Conditional-Fault Diagnosability of Multiprocessor Systems with an Efficient Local Diagnosis Algorithm under the PMC Model

Cheng-Kuan Lin, Tzu-Liang Kung, and Jimmy J.M. Tan

Abstract—Diagnosis is an essential subject for the reliability of multiprocessor systems. Under the PMC diagnosis model, Dahbura and Masson [12] proposed a polynomial-time algorithm with time complexity $O(N^{2.5})$ to identify all the faulty processors in a system with N processors. In this paper, we present a novel method to diagnose a conditionally faulty system by applying the concept behind the local diagnosis, introduced by Somani and Agarwal [30], and formalized by Hsu and Tan [18]. The goal of local diagnosis is to identify the fault status of any single processor correctly. Under the PMC diagnosis model, we give a sufficient condition to estimate the local diagnosability of a given processor. Furthermore, we propose a helpful structure, called the augmenting star, to efficiently determine the fault status of each processor. For an N-processor system in which every processor has an $O(\log N)$ degree, the time complexity of our algorithm to diagnose any given processor is $O((\log N)^2)$, provided that each processor can construct an augmenting star structure of full order in time $O((\log N)^2)$ and the time for a processor to test another one is constant. Therefore, the time totals to $O(N(\log N)^2)$ for diagnosing the whole system.

Index Terms—Fault diagnosis, PMC model, diagnosability, reliability, diagnosis algorithm.

1 Introduction

Recently, high-speed multiprocessor systems have become more and more popular in computer technology. A multiprocessor system consists of processors and communication links between processors. The reliability of processors is crucial since even a few malfunctioning processors may lead to a severe system breakdown. Whenever processors are found to be faulty, they should be replaced with fault-free ones as soon as possible to guarantee the system can work properly.

Identifying all the faulty processors in a system is known as *system-level diagnosis*. Preparata et al. [27] distinguished two types of self-diagnosable systems: one-step diagnosable systems and sequentially diagnosable systems. A system is said to be *one-step t-diagnosable* if all its faulty processors can be precisely pointed out by one application of a diagnostic process provided that the total number of faulty processors does not exceed t, whereas a system is *sequentially t-diagnosable* if at least one faulty processor can be identified provided that the total number of faulty processors does not exceed t. In this paper, we focus on one-step diagnosis only. The maximum number of faulty processors that can be correctly identified is an important parameter, known as the *one-step diagnosability* of a system. In other words, the one-step diagnosability of a system G is just equal to the maximum integer t such that G can be one-step t-diagnosable.

In practice, some multiprocessor systems are based on an underlying bus structure, or fabric, and are perfectly feasible for a centralized test controller (an independent processor acting as a controller) to check each processor in the system. In such a scheme, the centralized controller itself can be tested externally. Some research is related to the issue of network-on-chip (NoC); for example, Pande et al. [26] developed an evaluation methodology to compare the performance and characteristics of a variety of NoC topologies; Bartic et al. [4] presented an NoC design which is suitable for building networks with irregular topologies. Instead, a self-diagnosable system contains no centralized test controller. In a self-diagnosable system, a testing signal is supposed to be delivered from a processor to another one through the communication bus at one time. Then the system performs self-diagnosis by making each processor act as a tester to test each of the directly connected ones. This paper is concerned with the self-diagnosis.

1.1 Diagnosis Models

The problem of system-level diagnosis has been widely discussed by many researchers [9], [12], [13], [15], [16], [18], [20], [23], [24], [27], [29]. Several well-known approaches have been developed. One classic approach, called the PMC diagnosis model (or PMC model for short), was first proposed by Preparata et al. [27]. This model makes diagnosis by sending a test signal from a processor to another linked one and then receiving a response in the reverse direction. According to the collection of all test outcomes, the fault status of every processor can be identified. The fundamental assumption of the PMC model is that a test outcome is reliable if and only if the testing processor is fault-free. Another diagnostic model, called the BGM model [3], was defined by

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Barsi, Grandoni, and Maestrini with asymmetric interpretation of test outcomes reported by faulty units. In the BGM model, any test that is complete for a given class of faults in a unit necessarily consists of a sequence of a large number of stimuli. Supposedly, there must be at least one mismatch between actual and expected reaction to the stimuli whenever the tested unit is faulty, even if the testing unit itself is faulty.

In this paper we address the PMC model. Following this model, Hakimi and Amin [16] proved that a system is one-step t-diagnosable if it is t-connected with at least $2t+1$ nodes. They also posed a sufficient and necessary condition for verifying if a system is one-step t-diagnosable under the PMC model. More practically, Dahbura and Masson [12] presented an $O(N^{2.5})$ diagnosis algorithm to identify all the faulty processors in a system with N processors. It is noticed that only processors with direct connections are allowed to test each other. In particular, if all the neighboring processors of a processor v are faulty simultaneously, then it seems unlikely to determine whether processor v is fault-free or faulty. In this way, the one-step diagnosability of a system G is trivially bounded by the minimum degree of G. For most practical systems that are sparsely connected, only a small number of faulty processors can be recognized under the PMC diagnosis model. Therefore, it has long been an intriguing issue to explore some measure that can better reflect fault patterns in a real system. For example, Das et al. [13] investigated fault diagnosis under local constraints; Lai et al. [20] proposed a new measure of diagnosis capability, namely conditional diagnosability, by restricting that for each processor in a system, all its neighboring processors do not fail at the same time. Recently, Xu et al. [31] investigated the conditional diagnosability with respect to a class of matching composition networks. However, these works did not provide any diagnosis algorithm, so it is not clear how to identify faults efficiently in such a situation. In this paper, we will relax that condition imposed in [20], [31] by assuming that every fault-free processor can have at least one fault-free neighbor. Under this assumption, not only can the diagnosis capability be proved theoretically, but also it is guaranteed in an algorithmic point of view.

1.2 The State-of-the-Art

Many previous studies about system-level diagnosis were devoted to the diagnosis capability in a global sense but ignored some local connection. For instance, it is likely to correctly point out all the faulty processors in a t-diagnosable system even when the number of faulty processors has been already greater than t. Consider two hypercube systems $\mathbf{Q}_m$ and $\mathbf{Q}_n$, which are known to be m-diagnosable and n-diagnosable [19], respectively, where m and n are two integers with $m \gg n$. A new system G can be built by integrating these two systems with a few communication links in some way that makes the new system have one-step diagnosability limited by n. See Fig. 1 for illustration. Consider the following scenario: There are m faulty processors within $\mathbf{Q}_m$. Then these m faulty processors can be correctly identified because $\mathbf{Q}_m$ is m-diagnosable. Even though this new system is only n-diagnosable, it is of high probability that the correct diagnosis can be made when the total number of faulty processors is between m and n.

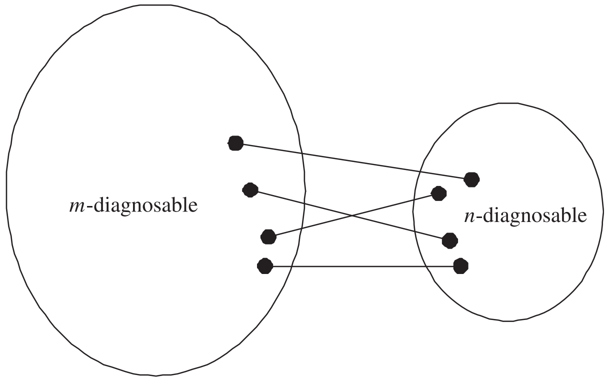


Fig. 1. An n-diagnosable system obtained by integrating an n-diagnosable subsystem and m-diagnosable subsystem.

In the last two decades, a variety of methods were developed to achieve system-level diagnosis for various interconnected structures. For example, Chessa and Maestrini [10] introduced a correct and almost complete diagnosis method for square grids. Later, Caruso et al. [6], [7], [8] presented two correct and almost complete diagnosis algorithms, called EDARS and NDA, respectively. The two algorithms have time complexity $O(kN)$ when applied to k-regular systems of N units. A lower bound to the worst-case diagnosis completeness for regular graphs under the PMC model is shown in [9]. Recently, Mánik and Gramatová [22], [23] proposed the Boolean formalization of the PMC model for the syndrome-decoding process. When this approach is applied to regular systems, the computation time of fault diagnosis can be significantly reduced. In addition, Somani and Agarwal [30] developed a distributed diagnosis algorithm for regular systems based on the concept of local diagnosis. Later, Altmann et al. [2] addressed an event-driven distributed approach to multiprocessor diagnosis, and Masuyama and Miyoshi [24] presented a nonadaptive distributed system-level diagnosis method for computer networks.

In some circumstances, however, we are only concerned about some substructure of a multiprocessor system, which is implementable in very large-scale integration (VLSI). Such a substructure, for example, can be a ring, a path, a tree, a mesh and so on. If all processors in these substructures can be guaranteed to be fault-free, a procedure is still workable even though there are many faulty processors in the remaining part of the system. Thus, the local substructure plays a more critical role than the global fault status of the entire system. Motivated by such a concept, Hsu and Tan [18] presented an elegant measure of diagnosis capability, known as local diagnosability, to identify the one-step diagnosability of a system by computing the local diagnosability with respect to each individual processor. For any processor in a system, two useful structures [18] were presented to determine its local diagnosability under the PMC model. Hence, in this paper, we will extend the previous study and design an efficient diagnosis algorithm based on the proposed structure, named the augmenting star, provided that each fault-free processor has at least one fault-free neighboring processor. In short, our algorithm proceeds depending on the existence of the augmenting star structure. Moreover, the key difference between our work and the others is that we address the conditional-fault identification problem from a standpoint of local diagnosis. For many practical multiprocessor systems, the number of links incident to each processor is in the order of $\log N$, where N is the total number of processors. Accordingly, the time for diagnosing any given processor v can be bounded by $O((\log N)^2)$ if there exists an augmenting star structure rooted at processor v. So all the faulty processors can be identified one by one with time complexity $O(N(\log N)^2)$, provided that the augmenting star can also be constructed at each processor in time $O((\log N)^2)$.

