

Minimal Linear Combinations of the Inertia Parameters of a Manipulator

Shir-Kuan Lin

Abstract—This paper deals with the problem of identifying the inertia parameters of a manipulator. We begin by introducing the terminology of minimal linear combinations of the inertia parameters (MLC's) that are linearly independent of one another and determine the manipulator dynamics while keeping the number of linear combinations of the inertia parameters to a minimum. The problem is then to find an identification procedure for estimating the MLC's and to use the MLC's in the inverse dynamics for control. The recursive Newton-Euler formulation is rederived in terms of the MLC's. The resulting formulation is almost as efficient as the most efficient formulation in the literature. This formulation also provides a starting point from which to derive a recursive identification procedure. The identification procedure is simple and efficient, since it does not require symbolic closed-form equations and it has a recursive structure. The three themes concerning the dynamic modeling of a manipulator—the MLC's, the inverse dynamics in terms of the MLC's, and the identification procedure—are treated in sequence in this paper.

I. INTRODUCTION

THE dynamic model of a manipulator is highly nonlinear and requires knowledge of the kinematic parameters (relations between two adjacent links) and the inertia parameters (mass, center of mass and inertia tensor of each link). The kinematic parameters are usually provided by the manufacturer or can be calibrated precisely. However, the inertia parameters of industrial robots are almost all unavailable from manufacturers, because these values are not needed for commercial controllers. Yet, the values of the inertia parameters are required for most modern control schemes of manipulators that incorporate the inverse dynamics. To evaluate the inertia parameters of the manipulator dynamics, Armstrong *et al.* [1] disassembled a PUMA 560 robot and used a mechanical method to measure the parameters. This approach is tedious and does not yield precise results. Fortunately, Atkeson *et al.* [3] found that the actuator forces of a manipulator are linear functions of the inertia parameters (i.e., the dynamics of a manipulator can be expressed as linear equations with respect to the inertia parameters), provided that friction can be neglected or considered separately. Previous attempts to identify the inertia parameters have tried to formulate the

linear equations either explicitly [6], [9], [11], [15], [17] or implicitly [3], [29], [30], [31].

Identifying the inertia parameters is still difficult, however, for not all parameters can be estimated. Some parameters affect the manipulator dynamics jointly, not independently. Khosla and Kanade [17] intuitively regrouped the closed-form dynamic equations, and other researchers [9], [11], [15] developed regrouping rules to minimize the number of inertia parameters appearing in the linear equations. These approaches are not practical for a manipulator with six or more joints since the closed-form dynamic equations of a six-joint manipulator are difficult to analyze. Some authors [3], [7], [34] have presented numerical approaches such as singular value decomposition and the QR method. Because we lack knowledge of the physical meaning of the identified parameters, these parameters cannot be used effectively in computing the inverse dynamics.

Gautier *et al.* [8], [10], [12] developed a regrouping rule to eliminate redundant inertia parameters and to symbolically form a set of the minimal parameters needed to determine the dynamic model. Mayeda *et al.* [25]–[28] found the minimal parameters in closed form. Although the results of Gautier *et al.* and Mayeda *et al.* are substantially the same [12], the approach of Gautier *et al.* requires a regrouping process for each type of manipulator.

In this paper, we first show that the manipulator dynamics are uniquely determined by a set of minimal parameters which are linear combinations of the inertia parameters and are linearly independent. These parameters are termed the *minimal linear combinations* (MLC's) of the inertia parameters in this context. Although the notation of the MLC's is equivalent to that of minimal parameters in the literature, we must emphasize that the minimal parameters are in fact *linear combinations* of the original inertia parameters. A set of MLC's found by another approach in the present author's earlier work [23] will be used to interpret the concept of MLC's.

Finding the MLC's of the inertia parameters does not provide a complete solution for the dynamic modeling of a manipulator. The central problem is to find an efficient identification procedure for the set of MLC's. Application of the identified MLC's to the inverse or forward dynamics is also essential for manipulator control and simulation. These three problems have seldom been addressed together in the context of a single paper. This paper attempts to solve the three problems in sequence.

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S.-K. Lin is with the Institute of Control Engineering, National Chiao Tung University, Hsinchu 30050, Taiwan.
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A new version of the recursive Newton-Euler formulation in terms of the set of MLC's is derived in this paper. From the new formulation, we deduce an identification procedure for estimating the MLC's of the inertia parameters. The identification procedure is recursive from link n to link 1 and does not require symbolic closed-form dynamic equations. The identified inertia constants of the composite bodies $i + 1$ to n are used to numerically form the linear equations for the actuator force of joint i , so that the identifiable inertia constants (i.e., MLC's) of the composite body i can be estimated by the linear least squares method. This procedure is distinct from the one used in an earlier work by the author [23]. The latter strictly requires that only one joint move at a time, so it is an off-line procedure, although the same MLC's are estimated.

However, the identification procedure proposed here is limited in that friction must be treated separately from the dynamic model of the manipulator. The dominant dynamics of direct drive robots such as MIT DDArm [3] and CMU DDArm II [17] can be obtained from the standard Newton-Euler formulation [19], so the present identification method is valid for estimating the MLC's of these manipulators. For a manipulator with high-ratio gear trains, such as the PUMA arm, the present method is valid only when the viscous and static friction and the inertia of the motor actuators can be identified *a priori* (techniques for doing so can be found in the work of Leahy and Saridis [19]). In any case, this paper provides a starting point for future investigation of the identification problem for manipulators with high-ratio gear trains.

This paper is organized as follows. Section II describes the concept of minimal linear combinations (MLC's) of the inertia parameters. The new version of the recursive Newton-Euler formulation in terms of the MLC's is derived in Section III. Section IV presents the identification procedure.

II. MINIMAL LINEAR COMBINATIONS OF INERTIA PARAMETERS

Knowing that the dynamics of a manipulator (neglecting the effects of friction) can be formulated as linear equations with respect to the inertia parameters [3], we consider dynamic system with the linear deterministic form of

$$\mathbf{y} = \mathbf{A}(\theta)\mathbf{x} \quad (1)$$

where $\mathbf{y} \in \mathbb{R}^n$, $\theta \in \mathbb{R}^m$ are observable signals, $\mathbf{x} \in \mathbb{R}^p$ consists of the system parameters, $p > n$, and $\mathbf{A}(\theta): \mathbb{R}^m \rightarrow \mathbb{R}^{n \times p}$.

A set of columns $\mathbf{a}_i(\theta): \mathbb{R}^m \rightarrow \mathbb{R}^n$ is said to be linearly dependent over \mathbb{R}^m if there exist constants $\alpha_i, i = 1, \dots, n$, not all zero such that

$$\sum_{i=1}^n \alpha_i \mathbf{a}_i(\theta) = \mathbf{0}, \quad \forall \theta \in \mathbb{R}^m. \quad (2)$$

If $\alpha_i, i = 1, \dots, n$, are all zero, the set is said to be linearly independent over \mathbb{R}^m . By this definition we obtain the following lemma [22].

Lemma 1: The number of linearly independent columns of $\mathbf{A}(\theta)$ in (1) over \mathbb{R}^m is $k \leq p$ if and only if there exist $\bar{\mathbf{A}}(\theta): \mathbb{R}^m \rightarrow \mathbb{R}^{n \times k}$ whose columns are linearly independent over \mathbb{R}^m , and $\mathbf{w}(\mathbf{x}): \mathbb{R}^p \rightarrow \mathbb{R}^k$ whose elements are linear combinations of \mathbf{x} and are linearly independent over \mathbb{R}^p , such that

$$\mathbf{A}(\theta)\mathbf{x} = \bar{\mathbf{A}}(\theta)\mathbf{w}(\mathbf{x}), \quad \forall \theta \in \mathbb{R}^m \quad \text{and} \quad \mathbf{x} \in \mathbb{R}^p. \quad (3)$$

According to the least squares theory [18], not all system parameters of the system (1) can be identified if the columns of $\mathbf{A}(\theta)$ are linearly dependent over \mathbb{R}^m , i.e., \mathbf{A} is rank-deficient. Conversely, if \mathbf{A} is of full rank, all system parameters \mathbf{x} are identifiable. Lemma 1 then states that a set of linear combinations of the system parameters, $\mathbf{w}(\mathbf{x})$, is identifiable since $\bar{\mathbf{A}}$ is of full rank. $\mathbf{A}(\theta)\mathbf{x}$ fully determines \mathbf{y} , as does $\bar{\mathbf{A}}(\theta)\mathbf{w}(\mathbf{x})$. Hence knowledge of $\mathbf{w}(\mathbf{x})$ is sufficient to determine \mathbf{y} . The parameter identification problem of the deterministic system (1) turns out to be to find and identify $\mathbf{w}(\mathbf{x})$. To make use of this fact, we introduce the following definition.

Definition 1: A set $\mathbf{w}(\mathbf{x})$ is a set of *minimal linear combinations* (MLCs) of the system parameters for the system (1) if the set is linearly independent over the domain of \mathbf{w} and there exists $\bar{\mathbf{A}}(\theta)$ whose columns are linearly independent over the domain of $\bar{\mathbf{A}}$ such that (3) holds.

For a manipulator, we are concerned with the MLC's of the inertia parameters. Ha *et al.* [15] showed that the dynamic model of a manipulator can be formulated in a form like (3) by using intuitive regrouping rules. Lemma 1 gives a necessary condition for the number of linearly independent columns of $\mathbf{A}(\theta)$ and rigorously interprets the relation between the system parameters and the MLC's of the system parameters for the deterministic system (1). Since there are numerous methods for selecting $\bar{\mathbf{A}}(\theta)$ from $\mathbf{A}(\theta)$, the set of MLC's is not unique. The notations of *base parameters* [25]–[28] and *minimum inertia parameters* [8], [12] in the literature refer to the same basic idea as the MLC's of inertia parameters. The notation of MLC's, however, provides direct insight into the parameter identification of a manipulator. In the following, we use the minimal parameters found in [23] to elucidate the concept of MLC's.

We consider a manipulator with n low-pair joints, which are labeled joints 1 to n outward from the base. Assign a body-fixed frame on each joint (i.e., frame E_i is fixed on joint i) in accord with the normal driving-axis coordinate system [20] (known also as modified Denavit-Hartenberg notation [5]). The distance from the origin of E_i to that of E_j is designated j_i and that to the center of mass of link i is designated c_i .

In the normal driving-axis coordinate system (see Fig. 1), the z -axis of a body-fixed frame is the driving axis of the corresponding link, i.e., the unit vector along joint i is $\mathbf{u}_i^{(i)} = [0, 0, 1]^T$, where the superscript “ $\langle i \rangle$ ” denotes the representation of a vector with respect to frame E_i . The distance from the origin of frame E_{i-1} to frame E_i is shown to be

$${}_{i-1}^i \mathbf{s}^{(i-1)} = \begin{bmatrix} b_i \\ -d_i S \beta_i \\ d_i C \beta_i \end{bmatrix}, \quad \text{or} \quad {}_{i-1}^i \mathbf{s}^{(i)} = \begin{bmatrix} b_i C \theta_i \\ -b_i S \theta_i \\ d_i \end{bmatrix} \quad (4)$$

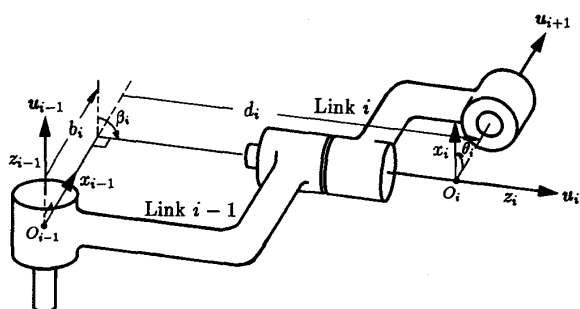


Fig. 1. Normal driving-axis coordinate system.

where $S\theta_i \equiv \sin \theta_i$, $C\theta_i \equiv \cos \theta_i$, and b_i , d_i , β_i , and θ_i are the geometrical parameters of the coordinate system shown in Fig. 1. The coordinate transformation matrix from E_{i-1} to E_i is then

$${}_{i-1}^i \mathbf{R} = \begin{bmatrix} C\theta_i & -S\theta_i & 0 \\ C\beta_i S\theta_i & C\beta_i C\theta_i & -S\beta_i \\ S\beta_i S\theta_i & S\beta_i C\theta_i & C\beta_i \end{bmatrix} \quad (5)$$

The *composite body* i is defined as the union of link i to link n . Let the mass of the composite body i and the first moment of the composite body about the origin of E_i be denoted by \hat{m}_i and \hat{c}_i , respectively, i.e., $\hat{m}_i = \sum_{j=i}^n m_j$ and $\hat{c}_i^{(i)} = \sum_{j=i}^n m_j ({}^j_i \mathbf{s}^{(i)} + \mathbf{c}_j^{(i)})$, where m_j is the mass of link j . According to the Huygen-Steiner formula [36], the inertia tensor of the composite body i about the origin of frame E_i is

$$\hat{\mathbf{J}}_i^{(i)} = \sum_{j=i}^n {}^j_i \mathbf{R} \mathbf{I}_j^{(j)} {}^j_i \mathbf{R}^T - m_j [({}^j_i \mathbf{s}^{(i)} + \mathbf{c}_j^{(i)}) \times] [({}^j_i \mathbf{s}^{(i)} + \mathbf{c}_j^{(i)}) \times] \quad (6)$$

where $\mathbf{I}_j^{(j)}$ is the representation of the inertia tensor of link j with respect to frame E_j and $[\mathbf{a} \times]$ denotes a skew-symmetric matrix representing the vector multiplication, i.e., $[\mathbf{a} \times] \mathbf{b} = \mathbf{a} \times \mathbf{b}$. In this paper, the hat symbol “ $\hat{\cdot}$ ” is used to denote the inertia parameters (mass, first moment, and inertia tensor) of a composite body.

