-$(n-m)$ principal submatrix of $A$ obtained by deleting its rows and columns indexed by $i_1, \ldots, i_m$. The *numerical radius* $w(A)$ and *generalized Crawford number* $w_0(A)$ of $A$ are, by definition, $\max\{|z| : z \in W(A)\}$ and $\min\{|z| : z \in \partial W(A)\}$, respectively. A diagonal matrix with diagonals $a_1, \ldots, a_n$ is denoted by diag$(a_1, \ldots, a_n)$. Our basic reference for properties of matrices is [3].

For an $n$-by-$n$ matrix $A$, consider the degree-$n$ homogeneous polynomial $p_A(x, y, z) = \det(x\text{Re } A + y\text{Im } A + zI_n)$. A result of Kippenhahn [5, p. 199] says that the numerical range $W(A)$ is the convex hull of the real points in the dual of the curve $p_A(x, y, z) = 0$, that is, $W(A) = \{a + ib \in \mathbb{C} : a, b \text{ real}, ax + by + z = 0 \text{ is tangent to } p_A(x, y, z) = 0\}^\wedge$. Here, for any subset $\triangle$ of $\mathbb{C}$, $\triangle^\wedge$ denotes its convex hull, that is, $\triangle^\wedge$ is the smallest convex set containing $\triangle$.

For any nonzero complex number $z = x + iy$ ($x$ and $y$ real), arg $z$ is the angle $\theta$, $0 \leqslant \theta < 2\pi$, from the positive $x$-axis to the vector $(x, y)$. If $z = 0$, then arg $z$ can be an arbitrary real number. In the following, let $\omega_n = e^{2\pi i/n}$ for $n \geqslant 1$.

The main result of this paper is the following.

**Theorem 1.** *Let $A$ be an $n$-by-$n$ ($n \geqslant 3$) weighted shift matrix with nonzero weights $a_1, \ldots, a_n$. Assume that $|a_j| = |a_{k+j}| = \cdots = |a_{(m-1)k+j}|$ for all $1 \leqslant j \leqslant k$, where $n = km$ for some $k$ and $m$, $k, m \geqslant 2$. Then*

*(a) $p_A$ is reducible and $W(A) = W(B)$, where $B = C \oplus (e^{i\theta}C) \oplus \cdots \oplus (e^{i(m-1)\theta}C)$ and $C$ is the $k$-by-$k$ weighted shift matrix with weights $a_1, \ldots, a_{k-1}, \alpha a_k$, $\alpha = (a_1 \cdots a_n)^{1/m}/(a_1 \cdots a_k)$ and $\theta = 2\pi/n$.*

*(b) $\partial W(A)$ has a line segment $L$ and dist$(0, L) = w_0(A) = w(A[i]) =$ maximum zero of $\det(\lambda I_{n-1} - \text{Re } A[i])$ for every $i$, $1 \leqslant i \leqslant n$.*

Note that $\partial W(A)$ has a line segment for $k = 1$ by [6, Proposition 4].

An easy consequence of the preceding theorem is the following:

**Corollary 2.** *Let $A$ be an $n$-by-$n$ ($n \geqslant 3$) weighted shift matrix with weights $a_1, \ldots, a_n$. Suppose that $n - 2$ of the $a_j$'s have equal absolute value and the remaining two terms are $a_k$ and $a_l$. Then $\partial W(A)$ has a line segment if and only if all the $a_i$'s are nonzero and either*