The rest of this paper is organized as follows: Section 2 provides preliminary background for system-level diagnosis and graph-theoretic terminology. Section 3 introduces how to diagnose a system with random faults. A diagnosis algorithm based on the augmenting star structure is presented in Section 4. Some examples are shown in Section 5. Finally, our conclusions are given in Section 6.

2 Preliminaries

The underlying topology of a multiprocessor system is usually modeled as a graph, whose vertex set and edge set represent the set of all processors and the set of all communication links between processors, respectively. Throughout this paper graphs are finite, simple, and unless specified otherwise, undirected. Some important graph-theoretic definitions and notations will be introduced in advance. For those not defined here, however, we follow the standard terminology given by Bondy and Murty [5].

An undirected graph G is an ordered pair (V, E), where V is a nonempty set, and E is a subset of $\{\{u, v\} \mid \{u, v\}$ is a 2-element subsets of $V\}$.[1] The set V is called the *vertex set* of G, and the set E is called the *edge set* of G. For convenience, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. Two vertices, u and v, in graph G are *adjacent* if $\{u, v\} \in E(G)$; we say u is a *neighbor* of v, and vice versa. The degree of a vertex v in G, denoted by $deg_G(v)$, is the number of edges incident to v. The neighborhood of vertex v, denoted by $N_G(v)$, is the set of vertices adjacent to v. For a set $S \subset V$, the notation $G - S$ represents the graph obtained by removing every vertex in S from G and deleting those edges incident to at least one vertex in S. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The *components* of a graph G are its maximal connected subgraphs. A component is trivial if it has no edges; otherwise, it is nontrivial.

In the PMC model [27], adjacent units are capable of performing tests on each other. A testing unit u_i specifies some test sequence to a tested unit u_j and receives a response sequence from u_j. The testing unit outputs a test outcome $a_{i,j} = 1$ if the actual response sequence mismatches the expected one; otherwise, $a_{i,j} = 0$. Let an undirected graph $G = (V, E)$ denote the underlying topology of a multiprocessor system. For any two adjacent vertices $u, v \in V$, the ordered pair (u, v) represents the *test* that processor u diagnoses processor v. In this situation, u is a *tester*, and v is a *testee*. The outcome of a test (u, v) is 1 (respectively, 0) if u evaluates v to be faulty (respectively, fault-free). The notation $u \xrightarrow{\gamma} v$ means that u tests v with outcome γ. Because the faults considered here are permanent, the outcome of a test is *reliable* if and only if the tester is fault-free. A *test assignment* for system G is a collection of tests and thus can be modeled as a directed graph $T = (V, L)$, where $(u, v) \in L$ and $(v, u) \in L$ if and only if $\{u, v\} \in E$. The collection of all test outcomes from the test assignment T is called a *syndrome*. Formally, a syndrome of T is a mapping $\sigma : L \to \{0, 1\}$. The set F of all faulty processors in G is called a *faulty set*. It is noticed that F can be any subset of V. The process of identifying all faulty vertices is said to be the system-level diagnosis. Furthermore, the maximum number of faulty vertices that can be correctly identified in a system G is called the *one-step diagnosability* of G, denoted by $\tau(G)$.

For any given syndrome σ resulting from a test assignment $T = (V, L)$, a subset of vertices $F \subseteq V$ is said to be *consistent* with σ if for any $(u, v) \in L$ with $u \in V - F$, then $\sigma(u, v) = 1$ if and only if $v \in F$. This corresponds to the assumption that fault-free testers always give correct test results, whereas faulty testers can lead to unreliable results. Therefore, a given set F of faulty vertices may be consistent with different syndromes. Let $\sigma(F)$ denote the set of all possible syndromes with which the faulty set F can be consistent. Then two distinct faulty sets $F_1, F_2 \subset V$ are said to be *distinguishable* if $\sigma(F_1) \cap \sigma(F_2) = \emptyset$; otherwise, F_1 and F_2 are said to be *indistinguishable*. That is, (F_1, F_2) is a *distinguishable pair* (respectively, an *indistinguishable pair*) of faulty sets if $\sigma(F_1) \cap \sigma(F_2) = \emptyset$ (respectively, $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$).

Lemma 1 [12]. *A system G is one-step t-diagnosable if and only if for any two distinct faulty sets $F_1, F_2 \subset V(G)$ with $|F_1| \leq t$ and $|F_2| \leq t$, (F_1, F_2) is a distinguishable pair.*

Let F_1, F_2 be two distinct sets, and let $F_1 \triangle F_2 = (F_1 - F_2) \cup (F_2 - F_1)$ denote the *symmetric difference* between F_1 and F_2. Dahbura and Masson [12] presented a sufficient and necessary characterization of one-step t-diagnosable systems and exploited it to design a polynomial-time algorithm for identifying the set of faulty processors.

Lemma 2 [12]. *Let $G = (V, E)$ be a graph. For any two distinct faulty sets $F_1, F_2 \subset V$, (F_1, F_2) is a distinguishable pair if and only if there exists a vertex $u \in V - (F_1 \cup F_2)$ and a vertex $v \in F_1 \triangle F_2$ such that $\{u, v\} \in E$.*

3 Random-Fault Diagnosis

For a multiprocessor system, the random-fault model assumes that the probabilities of processor failures are identical and independent. Let v be any vertex in a graph G. It is intuitive to observe that $(N_G(v), \{v\} \cup N_G(v))$ forms an indistinguishable pair of faulty sets. That is, the conventional one-step diagnosability is mainly concerned with the global status of a system under the random-fault model. Instead, Hsu and Tan [18] turned their attention to the local connective substructure in a system. More precisely, given any single vertex v in a graph it is only required to determine whether v is faulty or not. The following concept is proposed in [18].

1. We denote $\{u, v\}$ and (u, v) an undirected edge and a directed edge from u to v, respectively.

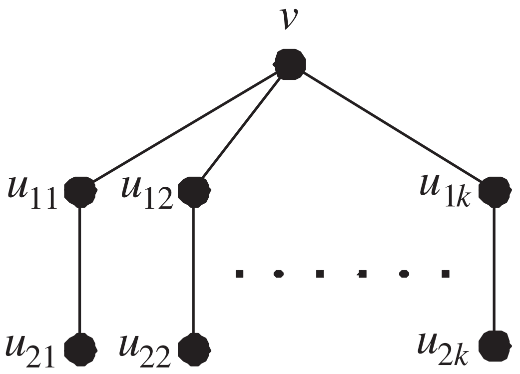


Fig. 2. The extending star $\mathbb{T}_G(v;k)$ consists of $2k+1$ vertices and $2k$ edges.

Definition 1 [18]. *Let G be a graph and v denote any one of its vertices. Then G is locally t-diagnosable at vertex v if, given a syndrome σ_F produced by a set of faulty vertices $F \subseteq V$ with $v \in F$ and $|F| \leq t$, every faulty set of at most t vertices that is also consistent with σ_F must contain vertex v.*

By Definition 1, Hsu and Tan [18] further proved that a graph G is locally t-diagnosable at vertex v if and only if for any two distinct sets of vertices $F_1, F_2 \subset V(G)$ such that $|F_1|, |F_2| \leq t$ and $v \in F_1 \triangle F_2$, (F_1, F_2) is a distinguishable pair. It was also shown that a graph G is one-step t-diagnosable if and only if it is locally t-diagnosable at every vertex. Moreover, the *local diagnosability* of a vertex v in G, denoted by $\pi_G(v)$, is defined to be the maximum integer of t such that G is locally t-diagnosable at vertex v. The relationship between one-step diagnosability and local diagnosability is revealed in the next lemma.

Lemma 3 [18]. *Let G denote the underlying topology of a multiprocessor system. Then $\tau(G) = \min\{\pi_G(v) \mid v \in V(G)\}$.*

In [18], the following structure is presented to compute the local diagnosability with respect to any given vertex under the PMC model.

Definition 2 [18]. *Letting $G = (V, E)$ be a graph, $v \in V$ be any vertex, and k be an integer greater than or equal to 1, an extending star of order k rooted at vertex v is defined to be the subgraph of G, denoted by $\mathbb{T}_G(v;k) = (V(v;k), E(v;k))$, where $V(v;k) = \{v\} \cup \{u_{ij} \mid 1 \leq i \leq 2, 1 \leq j \leq k\}$ and $E(v;k) = \{\{v, u_{1j}\}, \{u_{1j}, u_{2j}\} \mid 1 \leq j \leq k\}$. An extending star of order k is said to be of full order if $k = deg_G(v)$. See Fig. 2 for illustration.*

In practice, it is more applicable to have an efficient procedure that is capable of identifying the fault status of a given vertex. Based on the extending star, a polynomial-time algorithm, namely Diagnose-Vertex-In-Random-Faults (**DVRF**, abbreviated for short), is proposed to determine whether any given vertex is faulty or not.

Algorithm. DVRF(G, v)

Input: Any vertex v in a graph G, in which there exists an extending star of full order rooted at v.

Output: The fault status of vertex v. As a convention, the algorithm output is 0 or 1 if vertex v is fault-free or faulty, respectively.