In the following, the notation $(\cdot)_{ij}$ denotes the (i, j) th entry of a matrix and $(\cdot)_x$ the x -component of a three-dimensional vector. Define

$$K_i^* \equiv (1 - K_i) \equiv \begin{cases} 1, & \text{for rotational joint } i, \\ 0, & \text{for translational joint } i. \end{cases} \quad (7)$$

By the principle of mathematical induction, it has been shown [23] that the first moment and the inertia tensor of the composite body i can be expressed as the sum of a constant vector (\mathbf{k}_i or \mathbf{U}_i) and a varying vector (ℓ_i or \mathbf{V}_i) as follows:

$$\hat{c}_i^{(i)} = \mathbf{k}_i + \ell_i \quad (8)$$

$$\hat{\mathbf{J}}_i^{(i)} = \mathbf{U}_i + \mathbf{V}_i \quad (9)$$

where $\mathbf{k}_n = m_n \mathbf{c}_n^{(n)}$, $\ell_n = \mathbf{0}$, $\mathbf{U}_n = \mathbf{I}_n^{(n)} - m_n [\mathbf{c}_n^{(n)} \times] [\mathbf{c}_n^{(n)} \times]$, $\mathbf{V}_n = \mathbf{0}$ and

$$\mathbf{k}_i = m_i \mathbf{c}_i^{(i)} + K_{i+1}^* \left(\hat{m}_{i+1} {}^{i+1}_i \mathbf{s}^{(i)} + {}^{i+1}_i \mathbf{R} \begin{bmatrix} 0 \\ 0 \\ (\mathbf{k}_{i+1})_z \end{bmatrix} \right) + K_{i+1} \left(\hat{m}_{i+1} \begin{bmatrix} ({}^{i+1}_i \mathbf{s}^{(i)})_x \\ 0 \\ 0 \end{bmatrix} + {}^{i+1}_i \mathbf{R} \mathbf{k}_{i+1} \right) \quad (10)$$

$$\ell_i = K_{i+1}^* {}^{i+1}_i \mathbf{R} \begin{bmatrix} (\hat{c}_{i+1}^{(i+1)})_x \\ (\hat{c}_{i+1}^{(i+1)})_y \\ (\ell_{i+1})_z \end{bmatrix} + K_{i+1} {}^{i+1}_i \mathbf{R} \left(\ell_{i+1} + \hat{m}_{i+1} \begin{bmatrix} 0 \\ 0 \\ ({}^{i+1}_i \mathbf{s}^{(i+1)})_z \end{bmatrix} \right) \quad (11)$$

If joint $i+1$ is a rotational joint, then

$$\mathbf{U}_i = \mathbf{I}_i^{(i)} - m_i [\mathbf{c}_i^{(i)} \times] [\mathbf{c}_i^{(i)} \times] - \hat{m}_{i+1} [{}^{i+1}_i \mathbf{s}^{(i)} \times] [{}^{i+1}_i \mathbf{s}^{(i)} \times] + {}^{i+1}_i \mathbf{R} \begin{bmatrix} (\mathbf{U}_{i+1})_{22} & 0 & 0 \\ 0 & (\mathbf{U}_{i+1})_{22} & 0 \\ 0 & 0 & (\mathbf{U}_{i+1})_{33} \end{bmatrix} {}^{i+1}_i \mathbf{R}^T - [{}^{i+1}_i \mathbf{s}^{(i)} \times] [({}^{i+1}_i \mathbf{R}_b (\mathbf{k}_{i+1})_z) \times] - [({}^{i+1}_i \mathbf{R}_b (\mathbf{k}_{i+1})_z) \times] [{}^{i+1}_i \mathbf{s}^{(i)} \times] \quad (12)$$

$$\mathbf{V}_i = {}^{i+1}_i \mathbf{R} (\mathbf{V}_{i+1} + \mathbf{U}_{i+1} - \begin{bmatrix} (\mathbf{U}_{i+1})_{22} & 0 & 0 \\ 0 & (\mathbf{U}_{i+1})_{22} & 0 \\ 0 & 0 & (\mathbf{U}_{i+1})_{33} \end{bmatrix}) {}^{i+1}_i \mathbf{R}^T - [{}^{i+1}_i \mathbf{s}^{(i)} \times] [\ell_i \times] - [\ell_i \times] [{}^{i+1}_i \mathbf{s}^{(i)} \times] \quad (13)$$

whereas for translational joint $i+1$,

$$\mathbf{U}_i = \mathbf{I}_i^{(i)} - m_i [\mathbf{c}_i^{(i)} \times] [\mathbf{c}_i^{(i)} \times] + {}^{i+1}_i \mathbf{R} \mathbf{U}_{i+1} {}^{i+1}_i \mathbf{R}^T - \hat{m}_{i+1} \times [\mathbf{b}_{i+1}^{(i)} \times] [\mathbf{b}_{i+1}^{(i)} \times] - [\mathbf{b}_{i+1}^{(i)} \times] [({}^{i+1}_i \mathbf{R} \mathbf{k}_{i+1}) \times] - [({}^{i+1}_i \mathbf{R} \mathbf{k}_{i+1}) \times] [\mathbf{b}_{i+1}^{(i)} \times] \quad (14)$$

$$\mathbf{V}_i = {}^{i+1}_i \mathbf{R} (\mathbf{V}_{i+1} - \hat{m}_{i+1} [\mathbf{d}_{i+1}^{(i+1)} \times] [\mathbf{d}_{i+1}^{(i+1)} \times] - \hat{m}_{i+1} [\mathbf{d}_{i+1}^{(i+1)} \times] [\mathbf{b}_{i+1}^{(i+1)} \times] - \hat{m}_{i+1} [\mathbf{b}_{i+1}^{(i+1)} \times] [\mathbf{d}_{i+1}^{(i+1)} \times] - [\mathbf{d}_{i+1}^{(i+1)} \times] [\hat{c}_{i+1}^{(i+1)} \times] - [\hat{c}_{i+1}^{(i+1)} \times] [\mathbf{d}_{i+1}^{(i+1)} \times] - [\mathbf{b}_{i+1}^{(i+1)} \times] [\ell_{i+1} \times] - [\ell_{i+1} \times] [\mathbf{b}_{i+1}^{(i+1)} \times]) {}^{i+1}_i \mathbf{R}^T \quad (15)$$

Note that ${}^{i+1}_i \mathbf{R}_b$ is the third column of ${}^{i+1}_i \mathbf{R}$ (i.e., ${}^{i+1}_i \mathbf{R}_b = [0, -S\beta_{i+1}, C\beta_{i+1}]^T$), $\mathbf{b}_{i+1}^{(i)} = [b_{i+1}, 0, 0]^T$ and $\mathbf{d}_{i+1}^{(i+1)} = [0, 0, d_{i+1}]^T$ (i.e., ${}^{i+1}_i \mathbf{s}^{(i)} = \mathbf{b}_{i+1}^{(i)} + \mathbf{d}_{i+1}^{(i+1)}$).

\hat{m}_i , the vectors \mathbf{k}_i in (10) and the matrices \mathbf{U}_i in (12) and (14) are invariant to manipulator motion; we shall refer to these as *inertia constants of composite bodies*. It should be remarked that these constants are different from *Renaud's*

TABLE I

Present	Gautier et al. [8], [10], [12]
$\hat{m}_{i-1} = m_{i-1} + m_i + \dots + m_n$	$mR_{i-1} = m_{i-1}$
$\mathbf{k}_{i-1} = m_{i-1} \mathbf{c}_{i-1}^{<i-1>} + \hat{m}_i \begin{bmatrix} b_i \\ 0 \\ 0 \end{bmatrix} + {}_{i-1}^i \mathbf{R} \mathbf{k}_i$	$\begin{bmatrix} mX R_{i-1} \\ mY R_{i-1} \\ mZ R_{i-1} \end{bmatrix} = m_{i-1} \mathbf{c}_{i-1}^{<i-1>}$
$\hat{m}_i = m_i + m_{i+1} + \dots + m_n$	$mR_i = m_i + m_{i+1} + \dots + m_n$
$\mathbf{k}_i = m_i \mathbf{c}_i^{<i>} + \hat{m}_{i+1} {}_{i+1}^i \mathbf{s}^{<i>}$	$\begin{bmatrix} mX R_i \\ mY R_i \\ mZ R_i \end{bmatrix} = m_i \mathbf{c}_i^{<i>} + mR_{i+1} {}_{i+1}^i \mathbf{s}^{<i>}$
$+ \begin{bmatrix} 0 \\ -S\beta_{i+1} \\ C\beta_{i+1} \end{bmatrix} (\mathbf{k}_{i+1})_z$	$+ \begin{bmatrix} 0 \\ -S\beta_{i+1} \\ C\beta_{i+1} \end{bmatrix} mZ R_{i+1}$

inertia constants (i.e., the first moment and the inertia tensor of an augmented body when the composite body contains only rotational joints [32], [33]). The main difference is that the varying terms in $\hat{\mathbf{c}}_i^{(i)}$ and $\hat{\mathbf{J}}_i^{(i)}$ can be calculated with only some (not all) of the inertia constants of composite bodies [23]. This property allows us to set forth the following theorem.

Theorem 2: For a manipulator with n low-pair joints, in which joint r is the first rotational joint counting from the base and joint s is the nearest rotational joint not parallel to joint r , a set of MLC's for determining the actuator forces τ is the set \mathcal{S} consisting of all nonzero elements of

- 1) $K_j^*(\mathbf{U}_j)_{33}$, $\delta_j K_j^*(\mathbf{k}_j)_x$, $\delta_j K_j^*(\mathbf{k}_j)_y$ for $r \leq j < s$,
- 2) $K_j^*((\mathbf{U}_j)_{11} - (\mathbf{U}_j)_{22})$, $K_j^*(\mathbf{U}_j)_{33}$, $K_j^*(\mathbf{U}_j)_{12}$, $K_j^*(\mathbf{U}_j)_{13}$, $K_j^*(\mathbf{U}_j)_{23}$, $K_j^*(\mathbf{k}_j)_x$, $K_j^*(\mathbf{k}_j)_y$ for $s \leq j \leq n$,
- 3) $K_i \hat{m}_i$ for $i = 1, \dots, n$,
- 4) $K_i(\mathbf{k}_i)_x$, $K_i(\mathbf{k}_i)_y$, $K_i(\mathbf{k}_i)_z$ for $s < i \leq n$,
- 5) $\sigma_i K_i \{ -(\mathbf{u}_r^{(i)})_z [(\mathbf{u}_r^{(i)})_x (\mathbf{k}_i)_x + (\mathbf{u}_r^{(i)})_y (\mathbf{k}_i)_y] + [1 - (\mathbf{u}_r^{(i)})_z^2] (\mathbf{k}_i)_z \}$, $\sigma_i K_i \{ -(\mathbf{u}_r^{(i)})_y (\mathbf{k}_i)_x + (\mathbf{u}_r^{(i)})_x (\mathbf{k}_i)_y \}$ for $r < i < s$,

where $\delta_j = 0$ for the case where $\mathbf{u}_r \parallel \mathbf{u}_k \parallel \mathbf{g}$, $\forall k < j < s$, and $\frac{j}{m} s$ (when $j > r$) is zero or parallel to \mathbf{u}_r for every rotational joint m , $r \leq m < j$, otherwise $\delta_j = 1$; and $\sigma_i = 0$ for the case of $\mathbf{u}_i \parallel \mathbf{u}_r$, $r < i < s$, otherwise $\sigma_i = 1$.

Remark 1: In [23], it was shown that knowledge of the set \mathcal{S} in Theorem 2 is sufficient to determine the actuator forces of a manipulator and that all elements of \mathcal{S} are identifiable. According to Lemma 1, we can say that the set \mathcal{S} is a set of the MLC's. However, a direct and rigorous method should show that the dynamic equations of a manipulator can be reformulated as (3) in Lemma 1 with the elements of \mathcal{S} as \mathbf{w} . Such a method can be found in [22]. The advantage of this method is that it provides a systematic way of finding the MLC's.

Remark 2: The set of the MLC's of the inertia parameters in Theorem 2 is different from the results of Gautier *et al.* [8], [10], [12] and Mayeda *et al.* [25]–[28] only in some minor terms. In particular, the present result and theirs are almost the same (the \mathbf{U}_i are slightly different) when a manipulator has only rotational joints. Suppose that joint i is a translational

joint and joints $i + 1, \dots, n$ are rotational joints; then \hat{m}_{i-1} and \mathbf{k}_{i-1} in this paper can be compared with their counterparts mR_{i-1} and $[mX R_{i-1}, mY R_{i-1}, mZ R_{i-1}]^T$ in [8], [10], [12] as shown in Table I. It is apparent that the present set of MLC's is not identical to that of Gautier *et al.* for a manipulator with translational joints. Note that \mathbf{U}_{i-1} is also different since it contains $(\mathbf{k}_i)_z$. The merit of Theorem 2 is that it clearly describes the set of MLC's by introducing joints r and s .

III. INVERSE DYNAMICS

Khalil and Kleinfinger [16] modified the recursive Newton-Euler formulation [24] by using the first moments ($m_i \mathbf{c}_i^{(i)}$) and the inertia tensors about the origin on the driving joint ($\mathbf{J}_i^{(i)}$) instead of the centers of mass ($\mathbf{c}_i^{(i)}$) and the inertia tensors about the center of mass ($\mathbf{I}_i^{(i)}$). Their set of MLC's [8], [10], [12] can then be used in this modified formulation by replacing the first moment and the inertia tensor of each link with their counterparts in the MLC's (if the counterparts are not redundant) or with zero (if they are redundant, i.e., not in the MLC's). This approach draws on the work of Atkeson *et al.* [3], who suggested that the value of each linear combination in the MLC's be kept the same while one original inertia parameter in the linear combination is assigned the same value as the linear combination and the other inertia parameters are set to zero. Since the dynamic model is linear with respect to the MLC's, the same values for the MLC's determine the same dynamic model. However, this property does not hold for all sets of MLC's.