*(a) $n$ is even, $|k - l| = n/2$, $|a_k| = |a_l|$, or*

*(b) all the $a_i$'s have the same absolute values.*

**Proof.** The sufficiency follows easily from Theorem 1. Now we prove the necessity. By [6, Theorem 1], we have that the $a_i$'s are nonzero and $W(A[1]) = \cdots = W(A[n])$. If $n$ is even and $|k-l| = n/2$, then we may assume that $k = n/2, l = n, a_i = a_j > 0$ for $1 \leqslant i < j \leqslant n-1, i,j \neq n/2$ and $a_{n/2}$, $a_n > 0$ by [6, Lemma 2(1) and (2)]. Also, $W(A[n/2]) = W(A[n])$ implies that $|a_{n/2}| = |a_n|$ by [6, Lemma 5]. Otherwise, we may assume that $1 \leqslant k < n/2, l = n, a_i = a_j > 0$ for $1 \leqslant i < j \leqslant n-1$, $i, j \neq k$ and $a_k, a_n > 0$ by [6, Lemma 2(1) and (2)]. Let $a \equiv a_i$, where $i \neq k, n$. Note that we have $W(A[k]) = W(A[2k]) = W(A[n-k]) = W(A[n])$ and $A[n]$ is the $(n-1)$-by-$(n-1)$ matrix $[s_{ij}]_{i,j=1}^{n-1}$, where $s_{i,i+1} = a$ for $1 \leqslant i \leqslant n-2, i \neq k, s_{k,k+1} = a_k$ and $s_{i,j} = 0$ otherwise. By [6, Lemma 2(1)], we may assume that $A[k]$ is the $(n-1)$-by-$(n-1)$ matrix $[t_{ij}]_{i,j=1}^{n-1}$, where $t_{i,i+1} = a$ for $1 \leqslant i \leqslant n-2, i \neq n-k, t_{n-k,n-k+1} = a_n$ and $t_{i,j} = 0$ otherwise. For the orders of $\{a_n, a_k, a\}$, consider the following three cases:

(1) $a_n \geqslant a \geqslant a_k$ or $a_n \leqslant a \leqslant a_k$. Since $W(A[n]) = W(A[k])$, by [6, Lemma 5(2)], we infer that $a_n = a = a_k$.
(2) $a_n \geqslant a_k \geqslant a$ or $a_n \leqslant a_k \leqslant a$. By [6, Lemma 2(1)], we may assume that $A[n-k]$ is the $(n-1)$-by-$(n-1)$ matrix $[u_{ij}]_{i,j=1}^{n-1}$, where $u_{i,i+1} = a$ for $1 \leqslant i \leqslant n-2, i \neq k, 2k, u_{k,k+1} = a_n$, $u_{2k,2k+1} = a_k$ and $u_{i,j} = 0$ otherwise. Since $W(A[n]) = W(A[n-k])$, by [6, Lemma 5(2)], we also infer that $a_n = a = a_k$.
(3) $a_k \geqslant a_n \geqslant a$ or $a_k \leqslant a_n \leqslant a$. By [6, Lemma 2(1)], we may assume that $A[2k]$ is the $(n-1)$-by-$(n-1)$ matrix $[v_{ij}]_{i,j=1}^{n-1}$, where $v_{i,i+1} = a$ for $1 \leqslant i \leqslant n-2, i \neq n-k, n-2k$, $v_{n-2k,n-2k+1} = a_n, v_{n-k,n-k+1} = a_k$ and $v_{i,j} = 0$ otherwise. Since $W(A[k]) = W(A[2k])$, by [6, Lemma 5(2)], we obtain that $a_n = a = a_k$ and complete the proof. □

We are now ready to prove Theorem 1.