BEGIN

1) $t \leftarrow deg_G(v)$.
2) Construct an extending star of order t rooted at vertex v, $\mathbb{T}_G(v;t)$, as illustrated in Fig. 2.
3) $n_0 \leftarrow |\{1 \leq j \leq t \mid (\sigma(u_{2j}, u_{1j}), \sigma(u_{1j}, v)) = (0,0)\}|$
   $n_1 \leftarrow |\{1 \leq j \leq t \mid (\sigma(u_{2j}, u_{1j}), \sigma(u_{1j}, v)) = (0,1)\}|$
4) **if** $n_0 \geq n_1$
   then return 0
   else return 1

END

Theorem 1. *Let G be a graph, $v \in V(G)$, and $t = deg_G(v)$. Suppose that there exists an extending star of full order rooted at vertex v, $\mathbb{T}_G(v;t)$. Then the algorithm **DVRF**(G, v) correctly identifies the fault status of vertex v if the total number of faulty vertices in $\mathbb{T}_G(v;t)$ does not exceed t.*

Proof. Let

$$n_0 = |\{1 \leq j \leq t \mid (\sigma(u_{2j}, u_{1j}), \sigma(u_{1j}, v)) = (0,0)\}|,$$
$$n_1 = |\{1 \leq j \leq t \mid (\sigma(u_{2j}, u_{1j}), \sigma(u_{1j}, v)) = (0,1)\}|,$$
$$n_2 = |\{1 \leq j \leq t \mid (\sigma(u_{2j}, u_{1j}), \sigma(u_{1j}, v)) = (1,0)\}|,$$

and

$$n_3 = |\{1 \leq j \leq t \mid (\sigma(u_{2j}, u_{1j}), \sigma(u_{1j}, v)) = (1,1)\}|.$$

Obviously, we have $t = n_0 + n_1 + n_2 + n_3$.

First, we consider the case that vertex v is faulty. Suppose, by contradiction, that $n_0 \geq n_1$. Then the total number of faulty vertices in $\mathbb{T}_G(v;t)$ amounts to at least $2n_0 + n_2 + n_3 + 1 \geq n_0 + n_1 + n_2 + n_3 + 1 = t + 1$. This contradicts the assumption that the total number of faulty vertices in $\mathbb{T}_G(v;t)$ does not exceed t. Hence, n_0 is strictly less than n_1, and the proposed algorithm outputs a correct diagnosis result.

Next, we consider the case that vertex v is fault-free. Again, we assume, by contradiction, that $n_0 < n_1$. Then the total number of faulty vertices in $\mathbb{T}_G(v;t)$ amounts to at least $2n_1 + n_2 + n_3 + 1 \geq n_0 + n_1 + n_2 + n_3 + 1 = t + 1$, contradicting the assumption that the total number of faulty vertices in $\mathbb{T}_G(v;t)$ does not exceed t. Hence, n_0 needs to be greater than or equal to n_1, and the proposed algorithm correctly diagnoses the given vertex v.

Therefore, the proof is completed. □

The extending star structure can be constructed in many multiprocessor systems and interconnection networks, such as hypercubes [28], crossed cubes [14], augmented cubes [11], star graphs [1], etc. Among various kinds of network topologies, the hypercube is one of the most popular networks for parallel and distributed computation. Not only is it ideally suited to both special-purpose and general-purpose tasks, but it can efficiently simulate many other networks [21]. Hence, we describe here how to construct an extending star of full order in the hypercube.

Let $\mathbf{v} = b_n \ldots b_i \ldots b_1$ be an n-bit binary string. For $1 \leq i \leq n$, we use $(\mathbf{v})^i$ to denote the binary string $b_n \ldots \bar{b}_i \ldots b_1$. Moreover, we use $[\mathbf{v}]_i$ to denote the ith bit b_i of $\mathbf{v}$. The n-dimensional hypercube (or n-cube for short), denoted by $\mathbf{Q}_n$, consists of 2^n vertices and $n2^{n-1}$ edges. Each vertex corresponds to an n-bit binary string. Two vertices, $\mathbf{u}$ and $\mathbf{v}$, are adjacent if and only if $\mathbf{v} = (\mathbf{u})^i$ for

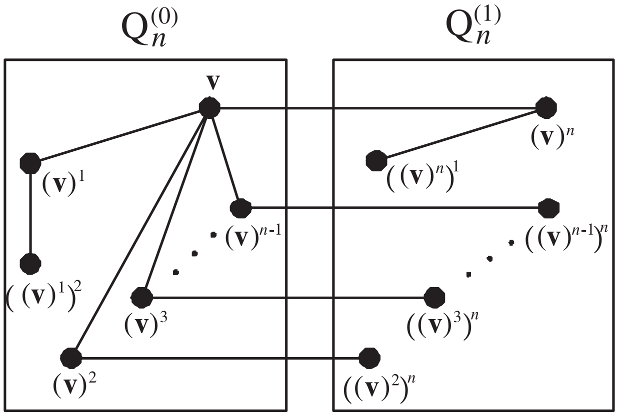


Fig. 3. An extending star of full order rooted at a vertex $\mathbf{v} \in V(\mathbf{Q}_n^{(0)})$.

some i. An n-cube can be constructed recursively. Let $\mathbf{Q}_n^{(0)}$ and $\mathbf{Q}_n^{(1)}$ denote two subgraphs of $\mathbf{Q}_n$ induced by vertex subsets $\{\mathbf{v} \in V(\mathbf{Q}_n) \mid [\mathbf{v}]_n = 0\}$ and $\{\mathbf{v} \in V(\mathbf{Q}_n) \mid [\mathbf{v}]_n = 1\}$, respectively. For $n \geq 2$, $\mathbf{Q}_n^{(0)}$ and $\mathbf{Q}_n^{(1)}$ are isomorphic to $\mathbf{Q}_{n-1}$. Then an extending star of full order rooted at any vertex $\mathbf{v}$ in the n-cube can be formed by the graph $\mathbb{T}_{\mathbf{Q}_n}(v, n)$, whose vertex set and edge set are $\{\mathbf{v}, (\mathbf{v})^1, ((\mathbf{v})^1)^2, (\mathbf{v})^n, ((\mathbf{v})^n)^1\} \cup \bigcup_{i=2}^{n-1} \{(\mathbf{v})^i, ((\mathbf{v})^i)^n\}$ and $\{\{\mathbf{v}, (\mathbf{v})^1\}, \{(\mathbf{v})^1, ((\mathbf{v})^1)^2\}, \{\mathbf{v}, (\mathbf{v})^n\}, \{(\mathbf{v})^n, ((\mathbf{v})^n)^1\}\} \cup \bigcup_{i=2}^{n-1} \{\{\mathbf{v}, (\mathbf{v})^i\}, \{(\mathbf{v})^i, ((\mathbf{v})^i)^n\}\}$, respectively. See Fig. 3 for illustration.

We now measure the time complexity of the proposed algorithm. For most of the practical systems G with N vertices, the degree of each vertex is in the order of $\log N$, and the extending star structure of full order can be constructed in time $O(\log N)$. For example, both the n-cube and n-dimensional crossed cube have $N = 2^n$ vertices, and the degree of each vertex is $n = \log N$. Under the PMC model we assume that the time for a vertex to test another one is a constant c. Given an extending star $\mathbb{T}_G(v, n)$ rooted at a vertex v in system G, the time needed for determining the fault status of vertex v is $2c \log N = O(\log N)$. As a result, the total time for diagnosing the whole system G is $O(N \log N)$.

4 Conditional-Fault Diagnosis

The underlying topologies of many multiprocessor systems are usually regular and even vertex-symmetric. By definition [1], a graph is *vertex-symmetric* if for every pair u, v of vertices, there exists an automorphism of the graph that maps u into v.

Consider a vertex-symmetric graph G with one-step diagnosability $\tau(G) = t$; so G is one-step t-diagnosable but not $(t+1)$-diagnosable. However, the only case that stops it from being $(t+1)$-diagnosable is usually that there exists a vertex v whose neighbors are all faulty simultaneously. For example, members in the cube family are so. A system is known to be *strongly t-diagnosable* if it is one-step t-diagnosable and can achieve $(t+1)$-diagnosability, except for the case where a node's neighbor are all faulty. Recently, Hsieh and Chuang [17] studied the strong diagnosability of regular networks and product networks under the PMC model. We are, however, led to the following question: How large can the maximum value of t be such that G remains t-diagnosable under the additional condition that every fault-free vertex has at least one fault-free neighbor?

For a classical measurement of diagnosis capability, it is usually assumed that processor failures are statistically independent. It does not reflect the total number of processors in the system and the probabilities of processor failures. Najjar and Gaudiot [25] proposed the *network resilience* as the maximum number of failures that can be sustained while the network remains connected with a reasonably high probability. For a hypercube, the fault resilience is shown to be 25 percent for the four-dimensional hypercube $\mathbf{Q}_4$, and it increases to 33 percent for the 10-dimensional hypercube $\mathbf{Q}_{10}$. More particularly, the 10-dimensional hypercube $\mathbf{Q}_{10}$ still remains connected with a probability higher than 0.99 even when 33 percent of its processors fail. They also drew a conclusion that large-scale systems with a constant degree are more susceptible to failures by disconnection than smaller networks. Intuitively, a connected network should have better diagnosis capability.

Let G be a graph. A set $F \subset V(G)$ is called *conditionally faulty* if $N_G(v) \not\subseteq F$ for every vertex $v \in V(G) - F$. A graph is *conditionally faulty* if its faulty vertices form a conditionally faulty set. Furthermore, G is said to be *conditionally t-diagnosable* if for any two conditionally faulty sets $F_1, F_2 \subset V((G)$ with $F_1 \neq F_2$ and $|F_1|, |F_2| \leq t$, (F_1, F_2) is a distinguishable pair. We propose the following concept.