Consider a manipulator with rotational joint $i - 1$ and translational joint i , $i > s$. The set of MLC's in Theorem 2 for the manipulator contains the x - and y -components of \mathbf{k}_{i-1} and all three components of \mathbf{k}_i . Since joint i is a translational joint, \mathbf{k}_{i-1} has the contribution of \mathbf{k}_i (see Table I). $m_{i-1} \mathbf{c}_{i-1}^{(i-1)}$ should not be assigned the same value as \mathbf{k}_{i-1} , otherwise \mathbf{k}_i must be zero, which contradicts the principle that the values of MLCs should be preserved while distributing their values to the original inertia parameters. Hence the modified formulation [16] cannot be applied to the general case, where the values of the MLC's are not preserved while using

Atkeson's technique, and so it cannot be applied to the present set of MLC's.

Thus it is necessary to derive a new version of the recursive Newton-Euler formulation in terms of the present set of MLC's. The derivation process presented in this section could also be used for other sets of MLC's. At the end of this section, it will be seen that the result is identical to that of Khalil and Kleinfinger [16] when the manipulator has only rotational joints. This identity is because the two sets of MLC's are almost the same for a manipulator with only rotational joints. This also verifies the result of Khalil and Kleinfinger. Since these two formulations both use minimal parameters, it will be found that they are about equally efficient for a manipulator with one or more translational joints. However, the derivation of the new Newton-Euler formulation in terms of the present MLC's is tediously long. In the sense of deriving the formulation for the inverse dynamics, the MLC's of Gautier *et al.* [8], [10], [12] is superior, since the formulation of Khalil and Kleinfinger [16] is applied to them.

On the other hand, the formulation of the forward dynamics in terms of the present MLC's is easy to derive [23]. Renaud's formulation of the forward dynamics [32], [33] uses the masses, first moments, and inertia tensors of composite bodies (\hat{m}_i , $\hat{c}_i^{(i)}$ and $\hat{J}_i^{(i)}$), which are directly related to the set of MLCs in the form of (8)–(15). Since all possible constant terms of $\hat{c}_i^{(i)}$ and $\hat{J}_i^{(i)}$ are concatenated to be \mathbf{k}_i and \mathbf{U}_i , respectively, the computation of $\hat{c}_i^{(i)}$ and $\hat{J}_i^{(i)}$ in terms of the present set of MLC's is more efficient than that in terms of the set of Gautier *et al.* When joint $i+1$ is a translational joint and joint i is a rotational joint, the present set requires $8M + 7A$ and $50M + 40A$ for computing $\hat{c}_i^{(i)}$ and $\hat{J}_i^{(i)}$, respectively (see (11) in this paper and (36) in [23]), while the set of Gautier *et al.* requires $11M + 9A$ and $55M + 41A$ (see (A7) and (A8) in [21]). Note that M denotes multiplications and A additions/subtractions.

In the following, we first derive a formulation in terms of the inertia constants of composite bodies, and then reduce this formulation so that it is expressed in terms of the present set of MLC's.

A. Formulation

We start with the recursive Newton-Euler formulation [24]. Let $\dot{\omega}_i$ and ω_i be the angular acceleration and velocity of link i , \mathbf{a}_i the acceleration of the origin of frame E_i , \mathbf{g} the gravitational acceleration, q_i the joint displacement of joint i , \mathbf{f}_{T_i} and \mathbf{t}_{T_i} the inertia force and torque of link i , \mathbf{f}_i and \mathbf{t}_i the force and torque exerted on link i by joint i , and τ_i the actuator force (or torque) of joint i . The Newton-Euler formulation based on the normal driving-axis coordinate system is [5], [20]:

$$\dot{\omega}_i^{(i)} = {}^{i-1}\mathbf{R}\omega_{i-1}^{(i-1)} + K_i^* \mathbf{u}_i^{(i)} \dot{q}_i \quad (16)$$

$$\dot{\omega}_i^{(i)} = {}^{i-1}\mathbf{R}\dot{\omega}_{i-1}^{(i-1)} + K_i^* (\mathbf{u}_i^{(i)} \dot{q}_i + \omega_{i-1}^{(i)} \times \mathbf{u}_i^{(i)} \dot{q}_i) \quad (17)$$

$$\begin{aligned} \mathbf{a}_i^{(i)} = & {}^{i-1}\mathbf{R}[\mathbf{a}_{i-1}^{(i-1)} + \dot{\omega}_{i-1}^{(i-1)} \times {}^{i-1}\mathbf{s}^{(i-1)} \\ & + \omega_{i-1}^{(i-1)} \times (\omega_{i-1}^{(i-1)} \times {}^{i-1}\mathbf{s}^{(i-1)})] \\ & + K_i (\mathbf{u}_i^{(i)} \ddot{q}_i + 2\omega_i^{(i)} \times \mathbf{u}_i^{(i)} \dot{q}_i) \end{aligned} \quad (18)$$

$$\begin{aligned} \mathbf{f}_{T_i}^{(i)} = & -m_i [\mathbf{a}_i^{(i)} + \dot{\omega}_i^{(i)} \times \mathbf{c}_i^{(i)} \\ & + \omega_i^{(i)} \times (\omega_i^{(i)} \times \mathbf{c}_i^{(i)})] \end{aligned} \quad (19)$$

$$\mathbf{t}_{T_i}^{(i)} = -\mathbf{I}_i^{(i)} \dot{\omega}_i^{(i)} - \omega_i^{(i)} \times (\mathbf{I}_i^{(i)} \omega_i^{(i)}) \quad (20)$$

$$\mathbf{f}_i^{(i)} = {}^{i+1}\mathbf{R}\mathbf{f}_{i+1}^{(i+1)} - m_i \mathbf{g}^{(i)} - \mathbf{f}_{T_i}^{(i)} \quad (21)$$

$$\begin{aligned} \mathbf{t}_i^{(i)} = & {}^{i+1}\mathbf{R}\mathbf{t}_{i+1}^{(i+1)} - \mathbf{t}_{T_i}^{(i)} - \mathbf{c}_i^{(i)} \\ & \times (m_i \mathbf{g}^{(i)} + \mathbf{f}_{T_i}^{(i)}) + {}^{i+1}\mathbf{s}^{(i)} \times \mathbf{f}_{i+1}^{(i+1)} \end{aligned} \quad (22)$$

$$\tau_i = \mathbf{u}_i^{(i)} \cdot (K_i \mathbf{f}_i^{(i)} + K_i^* \mathbf{t}_i^{(i)}) \quad (23)$$

Note that $\mathbf{f}_{n+1}^{(n)} = -\mathbf{f}_E^{(n)}$, $\mathbf{t}_{n+1}^{(n)} = -\mathbf{t}_E^{(n)}$ and ${}^{n+1}\mathbf{s}^{(n)} = \frac{E}{n} \mathbf{s}^{(n)}$, where $\mathbf{f}_E^{(n)}$ and $\mathbf{t}_E^{(n)}$ are the external force and torque acting on the gripping point of the end-effector, respectively, and $\frac{E}{n} \mathbf{s}^{(n)}$ is the distance from the origin of frame E_n to the gripping point of the end-effector.

We define the following notation:

$$\Omega_i^{(i)} \equiv [\dot{\omega}_i^{(i)} \times] + [\omega_i^{(i)} \times] [\omega_i^{(i)} \times] \quad (24)$$

$$\Psi(\alpha, \omega) \equiv \begin{bmatrix} \alpha_x & -\omega_y \omega_z & \omega_y \omega_z & \alpha_y - \omega_x \omega_z \\ \omega_x \omega_z & \alpha_y & -\omega_x \omega_z & \alpha_x + \omega_y \omega_z \\ -\omega_x \omega_y & \omega_x \omega_y & \alpha_z & \omega_x^2 - \omega_y^2 \\ \alpha_z + \omega_x \omega_y & \omega_y^2 - \omega_z^2 & & \\ \omega_z^2 - \omega_x^2 & \alpha_z - \omega_x \omega_y & & \\ \alpha_x - \omega_y \omega_z & \alpha_y + \omega_x \omega_z & & \end{bmatrix} \quad (25)$$

$$\psi(\mathbf{A}) \equiv \begin{bmatrix} (\mathbf{A})_{11} \\ (\mathbf{A})_{22} \\ (\mathbf{A})_{33} \\ (\mathbf{A})_{12} \\ (\mathbf{A})_{13} \\ (\mathbf{A})_{23} \end{bmatrix} \quad (26)$$

where $\mathbf{A} \in \mathbb{R}^{3 \times 3}$ is a symmetric matrix and α_x and ω_x are the x -components of α and ω , respectively. The purpose of this notation is to rewrite (19) and (20) in matrix form,

$$\mathbf{f}_{T_i}^{(i)} = -m_i (\mathbf{a}_i^{(i)} + \Omega_i^{(i)} \mathbf{c}_i^{(i)}) \quad (27)$$

$$\mathbf{t}_{T_i}^{(i)} = -\Psi(\dot{\omega}_i^{(i)}, \omega_i^{(i)}) \psi(\mathbf{I}_i^{(i)}) \quad (28)$$

The following three equalities are relevant to the derivation of the new formulation. The proof of them can be found in the Appendix.

$$\mathbf{a} \times (\Omega_i^{(i)} \mathbf{b}) + \mathbf{b} \times (\Omega_i^{(i)} \mathbf{a}) = \Psi(\dot{\omega}_i^{(i)}, \omega_i^{(i)}) \psi(\mathbf{A}) \quad (29)$$

$$\Omega_i^{(i)} \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} = \Omega_{i-1}^{(i)} \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} \quad (30)$$

$$\begin{aligned} & \Psi(\dot{\omega}_i^{(i)}, \omega_i^{(i)}) \psi(\text{diag}[a, a, 0]) \\ & = \Psi(\dot{\omega}_{i-1}^{(i)}, \omega_{i-1}^{(i)}) \psi(\text{diag}[a, a, 0]) \\ & = {}^{i-1}\mathbf{R}^T \Psi(\dot{\omega}_{i-1}^{(i-1)}, \omega_{i-1}^{(i-1)}) \psi({}^{i-1}\mathbf{R} \text{diag}[a, a, 0] {}^{i-1}\mathbf{R}^T) \end{aligned} \quad (31)$$

where $\mathbf{A} = -[\mathbf{a} \times][\mathbf{b} \times] - [\mathbf{b} \times][\mathbf{a} \times]$, k is a constant, and $\text{diag}[a, b, c]$ denotes a diagonal matrix with entries of a , b , and c .

Lemma 3: The joint force of joint i acting on link i by joint i is

$$\mathbf{f}_i^{(i)} = \hat{m}_i (\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)}) + \mathbf{\Omega}_i^{(i)} \mathbf{k}_i + \mu_{i+1}^{(i)} \quad (32)$$

where $\mu_{n+1}^{(n)} = -\mathbf{f}_E^{(n)}$ and

$$\begin{aligned} \mu_i^{(i)} &= -\mathbf{f}_E^{(i)} + \sum_{j=i}^n {}^j \mathbf{R} \\ &\left\{ K_j^* \mathbf{\Omega}_j^{(j)} \begin{bmatrix} (\mathbf{k}_j)_x \\ (\mathbf{k}_j)_y \\ 0 \end{bmatrix} + K_j \hat{m}_j \left(\mathbf{\Omega}_j^{(j)} \begin{bmatrix} 0 \\ 0 \\ d_j \end{bmatrix} \right. \right. \\ &\left. \left. + \mathbf{u}_j^{(j)} \ddot{q}_j + 2\omega_j^{(j)} \times \mathbf{u}_j^{(j)} \dot{q}_j \right) \right\} \\ &= \mu_{i+1}^{(i)} + K_i^* \mathbf{\Omega}_i^{(i)} \begin{bmatrix} (\mathbf{k}_i)_x \\ (\mathbf{k}_i)_y \\ 0 \end{bmatrix} \\ &\left. + K_i \hat{m}_i \left(\mathbf{\Omega}_i^{(i)} \begin{bmatrix} 0 \\ 0 \\ d_i \end{bmatrix} + \mathbf{u}_i^{(i)} \ddot{q}_i + 2\omega_i^{(i)} \times \mathbf{u}_i^{(i)} \dot{q}_i \right) \right\} \quad (33) \end{aligned}$$

Proof: According to the recursive Newton-Euler formulation (16)–(23), we have

$$\begin{aligned} \mathbf{f}_i^{(i)} &= -\mathbf{f}_E^{(i)} - \sum_{j=i}^n (\mathbf{f}_{T_j}^{(i)} + m_j \mathbf{g}^{(i)}) \\ &= -\mathbf{f}_E^{(i)} + \sum_{j=i}^n {}^j \mathbf{R} m_j (\mathbf{a}_j^{(j)} - \mathbf{g}^{(j)} + \mathbf{\Omega}_j^{(j)} \mathbf{c}_j^{(j)}) \\ &= -\mathbf{f}_E^{(i)} + \sum_{j=i}^n m_j \left({}^j \mathbf{R} \mathbf{\Omega}_j^{(j)} \mathbf{c}_j^{(j)} - \mathbf{g}^{(i)} + \mathbf{a}_i^{(i)} \right. \\ &\quad \left. + \sum_{k=i}^{j-1} {}^k \mathbf{R} \mathbf{\Omega}_k^{(k)} {}^{k+1} \mathbf{s}^{(k)} \right) \\ &\quad + \sum_{j=i+1}^n m_j \sum_{k=i+1}^j K_k {}^k \mathbf{R} (\mathbf{u}_k^{(k)} \ddot{q}_k + 2\omega_k^{(k)} \times \mathbf{u}_k^{(k)} \dot{q}_k) \quad (34) \end{aligned}$$

where \sum_i^j for $j < i$ is assumed to be zero. It can be shown that

$$\sum_{j=i}^n \sum_{k=i}^{j-1} = \sum_{k=i}^{n-1} \sum_{j=k+1}^n, \quad \sum_{j=i+1}^n \sum_{k=i+1}^j = \sum_{k=i+1}^n \sum_{j=k}^n \quad (35)$$

$$\begin{aligned} &\hat{m}_{j+1} {}^{j+1} \mathbf{s}^{(j)} + m_j \mathbf{c}_j^{(j)} \\ &= \mathbf{k}_j - {}^{j+1} \mathbf{R} \left\{ K_{j+1}^* \begin{bmatrix} 0 \\ 0 \\ (\mathbf{k}_{j+1})_z \end{bmatrix} \right. \\ &\quad \left. + K_{j+1} \left(\mathbf{k}_{j+1} - \hat{m}_{j+1} \begin{bmatrix} 0 \\ 0 \\ d_{j+1} \end{bmatrix} \right) \right\} \text{ for } j < n \quad (36) \end{aligned}$$

in which the latter follows from (10). Since the angular velocity and acceleration of link j and those of link $j+1$ are the same for the case where joint $j+1$ is a translational joint, we obtain

$$K_{j+1} \mathbf{\Omega}_j^{(j)} {}^{j+1} \mathbf{R} \mathbf{k}_{j+1} = K_{j+1} {}^{j+1} \mathbf{R} \mathbf{\Omega}_{j+1}^{(j+1)} \mathbf{k}_{j+1} \quad (37)$$