**Proof of Theorem 1**

(a) Let $B = C \oplus (e^{i\theta}C) \oplus \cdots \oplus (e^{i(m-1)\theta}C)$, where $C$ is the $k$-by-$k$ weighted shift matrix with weights $a_1, \ldots, a_{k-1}, \alpha a_k$, $\alpha = (a_1 \cdots a_n)^{1/m}/(a_1 \cdots a_k)$ and $\theta = 2\pi/n$. Since $|a_j| = |a_{k+j}| = \cdots = |a_{(m-1)k+j}| \neq 0$ for $1 \leqslant j \leqslant k$ and $\arg\,(a_1 \cdots a_n)/((a_1 \cdots a_k)^m \alpha^m) = 0$, we may assume that $A$ is the $n$-by-$n$ weighted shift matrix with periodic weights $a_1, \ldots, a_{k-1}, \alpha a_k, \ldots, a_1, \ldots, a_{k-1}, \alpha a_k$ by [6, Lemma 2(2)]. Let the matrix $x\mathrm{Re}\,A + y\mathrm{Im}\,A + zI_n$ be partitioned as $\begin{bmatrix} C_{11} & \cdots & C_{1m} \\ \vdots & & \vdots \\ C_{m1} & \cdots & C_{mm} \end{bmatrix}$ with $C_{ij}$ of sizes $k$-by-$k$ for all $i, j$, $1 \leqslant i,j \leqslant n$. Since $C_{1j} + \cdots + C_{mj} = x\mathrm{Re}\,C + y\mathrm{Im}\,C + zI_k$, for all $j$, $1 \leqslant j \leqslant m$, we have

$$
\begin{aligned}
p_A(x,y,z) &= \det(x\mathrm{Re}\,A + y\mathrm{Im}\,A + zI_n) \\
&= \det \begin{bmatrix} C_{11} + \cdots + C_{m1} & C_{12} + \cdots + C_{m2} & \cdots & C_{1m} + \cdots + C_{mm} \\ C_{21} & C_{22} & \cdots & C_{2m} \\ \vdots & \vdots & & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{mm} \end{bmatrix} \\
&= \det \begin{bmatrix} x\mathrm{Re}\,C + y\mathrm{Im}\,C + zI_k & 0 & \cdots & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}.
\end{aligned}
$$

Hence $p_C | p_A$. Since $A$ and $\omega_n^j A$ are unitarily equivalent for all integer $j$, then $p_C | p_{\omega_n^j A}$ for all integer $j$, consequently, $p_{\omega_n^j C} | p_A$ for all $j = 0, 1, \ldots, m-1$. Note that the real foci of the curve $p_{\omega_n^j C} = 0$ are eigenvalues of $\omega_n^j C$ for each $j$. Since $\sigma(C) = \{\lambda, \omega_k\lambda, \ldots, \omega_k^{k-1}\lambda\}$, where $\lambda = (a_1 \cdots a_k)^{1/k}$, it follows that $\sigma(\omega_n^i C) \cap \sigma(\omega_n^j C) = \emptyset$ for any $0 \leqslant i < j \leqslant m-1$, thus the homogeneous polynomials $p_{\omega_n^i C}$ and $p_{\omega_n^j C}$ have no common factor for any $0 \leqslant i < j \leqslant m-1$. Therefore, we deduce that $p_A = \prod_{j=0}^{m-1} p_{\omega_n^j C}$ or $W(A) = W(B)$. This completes the proof.

(b) By [6, Proposition 3(6)] and its proof, we need only prove that $\partial W(A)$ has a line segment. Since the $a_j$'s are nonzero, from [6, Lemma 2(2)], we may assume that $a_j > 0$ for all $j$. Then $C$ is a $k$-by-$k$ weighted shift matrix with positive weights, thus $w(C) \in W(C)$. Let $D = \{z \in \mathbb{C} : |z| \leqslant w(C)\}$ and $B = C \oplus B'$, where $B' = \omega_n C \oplus \omega_n^2 C \oplus \cdots \oplus \omega_n^{m-1} C$. For each $j = 0, 1, \ldots, m-1$, by [6, Proposition 3(4)], we have $W(\omega_n^j C) \subseteq D$ and $W(\omega_n^j C) \cap \partial D = \{\omega_n^j \omega_k^i w(C) : i = 0, 1, \ldots, k-1\}$. Since each point on $\partial D$ is an extreme point of $D$, thus $W(B') \subseteq D$ and $W(B') \cap \partial D = \{\omega_n^j w(C) : j = 0, 1, \ldots, n-1\} \setminus \{\omega_k^j w(C) : j = 0, 1, \ldots, k-1\}$. It follows that $w(C) \in W(C) \setminus W(B')$ and $\omega_n w(C) \in W(B') \setminus W(C)$, that is, $W(C) \nsubseteq W(B')$ and $W(B') \nsubseteq W(C)$. Therefore, $W(B')$ and $W(C)$ have a common supporting line, says, $\cos(t)x + \sin(t)y = r$. This implies $p_C(\cos(t), \sin(t), -r) = 0 = p_{B'}(\cos(t), \sin(t), -r)$ and $r = \max\sigma(\mathrm{Re}\,(e^{-it}C)) = \max\sigma(\mathrm{Re}\,(e^{-it}B')) = \max\sigma(\mathrm{Re}\,(e^{-it}B))$. Since $p_A = p_B = p_C p_{B'}$ from Theorem 1(a) and its proof, it follows that $r$ is the maximal eigenvalue of $\mathrm{Re}\,(e^{-it}A)$ with multiplicity at least two. By [6, Lemma 11], we obtain that the boundary of $W(A)$ contains a line segment as asserted. □