Definition 3. *Let G be a graph and v denote any vertex in G. Then G is conditionally t-diagnosable locally at vertex v if, given a syndrome σ_F produced by any conditionally faulty set of vertices $F \subseteq V(G)$ with $v \in F$ and $|F| \leq t$, the vertex v must be an element of every conditionally faulty set of at most t vertices that is consistent with σ_F.*

The following theorem is another standpoint for characterizing whether a system is conditionally t-diagnosable locally at its vertex v.

Theorem 2. *A graph G is conditionally t-diagnosable locally at vertex $v \in V(G)$ if F_1 and F_2 form a distinguishable pair for any two conditionally faulty sets $F_1, F_2 \subset V(G)$ such that $F_1 \neq F_2$, $v \in F_1 \triangle F_2$, $|F_1| \leq t$, and $|F_2| \leq t$.*

Proof. Let $S_1 \subset V(G)$ be any conditionally faulty set with $|S_1| \leq t$ and $v \in S_1$. Furthermore, let $S_2 \subset V(G)$ denote any conditionally faulty set with $|S_2| \leq t$ and $v \notin S_2$. Suppose that any two distinct conditionally faulty sets F_1, F_2 of G, with $v \in F_1 \triangle F_2$ and $|F_1|, |F_2| \leq t$, form a distinguishable pair. Then we have $\sigma(S_1) \cap \sigma(S_2) = \emptyset$. Thus, S_2 is not consistent with any syndrome in $\sigma(S_1)$. It follows from contraposition that any conditionally faulty set $X \subset V(G)$, which has at most t elements and can be consistent with a syndrome in $\sigma(S_1)$, must contain vertex v. By Definition 3, G is conditionally t-diagnosable locally at vertex v. □

The *edge-degree* of an edge $\{u, v\}$ in a graph G, denoted by $\epsilon_G(\{u, v\})$, is the number of distinct vertices of $V(G) - \{u, v\}$ adjacent to u or v. For any vertex $v \in V(G)$, let $\xi_G(v) = \min\{\epsilon_G(\{u, v\}) \mid \{u, v\} \in E(G)\}$ denote the *minimum edge-degree of all the edges incident to vertex v*.

Theorem 3. *Let G be a graph, $v \in V(G)$ denote a vertex, and t be any positive integer less than or equal to $\xi_G(v) + 1$. Then G is conditionally t-diagnosable locally at vertex v if for every* *conditionally faulty set* $F \subset V(G) - \{v\}$ *with* $0 \leq |F| \leq t-1$, *the connected component of* $G - F$, *which contains vertex* v, *either has at least* $2(t - |F|) + 1$ *vertices or consists of only two vertices that are adjacent to each other.*

Proof. We prove the sufficiency by contradiction. Suppose that G is not conditionally t-diagnosable locally at vertex v if the sufficient condition holds. By Theorem 2, there exists an indistinguishable pair of conditionally faulty sets (F_1, F_2), where $F_1 \neq F_2$, $|F_1| \leq t$, $|F_2| \leq t$, and $v \in F_1 \triangle F_2$. It follows from Lemma 2 that there is no edge between $V(G) - (F_1 \cup F_2)$ and $F_1 \triangle F_2$.

Let $F = F_1 \cap F_2$ and $p = |F|$. Because both F_1 and F_2 are conditionally faulty, F is conditionally faulty too. Moreover, we have $0 \leq p \leq t - 1$ and $v \notin F$. Hence, $F_1 \triangle F_2$ is disconnected from other parts after removing all the vertices in F from G. We observe that $|F_1 \triangle F_2| \leq 2(t-p)$. Thus, the connected component C_v of $G - F$, which contains vertex v, has at most $2(t-p)$ vertices. It is noticed that $|V(C_v)| \geq 2$ since $N_G(v) \not\subseteq F$. Hence, we distinguish the following two cases.

Case 1: Suppose that $|V(C_v)| \geq 3$. Since $|V(C_v)| \leq |F_1 \triangle F_2| \leq 2(t-p)$, this contradicts the assumption that the component C_v has at least $2(t-p)+1$ vertices.

Case 2: Suppose that $|V(C_v)| = 2$. For convenience, let $V(C_v) = \{u, v\}$. Without loss of generality, we assume that $v \in F_1$. It is easy to see that $N_G(v) \cup N_G(u) - \{u, v\} \subseteq F$. Thus, we have $|F| \geq \epsilon_G(\{u, v\})$. However, because F_1 is conditionally faulty, vertex u is also in F_1. Hence, we have $|F_1| \geq |F \cup \{u, v\}| = |F| + 2 \geq \epsilon_G(\{u, v\}) + 2$, contradicting the assumption that $|F_1| \leq t \leq \xi_G(v) + 1 \leq \epsilon_G(\{u, v\}) + 1$.

By contradiction, G is really conditionally t-diagnosable locally at vertex v. □

We now propose a helpful structure, called augmenting star, to identify whether a given vertex is fault-free in a conditionally faulty system.

Definition 4. *Letting* $G = (V, E)$ *be a graph,* $v \in V$ *be any vertex, and* k *be an integer greater than or equal to 2, an augmenting star of order* k *rooted at vertex* v *is defined to be the subgraph* $\mathbb{A}_G(v;k) = (V(v;k), E(v;k))$ *of* G, *where* $V(v;k) = \{v\} \cup \{u_i \mid 1 \leq i \leq k\} \cup \{x_{i,j}, y_{i,j}, z_{i,j} \mid 1 \leq i \leq k, 1 \leq j \leq k-1\}$ *and*

$$E(v;k) = \{\{v, u_i\} \mid 1 \leq i \leq j\} \cup \{\{u_i, x_{i,j}\} \mid 1 \leq i \leq k, 1 \leq j \leq k-1\} \cup \{\{x_{i,j}, y_{i,j}\}, \{y_{i,j}, z_{i,j}\} \mid 1 \leq i \leq k, 1 \leq j \leq k-1\}.$$

An augmenting star of order k *is said to be of full order if* $k = deg_G(v)$. *See Fig. 4 for illustration. For any* $1 \leq i \leq k$, *the subgraph of* $\mathbb{A}_G(v;k)$ *induced by the vertex set* $\{u_i, x_{i,j}, y_{i,j}, z_{i,j} \mid 1 \leq j \leq k-1\}$ *is denoted by* $\mathbb{A}_G^{(i)}(v;k)$.

Theorem 4. *Let* G *be a graph and* $v \in V(G)$ *denote a vertex. Suppose that the degree* t *of vertex* v *is at least 2; i.e.,* $t \geq 2$. *Then* G *is conditionally* $(2t-1)$*-diagnosable locally at vertex* v *if it contains an augmenting star of full order rooted at vertex* v *as a subgraph.*

Proof. Suppose that G contains an augmenting star of full order rooted at v, $\mathbb{A}_G(v;t)$, as a subgraph. Then it is easy to see that $\xi_G(v) \geq 2t - 2$. Therefore, we can apply Theorem 3 to prove the result.

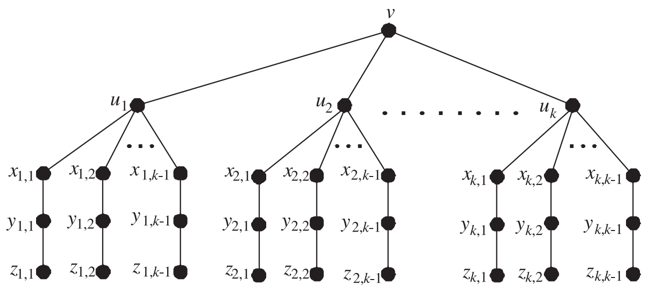


Fig. 4. The augmenting star $\mathbb{A}_G(v;k)$ consists of $3k^2 - 2k + 1$ vertices and $3k^2 - 2k$ edges.

Let $F \subset V(G) - \{v\}$ be a conditionally faulty set of p vertices for $0 \leq p \leq 2t-2$. Then we have to show that the connected component C_v of $G - F$, which contains vertex v, either has at least $2(2t-1-p)+1 = 4t-2p-1$ vertices or consists of only two adjacent vertices. For convenience, let $r = |\{u_1, u_2, \ldots, u_t\} \cap F|$. Because F is conditionally faulty and $v \notin F$, we have $0 \leq r \leq \min\{t-1, p\}$. Without loss of generality, we can assume that $\{u_{r+1}, \ldots, u_t\} \cap F = \emptyset$. Let $L_{i,j} = \{u_i, x_{i,j}, y_{i,j}, z_{i,j}\}$ for $1 \leq i \leq t$ and $1 \leq j \leq t-1$. Hence, there are at least $(t-r)(t-1) - (p-r)$ $L_{i,j}$'s with $L_{i,j} \cap F = \emptyset$. Thus, C_v has at least $3[(t-r)(t-1)-(p-r)] + (t-r) + 1$ vertices. Comparing $3[(t-r)(t-1)-(p-r)] + (t-r) + 1$ with $4t - 2p - 1$, we set

$$\Delta \stackrel{\text{def}}{=} \{3[(t-r)(t-1)-(p-r)] + (t-r) + 1\} - (4t-2p-1). \quad (1)$$

First, we assume that $0 \leq p \leq 2t-3$ or $r \leq t-2$. If $0 \leq p \leq 2t-3$, then

$$\begin{aligned}\Delta &= 3(t-r)(t-1) - 3t + 2r - p + 2 \\ &\geq 3(t-r)(t-1) - 3t + 2r - (2t-3) + 2 \\ &= (t-r-1)(3t-5) \geq 0;\end{aligned}$$

if $r \leq t-2$, then $\Delta \geq 3(t-r)(t-1) - 5t + 2r + 4 = (t-r-2)(3t-2) + 3r \geq 0$. Thus, C_v has at least $4t - 2p - 1$ vertices.