Substituting (30) and (35)–(37) into (34) yields (32). \square

As was shown by (23), the actuator force requires only the z -component of the joint force or the joint torque. On the other hand, the joint torque of a translational joint need not be calculated if its effect on the actuator torque of the rotational joint in front of it can be merged in the inertia torque of the rotational joint. To achieve this, we introduce the following notation:

$$\begin{aligned} \mathbf{D}_i^{(i)} &= \mathbf{U}_i^* - \text{diag}[(\mathbf{U}_i^*)_{22}, (\mathbf{U}_i^*)_{22}, 0] \\ &\quad - \sum_{k=i+1}^{m_i-1} {}^k \mathbf{R} ([\mathbf{d}_k^{(k)} \times] [\mathbf{k}_k \times] + [\mathbf{k}_k \times] [\mathbf{d}_k^{(k)} \times]) {}^k \mathbf{R}^T \quad (38) \end{aligned}$$

where

$$\begin{aligned} \mathbf{U}_i^* &= \mathbf{U}_i + {}^{m_i} \mathbf{R} \mathbf{G}_{m_i}^{(m_i)} {}^{m_i} \mathbf{R}^T \quad (39) \\ \mathbf{G}_{m_i}^{(m_i)} &= \begin{bmatrix} (\mathbf{U}_{m_i}^* - \mathbf{U}_{m_i})_{22} & 0 & 0 \\ 0 & (\mathbf{U}_{m_i}^* - \mathbf{U}_{m_i})_{22} & 0 \\ 0 & 0 & -(\mathbf{U}_{m_i})_{33} \end{bmatrix} \quad (40) \end{aligned}$$

and joint m_i is the nearest rotational joint behind joint i , i.e., joints $i+1$ to m_i-1 are translational joints. The intention of this notation can be realized by observing (47) and (52) below. It should be noted that

$$\begin{aligned} (\mathbf{U}_i^* - \mathbf{U}_i)_{22} &= (\mathbf{G}_{m_i}^{(i)})_{22} \\ &= ({}^{m_i} \mathbf{R} \mathbf{G}_{m_i}^{(m_i)} {}^{m_i} \mathbf{R}^T)_{22} \quad (41) \end{aligned}$$

Equation (41), and thus $\mathbf{G}_i^{(i)}$, can be calculated for all rotational joints even if $(\mathbf{U}_i)_{22}$, $i = 1, \dots, n$, are unknown, since $\bar{\mathbf{U}}_j = \mathbf{U}_j$ and $(\mathbf{U}_j^* - \mathbf{U}_j)_{22} = 0$ for the outermost rotational joint j . It follows from (39)–(41) that

$$\begin{aligned} &\mathbf{U}_i^* - \text{diag}[(\mathbf{U}_i^*)_{22}, (\mathbf{U}_i^*)_{22}, 0] \\ &= \mathbf{U}_i - \text{diag}[(\mathbf{U}_i)_{22}, (\mathbf{U}_i)_{22}, 0] \\ &\quad + \mathbf{G}_{m_i}^{(i)} - \text{diag}[(\mathbf{G}_{m_i}^{(i)})_{22}, (\mathbf{G}_{m_i}^{(i)})_{22}, 0] \quad (42) \\ \mathbf{G}_{m_i}^{(m_i)} &= \begin{bmatrix} (\mathbf{G}_{m_i}^{(i)})_{22} & 0 & 0 \\ 0 & (\mathbf{G}_{m_i}^{(i)})_{22} & 0 \\ 0 & 0 & -(\mathbf{U}_{m_i})_{33} \end{bmatrix} \quad (43) \end{aligned}$$

Hence $\mathbf{D}_i^{(i)}$ is expressed in terms of the MLC's in Theorem 2.

Lemma 4: If joint i is a rotational joint, the z -component of the joint torque acting on link i by joint i is

$$(\mathbf{t}_i^{(i)})_z = (\zeta_i^{(i)})_z \quad (44)$$

where

$$\begin{aligned} \zeta_i^{(i)} &= \zeta_{i+1}^{(i)} + {}^{i+1}i\mathbf{s}^{(i)} \times \mu_{i+1}^{(i)} \\ &+ K_i^* \left\{ \begin{bmatrix} (\mathbf{k}_i)_x \\ (\mathbf{k}_i)_y \\ 0 \end{bmatrix} \times (\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)}) \right. \\ &\quad \left. + \Psi(\dot{\omega}_i^{(i)}, \omega_i^{(i)}) \psi(\mathbf{D}_i^{(i)}) \right\} \\ &+ K_i \left\{ \hat{m}_i \begin{bmatrix} 0 \\ 0 \\ d_i \end{bmatrix} \times \left(\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)} - \Omega_i^{(i)} \begin{bmatrix} 0 \\ 0 \\ d_i \end{bmatrix} \right) \right. \\ &\quad \left. + \left(\mathbf{k}_i - \hat{m}_i \begin{bmatrix} 0 \\ 0 \\ d_i \end{bmatrix} \right) \times (\mathbf{u}_i^{(i)} \ddot{q}_i + 2\omega_i^{(i)} \times \mathbf{u}_i^{(i)} \dot{q}_i) \right\} \end{aligned} \quad (45)$$

Note that $\mu_{n+1}^{(n)} = -\mathbf{f}_E^{(n)}$ and $\zeta_{n+1}^{(n)} = -\mathbf{t}_E^{(n)} - {}_E^n\mathbf{s}^{(n)} \times \mathbf{f}_E^{(n)}$.

Proof: It can be shown (see the Appendix) that

$$\begin{aligned} \mathbf{t}_i^{(i)} &= \sum_{j=i}^n {}^j_i\mathbf{R} \Psi(\dot{\omega}_j^{(j)}, \omega_j^{(j)}) \psi(\mathbf{A}_j^{(j)}) \\ &+ \mathbf{k}_i \times (\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)}) + \eta_{i+1}^{(i)} \end{aligned} \quad (46)$$

where $\mathbf{A}_n^{(n)} = \mathbf{I}_n^{(n)} - m_n [\mathbf{c}_n^{(n)} \times] [\mathbf{c}_n^{(n)} \times]$ and

$$\begin{aligned} \mathbf{A}_j^{(j)} &= \mathbf{U}_j - K_{j+1}^* \\ &\times {}^{j+1}j\mathbf{R} \begin{bmatrix} (\mathbf{U}_{j+1})_{22} & 0 & 0 \\ 0 & (\mathbf{U}_{j+1})_{22} & 0 \\ 0 & 0 & (\mathbf{U}_{j+1})_{33} \end{bmatrix} {}^{j+1}j\mathbf{R}^T \\ &- K_{j+1} {}^{j+1}j\mathbf{R} (\mathbf{U}_{j+1} + [d_{j+1}^{(j+1)} \times] [\mathbf{k}_{j+1} \times] \\ &+ [\mathbf{k}_{j+1} \times] [d_{j+1}^{(j+1)} \times]) {}^{j+1}j\mathbf{R}^T \end{aligned} \quad (47)$$

$$\begin{aligned} \eta_i^{(i)} &= \sum_{j=i}^n {}^j_i\mathbf{R} \left\{ {}^{j-1}j\mathbf{s}^{(j)} \times \mu_j^{(j)} \right. \\ &+ K_j^* \begin{bmatrix} (\mathbf{k}_j)_x \\ (\mathbf{k}_j)_y \\ 0 \end{bmatrix} \times (\mathbf{a}_j^{(j)} - \mathbf{g}^{(j)}) \\ &+ K_j \left[\hat{m}_j \begin{bmatrix} 0 \\ 0 \\ d_j \end{bmatrix} \times \left(\mathbf{a}_j^{(j)} - \mathbf{g}^{(j)} - \Omega_j^{(j)} \begin{bmatrix} 0 \\ 0 \\ d_j \end{bmatrix} \right) \right. \\ &\quad \left. + \left(\mathbf{k}_j - \hat{m}_j \begin{bmatrix} 0 \\ 0 \\ d_j \end{bmatrix} \right) \times (\mathbf{u}_j^{(j)} \ddot{q}_j + 2\omega_j^{(j)} \times \mathbf{u}_j^{(j)} \dot{q}_j) \right\} \\ &- {}_i^n\mathbf{R} (\mathbf{t}_E^{(n)} + {}_E^n\mathbf{s}^{(n)} \times \mathbf{f}_E^{(n)}) \end{aligned} \quad (48)$$

Consider the case where joint i is a rotational joint and joint m_i is the nearest rotational joint behind joint i . Thus

$$\dot{\omega}_i^{(i)} = \dot{\omega}_{i+1}^{(i)} = \dots = \dot{\omega}_{m_i-1}^{(i)} \quad (49)$$

$$\omega_i^{(i)} = \omega_{i+1}^{(i)} = \dots = \omega_{m_i-1}^{(i)} \quad (50)$$

which imply that

$$\begin{aligned} &\sum_{k=i}^{m_i-1} {}^k_i\mathbf{R} \Psi(\dot{\omega}_k^{(k)}, \omega_k^{(k)}) \psi(\mathbf{A}_k^{(k)}) \\ &= \Psi(\dot{\omega}_i^{(i)}, \omega_i^{(i)}) \psi \left(\sum_{k=i}^{m_i-1} \mathbf{A}_k^{(i)} \right) \end{aligned} \quad (51)$$

Comparing (47) and (38) yields

$$\begin{aligned} \sum_{k=i}^{m_i-1} \mathbf{A}_k^{(i)} &= \mathbf{D}_i^{(i)} + \text{diag}[(\mathbf{U}_i^*)_{22}, (\mathbf{U}_i^*)_{22}, 0] \\ &- {}^{m_i}i\mathbf{R} \text{diag}[(\mathbf{U}_{m_i}^*)_{22}, (\mathbf{U}_{m_i}^*)_{22}, 0] {}^{m_i}i\mathbf{R}^T \end{aligned} \quad (52)$$

Substituting (51) and (52) into (46) and using (31), we obtain the joint torque of rotational joint i ,

$$\begin{aligned} \mathbf{t}_i^{(i)} &= \eta_i^{(i)} + \begin{bmatrix} 0 \\ 0 \\ (\mathbf{k}_i)_z \end{bmatrix} \times (\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)}) \\ &- {}_{i-1}i\mathbf{s}^{(i)} \times \mu_i^{(i)} + \Psi(\dot{\omega}_i^{(i)}, \omega_i^{(i)}) \psi(\mathbf{D}_i^{(i)}) \\ &+ \text{diag}[(\bar{\mathbf{U}}_i)_{22}, (\mathbf{U}_i^*)_{22}, 0] \\ &+ \sum_{j=i+1}^n K_j^* {}^j_i\mathbf{R} \Psi(\dot{\omega}_j^{(j)}, \omega_j^{(j)}) \psi(\mathbf{D}_j^{(j)}) \end{aligned} \quad (53)$$

The rule of the vector product and (25) imply that

$$\begin{aligned} &\left(\begin{bmatrix} 0 \\ 0 \\ (\mathbf{k}_i)_z \end{bmatrix} \times (\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)}) \right)_z = 0 \\ &(\Psi(\dot{\omega}_i^{(i)}, \omega_i^{(i)}) \psi(\mathbf{D}_i^{(i)} + \text{diag}[(\mathbf{U}_i^*)_{22}, (\mathbf{U}_i^*)_{22}, 0]))_z \\ &= (\Psi(\dot{\omega}_i^{(i)}, \omega_i^{(i)}) \psi(\mathbf{D}_i^{(i)}))_z \end{aligned} \quad (54)$$

$\eta_i^{(i)}$ in (48) is related to $\zeta_i^{(i)}$ as follows:

$$\eta_i^{(i)} = \zeta_i^{(i)} + {}_{i-1}i\mathbf{s}^{(i)} \times \mu_i^{(i)} - \sum_{j=i}^n K_j^* {}^j_i\mathbf{R} \Psi(\dot{\omega}_j^{(j)}, \omega_j^{(j)}) \psi(\mathbf{D}_j^{(j)}) \quad (56)$$

Applying (54)–(56) to the z -component of $\mathbf{t}_i^{(i)}$ in (53) yields (44). \square

According to Lemmas 3 and 4, we develop a new version of the recursive Newton-Euler formulation in terms of the inertia constants of composite bodies as follows.