The next theorem gives a necessary and sufficient condition for an $n$-by-$n$ weighted shift matrix $A$ to have a noncircular elliptic arc in $\partial W(A)$. Moreover, in this case, $\partial W(A)$ also has a line segment.

**Theorem 3.** *Let $A$ be an $n$-by-$n$ ($n \geqslant 3$) weighted shift matrix with weights $a_1, \ldots, a_n$. Then $\partial W(A)$ has a noncircular elliptic arc if and only if the $a_j$'s are nonzero, $n$ is even, $|a_1| = |a_3| = \cdots = |a_{n-1}|$, $|a_2| = |a_4| = \cdots = |a_n|$ and $|a_1| \neq |a_2|$. In this case, $W(A) = W(B)$, where $B = C \oplus (e^{i\theta}C) \oplus \cdots \oplus (e^{((n/2)-1)\theta}C)$, $C = \begin{bmatrix} 0 & a_1 \\ \alpha a_2 & 0 \end{bmatrix}$, $\alpha = (a_1 \cdots a_n)^{2/n}/(a_1 a_2)$ and $\theta = 2\pi/n$, and $\partial W(A)$ has a line segment.*

**Proof.** The sufficiency follows easily from Theorem 1(a) and the fact that $W\left(\begin{bmatrix} 0 & a_1 \\ \alpha a_2 & 0 \end{bmatrix}\right)$ is a noncircular elliptic disc as $|a_1| \neq |\alpha a_2|$ and both are nonzero, where $\alpha = (a_1 \cdots a_n)^{2/n}/(a_1 a_2)$.