Second, we consider that $p = 2t-2$ and $r = t-1$. It is noticed that $u_1, u_2, \ldots, u_{t-1} \in F$ and $u_t \notin F$. Then we distinguish the following two cases.

Case 1: Suppose that $N_G(u_t) - \{v\} \subseteq F$. Obviously, we have $F = \bigcup_{k=1}^{t-1}\{u_k, x_{t,k}\}$. Hence, C_v contains only two adjacent vertices v and u_t.

Case 2: Suppose that $N_G(u_t) - \{v\} \not\subseteq F$. Accordingly, C_v has at least three vertices, i.e., $|V(C_v)| \geq 3 = 4t - 2p - 1 = 4t - 2(2t-2) - 1$.

Hence, the theorem holds. □

Using the augmenting star structure we can design an efficient algorithm, namely Diagnose-The-Given-Vertex-In-Conditional-Faults (**DVCF**, for short), to diagnose any vertex in a conditionally faulty system.

Algorithm. DVCF(G, v)

Input: Any vertex v in a conditionally faulty graph G, in which there exists an augmenting star of full order rooted at v.

Output: The fault status of vertex v. As a convention, the algorithm output is 0 or 1 if vertex v is fault-free or faulty, respectively.

BEGIN
1) $t \leftarrow deg_G(v)$.
2) Construct an augmenting star of order t rooted at v, $\mathbb{A}_G(v;t)$, as illustrated in Fig. 4.
3) $S \leftarrow \bigcup_{1\le i\le t}\{u_i \mid \mathbf{DVRF}(\mathbb{A}_G^{(i)}(v;t), u_i) \text{ outputs } 0\}$
   $m_{i,0} \leftarrow |\{1\le j\le t-1 \mid (\sigma(y_{i,j},x_{i,j}),\sigma(x_{i,j},u_i))=(0,0)\}|$
   $m_{i,1} \leftarrow |\{1\le j\le t-1 \mid (\sigma(y_{i,j},x_{i,j}),\sigma(x_{i,j},u_i))=(0,1)\}|$
   $m_{i,2} \leftarrow |\{1\le j\le t-1 \mid (\sigma(y_{i,j},x_{i,j}),\sigma(x_{i,j},u_i))=(1,0)\}|$
   $m_{i,3} \leftarrow |\{1\le j\le t-1 \mid (\sigma(y_{i,j},x_{i,j}),\sigma(x_{i,j},u_i))=(1,1)\}|$
4) **if** $|S|\ge 3$
   then $n_0 \leftarrow |\{w\in S \mid \sigma(w,v)=0\}|$
   $n_1 \leftarrow |\{w\in S \mid \sigma(w,v)=1\}|$
   if $n_0 > n_1$
   then return 0
   else return 1
5) **if** $|S|=2$
   then let $u_p, u_q \in S$
   if $m_{p,0}-m_{p,1} \ge m_{q,0}-m_{q,1}$
   then return $\sigma(u_p,v)$
   else return $\sigma(u_q,v)$
6) **if** $|S|=1$
   then let $u_p\in S$ and **return** $\sigma(u_p,v)$
7) **if** $|S|=0$
   then if $m_{i,1}-m_{i,0}\ge 2$ for every $1\le i\le t$
   then return 1
   else let p be an integer such that $m_{p,1}-m_{p,0}$ is equal to 1
   $r \leftarrow |\{1\le j\le t-1 \mid (\sigma(z_{p,j},y_{p,j}), \sigma(y_{p,j},x_{p,j}),\sigma(x_{p,j},u_p))=(1,0,1)\}|$
   if $r\ge 1$
   then return $\sigma(u_p,v)$
   else return 1

END

Theorem 5. *Let G be a conditionally faulty graph, $v\in V(G)$ denote any vertex, and $t=deg_G(v)$. Suppose that there exists an augmenting star of full order rooted at vertex v, $\mathbb{A}_G(v;t)$. Then the proposed algorithm **DVCF**(G,v) can identify the fault status of vertex v correctly if $t\ge 4$ and the total number of faulty vertices in $\mathbb{A}_G(v;t)$ does not exceed $2t-1$.*

Proof. Let $\{u_1,u_2,\dots,u_t\}$ denote the set of neighbors of vertex v and $S\subseteq\{u_1,u_2,\dots,u_t\}$ be the set used in step (3) of the proposed algorithm. For convenience, we denote the set of all faulty vertices in $\mathbb{A}_G(v;t)$ by F. Let A and B denote two subsets of neighbors of vertex v as follows:

$$A=\left(\bigcup_{1\le i\le t}\{u_i \mid \mathbf{DVRF}(\mathbb{A}_G^{(i)}(v;t),u_i) \text{ outputs } 0\}\right)\cap F$$

$$B=\left(\bigcup_{1\le i\le t}\{u_i \mid \mathbf{DVRF}(\mathbb{A}_G^{(i)}(v;t),u_i) \text{ outputs } 1\}\right)-F$$

First of all, we claim that $|A|+|B|\le 1$. By Theorem 1, the algorithm $\mathbf{DVRF}(\mathbb{A}_G^{(i)}(v;t),u_i)$ correctly identifies the faulty/fault-free status of vertex u_i, $1\le i\le t$, in $\mathbb{A}_G^{(i)}(v;t)$ if the number of faulty vertices in $\mathbb{A}_G^{(i)}(v;t)$ does not exceed $t-1$. We assume, by contradiction, that $|A|+|B|\ge 2$. Then we have $|F|\ge t|A|+t|B|=(|A|+|B|)t\ge 2t$, contradicting the condition that $|F|\le 2t-1$. Hence, the claim holds.

We now consider the following four cases according to the number of vertices in S. For convenience, we use $f(H)$ to denote the number of faulty vertices in a graph H.

Case 1: Suppose that $|S|\ge 3$. Since $|A|+|B|\le 1$, at most one vertex in S is likely to be faulty. Thus, more than half of the vertices in S can correctly diagnose vertex v. Let $n_0=|\{w\in S\mid\sigma(w,v)=0\}|$ and $n_1=|\{w\in S\mid\sigma(w,v)=1\}|$. Then vertex v is fault-free if and only if $n_0>n_1$.

Case 2: Suppose that $|S|=2$. Let $S=\{u_p,u_q\}$ with some $1\le p,q\le t$. Then we claim that u_p is fault-free if $m_{p,0}-m_{p,1}\ge m_{q,0}-m_{q,1}$. Suppose, by contradiction, that u_p is faulty. Moreover, because at most one vertex in S is likely to be faulty, vertex u_q has to be fault-free. We further claim that $m_{q,1}=0$ and $m_{q,2}+m_{q,3}\le 1$. We assume, by contradiction, that $m_{q,1}\ge 1$ or $m_{q,2}+m_{q,3}\ge 2$. Accordingly, the number of faulty vertices can be counted as follows:

$$\begin{aligned}|F| &\ge |\{u_i \mid 1\le i\le t, i\notin\{p,q\}\}| + f(\mathbb{A}_G^{(p)}(v;t)) \\ &\quad + f(\mathbb{A}_G^{(q)}(v;t)) \\ &\ge (t-2)+t+(2m_{q,1}+m_{q,2}+m_{q,3}) \\ &\ge (t-2)+t+2 \\ &= 2t,\end{aligned}$$

which contradicts the assumption that $|F|\le 2t-1$. Hence, the claim of $m_{q,1}=0$ and $m_{q,2}+m_{q,3}\le 1$ is true. Since $m_{q,0}+m_{q,1}+m_{q,2}+m_{q,3}=t-1$, we have $m_{p,0}-m_{p,1}\ge m_{q,0}-m_{q,1}=m_{q,0}\ge t-2\ge 2$ for $t\ge 4$. As a result, the number of faulty vertices is estimated as follows:

$$\begin{aligned}|F| &\ge |\{u_i \mid 1\le i\le t, i\notin\{p,q\}\}| + f(\mathbb{A}_G^{(p)}(v;t)) \\ &\ge (t-2)+(1+2m_{p,0}+m_{p,2}+m_{p,3}) \\ &= (t-2)+(1+m_{p,0}+m_{p,0}+m_{p,2}+m_{p,3}) \\ &\ge (t-2)+(1+m_{p,1}+2+m_{p,0}+m_{p,2}+m_{p,3}) \\ &= (t-2)+(3+t-1) \\ &= 2t,\end{aligned}$$

which contradicts the requirement of $|F|\le 2t-1$. Such a contradiction results from the original assumption that u_p is faulty. In other words, vertex u_p is really fault-free and able to make a correct diagnosis if $m_{p,0}-m_{p,1}\ge m_{q,0}-m_{q,1}$.

Case 3: Suppose that $|S|=1$. Let $S=\{u_p\}$ with some $1\le p\le t$. Then we claim that vertex u_p is fault-free. Suppose, by contradiction, that u_p is faulty; i.e., $u_p\in A$. However, Theorem 1 ensures that the algorithm $\mathbf{DVRF}(\mathbb{A}_G^{(p)}(v;t),u_p)$ correctly identifies the fault status of vertex u_p in $\mathbb{A}_G^{(p)}(v;t)$ if the number of faulty vertices in $\mathbb{A}_G^{(p)}(v;t)$ does not exceed $t-1$. Therefore, the set of faulty vertices in $\mathbb{A}_G^{(p)}(v;t)$, denoted by X, has cardinality at least t. Furthermore, since $|A|+|B|\le 1$, the remaining $t-1$ neighbors of vertex v (i.e., u_i, $1\le i\ne p\le t$) are faulty too. In a conditionally faulty graph, every fault-free vertex needs to have at least one fault-free neighbor. As a consequence, vertex v is also faulty. In short, we have $\{v\} \cup N_G(v) \cup X \subseteq F$ so that $|F| \geq |\{v\} \cup N_G(v) \cup X| = |\{v\}| + |N_G(v) \cup X| \geq 1 + t + (t-1) = 2t$. Again, this contradicts the assumption that $|F| \leq 2t-1$; that is, the claim holds.