• Forward recursion:

$$\omega_i^{(i)} = {}^{i-1}i\mathbf{R} \omega_{i-1}^{(i-1)} + K_i^* \mathbf{u}_i^{(i)} \dot{q}_i \quad (57)$$

$$\dot{\omega}_i^{(i)} = {}^{i-1}i\mathbf{R} \dot{\omega}_{i-1}^{(i-1)} + K_i^* (\mathbf{u}_i^{(i)} \ddot{q}_i + \omega_{i-1}^{(i)} \times \mathbf{u}_i^{(i)} \dot{q}_i) \quad (58)$$

$$\Omega_i^{(i)} = [\dot{\omega}_i^{(i)} \times] + [\omega_i^{(i)} \times] [\omega_i^{(i)} \times] \quad (59)$$

$$\begin{aligned} \Psi(\alpha = \dot{\omega}_i^{(i)}, \beta = \omega_i^{(i)}) &= \begin{bmatrix} \alpha_x & -\beta_y \beta_z & \beta_y \beta_z \\ \beta_x \beta_z & \alpha_y & -\beta_x \beta_z \\ -\beta_x \beta_y & \beta_x \beta_y & \alpha_z \\ \alpha_y - \beta_x \beta_z & \alpha_z + \beta_x \beta_y & \beta_y^2 - \beta_z^2 \\ \alpha_x + \beta_y \beta_z & \beta_z^2 - \beta_x^2 & \alpha_z - \beta_x \beta_y \\ \beta_x^2 - \beta_y^2 & \alpha_x - \beta_y \beta_z & \alpha_y + \beta_x \beta_z \end{bmatrix} \end{aligned} \quad (60)$$

$$\ddot{\mathbf{h}}_i^{(i)} \equiv K_i (\mathbf{u}_i^{(i)} \ddot{q}_i + 2\omega_i^{(i)} \times \mathbf{u}_i^{(i)} \dot{q}_i) \quad (61)$$

$$\mathbf{a}_i^{(i)} = {}^{i-1}\mathbf{R} \left\{ \mathbf{a}_{i-1}^{(i-1)} + \Omega_{i-1}^{(i-1)} \left(K_i^* {}^{i-1}\mathbf{s}^{(i-1)} K_i \begin{bmatrix} b_i \\ 0 \\ 0 \end{bmatrix} \right) \right\} \\ + K_i \left(\Omega_i^{(i)} \begin{bmatrix} 0 \\ 0 \\ d_i \end{bmatrix} + \ddot{\mathbf{h}}_i^{(i)} \right) \quad (62)$$

$$\rho_i \equiv K_i \left[\hat{m}_i (\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)})_z + (\Omega_i^{(i)} \mathbf{k}_i)_z \right] \quad (63)$$

$$\bar{\mu}_i^{(i)} \equiv K_i^* \Omega_i^{(i)} \begin{bmatrix} (\mathbf{k}_i)_x \\ (\mathbf{k}_i)_y \\ 0 \end{bmatrix} + K_i \hat{m}_i \left(\Omega_i^{(i)} \begin{bmatrix} 0 \\ 0 \\ d_i \end{bmatrix} + \ddot{\mathbf{h}}_i^{(i)} \right) \quad (64)$$

$$\bar{\zeta}_i^{(i)} \equiv K_i^* \left\{ \begin{bmatrix} (\mathbf{k}_i)_x \\ (\mathbf{k}_i)_y \\ 0 \end{bmatrix} \times (\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)}) \right. \\ \left. + \Psi(\dot{\omega}_i^{(i)}, \omega_i^{(i)}) \psi(\mathbf{D}_i^{(i)}) \right\} + K_i \left\{ \hat{m}_i \begin{bmatrix} 0 \\ 0 \\ d_i \end{bmatrix} \right. \\ \left. \times {}^{i-1}\mathbf{R} \left(\mathbf{a}_{i-1}^{(i-1)} - \mathbf{g}^{(i-1)} + \Omega_{i-1}^{(i-1)} \begin{bmatrix} b_i \\ 0 \\ 0 \end{bmatrix} \right) \right. \\ \left. + \mathbf{k}_i \times \ddot{\mathbf{h}}_i^{(i)} \right\} \quad (65)$$

• Backward recursion:

$$\mu_i^{(i)} = \bar{\mu}_i^{(i)} + {}^{i+1}\mathbf{R} \mu_{i+1}^{(i+1)} \quad (66)$$

$$\zeta_i^{(i)} = \bar{\zeta}_i^{(i)} + {}^{i+1}\mathbf{s}^{(i)} \times \mu_{i+1}^{(i)} + {}^{i+1}\mathbf{R} \zeta_{i+1}^{(i+1)} \quad (67)$$

$$\tau_i = K_i (\rho_i + (\mu_{i+1}^{(i)})_z) + K_i^* (\zeta_i^{(i)})_z \quad (68)$$

It should be remarked that $\mu_{n+1}^{(n)} = -\mathbf{f}_E^{(n)}$ and $\zeta_{n+1}^{(n)} = -\mathbf{t}_E^{(n)} - \frac{E}{n} \mathbf{s}^{(n)} \times \mathbf{f}_E^{(n)}$. Also note that (62) follows from (18) and $K_i \omega_i^{(i)} = K_i \omega_{i-1}^{(i)}$, in order to save some computation. In this formulation, (57)–(62) are exactly (16)–(18) of the original Newton-Euler formulation in an alternative form. The inertia force and torque ($\mathbf{f}_{T_i}^{(i)}$ and $\mathbf{t}_{T_i}^{(i)}$) are replaced by ρ_i , $\bar{\mu}_i^{(i)}$, and $\bar{\zeta}_i^{(i)}$ in (63)–(65). The backward recursion (66)–(68) is still similar to that of the original formulation. If in (63)–(65) we set $K_i = 0$, this formulation is identical to that of Khalil and Kleinfinger [16] for a manipulator with only rotational joints.

Although the derivation of this formulation is considerably long, the result is not complicated, since some terms in (64) and (65) are parts of $\mathbf{a}_i^{(i)}$ in (62). For instance, (62) can be decoupled into two parts and replaced by

$$\ddot{\mathbf{h}}_i^* \equiv K_i \left(\ddot{\mathbf{h}}_i^{(i)} + \Omega_i^{(i)} \begin{bmatrix} 0 \\ 0 \\ d_i \end{bmatrix} \right) \quad (69)$$

$$\mathbf{a}_i^{*(i-1)} \equiv \mathbf{a}_{i-1}^{(i-1)} + \Omega_{i-1}^{(i-1)} \left(K_i^* {}^{i-1}\mathbf{s}^{(i-1)} + K_i \begin{bmatrix} b_i \\ 0 \\ 0 \end{bmatrix} \right) \quad (70)$$

$$\mathbf{a}_i^{(i)} = {}^{i-1}\mathbf{R} \mathbf{a}_i^{*(i-1)} + K_i \ddot{\mathbf{h}}_i^* \quad (71)$$

Thus (64) and (65) can be rewritten in more compact form as follows:

$$\bar{\mu}_i^{(i)} = K_i^* \Omega_i^{(i)} \begin{bmatrix} (\mathbf{k}_i)_x \\ (\mathbf{k}_i)_y \\ 0 \end{bmatrix} + K_i \hat{m}_i \ddot{\mathbf{h}}_i^* \quad (72)$$

$$\bar{\zeta}_i^{(i)} = K_i^* \left\{ \begin{bmatrix} (\mathbf{k}_i)_x \\ (\mathbf{k}_i)_y \\ 0 \end{bmatrix} \times \mathbf{a}_i^{(i)} + \Psi(\dot{\omega}_i^{(i)}, \omega_i^{(i)}) \psi(\mathbf{D}_i^{(i)}) \right\} \\ + K_i \left\{ \hat{m}_i \begin{bmatrix} 0 \\ 0 \\ d_i \end{bmatrix} \times \mathbf{a}_i^* + \mathbf{k}_i \times \ddot{\mathbf{h}}_i^{(i)} \right\} \quad (73)$$

Furthermore, a technique can be used to save some computation for $\zeta_i^{(i)}$. Define

$$\zeta_{i+1}^{*(i+1)} \equiv \zeta_{i+1}^{(i+1)} + \begin{bmatrix} 0 \\ 0 \\ d_{i+1} \end{bmatrix} \times \mu_{i+1}^{(i+1)} \quad (74)$$

Then (67) can be reformulated as

$$\zeta_i^{(i)} = \bar{\zeta}_i^{(i)} + {}^{i+1}\mathbf{R} \zeta_{i+1}^{*(i+1)} + \begin{bmatrix} b_{i+1} \\ 0 \\ 0 \end{bmatrix} \times \mu_{i+1}^{(i)} \quad (75)$$

B. Algorithm

The above formulation is suited for any joint of a manipulator. However, it requires knowledge of some parameters other than the MLC's for joint i , $i < s$. In order to replace the inertia constants with the MLC's, the individual joints must be taken into account. As usual, we set

$$\mathbf{a}_0^{(0)} =: -\mathbf{g}^{(0)} \quad (76)$$

In accord with Theorem 2, the joints are classified into three groups.

- 1) Joints remaining in front of the first rotational joint r , i.e., $i < r$.

The angular velocity and acceleration of the links remaining in front of joint r are all zero and $\zeta_i^{(i)}$ are not required for τ_i . Therefore, (57)–(61), (65), and (67) are redundant, while (62)–(64) can be replaced by

$$\mathbf{a}_i^{(i)} = {}^{i-1}\mathbf{R} \mathbf{a}_{i-1}^{(i)} + \mathbf{u}_i^{(i)} \ddot{q}_i \quad (77)$$

$$\rho_i = \hat{m}_i (\mathbf{a}_i^{(i)})_z \quad (78)$$

$$\bar{\mu}_i^{(i)} = \hat{m}_i \mathbf{u}_i^{(i)} \ddot{q}_i \quad (79)$$

in which $\bar{\mu}_1^{(1)}$ is also redundant (see (67)). As a result, only the members of the MLC's (i.e., \hat{m}_i) are required in the algorithm for $i < r$.

- 2) Joints remaining between joint $r - 1$ and joint s , i.e., $r \leq i < s$:

The rotational joints remaining in front of joint s are parallel to one another, so that $\bar{\zeta}_i^{(i)} = \pm \bar{\zeta}_i^{(r)}$ for rotational joint i (and then only its z -component must be computed) and

$$\omega_i^{(i)} = \mathbf{u}_r^{(i)} (\omega_i^{(r)})_z \\ = \mathbf{u}_r^{(i)} \{ (\omega_{i-1}^{(r)})_z + K_i^* (\mathbf{u}_r^{(i)})_z \dot{q}_i \} \quad (80)$$

$$\dot{\omega}_i^{(i)} = \mathbf{u}_r^{(i)} (\dot{\omega}_i^{(r)})_z \\ = \mathbf{u}_r^{(i)} \{ (\dot{\omega}_{i-1}^{(r)})_z + K_i^* (\mathbf{u}_r^{(i)})_z \ddot{q}_i \} \quad (81)$$

which replace (57) and (58). Note that $\bar{\mu}_r^{(r)}$ is redundant if $r = 1$.

TABLE II

	$i < r$	$i = r$		$r < i < s$		$s < i \leq n$	
		$r = 1$	$r \neq 1$	R	T	R	T
$\omega_i^{<i>}$	-	0		1A	3M	8M5A	8M4A
$\dot{\omega}_i^{<i>}$	-	0		1A	3M	10M7A	8M4A
$\Omega_i^{<i>}$	-	1M		1M	6M9A	6M9A	6M9A
Ψ	-	-		-	-	3A	3A
$\ddot{h}_i^{<i>}$	-	-		-	5M4A	-	5M4A
$a_i^{<i>}$	8M5A	3M	8M4A	12M8A	14M10A	17M13A	14M10A
ρ_i	1M	-		-	4M2A	-	4M3A
$\bar{\mu}_i^{<i>}$	1M	0	4M2A	4M2A	3M	6M3A	3M
$\zeta_i^{<i>}$	-	3M2A		3M2A	6M3A	22M19A	9M5A
$\mu_i^{<i>}$	-	8M4A	8M7A	8M7A	8M7A	$\begin{cases} 8M7A \\ (3A \text{ for } i = n) \end{cases}$	
$\zeta_i^{<i>}$	-	2M3A [†]		3M4A [†]	4M5A [†]	$\begin{cases} 12M11A \\ (6M \ 9A \text{ for } i = n) \end{cases}$	
τ_i	1A	0		0	1A	0	1A

[†] It additionally requires 3M2A for rotational joint s-1 or 11M6A for translational joint s-1.

As was mentioned above, (62) is replaced by (69)–(71). Applying (80) and (81), we modify (61) and (63)–(65) as follows:

$$\ddot{h}_i^{(i)} = K_i \left(\begin{bmatrix} 0 \\ 0 \\ \dot{q}_i \end{bmatrix} + 2 \begin{bmatrix} (\omega_i^{(i)})_y \\ -(\omega_i^{(i)})_x \\ 0 \end{bmatrix} \dot{q}_i \right) \quad (82)$$

$$\rho_i = K_i [\hat{m}_i (a_i^{(i)})_z + \kappa_{1i} (\dot{\omega}_i^{(r)})_z - \kappa_{2i} (\omega_i^{(r)})_z^2] \quad (83)$$

$$\bar{\mu}_i^{(i)} = \delta_i K_i^* \Omega_i^{(i)} \begin{bmatrix} (k_i)_x \\ (k_i)_y \\ 0 \end{bmatrix} + K_i \hat{m}_i \ddot{h}_i^* \quad (84)$$

$$\begin{aligned} (\bar{\zeta}_i^{(r)})_z &= K_i^* \left\{ \delta_i (\mathbf{u}_r^{(i)})_z \begin{bmatrix} (a_i^{(i)})_y \\ -(\mathbf{a}_i^{(i)})_x \end{bmatrix} \cdot \begin{bmatrix} (k_i)_x \\ (k_i)_y \end{bmatrix} \right. \\ &\quad \left. + (\dot{\omega}_i^{(r)})_z (D_i^{(i)})_{33} \right\} \\ &\quad + K_i \left\{ \begin{bmatrix} (\mathbf{u}_r^{(i)})_x \\ (\mathbf{u}_r^{(i)})_y \end{bmatrix} \cdot \begin{bmatrix} -(\mathbf{a}_i^*)_y \\ (\mathbf{a}_i^*)_x \end{bmatrix} \hat{m}_i d_i \right. \\ &\quad \left. + \kappa_{1i} \dot{q}_i + 2\kappa_{2i} (\omega_i^{(r)})_z \dot{q}_i \right\} \quad (85) \end{aligned}$$

where the constants are

$$\kappa_{1i} \equiv -(\mathbf{u}_r^{(i)})_y (k_i)_x + (\mathbf{u}_r^{(i)})_x (k_i)_y \quad (86)$$

$$\begin{aligned} \kappa_{2i} &\equiv -(\mathbf{u}_r^{(i)})_z [(\mathbf{u}_r^{(i)})_x (k_i)_x + (\mathbf{u}_r^{(i)})_y (k_i)_y] \\ &\quad + [1 - (\mathbf{u}_r^{(i)})_z^2] (k_i)_z \quad (87) \end{aligned}$$

which are the elements of the set of MLC's. Note that $\mathbf{u}_r^{(i)}$, $i < s$, are constant vectors. The reason for (84) is that the components of \mathbf{k}_i are not MLC's and thus are redundant for determining the actuator forces if $\delta_i = 0$ (see Theorem 2). Furthermore, (60) is redundant since (85) does not require it to compute $(\bar{\zeta}_i^{(i)})_z$.