To prove the necessity, by [6, Lemma 2(2)], we have that $A$ is unitarily equivalent to $e^{i\phi}A'$, where $\phi = (\sum_{j=1}^n \arg a_j)/n$ and $A'$ is the $n$-by-$n$ weighted shift matrix with weights $|a_1|, \ldots, |a_n|$. Then $\sigma(A) = \{|a_1 \cdots a_n|^{1/n}\omega_n^j : j = 0, 1, \ldots, n-1\}$. Since $\partial W(A)$ has a noncircular elliptic arc, by [1, Theorem], there is a 2-by-2 matrix $C_1$ such that $p_{C_1} | p_{A'}$ and $\sigma(C_1) \subseteq \sigma(A')$, say, $\sigma(C_1) = \{\beta, \gamma\}$. From [6, Proposition 3(1)], we infer that $p_{\omega_n^j C_1} | p_{A'}$ and $\sigma(\omega_n^j C_1) \subseteq \sigma(A')$ for all $j = 0, 1, \ldots, n-1$. Therefore, $\sigma(A') \supseteq \{\omega_n^j\beta : j = 0, \ldots, n-1\} \cup \{\omega_n^j\gamma : j = 0, \ldots, n-1\}$. Since these sets $\sigma(A')$, $\{\omega_n^j\beta : j = 0, \ldots, n-1\}$ and $\{\omega_n^j\gamma : j = 0, \ldots, n-1\}$ consist of $n$ distinct elements, we deduce that $\sigma(A') = \{\omega_n^j\beta : j = 0, \ldots, n-1\} = \{\omega_n^j\gamma : j = 0, \ldots, n-1\}$. Therefore, we may assume that $\beta = |a_1 \cdots a_n|^{1/n}$ and $\gamma = \omega_n^{j_0}\beta$ for some $j_0$. Now, if $\omega_n^{j_0} \neq -1$ or $n$ is odd, then these irreducible homogeneous polynomials $p_{C_1}, p_{\omega_n C_1}, \ldots, p_{\omega_n^{\lfloor n/2 \rfloor} C_1}$ are distinct, it follows that $p_{A'}$ can be divided by the homogeneous polynomial $\prod_{j=0}^{\lfloor n/2 \rfloor} p_{\omega_n^j C_1}$ of degree $2(\lfloor n/2 \rfloor + 1) > n$, this contradicts to the fact that $p_{A'}$ is of degree $n$. Therefore, we deduce that $\omega_n^{j_0} = -1$ and $n$ is even.

Moreover, $p_{A'} = \prod_{j=0}^{(n/2)-1} p_{\omega_n^j C_1}$. On the other hand, since $C_1$ is 2-by-2 with eigenvalues $\pm|a_1 \cdots a_n|^{1/n}$, by unitarily equivalence, we may assume that $C_1 = \begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix}$, where $b_1, b_2 > 0$, $b_1 \neq b_2$ and $b_1 b_2 = |a_1 \cdots a_n|^{2/n}$. Let $B' = C_1 \oplus \omega_n C_1 \oplus \cdots \oplus \omega_n^{(n/2)-1} C_1$ and $B_1$ be the $n$-by-$n$ weighted shift matrix with periodic weights $b_1, b_2, b_1, b_2, \ldots, b_1, b_2$. By Theorem 1(a), we have $p_{B_1} = p_{B'} = p_{A'}$. Compute now the coefficients of $(x^2 + y^2)z^{n-2}$ and $y^n$ of $p_{A'}$ and $p_{B_1}$. Since $p_{A'} = p_{B_1}$, we have $\sum_{j=1}^n |a_j|^2 = (b_1^2 + b_2^2)n/2$ and $(\prod_{j=1}^{n/2} |a_{2j-1}| - \prod_{j=1}^{n/2} |a_{2j}|)^2 = (b_1^{n/2} - b_2^{n/2})^2$. Hence we may assume that $b_1^{n/2} - b_2^{n/2} = \prod_{j=1}^{n/2} |a_{2j-1}| - \prod_{j=1}^{n/2} |a_{2j}|$. In addition, $b_1 b_2 = |a_1 \cdots a_n|^{2/n}$ implies that $b_1^{n/2} = \prod_{j=1}^{n/2} |a_{2j-1}|$ and $b_2^{n/2} = \prod_{j=1}^{n/2} |a_{2j}|$. We also have

$$\sum_{j=1}^{n} |a_j|^2 = \sum_{j=1}^{\frac{n}{2}} |a_{2j-1}|^2 + \sum_{j=1}^{\frac{n}{2}} |a_{2j}|^2$$
$$\geqslant \frac{n}{2}\left(\prod_{j=1}^{\frac{n}{2}} |a_{2j-1}|^2\right)^{\frac{2}{n}} + \frac{n}{2}\left(\prod_{j=1}^{\frac{n}{2}} |a_{2j}|^2\right)^{\frac{2}{n}} = \frac{n}{2}\left(b_1^2 + b_2^2\right).$$