Case 4: Suppose that $|S| = 0$. Obviously, we have $m_{i,1} - m_{i,0} \geq 1$ for every $1 \leq i \leq t$. In this case we first claim that for any $1 \leq i \leq t$, vertex u_i is faulty if $m_{i,1} - m_{i,0} \geq 2$. Suppose, by contradiction, that u_i is fault-free. Since $|A| + |B| \leq 1$, u_i is the only fault-free neighbor of v. Accordingly, the number of faulty vertices can be counted as follows:

$$\begin{aligned}|F| &\geq |\{u_j \mid 1 \leq j \neq i \leq t\}| + f(\mathbb{A}_G^{(i)}(v;t))\\ &\geq (t-1) + 2m_{i,1} + m_{i,2} + m_{i,3}\\ &\geq (t-1) + (2 + m_{i,0}) + m_{i,1} + m_{i,2} + m_{i,3}\\ &= (t-1) + 2 + (t-1)\\ &= 2t,\end{aligned}$$

which contradicts the assumption that $|F| \leq 2t-1$. So vertex u_i is faulty if $m_{i,1} - m_{i,0} \geq 2$. With this claim, it is easy to see that vertex v is faulty if $m_{i,1} - m_{i,0} \geq 2$ for every $1 \leq i \leq t$.

Suppose that there exists an integer p, $1 \leq p \leq t$, such that $m_{p,1} - m_{p,0} = 1$. Let r be an integer defined as follows:

$$r = |\{1 \leq j \leq t-1 \mid (\sigma(z_{p,j}, y_{p,j}), \sigma(y_{p,j}, x_{p,j}), \sigma(x_{p,j}, u_p)) = (1,0,1)\}|.$$

We claim that vertex u_p happens to be fault-free (respectively, faulty) if $r \geq 1$ (respectively, $r = 0$).

Subcase 4.1: Assume that $r \geq 1$. Suppose, by contradiction, that u_p is faulty. If u_i is faulty for every $1 \leq i \leq t$, then v has to be faulty. Accordingly, the total number of faulty vertices can be counted as follows:

$$\begin{aligned}|F| &\geq |\{u_i \mid 1 \leq i \neq p \leq t\} \cup \{v\}| + f(\mathbb{A}_G^{(p)}(v;t))\\ &\geq t + (1 + 2m_{p,0} + m_{p,2} + m_{p,3} + 1)\\ &= t + m_{p,0} + m_{p,1} + m_{p,2} + m_{p,3} + 1\\ &= t + (t-1) + 1\\ &= 2t,\end{aligned}$$

which contradicts the assumption that $|F| \leq 2t-1$. On the other hand, if v has a fault-free neighbor, say u_q ($q \neq p$), then the number of faulty vertices is counted as follows:

$$\begin{aligned}|F| &\geq |\{u_i \mid 1 \leq i \neq q \leq t\}| + f(\mathbb{A}_G^{(q)}(v;t)) + f(\mathbb{A}_G^{(p)}(v;t))\\ &\geq (t-1) + t + (2m_{p,0} + m_{p,2} + m_{p,3} + 1)\\ &\geq 2t,\end{aligned}$$

which contradicts the assumption that $|F| \leq 2t-1$. Anyway, vertex u_p is fault-free and can be an adequate tester if $r \geq 1$.

Subcase 4.2: Assume that $r = 0$. Suppose, by contradiction, that u_p is fault-free. Then the total number of faulty vertices can be counted as follows:

$$\begin{aligned}|F| &\geq |\{u_i \mid 1 \leq i \neq p \leq t\}| + f(\mathbb{A}_G^{(p)}(v;t))\\ &\geq (t-1) + 3m_{p,1} + m_{p,2} + m_{p,3}\\ &= (t-1) + 1 + m_{p,0} + 2m_{p,1} + m_{p,2} + m_{p,3}\\ &= (t-1) + t + m_{p,1}\\ &\geq 2t,\end{aligned}$$

which contradicts the assumption that $|F| \leq 2t-1$. That is, vertex u_p is faulty, and so is vertex v if $r = 0$.

The proof is completed. □

We end with estimating the time complexity of the proposed algorithm. As described in Section 3, many interconnected systems with N vertices have degree in the order of $\log N$ for each vertex. In these systems, an augmenting star structure can be constructed with time complexity $O((\log N)^2)$. However, for some unstructured networks, such as the ad hoc network, it is possible to build such a structure with a greater time complexity. Given an augmenting star $\mathbb{A}_G(v,n)$ rooted at a vertex v in a system G, the time taken in step (3) of the algorithm is $O((\log N)^2)$, because the time complexity of the algorithm **DVRF** is $O(\log N)$ for $\mathbb{A}_G^{(i)}(v,n)$, and it is run $O(\log N)$ times. As a result, the time complexity of **DVCF** algorithm is $O((\log N)^2)$ when an augmenting star of full order is obtained in time $O((\log N)^2)$. Based on the symmetry of most practical multiprocessor systems, the time for system-level diagnosis is $O(N(\log N)^2)$.

5 Examples

In this section, we show the proposed diagnosis algorithm can be applied to some well-known multiprocessor interconnected systems.

5.1 Construction of Augmenting Stars

As the first example, we show that the star graph [1] with dimension of five or more contains an augmenting star structure of full order rooted at each vertex as a subgraph. Let n be a positive integer. The *n-dimensional star graph*, denoted by $\mathbb{S}_n$, is a graph whose vertex set consists of all permutations of $\{1, 2, \ldots, n\}$. Each vertex is uniquely assigned a permutation $x_1x_2\ldots x_n$ and is adjacent to $(n-1)$ vertices $x_ix_2\ldots x_{i-1}x_1x_{i+1}\ldots x_n$ for $2 \leq i \leq n$, which are vertices obtained by a transposition of the first digit with the ith one. Consequently, there are $n!$ vertices in an n-dimensional star graph, and each vertex has degree $n-1$.

For clarity, we use boldface letters to denote vertices of a star graph. For any vertex $\mathbf{v} \in V(\mathbb{S}_n)$, its *i-neighbor*, denoted by $(\mathbf{v})^i$, is just the vertex obtained by a transposition of the first digit with the ith one of vertex $\mathbf{v}$. For convenience of description, we say that vertices $\mathbf{v}$ and $(\mathbf{v})^i$ are adjacent to each other with a $(1i)$ edge.

To construct an augmenting star of full order rooted at any vertex v in $\mathbb{S}_n$, we need to use a topological property of star graphs. For any $1 \leq i \leq n$, let V_i denote a subset of permutations of $\{1, 2, \ldots, n\}$, whose elements have symbol i in the nth digit. Clearly, we have $V(\mathbb{S}_n) = \bigcup_{i=1}^n V_i$. Moreover, it is shown in [1] that the subgraph of $\mathbb{S}_n$ induced by V_i is isomorphic to an $(n-1)$-dimensional star graph $\mathbb{S}_{n-1}$. We denote this subgraph by $\mathbb{S}_n^{\{i\}}$.

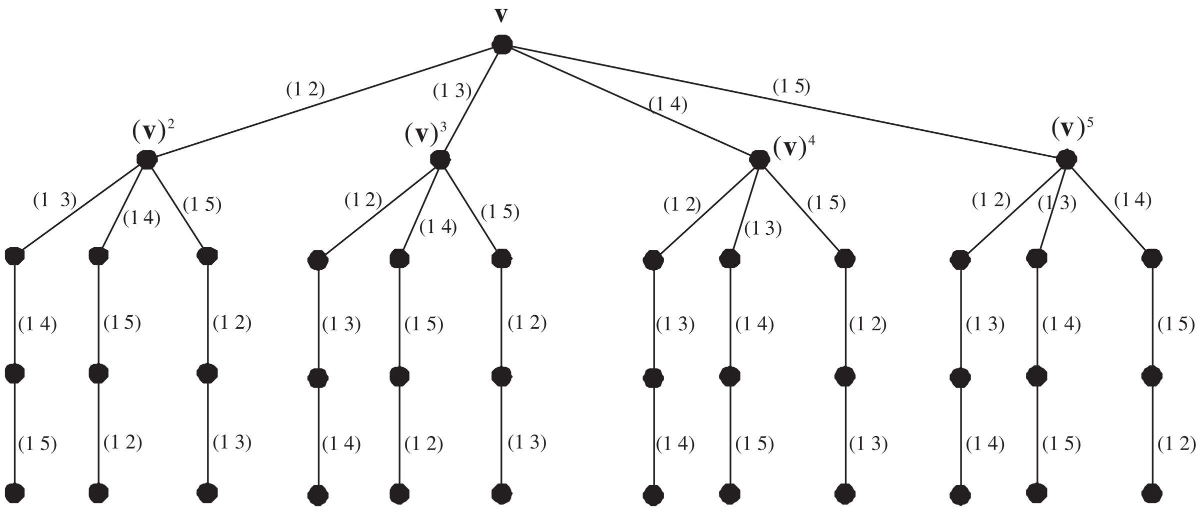


Fig. 5. An augmenting star $\mathbb{A}_{\mathbb{S}_5}(\mathbf{v};4)$ rooted at any vertex $\mathbf{v}$ in $\mathbb{S}_5$.