In the backward recursion, only (67) needs to be modified since only the z -component of $\zeta_i^{(i)}$ must be computed. According to (74) and (75), (67) is replaced by

$$(\zeta_s^{*(r)})_z = \mathbf{u}_r^{(s-1)} \cdot {}_{s-1}^s \mathbf{R} \left(\zeta_s^{(s)} + \begin{bmatrix} 0 \\ 0 \\ d_s \end{bmatrix} \times \mu_s^{(s)} \right) \quad (88)$$

for $i+1 = s$

$$(\zeta_{i+1}^{*(r)})_z = (\zeta_{i+1}^{(r)})_z + \mathbf{u}_r^{(i+1)} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ d_{i+1} \end{bmatrix} \times \mu_{i+1}^{(i+1)} \right) \quad (89)$$

for $i+1 < s$

$$\begin{aligned} (\zeta_i^{(r)})_z &= (\bar{\zeta}_i^{(r)})_z + (\zeta_{i+1}^{*(r)})_z \\ &\quad + \mathbf{u}_r^{(i)} \cdot \left(\begin{bmatrix} b_{i+1} \\ 0 \\ 0 \end{bmatrix} \times \mu_{i+1}^{(i)} \right) \quad (90) \end{aligned}$$

An examination of (83)–(85) indicates that all inertia constants in the formulation have been replaced by the MLC's described in Theorem 2.

- 3) Joints remaining behind joint $s-1$, i.e., $i \geq s$: None of (57)–(68) needs to be further modified since joints in this group fall under the general case. However, (62), (64), (65), and (67) are replaced by (69)–(75), respectively.

In addition, $D_i^{(i)}$ in (38) varies and must be calculated for each rotational joint each recursion. To save computation, (38) can be rewritten in the following more compact form:

$$\begin{aligned} D_i^{(i)} &= U_i^* - \text{diag}[(U_i^*)_{22}, (U_i^*)_{22}, 0] \\ &\quad + \sum_{k=i+1}^{m_i-1} d_k \mathbf{W}_k^{(i)} \quad \text{for } i \geq s \quad (91) \end{aligned}$$

TABLE III

Method	General Robot with n Rotational Joints	Stanford Arm	
		General	Simplified
Khalil and Kleinfinger [16]	M: $101n - 129$ A: $90n - 118$	240 227	142 99
Balafoutis et al. [4]	M: $93n - 69$ A: $81n - 66$	- -	- -
Gautier and Khalil [10]	M: $92n - 127$ A: $81n - 117$	232 218	142 99
Present	M: $89n - 82$	231	139
	A: $77n - 71$	219	101

where the first two terms on the right-hand side are calculated by using (42) and

$$\mathbf{W}_k^{(i)} \equiv {}^k_i \mathbf{R} \begin{bmatrix} 2(\mathbf{k}_k)_z & 0 & -(\mathbf{k}_k)_x \\ 0 & 2(\mathbf{k}_k)_z & -(\mathbf{k}_k)_y \\ -(\mathbf{k}_k)_x & -(\mathbf{k}_k)_y & 0 \end{bmatrix} {}^k_i \mathbf{R}^T \quad (92)$$

which is a constant symmetric matrix since ${}^k_i \mathbf{R}$, $i < k < m_i$, is constant. Note that \mathbf{U}_i^* is also constant. For a rotational joint in front of joint s , only the (3, 3)th entry of $\mathbf{D}_i^{(i)}$ is required, which is

$$(\mathbf{D}_i^{(i)})_{33} = (\mathbf{U}_i^*)_{33} + \sum_{k=i+1}^{m_i-1} 2\kappa_{2k} d_k \quad \text{for } i < s \quad (93)$$

This algorithm has been verified by a FORTRAN program. The number of operations for each variable in the algorithm is listed in Table II. If we consider a manipulator with n rotational joints whose second joint is not parallel to the first joint, the total number of computations of the present algorithm is $(89n - 82)\text{M}$ and $(77n - 71)\text{A}$, where "M" and "A" denote multiplication and addition/subtraction, respectively. The number of computations for the coordinate transformation matrices ${}^{i-1}_i \mathbf{R}$ and the distance between two frames ${}^{i-1}_i \mathbf{s}^{(i-1)}$ has not yet been taken into account; these total $4n\text{M}$ and n pairs of sin and cos for n rotational joints. In most industrial robots, adjacent joints are either parallel or perpendicular to each other, which reduces the number of computations for the product of a coordinate transformation and a vector to $4\text{M}+2\text{A}$. The total number of computations for an industrial robot is then $5(n-1)(4\text{M}+2\text{A}) + 2\text{M} + 1\text{A}$ less than the number for a general manipulator since only the z -component of $\zeta_1^{(1)}$ needs to be computed.

The efficiency of the present algorithm is compared with that of the other algorithms [4], [10], [16] in Table III. The algorithm of Gautier and Khalil [10] is a reformulation of Khalil and Kleinfinger [16] using only the MLC's. As was mentioned above, the present formulation is identical to that in [16] for a general manipulator with only rotational joints, so these two formulations should have same efficiency in this case. The difference in their efficiency shown in Table III may come from different programming techniques or

different assumptions. For instance, if joint 1 is parallel to the direction of gravity, the operation count of the present formulation listed in Table III can be further reduced. Nevertheless, Table III shows that the present formulation is almost as efficient as the most efficient formulation in the literature. This is demonstrated by the comparison for the Stanford arm, which has one translational joint.

IV. IDENTIFICATION

Our goal is to formulate the linear equations in a recursive form, so that the procedure for identifying the inertia constants of the composite body i can be executed on the basis of knowledge of the inertia constants of the composite bodies $i+1$ to n .

Substituting (63)–(67) into (68), we obtain

$$\begin{aligned} \tau_i = & K_i \left\{ \mathbf{v}_{Ti}^T \mathbf{w}_{Ti} + (\mu_{i+1}^{(i)})_z \right\} \\ & + K_i^* \left\{ \mathbf{v}_{Ri}^T \mathbf{w}_{Ri} + \left[\Psi(\dot{\omega}_i^{(i)}, \omega_i^{(i)}) \psi(\bar{\mathbf{D}}_i^{(i)}) \right. \right. \\ & \left. \left. + {}^{i+1}_i \mathbf{s}^{(i)} \times \mu_{i+1}^{(i)} + \zeta_{i+1}^{(i)} \right]_z \right\} \quad (94) \end{aligned}$$

where

$$\mathbf{w}_{Ti} \equiv \begin{bmatrix} \hat{m}_i \\ \mathbf{k}_i \end{bmatrix} \quad (95)$$

$$\mathbf{w}_{Ri} \equiv \begin{bmatrix} (\mathbf{k}_i)_x \\ (\mathbf{k}_i)_y \\ (\mathbf{U}_i)_{11} - (\mathbf{U}_i)_{22} \\ (\mathbf{U}_i)_{33} \\ (\mathbf{U}_i)_{12} \\ (\mathbf{U}_i)_{13} \\ (\mathbf{U}_i)_{23} \end{bmatrix} \quad (96)$$

$$\mathbf{v}_{Ti} \equiv \begin{bmatrix} (\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)})_z \\ -(\dot{\omega}_i^{(i)})_y + (\omega_i^{(i)})_x (\omega_i^{(i)})_z \\ (\dot{\omega}_i^{(i)})_x + (\omega_i^{(i)})_y (\omega_i^{(i)})_z \\ -(\omega_i^{(i)})_x^2 - (\omega_i^{(i)})_y^2 \end{bmatrix} = \begin{bmatrix} (\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)})_z \\ (\Omega_i^{(i)})_{31} \\ (\Omega_i^{(i)})_{32} \\ (\Omega_i^{(i)})_{33} \end{bmatrix} \quad (97)$$

$$\mathbf{v}_{Ri} \equiv \begin{bmatrix} (\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)})_y \\ -(\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)})_x \\ -(\dot{\omega}_i^{(i)})_x (\omega_i^{(i)})_y \\ (\dot{\omega}_i^{(i)})_z \\ (\omega_i^{(i)})_x^2 - (\omega_i^{(i)})_y^2 \\ (\dot{\omega}_i^{(i)})_x - (\omega_i^{(i)})_y (\omega_i^{(i)})_z \\ (\dot{\omega}_i^{(i)})_y + (\omega_i^{(i)})_x (\omega_i^{(i)})_z \end{bmatrix} = \begin{bmatrix} (\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)})_y \\ -(\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)})_x \\ \left(\Psi(\dot{\omega}_i^{(i)}, \omega_i^{(i)}) \right)_{31} \\ \left(\Psi(\dot{\omega}_i^{(i)}, \omega_i^{(i)}) \right)_{33} \\ \left(\Psi(\dot{\omega}_i^{(i)}, \omega_i^{(i)}) \right)_{34} \\ \left(\Psi(\dot{\omega}_i^{(i)}, \omega_i^{(i)}) \right)_{35} \\ \left(\Psi(\dot{\omega}_i^{(i)}, \omega_i^{(i)}) \right)_{36} \end{bmatrix} \quad (98)$$

$$\begin{aligned} \tilde{\mathbf{D}}_i^{(i)} &\equiv \mathbf{D}_i^{(i)} - \begin{bmatrix} (\mathbf{U}_i)_{11} - (\mathbf{U}_i)_{22} & (\mathbf{U}_i)_{12} & (\mathbf{U}_i)_{13} \\ (\mathbf{U}_i)_{12} & 0 & (\mathbf{U}_i)_{23} \\ (\mathbf{U}_i)_{13} & (\mathbf{U}_i)_{23} & (\mathbf{U}_i)_{33} \end{bmatrix} \\ &= \mathbf{G}_{m_i}^{(i)} - \text{diag}[(\mathbf{G}_{m_i}^{(i)})_{22}, (\mathbf{G}_{m_i}^{(i)})_{22}, 0] \\ &\quad + \sum_{k=i+1}^{m_i-1} d_k \mathbf{W}_k^{(i)} \end{aligned} \quad (99)$$

provided joint m_i is the nearest rotational joint behind joint i . Note that $\mathbf{G}_{m_i}^{(i)}$ in (99) is defined in (40).

For the case of $r \leq i < s$, the angular velocity and acceleration of link i are in alignment with the direction of joint r (see (80) and (81)). Equation (94) can be reduced to

$$\begin{aligned} \tau_i &= K_i \left\{ \bar{\mathbf{v}}_{Ti}^T \bar{\mathbf{w}}_{Ti} + (\mu_{i+1}^{(i)})_z \right\} \\ &\quad + K_i^* \left\{ \bar{\mathbf{v}}_{Ri}^T \bar{\mathbf{w}}_{Ri} + (\dot{\omega}_i^{(i)})_z (\tilde{\mathbf{D}}_i^{(i)})_{33} \right. \\ &\quad \left. + \left({}^{i+1}\mathbf{s}^{(i)} \times \mu_{i+1}^{(i)} + \zeta_{i+1}^{(i)} \right)_z \right\} \quad \text{for } r \leq i < s \end{aligned} \quad (100)$$

where

$$\bar{\mathbf{w}}_{Ti} \equiv \begin{bmatrix} \hat{m}_i \\ \sigma_i \kappa_{1i} \\ \sigma_i \kappa_{2i} \end{bmatrix} \quad (101)$$

$$\bar{\mathbf{w}}_{Ri} \equiv \begin{bmatrix} \delta_i(\mathbf{k}_i)_x \\ \delta_i(\mathbf{k}_i)_y \\ (\mathbf{U}_i)_{33} \end{bmatrix} \quad (102)$$

$$\bar{\mathbf{v}}_{Ti} \equiv \begin{bmatrix} (\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)})_z \\ (\dot{\omega}_i^{(r)})_z \\ -(\omega_i^{(r)})_z^2 \end{bmatrix} \quad (103)$$

$$\bar{\mathbf{v}}_{Ri} \equiv \begin{bmatrix} (\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)})_y \\ -(\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)})_x \\ (\dot{\omega}_i^{(i)})_z \end{bmatrix} \quad (104)$$

$$(\tilde{\mathbf{D}}_i^{(i)})_{33} = (\mathbf{G}_{m_i}^{(i)})_{33} + \sum_{k=i+1}^{m_i-1} 2\kappa_{2k} d_k \quad (105)$$

It should be remarked that the parameters in the linear equations (94) and (100) (i.e., \mathbf{w}_{Ti} , \mathbf{w}_{Ri} , $\bar{\mathbf{w}}_{Ti}$, $\bar{\mathbf{w}}_{Ri}$) are MLC's.

For the case of $i < r$, (94) leads to

$$\tau_i = (\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)})_z \hat{m}_i + (\mu_{i+1}^{(i)})_z \quad \text{for } i < r \quad (106)$$

since the angular velocity and acceleration of link i are both zero. The fact that $(\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)})_z$, $i < r$, is not zero implies that \hat{m}_i can be identified by means of (106).