Therefore, the equality holds if and only if $b_1 = |a_{2j-1}| \neq 0$, $b_2 = |a_{2j}| \neq 0$ for all $j$, $1 \leqslant j \leqslant n/2$ and $b_1 \neq b_2$. Let $C = e^{i\phi}C_1$ and $B = e^{i\phi}B_1$. Then $C$ is unitarily equivalent to $\begin{bmatrix} 0 & a_1 \\ \alpha a_2 & 0 \end{bmatrix}$, where $\alpha = e^{i(2\phi - \arg a_1 - \arg a_2)} = (a_1 \cdots a_n)^{2/n}/(a_1 a_2)$ and $W(A) = e^{i\phi}W(A') = e^{i\phi}W(B_1) = W(B)$. This proves our assertion. In particular, it follows from Theorem 1(b) that $\partial W(A)$ has a line segment. □

Note that the weighted shift matrix $A$ in the above theorem is a special case of the ones considered in Theorem 1. The next theorem is another special case. Recall that a matrix $A$ is said to be *reducible* if it is unitarily equivalent to the direct sun of two other matrices; otherwise, $A$ is *irreducible*. We characterize those $n$-by-$n$ weighted shift matrices $A$ which are reducible in the following theorem.

**Theorem 4.** *Let $A$ be an $n$-by-$n$ ($n \geqslant 2$) weighted shift matrix with weights $a_1, \ldots, a_n$. Then $A$ is reducible if and only if one of the following cases holds:*

*(1) $a_i = a_j = 0$ for some $1 \leqslant i < j \leqslant n$,*
*(2) $n$ is odd, $|a_i| = |a_j| \neq 0$ for all $1 \leqslant i < j \leqslant n$,*
*(3) $n$ is even, $|a_i| = |a_{i+(n/2)}| \neq 0$ for all $1 \leqslant i \leqslant n/2$.*

*In case (1), $A$ is unitarily equivalent to $B_1 \oplus B_2$, where $B_1$ and $B_2$ are the weighted shift matrices with weights $a_{j+1}, \ldots, a_{i-1}, 0$ and $a_{i+1}, \ldots, a_{j-1}, 0$, respectively ($a_r \equiv a_{n+r}$ for $1 \leqslant r \leqslant n$, $B_1 \equiv [0]$ if $i = 1, j = n$ and $B_2 \equiv [0]$ if $i = j - 1$). Hence $W(A)$ is a circular disc centered at the origin. In case (2), $A$ is unitarily equivalent to $\mathrm{diag}(\alpha, \alpha\omega_n, \ldots, \alpha\omega_n^{n-1})$, where $\omega_n = e^{2\pi i/n}$ and $\alpha = (a_1 \cdots a_n)^{1/n}$. Hence $W(A)$ is a closed regular $n$-gonal region centered at the origin and the distance from the origin to its vertices equals $|a_1 \cdots a_n|^{1/n}$. In case (3), $A$ is unitarily equivalent to $A_1 \oplus e^{i\theta}A_1$, where $\theta = 2\pi/n$ and $A_1$ is an $(n/2)$-by-$(n/2)$ weighted shift matrix with weights $a_1, \ldots, a_{(n/2)-1}, \alpha a_{n/2}$, $\alpha = (a_1 \cdots a_n)^{1/2}/(a_1 \cdots a_{n/2})$. In particular, $\partial W(A)$ has a line segment.*

**Proof**

(1) Let $a_i = a_j = 0$ for some $i, j$, $1 \leqslant i < j \leqslant n$. Also, by [6, Lemma 2(1)], we may assume that $j = n$. Then $A = B_1 \oplus B_2$, where $B_1$ and $B_2$ are the weighted shift matrices with weights $a_{j+1}, \ldots, a_n, a_1, \ldots, a_{i-1}, 0$ and $a_{i+1}, \ldots, a_{j-1}, 0$, respectively ($a_r \equiv a_{n+r}$ for $1 \leqslant r \leqslant n$, $B_1 \equiv [0]$