As usual, we use $\mathbb{A}_{\mathbb{S}_n}(\mathbf{v};n-1)$ to denote an augmenting star structure of full order rooted at any given vertex $\mathbf{v}$ in $\mathbb{S}_n$. We depict $\mathbb{A}_{\mathbb{S}_5}(\mathbf{v};4)$ in Fig. 5. In order to construct an augmenting star structure in $\mathbb{S}_n$ for $n \geq 5$, we propose the following algorithm (see Fig. 6 for illustration).

Algorithm. Construct-Augmenting-Star-Of-Full-Order-In-Star-Graph($\mathbb{S}_n, x_1x_2\dots x_n$)
Input: An n-dimensional star graph $\mathbb{S}_n$, $n \geq 5$, and its any vertex $\mathbf{v} = x_1x_2\dots x_n$.
Output: An augmenting star of full order rooted at vertex $\mathbf{v} = x_1x_2\dots x_n$ in $\mathbb{S}_n$.
BEGIN

1) **if** $n < 5$
 then error "the dimensionality is illegal"
2) $(V,E) \leftarrow (\{\mathbf{v}\}, \emptyset)$
3) **if** $n = 5$
 then for $i \leftarrow 2$ to n
  do $\mathbf{u} \leftarrow (\mathbf{v})^i$
   $(V,E) \leftarrow (V \cup \{\mathbf{u}\}, E \cup \{\{\mathbf{v},\mathbf{u}\}\})$
   for $j \leftarrow 2$ to n
   do if $j \neq i$
    then for $k \leftarrow 0$ to 2
     do if $j + k \geq 6$
      then $\mathbf{w} \leftarrow (\mathbf{u})^{j+k-4}$
      else $\mathbf{w} \leftarrow (\mathbf{u})^{j+k}$
     $(V,E) \leftarrow (V \cup \{\mathbf{w}\}, E \cup \{\{\mathbf{u},\mathbf{w}\}\})$
     $\mathbf{u} \leftarrow \mathbf{w}$
 return the graph $G \leftarrow (V,E)$ (see Fig. 5 for illustration)
4) $(V,E) \leftarrow$ Construct-Augmenting-Star-Of-Full-Order-In-Star-Graph($\mathbb{S}_n^{\{x_n\}}, x_1x_2\dots x_n$)
5) **for** $i \leftarrow 2$ to $n-1$
 do $\mathbf{u} \leftarrow (\mathbf{v})^i$
  $V \leftarrow V \cup \{(\mathbf{u})^n, ((\mathbf{u})^n)^2, (((\mathbf{u})^n)^2)^3\}$
  $E \leftarrow E \cup \{\{\mathbf{u},(\mathbf{u})^n\}, \{(\mathbf{u})^n, ((\mathbf{u})^n)^2\}, \{((\mathbf{u})^n)^2, (((\mathbf{u})^n)^2)^3\}\}$
6) $(V,E) \leftarrow (V \cup \{(\mathbf{v})^n\}, E \cup \{\{\mathbf{v},(\mathbf{v})^n\}\})$
7) **for** $i \leftarrow 2$ to $n-1$
 do $\mathbf{u} \leftarrow (\mathbf{v})^n$

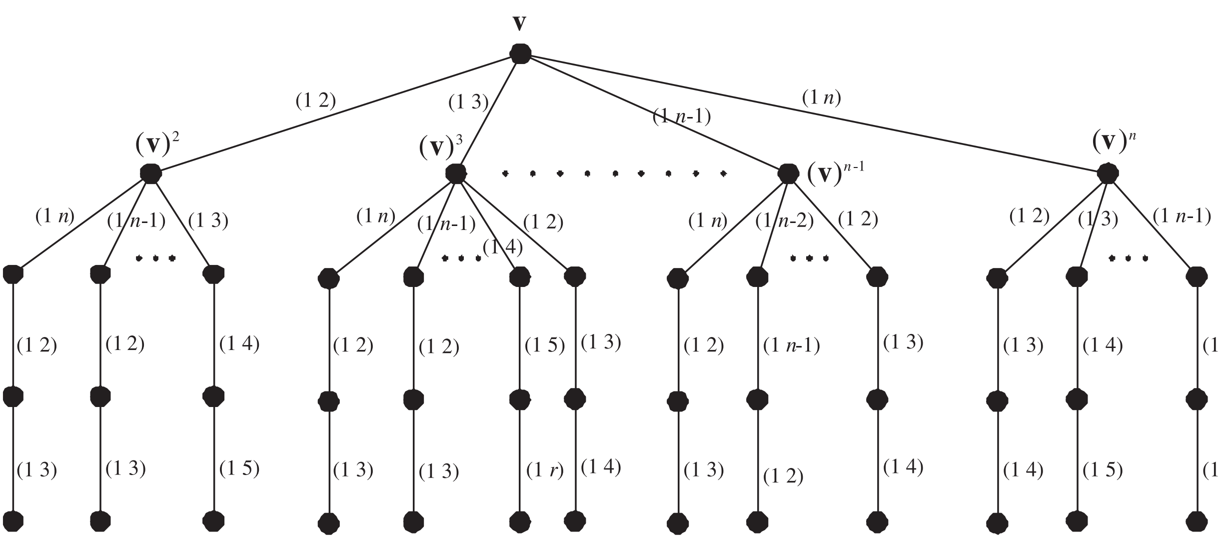


Fig. 6. An augmenting star $\mathbb{A}_{\mathbb{S}_n}(\mathbf{v};n-1)$ rooted at any vertex $\mathbf{v}$ in $\mathbb{S}_n$, in which $r = 2$ if $n = 6$, and $r = 6$ if $n \geq 7$.

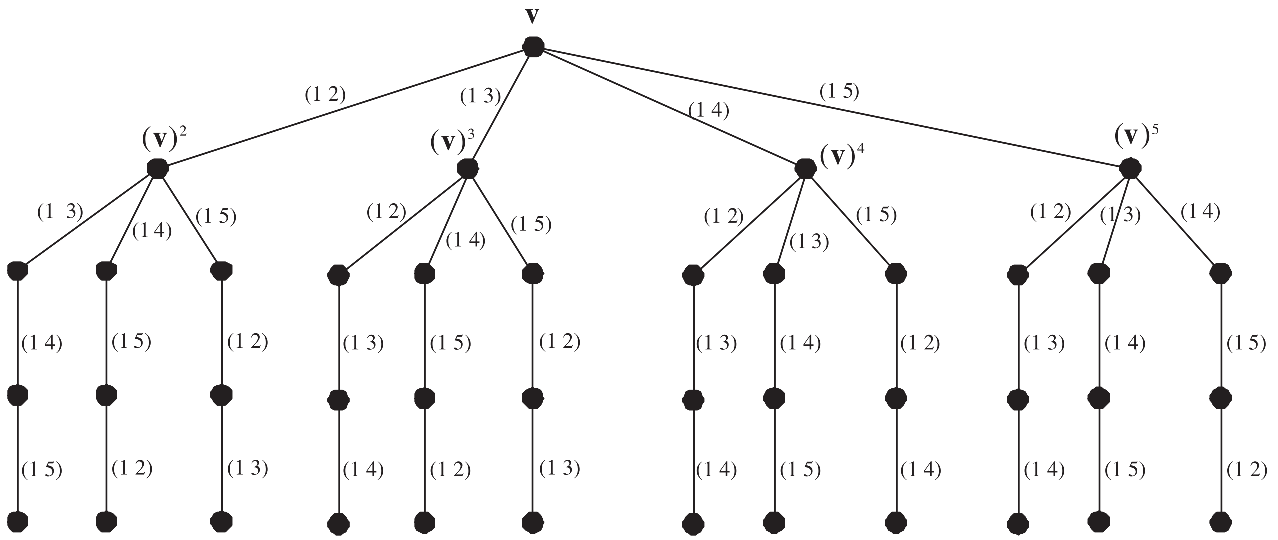


Fig. 7. An augmenting star $\mathbb{A}_{\mathbb{P}_5}(\mathbf{v};4)$ rooted at any vertex $\mathbf{v}$ in $\mathbb{P}_5$.

for $k \leftarrow 0$ to 2
  do if $i+k \geq n$
    then $\mathbf{w} \leftarrow (\mathbf{u})^{i+k-n+2}$
    else $\mathbf{w} \leftarrow (\mathbf{u})^{i+k}$
      $(V,E) \leftarrow (V \cup \{\mathbf{w}\}, E \cup \{\{\mathbf{u},\mathbf{w}\}\})$
   $\mathbf{u} \leftarrow \mathbf{w}$

8) **return** the graph $G \leftarrow (V,E)$ (see Fig. 6 for illustration)
END

As another example, we show that the pancake graph with dimension of 5 or more also contains an augmenting star of full order rooted at any vertex. The *n-dimensional pancake graph*, denoted by $\mathbb{P}_n$, has the same vertex set as an n-dimensional star graph, i.e., all permutations of $\{1,2,\ldots,n\}$. Its adjacency is defined as follows: vertex $x_1x_2\ldots x_i\ldots x_n$ is adjacent to vertex $y_1y_2\ldots y_i\ldots y_n$ through an i-dimensional edge, $2 \leq i \leq n$, if $y_j = x_{i-j+1}$ for all $1 \leq j \leq i$ and $y_j = x_j$ for all $i < j \leq n$. Because the pancake graph is algebraically similar to the star graph, an augmenting star of full order can be constructed in a way the same as that for the star graph, except for the case of $n=5$. For this reason, we only depict an augmenting star of full order in the five-dimensional pancake graph. See Fig. 7.