Define

$$\mathbf{A}_i \equiv \begin{bmatrix} (\mathbf{v}_i^{(1)})^T \\ (\mathbf{v}_i^{(2)})^T \\ \vdots \\ (\mathbf{v}_i^{(s)})^T \end{bmatrix} \quad (107)$$

$$\gamma_i \equiv \begin{bmatrix} \tau_i^{(1)} \\ \tau_i^{(2)} \\ \vdots \\ \tau_i^{(s)} \end{bmatrix} - \begin{bmatrix} \varphi_i^{(1)} \\ \varphi_i^{(2)} \\ \vdots \\ \varphi_i^{(s)} \end{bmatrix} \quad (108)$$

where

$$\mathbf{v}_i \equiv \begin{cases} \mathbf{v}_{Ti} & \text{for translational joint } i, i \geq s, \\ \mathbf{v}_{Ri} & \text{for rotational joint } i, i \geq s, \\ \bar{\mathbf{v}}_{Ti} & \text{for translational joint } i, r \leq i < s, \\ \bar{\mathbf{v}}_{Ri} & \text{for rotational joint } i, r \leq i < s, \end{cases} \quad (109)$$

$$\varphi_i \equiv \begin{cases} K_i (\mu_{i+1}^{(i)})_z + K_i^* \left[\Psi(\dot{\omega}_i^{(i)}, \omega_i^{(i)}) \psi(\tilde{\mathbf{D}}_i^{(i)}) \right. \\ \quad \left. + {}^{i+1}\mathbf{s}^{(i)} \times \mu_{i+1}^{(i)} + \zeta_{i+1}^{(i)} \right]_z & \text{for } s \leq i \leq n, \\ K_i (\mu_{i+1}^{(i)})_z + K_i^* \left[(\dot{\omega}_i^{(i)})_z (\tilde{\mathbf{D}}_i^{(i)})_{33} \right. \\ \quad \left. + ({}^{i+1}\mathbf{s}^{(i)} \times \mu_{i+1}^{(i)} + \zeta_{i+1}^{(i)})_z \right] & \text{for } r \leq i < s \end{cases} \quad (110)$$

and $\textcircled{2}$ denotes the i th sampling point. According to (94), (100), and (107)–(110), we have

$$\mathbf{A}_i \mathbf{w}_i = \gamma_i \quad \text{for } i \geq r \quad (111)$$

where

$$\mathbf{w}_i \equiv \begin{cases} \mathbf{w}_{Ti} & \text{for translational joint } i, i \geq s, \\ \mathbf{w}_{Ri} & \text{for rotational joint } i, i \geq s, \\ \bar{\mathbf{w}}_{Ti} & \text{for translational joint } i, r \leq i < s, \\ \bar{\mathbf{w}}_{Ri} & \text{for rotational joint } i, r \leq i < s. \end{cases} \quad (112)$$

The number of rows of \mathbf{A}_i is assumed to be greater than the number of columns, although the number of columns is different for different joints. The vector \mathbf{w}_i can be solved by a least squares method if the matrix \mathbf{A}_i is of full rank. Examining (97), (98), (103), and (104), we see that the elements of these vectors are individually linearly independent over the domain of $\{\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}\}$. Thus there exist N sampling points such that \mathbf{A}_i for $i \geq r$ are of full rank.

Assume there is a persistently exciting trajectory (i.e., \mathbf{A}_i , $i = 1, \dots, n$, are all of full rank along the trajectory). The linear equations (94), (100), and (106) provide us with the following identification procedure:

Identification Procedure:

Step 1: Compute $\dot{\omega}_i^{(i)}$, $\omega_i^{(i)}$, $\ddot{\mathbf{h}}_i^{(i)}$, and $\mathbf{a}_i^{(i)}$ from $i = 1$ to n for all sampling points (assume N points) of the persistently exciting trajectory by using (57)–(62). These values are saved in memory. If there is no rotational joint, i.e., $r > n$, go to Step 4.

Step 2: Use the values of $\dot{\omega}_n^{(n)}$, $\omega_n^{(n)}$, $\ddot{\mathbf{h}}_n^{(n)}$, $\mathbf{a}_n^{(n)}$ to calculate \mathbf{v}_n by (97), (98), (103) or (104) for each sampling point from point 1 to point N , and then form \mathbf{A}_n . Use the least squares method to solve \mathbf{w}_n from

$$\mathbf{A}_n \mathbf{w}_n = \begin{bmatrix} \tau_n^{(1)} \\ \tau_n^{(2)} \\ \vdots \\ \tau_n^{(N)} \end{bmatrix} \quad (113)$$

Step 3: If $r > n - 1$, go to Step 4; otherwise, set $\tilde{\mathbf{D}}_n^{(n)} = \mathbf{0}$ and do the following substeps for joint i recursively from $i = n - 1$ to $i = r$.

3.1 For joint i , do the following substeps for each sampling point from point 1 to point N .

3.1.1 If joint $i+1$ is a rotational joint, compute $\mathbf{D}_{i+1}^{(i+1)}$ for $i+1 \geq s$ by

$$\mathbf{D}_{i+1}^{(i+1)} = \tilde{\mathbf{D}}_{i+1}^{(i+1)} + \begin{bmatrix} (\mathbf{U}_{i+1})_{11} - (\mathbf{U}_{i+1})_{22} & (\mathbf{U}_{i+1})_{12} & (\mathbf{U}_{i+1})_{13} \\ (\mathbf{U}_{i+1})_{12} & 0 & (\mathbf{U}_{i+1})_{23} \\ (\mathbf{U}_{i+1})_{13} & (\mathbf{U}_{i+1})_{23} & (\mathbf{U}_{i+1})_{33} \end{bmatrix} \quad (114)$$

or compute $(\mathbf{D}_{i+1}^{(i+1)})_{33}$ for $i+1 < s$ by

$$(\mathbf{D}_{i+1}^{(i+1)})_{33} = (\tilde{\mathbf{D}}_{i+1}^{(i+1)})_{33} + (\mathbf{U}_{i+1})_{33} \quad (115)$$

3.1.2 Use the values of $\dot{\omega}_{i+1}^{(i+1)}$, $\omega_{i+1}^{(i+1)}$, $\ddot{\mathbf{h}}_{i+1}^{(i+1)}$, $\mathbf{a}_{i+1}^{(i+1)}$, \mathbf{w}_{i+1} (i.e., parts of $\hat{\mathbf{m}}_{i+1}$, \mathbf{k}_{i+1} and \mathbf{U}_{i+1}), and $\mathbf{D}_{i+1}^{(i+1)}$ to calculate $\mu_{i+1}^{(i+1)}$, $\bar{\zeta}_{i+1}^{(i+1)}$ by (59), (60), (64) and (65). Then compute $\mu_{i+1}^{(i+1)}$ and $\zeta_{i+1}^{(i+1)}$ by (66) and (67) and transform them to $\mu_{i+1}^{(i)}$ and $\zeta_{i+1}^{(i)}$.

3.1.3 If joint i is a rotational joint, compute $\tilde{\mathbf{D}}_i^{(i)}$ for $i \geq s$ by (99) or $(\tilde{\mathbf{D}}_i^{(i)})_{33}$ for $i < s$ by (105).

3.1.4 Compute φ_i by (110) and form \mathbf{v}_i by (97), (98), (103) or (104).

3.2 Compute γ_i by (108) and then form (111). Finally, use the least squares method to solve \mathbf{w}_i from (111).

Step 4: If $r > 1$, do the following substeps for joint i recursively from $i = r - 1$ to $i = 1$.

4.1 For joint i , do the following substep for each sampling point from point 1 to point N .

4.1.1 Use the values of $\dot{\omega}_{i+1}^{(i+1)}$, $\omega_{i+1}^{(i+1)}$ and $\ddot{\mathbf{h}}_{i+1}^{(i+1)}$ to calculate $\mu_{i+1}^{(i+1)}$ by (59) and (64). Then compute $\mu_{i+1}^{(i+1)}$ by (66) and transform it to $\mu_{i+1}^{(i)}$.

4.2 Form

$$\begin{bmatrix} (\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)})^{(1)} \\ (\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)})^{(2)} \\ \vdots \\ (\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)})^{(N)} \end{bmatrix} \hat{\mathbf{m}}_i = \begin{bmatrix} \tau_i^{(1)} - (\mu_{i+1}^{(i)})^{(1)} \\ \tau_i^{(2)} - (\mu_{i+1}^{(i)})^{(2)} \\ \vdots \\ \tau_i^{(N)} - (\mu_{i+1}^{(i)})^{(N)} \end{bmatrix} \quad (116)$$

and use the least squares method to solve $\hat{\mathbf{m}}_i$ from (116).

As just stated, the identification procedure requires a persistently exciting trajectory along which all modes of the system should be excited [35]. Since the actuator forces of a manipulator are bound, we can describe the persistently exciting trajectory in the following mathematical form.

Definition 2: The sampling signals $\{\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}\}$ of a trajectory are said to be *persistently exciting* for the least squares estimation of the system (111) if

$$\mathbf{M}_i = \sum_{k=1}^N \mathbf{v}_i^{(k)} (\mathbf{v}_i^{(k)})^T \quad (117)$$

is positive definite, where N is the number of sampling points in the trajectory.

It is apparent that $\mathbf{M}_i = \mathbf{A}_i^T \mathbf{A}_i$. If matrix \mathbf{A}_i is of full rank, $\mathbf{A}_i^T \mathbf{A}_i$ is symmetric and positive definite [14]. Examining (97), (98), (103), and (104), we find that \mathbf{A}_i is of full rank for most trajectories. This is why the trajectories arbitrarily selected in the literature [3], [15] are all persistently exciting. We performed computer simulations of the identification procedure on the Stanford arm for several persistently exciting trajectories, and the identified results matched the true values very closely. A detailed report on these simulations can be found in [22].

In practical identification, there are measurement errors, which were not taken into account in the computer simulation. The measured values, especially the joint velocities and accelerations, are perturbed within a certain error bound. In regression theory, the width of the prediction interval of the estimated values is proportional to the standard deviation of the residuals (i.e., the square root of the error mean square). Least squares theory [14], [18] indicates that the upper bound of the relative error, and thus that of the residuals, is about proportional to the condition number of the excitation matrix \mathbf{A}_i in (111). As a result, the accuracy of the least squares estimation depends on the condition number of the excitation matrix. An arbitrary, persistently exciting trajectory cannot ensure small condition numbers for \mathbf{A}_i , $i = 1, \dots, n$, so it is necessary to search for an optimal exciting trajectory. Two good references [2], [13] discuss this optimization problem.

V. CONCLUSION

This paper addresses the minimal linear combinations (MLC's) of the inertia parameters of a manipulator, the inverse dynamics in terms the MLC's, and an identification procedure for estimating the MLC's. These three themes are closely related to one another in the sense of the dynamic modeling of a manipulator, and so should be treated together. Knowledge of a set of MLC's facilitates parameter identification. The purpose of identifying the parameters is to use them in the inverse dynamics to control a manipulator. This paper formulates the inverse dynamics in terms of the set of MLC's; the proposed formulation is almost as efficient as the most efficient formulation of the inverse dynamics [10], [16]. It is interesting that the identification procedure is derived from the formulation of the inverse dynamics. The identification procedure is simple and efficient, since it

does not require symbolic closed-form equations and it has a recursive structure.

In the literature [3], [17], there have been successful examples of experimental identification of the parameters of a manipulator whose dominant dynamics can be described by the standard Newton-Euler formulation, e.g., a direct drive robot. It is reasonable to believe that the present identification procedure is valid in practice for direct drive robots since the procedure is also based on the Newton-Euler formulation.

The effects of friction from high-ratio gear trains on the manipulator dynamics will invalidate the proposed identification procedure for manipulators with high-ratio gear trains. Further investigation will be required to extract the friction terms from the dynamic model and to estimate them separately, so that the present procedure can be applied.

APPENDIX

Proof of (29)–(31): It is easy to show that

$$\begin{aligned} \mathbf{a} \times (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})) + \mathbf{b} \times (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{a})) \\ = -\boldsymbol{\omega} \times (\mathbf{a} \times (\mathbf{b} \times \boldsymbol{\omega}) + \mathbf{b} \times (\mathbf{a} \times \boldsymbol{\omega})) \end{aligned} \quad (\text{A1})$$

Expanding the left-hand side of (29) and applying (A1) and the equivalence property of (20) and (28), we then obtain (29).

Equation (30) is due to the assignment of the normal driving-axis coordinate system. Substituting (16) and (17) into (30) and expanding both sides while noting that $\mathbf{u}_i^{(i)} = [0, 0, 1]^T$, we find that (30) is true.

It is easy to show that for any constant matrix $\mathbf{A} \in \mathfrak{R}^{3 \times 3}$,

$$\begin{aligned} {}_{i-1}\mathbf{R}(\mathbf{A}\boldsymbol{\alpha} + \boldsymbol{\omega} \times (\mathbf{A}\boldsymbol{\omega})) \\ = ({}_{i-1}\mathbf{R}\mathbf{A}{}_{i-1}\mathbf{R}^T)_{i-1}\mathbf{R}\boldsymbol{\alpha} \\ + ({}_{i-1}\mathbf{R}[\boldsymbol{\omega} \times]_{i-1}\mathbf{R}^T)({}_{i-1}\mathbf{R}\mathbf{A}{}_{i-1}\mathbf{R}^T)_{i-1}\mathbf{R}\boldsymbol{\omega} \end{aligned} \quad (\text{A2})$$

Since $\text{diag}[a, a, 0] \in \mathfrak{R}^{3 \times 3}$ and ${}_{i-1}\mathbf{R}[\boldsymbol{\omega} \times]_{i-1}\mathbf{R}^T = [({}_{i-1}\mathbf{R}\boldsymbol{\omega}) \times]$, the second equality in (31) is true. Substituting (16) and (17) into (28) and replacing \mathbf{A} with $\text{diag}[a, a, 0]$, we have the first equality of (31).