if $i = 1, j = n$ and $B_2 \equiv [0]$ if $i = j - 1$). Hence $W(A)$ is a circular disc centered at the origin. Let $a_i = 0$ for some $i$, $1 \leqslant i \leqslant n$, and $a_j \neq 0$ for all $j \neq i$. Again, by [6, Lemma 2(1)], we may assume that $i = n$. Then for any orthogonal projection $P = [p_{ij}]_{i,j=1}^n$ such that $AP = PA$, we have $a_i(p_{i,i} - p_{i+1,i+1}) = a_{i+1}(p_{i+1,i+1} - p_{i+2,i+2}) = 0$ for $1 \leqslant i \leqslant n-1$. Thus $p_{1,1} = p_{2,2} = \cdots = p_{n,n}$. In addition, $AP = PA$ also implies that $a_i p_{i+1,1} = 0$ for $1 \leqslant i \leqslant n-1$. We substitute $p_{i+1,1} = 0$ in these equalities for $AP = PA$. Then $a_i p_{i+1,2} = a_1 p_{i,1} = 0$ for $2 \leqslant i \leqslant n-1$. Proceeding successively with the remaining equalities for $AP = PA$, we have $p_{i,j} = 0$ for $i > j$. Hence the assumption $P = P^* = P^2$ implies that $P = 0$ or $P = I_n$. Therefore, $A$ is irreducible.

(2) If $n$ is odd and $a_i \neq 0$ for all $1 \leqslant i \leqslant n$, then we may assume that $a_i > 0$ by [6, Lemma 2(2)]. For any orthogonal projection $P = [p_{ij}]_{i,j=1}^n$ such that $AP = PA$, we have $a_1(p_{1,1} - p_{2,2}) = a_2(p_{2,2} - p_{3,3}) = \cdots = a_n(p_{n,n} - p_{1,1}) = 0$. Thus $p_{1,1} = p_{2,2} = \cdots = p_{n,n}$. In addition, $AP = PA$ also implies that $a_i p_{i+1,i+2} = a_{i+1} p_{i,i+1}$ and $a_{i+1} p_{i+2,i+1} = a_i p_{i+1,i}$ for $1 \leqslant i \leqslant n$ ($p_{n,n+1} \equiv p_{n,1}$, $p_{n+1,n+2} \equiv p_{1,2}$, $p_{n+1,n} \equiv p_{1,n}$, $p_{n+2,n+1} \equiv p_{2,1}$, $a_{n+1} \equiv a_1$). Since $P = P^*$, we have $a_{i+1} p_{i+1,i+2} = a_i p_{i,i+1}$ for $1 \leqslant i \leqslant n$. Thus $p_{i,i+1} = 0$ for some $i$ or $a_1 = \cdots = a_n$. Hence $p_{i,i+1} = 0$ for every $i$, $1 \leqslant i \leqslant n$ or $a_1 = \cdots = a_n$. Since $n-1$ is even, by the same process, we have $p_{i,j} = 0$ for all $i < j$ or $a_1 = \cdots = a_n$. Thus $P = P^* = P^2$ implies that $P$ equals 0 or $I_n$, or $a_1 = \cdots = a_n$. That is, $A$ is reducible if and only if $|a_1| = \cdots = |a_n| \neq 0$. Hence the assertion on $W(A)$ follows from [6, Proposition 4].