5.2 Example of DVCF

Now we give an example of **DVCF** algorithm. Suppose that

$$F_1 = \{12345, 32145, 42315, 52341, 41325, 51342, 15342\}$$

is a set of seven faulty vertices in $\mathbb{S}_5$, and we are required to identify the fault status of vertex 12345. The test assignment for $\mathbb{A}_{\mathbb{S}_5}(12345;4)$ and its syndrome is illustrated in Fig. 8. In step 2 of **DVCF** algorithm, the method presented in the above section can be applied to obtain an augmenting star of full order rooted at vertex 12345. Next, in step 3, $S = \{21345\}$ is computed. Since $|S| = 1$, the procedure will goto step 6 and return the test outcome $\sigma(21345, 12345) = 1$ as its diagnosis output. That is, vertex 12345 is faulty.

In another case, we assume that

$$F_2 = \{32145, 42315, 52341, 31245, 41325, 51342, 15342\}$$

is a set of seven faulty vertices in $\mathbb{S}_5$. Again, we would like to identify the fault status of vertex 12345. The test assignment for $\mathbb{A}_{\mathbb{S}_5}(12345;4)$ and its syndrome is illustrated in Fig. 9. Now, in step (3), $S = \emptyset$ is determined. Since $|S| = 0$, the procedure will goto step (7). Accordingly, we have $u_p = 21345$ and $r = 1$ so that the test outcome $\sigma(21345, 12345) = 0$ is returned. That is, vertex 12345 is fault-free.

5.3 Simulation

Our simulation is aimed at measuring the time consuming of **DVCF** algorithm over the star graphs and the pancake graphs of different sizes. Because both the two graphs, $\mathbb{S}_n$ and $\mathbb{P}_n$, are vertex-symmetric and $(n-1)$-regular, we simulate the diagnosis process with respect to vertex $\mathbf{w}_n = 12\ldots n$. We carry out a round of simulation by randomly assigning a conditionally faulty set of $2(n-1)-1 = 2n-3$ vertices in the augmenting star structure rooted at $\mathbf{w}_n$ for 10,000 times and compute the average time for identifying the fault status of $\mathbf{w}_n$. Then such a round of simulation will be repeated 30 times to obtain the overall average. The hardware and software configuration include:

1. Intel Core 2 Quad CPU Q8300 2.5 GHz,
2. 4 GB RAM,
3. 64-bit Windows 7 OS, and
4. C++ Programming Language in Microsoft Visual Studio 2005.

The experimental results are shown in Fig. 10.

6 Concluding Remarks

The issue of identifying faulty processors is important for the design of multiprocessor interconnected systems, which are implementable with VLSI. The process of identifying all the faulty processors is the system-level diagnosis. In the random-fault probabilistic model of multiprocessor systems, processors are assumed to fail independently. Hence, the one-step diagnosability of a multiprocessor system is always upper bounded by its minimum degree. For many practical multiprocessor systems or interconnection

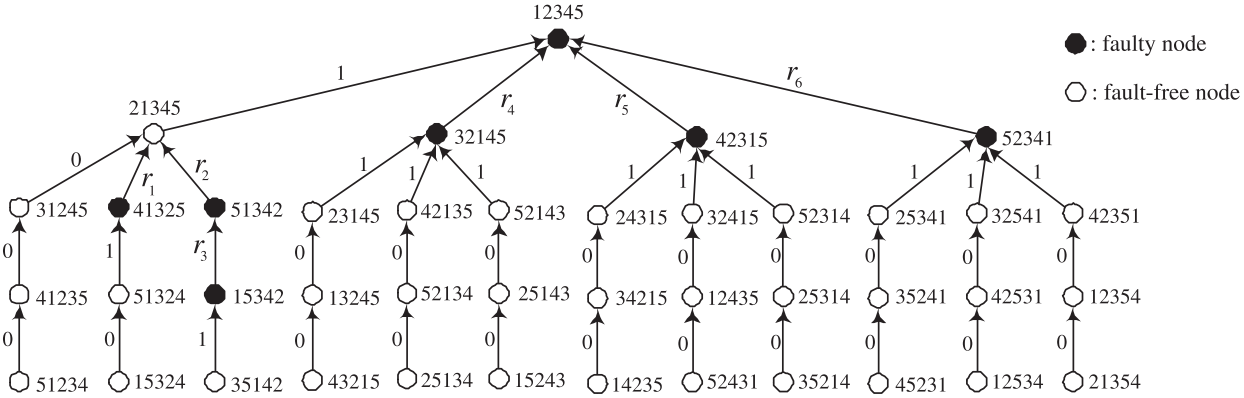


Fig. 8. The test assignment for $\mathbb{A}_{\mathbb{S}_5}(12345;4)$ and its syndrome, in which $r_1, r_2, r_3, r_4, r_5, r_6 \in \{0,1\}$.

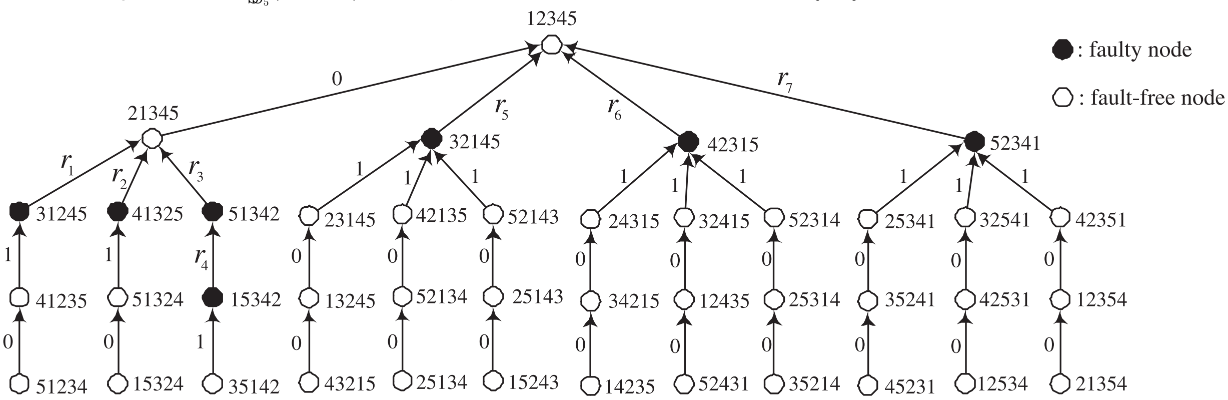
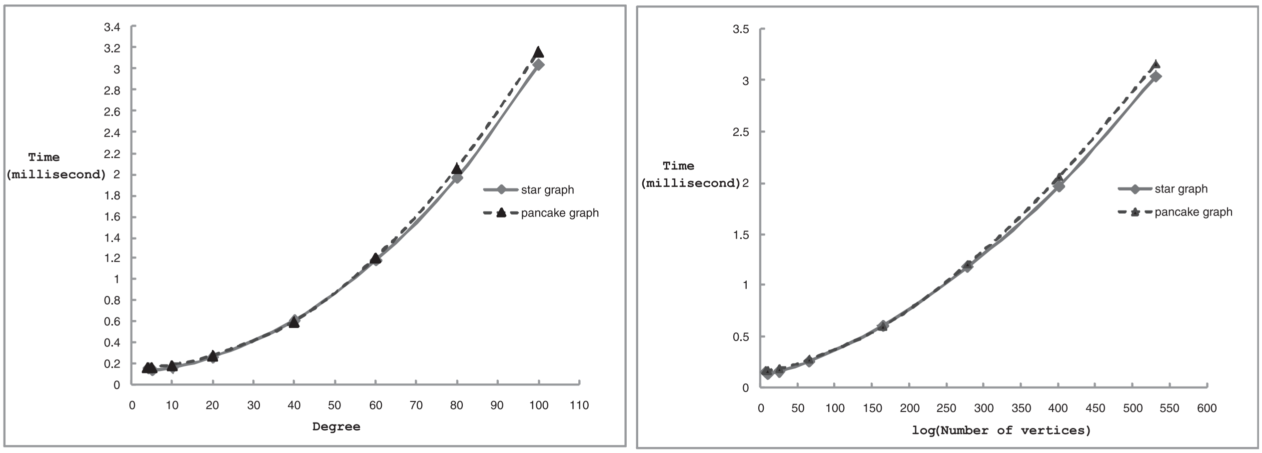


Fig. 9. The test assignment for $\mathbb{A}_{\mathbb{S}_5}(12345;4)$ and its syndrome, in which $r_3 = 1$, $r_4 = 0$, and $r_1, r_2, r_5, r_6, r_7 \in \{0,1\}$.



(a) (b)

Fig. 10. The time consuming of **DVCF** algorithm over the star graphs and the pancake graphs of different sizes.

networks, however, the probability that all the neighbors of a processor are faulty simultaneously is very small. In addition, the small diagnosability of a system is also owing to the fact that it only considers a global status of the entire system but ignores the unlikelihood of faulty processors occurring within a local substructure at the same time.

In this paper, we extend our previous research [18] and study the local diagnosis capability of a conditionally faulty system, in which every fault-free processor is required to have at least one fault-free neighbor. As shown in [18], estimating the local diagnosability with respect to each processor can also be thought of as a new strategy for checking the traditional one-step diagnosability of the whole system. Under the PMC model, we present a sufficient condition to estimate a given processor's local diagnosis capability in a conditionally faulty system. Moreover, we propose an efficient fault identification algorithm, provided that there can be an augmenting star structure of full order rooted at each processor and the time for a processor to test another one is a constant.

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