Proof of (46): According to the recursive Newton-Euler formulation (16)–(23), we have

$$\begin{aligned} \mathbf{t}_i^{(i)} = -\sum_{j=i}^n {}^j\mathbf{R} \left[\mathbf{t}_{T_j}^{(j)} + ({}^j\mathbf{s}^{(j)} + \mathbf{c}_j^{(j)}) \times (\mathbf{f}_{T_j}^{(j)} + m_j \mathbf{g}^{(j)}) \right] \\ - \mathbf{t}_E^{(i)} - \left({}^n\mathbf{s}^{(i)} + {}^E\mathbf{s}^{(i)} \right) \times \mathbf{f}_E^{(i)} \end{aligned} \quad (\text{A3})$$

It follows from (27)–(29) that

$$\begin{aligned} \mathbf{t}_{T_j}^{(j)} + \mathbf{c}_j^{(j)} \times (\mathbf{f}_{T_j}^{(j)} + m_j \mathbf{g}^{(j)}) \\ = -\Psi(\dot{\omega}_j^{(j)}, \omega_j^{(j)}) \psi(\mathbf{J}_j^{(j)}) - m_j \mathbf{c}_j^{(j)} \times (\mathbf{a}_j^{(j)} - \mathbf{g}^{(j)}) \end{aligned} \quad (\text{A4})$$

where $\mathbf{J}_j^{(j)} = \mathbf{I}_j^{(j)} - m_j [\mathbf{c}_j^{(j)} \times] [\mathbf{c}_j^{(j)} \times]$. It is easy to show that

$$\begin{aligned} \sum_{j=i}^n {}^j\mathbf{R} \left[{}^j\mathbf{s}^{(j)} \times (-\mathbf{f}_{T_j}^{(j)} - m_j \mathbf{g}^{(j)}) \right] - {}^n\mathbf{s}^{(i)} \times \mathbf{f}_E^{(i)} \\ = \sum_{j=i}^{n-1} {}^j\mathbf{R} ({}^{j+1}\mathbf{s}^{(j)} \times \mathbf{f}_{j+1}^{(j)}) \end{aligned} \quad (\text{A5})$$

We substitute (A4) and (A5) into (A3) to obtain

$$\begin{aligned} \mathbf{t}_i^{(i)} + \mathbf{t}_E^{(i)} + {}^E\mathbf{s}^{(i)} \times \mathbf{f}_E^{(i)} \\ = \mathbf{k}_i \times (\mathbf{a}_i^{(i)} - \mathbf{g}^{(i)}) + \eta_{i+1}^{(i)} \\ + \sum_{j=i}^n {}^j\mathbf{R} \Psi(\dot{\omega}_j^{(j)}, \omega_j^{(j)}) \psi(\mathbf{J}_j^{(j)}) \\ + \sum_{j=i}^{n-1} {}^j\mathbf{R} \left\{ {}^{j+1}\mathbf{s}^{(j)} \times (\boldsymbol{\Omega}_j^{(j)} \hat{m}_{j+1} {}^{j+1}\mathbf{s}^{(j)}) \right. \\ \left. + K_{j+1}^* \left[{}^{j+1}\mathbf{s}^{(j)} \times (\boldsymbol{\Omega}_j^{(j)} {}^{j+1}\mathbf{R}_b(\mathbf{k}_{j+1})_z) \right. \right. \\ \left. \left. + {}^{j+1}\mathbf{R}_b(\mathbf{k}_{j+1})_z \times (\boldsymbol{\Omega}_j^{(j)} {}^{j+1}\mathbf{s}^{(j)}) \right] \right. \\ \left. + K_{j+1} \left[{}^{j+1}\mathbf{s}^{(j)} \times (\boldsymbol{\Omega}_j^{(j)} {}^{j+1}\mathbf{R}\mathbf{k}_{j+1}) \right. \right. \\ \left. \left. + ({}^{j+1}\mathbf{R}\mathbf{k}_{j+1}) \times (\boldsymbol{\Omega}_j^{(j)} {}^{j+1}\mathbf{s}^{(j)}) \right] \right. \\ \left. - \hat{m}_{j+1} {}^{j+1}\mathbf{s}^{(j)} \times (\boldsymbol{\Omega}_j^{(j)} \mathbf{d}_{j+1}^{(j)}) \right. \\ \left. - \hat{m}_{j+1} \mathbf{d}_{j+1}^{(j)} \times (\boldsymbol{\Omega}_j^{(j)} {}^{j+1}\mathbf{s}^{(j)}) \right. \\ \left. + \hat{m}_{j+1} \mathbf{d}_{j+1}^{(j)} \times (\boldsymbol{\Omega}_j^{(j)} \mathbf{d}_{j+1}^{(j)}) \right\} \end{aligned} \quad (\text{A6})$$

by using (30), (32), (36), (37) and

$$\begin{aligned} \mathbf{a}_{j+1}^{(j)} = \mathbf{a}_j^{(j)} + \boldsymbol{\Omega}_j^{(j)} {}^{j+1}\mathbf{s}^{(j)} - K_{j+1} (\mathbf{u}_{j+1}^{(j)} \ddot{q}_{j+1} \\ + 2\boldsymbol{\omega}_{j+1}^{(j)} \times \mathbf{u}_{j+1}^{(j)} \dot{q}_{j+1}) \end{aligned} \quad (\text{A7})$$

Note that $\eta_i^{(i)}$ is defined in (48). By (29) and

$$\begin{aligned} [{}^{j+1}\mathbf{s}^{(j)} \times] [{}^{j+1}\mathbf{s}^{(j)} \times] \\ = [\mathbf{b}_{j+1}^{(j)} \times] [\mathbf{b}_{j+1}^{(j)} \times] + [\mathbf{b}_{j+1}^{(j)} \times] [\mathbf{d}_{j+1}^{(j)} \times] \\ + [\mathbf{d}_{j+1}^{(j)} \times] [\mathbf{b}_{j+1}^{(j)} \times] + [\mathbf{d}_{j+1}^{(j)} \times] [\mathbf{d}_{j+1}^{(j)} \times] \end{aligned} \quad (\text{A8})$$

(A6) turns out to be (46) with

$$\begin{aligned} \mathbf{A}_j^{(j)} = \mathbf{I}_j^{(j)} - m_j [\mathbf{c}_j^{(j)} \times] [\mathbf{c}_j^{(j)} \times] \\ - K_{j+1}^* \left\{ \hat{m}_{j+1} [{}^{j+1}\mathbf{s}^{(j)} \times] [{}^{j+1}\mathbf{s}^{(j)} \times] \right. \\ \left. + [{}^{j+1}\mathbf{s}^{(j)} \times] [({}^{j+1}\mathbf{R}_b(\mathbf{k}_{j+1})_z) \times] \right. \\ \left. + [({}^{j+1}\mathbf{R}_b(\mathbf{k}_{j+1})_z) \times] [{}^{j+1}\mathbf{s}^{(j)} \times] \right\} \\ - K_{j+1} \left\{ \hat{m}_{j+1} [\mathbf{b}_{j+1}^{(j)} \times] [\mathbf{b}_{j+1}^{(j)} \times] \right. \\ \left. + [{}^{j+1}\mathbf{s}^{(j)} \times] [({}^{j+1}\mathbf{R}\mathbf{k}_{j+1}) \times] \right. \\ \left. + [({}^{j+1}\mathbf{R}\mathbf{k}_{j+1}) \times] [{}^{j+1}\mathbf{s}^{(j)} \times] \right\} \end{aligned} \quad (\text{A9})$$

which is identical to (47) according to (12) and (14).

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REFERENCES

- [1] B. Armstrong, O. Khatib, and J. Burdick, "The explicit dynamic model and inertial parameters of the PUMA 560 arm," in *Proc. 1986 IEEE Conf. on Robot. and Automat.*, 1986, pp. 510-518.
- [2] B. Armstrong, "On finding exciting trajectories for identification experiments involving systems with nonlinear dynamics," *Int. J. Robot. Res.*, vol. 8, no. 6, pp. 28-48, 1989.
- [3] C. G. Atkeson, C. H. An, and J. M. Hollerbach, "Estimation of inertial parameters of manipulator loads and links," *Int. J. Robotics Res.*, vol. 5, no. 3, pp. 101-119, 1986.
- [4] C. A. Balafoutis, V. P. Rajnikant, and P. Mistra, "Efficient modeling and computation of manipulator dynamics using orthogonal Cartesian tensor," *IEEE J. Robot. and Automat.*, vol. 4, pp. 665-676, 1988.
- [5] J. J. Craig, *Introduction to Robotics: Manipulation and Control*. Reading, MA: Addison-Wesley, 1986.
- [6] M. Gautier, "Identification of robots dynamics," in *Proc. IFAC Symp. on Theory of Robot.*, 1986, pp. 351-356.
- [7] M. Gautier, "Numerical calculation of the base inertial parameters of robots," in *Proc. 1990 IEEE Conf. on Robot. and Automat.*, pp. 1020-1025, 1990.
- [8] M. Gautier, F. Bennis and W. Khalil, "The use of generalized links to determine the minimum inertial parameters of robots," *J. Robotic Systems*, pp. 225-242, 1990.
- [9] M. Gautier and W. Khalil, "On the identification of the inertial parameters of robots," in *Proc. 1988 IEEE Conf. on Decision and Cont.*, 1988, pp. 2264-2269.
- [10] M. Gautier and W. Khalil, "A direct determination of minimum inertial parameters of robots," in *Proc. 1988 IEEE Conf. on Robot. and Automat.*, 1988, pp. 1682-1687.
- [11] M. Gautier and W. Khalil, "Identification of the minimum inertial parameters of robots," in *Proc. 1989 IEEE Conf. on Robotics and Automation*, 1989, pp. 1529-1534.
- [12] M. Gautier and W. Khalil, "Direct calculation of minimum set of inertial parameters of serial robots," *IEEE Trans. Robot. and Automat.*, pp. 368-373, 1990.
- [13] M. Gautier and W. Khalil, "Exciting trajectories for the identification of base inertial parameters of robots," *Int. J. Robot. Res.*, vol. 11, no. 4, pp. 362-375, 1992.
- [14] G. H. Golub and C. F. Van Loan, *Matrix Computations*. London: Johns Hopkins Press, 1989.
- [15] I. J. Ha, M. S. Ko, and S. K. Kwon, "An efficient estimation algorithm for the model parameters of robotic manipulators," *IEEE Trans. Robot. and Automat.*, vol. 5, pp. 386-394, 1989.
- [16] W. Khalil and J. F. Kleinfinger, "Minimum operations and minimum parameters of the dynamic models of tree structure robots," *IEEE J. Robot. and Automat.*, vol. 3, pp. 517-526, 1987.
- [17] P. K. Khosla and T. Kanade, "Parameter identification of robot dynamics," in *Proc. IEEE Conf. on Decision and Cont.*, 1985, pp. 1754-1760.
- [18] C. H. Lawson and R. J. Hanson, *Solving Least Squares Problems*. Englewood Cliffs, NJ: Prentice-Hall, 1974.
- [19] M. B. Leahy and G. N. Saridis, "Compensation of industrial manipulator dynamics," *Int. J. Robot. Res.*, vol. 8, no. 4, pp. 73-84, 1989.
- [20] S. K. Lin, "Microprocessor implementation of the inverse dynamic system for industrial robot control," in *Proc. 10th IFAC World Congress on Automatic Control*, 1987, vol. 4, pp. 332-339.
- [21] S. K. Lin, "A new composite body method for manipulator dynamics," *J. Robot. Syst.*, vol. 8, pp. 197-219, 1991.
- [22] S. K. Lin, "Identification of the inertia parameters of an industrial robot," *NSC Project Report no. NSC80-0404-E-009-31*, National Science Council, Taiwan, 1991.
- [23] S. K. Lin, "An identification method for estimating the inertia parameters of a manipulator," *J. Robot. Syst.*, vol. 9, no. 4, pp. 505-528, 1992.
- [24] J. Y. S. Luh, M. W. Walker, and R. P. Paul, "Online computational scheme for mechanical manipulators," *Trans. ASME: J. Dynamic Syst., Meas., Cont.*, vol. 102, pp. 69-76, 1980.
- [25] H. Mayeda, K. Yoshida, and K. Osuka, "Base parameters of manipulator dynamics models," in *Proc. 1988 IEEE Conf. on Robot. and Automat.*, pp. 1367-1372, 1988.
- [26] H. Mayeda, K. Yoshida, and K. Osuka, "Base parameters of dynamics models for manipulators with rotational and translational joints," in *Proc. 1989 IEEE Conf. on Robot. and Automat.*, pp. 1523-1528, 1989.
- [27] H. Mayeda, K. Yoshida, and K. Osuka, "Base parameters of manipulator dynamics models," *IEEE Trans. Robot. and Automat.*, vol. 6, pp. 312-321, 1990.
- [28] H. Mayeda and K. Ohashi, "Base parameters of dynamic models for general open loop kinematic chains," in *Proc. 5th Int. Symp. on Robot. Res.*, 1990, pp. 271-278.
- [29] A. Mukerjee and D. H. Ballard, "Self-calibration in robot manipulations," in *Proc. 1985 IEEE Conf. on Robot. and Automat.*, 1985, pp. 1050-1057.
- [30] H. B. Olsen and G. A. Bekey, "Identification of parameters in models of robots with rotary joints," in *Proc. 1985 IEEE Conf. on Robot. and Automat.*, 1985, pp. 1045-1049.
- [31] H. B. Olsen and G. A. Bekey, "Identification of robot dynamics," in *Proc. 1986 IEEE Conf. on Robot. and Automat.*, 1986, pp. 1004-1010.
- [32] N. Renaud, "An efficient iterative analytical procedure for obtaining a robot manipulator dynamic model," in *Robotics Research*, M. Brady and R. Paul, Eds. Cambridge, MA: MIT Press, 1984, pp. 749-764.
- [33] N. Renaud, "A near minimum iterative analytical procedure for obtaining a robot manipulator dynamic model," in *Dynamics of Multibody Systems*, G. Bianchi and W. Schiehlen, Eds. Berlin: Springer-Verlag, 1986, pp. 201-212.
- [34] S. Y. Sheu and M. W. Walker, "Identifying the independent inertial parameter space of robot manipulators," *Int. J. Robot. Res.*, vol. 10, no. 6, pp. 668-683, 1991.
- [35] T. Söderström and Stoica, *System Identification*. New York: Prentice-Hall, 1989.
- [36] J. Wittenburg, *Dynamics of Systems of Rigid Bodies*. Stuttgart: B. G. Teubner, 1977.



Shir-Kuan Lin received the B.S. degree in aeronautical engineering from National Cheng Kung University, Taiwan, the M.S. degree in power mechanical engineering from National Tsing Hua University, Taiwan, and the Dr.-Ing. degree in manufacturing automation from Universität Erlangen-Nürnberg, Germany, in 1979, 1983, and 1988, respectively.

From 1984 to 1988, he was a recipient of the DAAD fellowship by the Deutschen Akademischen Austauschdienst. Since February 1989, he has been an Associate Professor in the Department and Institute of Control Engineering at the National Chiao Tung University, Taiwan. His major research interests include parameter identification, robotic control, multitask and multiprocessor system, servo control, and manufacturing automation.