(3) If $n$ is even and $a_i \neq 0$ for all $1 \leqslant i \leqslant n$, then we may assume that $a_i > 0$ by [6, Lemma 2(2)]. For any orthogonal projection $P = [p_{ij}]_{i,j=1}^n$ such that $AP = PA$, following a similar argument as in the proof of (2), we obtain $p_{1,1} = p_{2,2} = \cdots = p_{n,n}$ and $p_{i,j} = 0$ for all $i \neq j$, $|i-j| \neq n/2$. In addition, we also have $a_i p_{i+1,(n/2)+i+1} = a_{(n/2)+i} p_{i,(n/2)+i}$ and $a_{(n/2)+i} p_{(n/2)+i+1,i+1} = a_i p_{(n/2)+i,i}$ for every $i$, $1 \leqslant i \leqslant n/2$ ($p_{(n/2)+1,n+1} \equiv p_{(n/2)+1,1}$, $p_{n+1,(n/2)+1} \equiv p_{1,(n/2)+1}$). Hence $P = P^* = P^2$ implies that $P$ equals 0 or $I_n$, or $a_1 = a_{(n/2)+1}, \ldots, a_{n/2} = a_n$. Therefore, $A$ is reducible if and only if $|a_i| = |a_{i+(n/2)}|$ for all $i$, $1 \leqslant i \leqslant n/2$. Hence $\partial W(A)$ has a line segment by Theorem 1(b). Moreover, by [6, Lemma 2(2)], $A$ is unitarily equivalent to $e^{i\psi}B$, where $\psi = (\sum_{j=1}^n \arg\ a_j)/n$ and $B$ is the $n$-by-$n$ weighted shift matrix with weights $|a_1|, \ldots, |a_{n/2}|, |a_1|, \ldots, |a_{n/2}|$. Let $U = (1/\sqrt{2}) \begin{bmatrix} I_{n/2} & I_{n/2} \\ I_{n/2} & -I_{n/2} \end{bmatrix}$. Then $U^*BU = B_1 \oplus e^{i\theta}B_1$, where $\theta = 2\pi/n$ and $B_1$ is the $(n/2)$-by-$(n/2)$ weighted shift matrix with weights $|a_1|, \ldots, |a_{n/2}|$. Hence $A$ is unitarily equivalent to $(e^{i\psi}B_1) \oplus e^{i\theta}(e^{i\psi}B_1)$. Let $A_1 = e^{i\psi}B_1$. Then $A_1$ is the $(n/2)$-by-$(n/2)$ weighted shift matrix with weights $a_1, \ldots, a_{(n/2)-1}, \alpha a_{n/2}$, where $\alpha = e^{i\phi}$ and $\phi = (n/2)\theta - (\sum_{j=1}^{n/2} \arg\ a_j) = (n/2)(\sum_{j=1}^{n} \arg\ a_j)/n - (\sum_{j=1}^{n/2} \arg\ a_j) = (\sum_{j=1}^{n/2} \arg\ a_{(n/2)+j} - \sum_{j=1}^{n/2} \arg\ a_j)/2$. This proves our assertion. □

An immediate corollary of Theorem 4 and [1, Theorem] is the following:

**Corollary 5.** *Let $A$ be an $n$-by-$n$ ($n \geqslant 3$) weighted shift matrix with weights $a_1, \ldots, a_n$ and $a_i = 0$ for some $i$, $1 \leqslant i \leqslant n$. Then*

*(1) $p_A$ is reducible.*
*(2) $A$ is reducible if and only if $a_j = 0$ for some $j \neq i$, $1 \leqslant j \leqslant n$.*

Recall that the reducibility of an $n$-by-$n$ matrix $A$ implies the reducibility of $p_A$ but the converse is in general not true. We give two examples of weighted shift matrices $A$ for which $p_A$ is reducible but $A$ is irreducible.

**Example 6**

(1) If $A = J_n$ ($n \geqslant 3$), then $A$ is irreducible, $p_A$ is reducible and $\partial W(A)$ has no line segment.
(2) If $A$ is a 6-by-6 weighted shift matrix with weights 1, 2, 1, 2, 1, 2, then $A$ is irreducible, $p_A$ is reducible but $\partial W(A)$ has a line segment.

**Proof**

(1) From [6, Proposition 3(3)], we obtain that $W(A)$ is a circular disc centered at the origin. Hence the assertion follows directly from [1, Theorem] and Theorem 4.
(2) Follow directly from Theorems 1 and 4. □

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