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A comparison of similarity measures of fuzzy values

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Abstract

This paper extends the work of Pappis and Karacapilidis (1993) to present and compare the properties of several measures of similarity of fuzzy values. The measures examined in this paper are based on the geometric model, the set-theoretic approach, and the matching function S we presented in (Chen, 1988). It is shown that several properties are common to all measures and some properties do not hold for all of them.

Keywords: Fuzzy value; Similarity measure; Matching function

1. Introduction

Pappis [2] had discussed the notion of approximation of fuzzy values and introduced the definitions of proximity measure and approximate equal fuzzy sets. Pappis et al. [3] pointed out that approximation is inherent in fuzzy set theory and that approximation is implied when considering the multitude of solutions of the Inverse Problem [4]. Furthermore, in [3], they made an assessment of measures of similarity of fuzzy values. The measures examined in [3] include:

(1) The measure based on the operations of union and intersection.
(2) The measure based on the maximum difference.
(3) The measure based on the differences and the sum of grades of membership.

This investigation is important due to the fact that it can provide us some useful information to select a suitable similarity measure in applications of fuzzy sets.

In this paper, we extend the work of [3] to further investigate measures of similarity of fuzzy values. The measures examined in this paper are based on the geometric model, the set-theoretic approach [5], and the matching function S we presented in [1]. It is shown that some properties are common to these measures, and some properties do not hold for all of them, which may influence the choice of the measure to be used in fuzzy sets applications.

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2. Basic notations and definitions

In the following, we briefly review some basic notations and definitions of $\circ$ composition and α composition from [3]. Let A be a fuzzy set of the universe of discourse U, $U = \{u_1, u_2, \ldots, u_n\}$, and let $\bar{A}$ denote the complement of the fuzzy set A. Furthermore, let I, O, and M denote the unit, zero, and 0.5 fuzzy sets, i.e., the fuzzy sets with all grades of membership equal to 1.0, 0, or 0.5, respectively, where

$$I = \sum_{i=1}^{n} 1.0/u_i, \tag{1}$$

$$O = \sum_{i=1}^{n} 0/u_i, \tag{2}$$

$$M = \sum_{i=1}^{n} 0.5/u_i. \tag{3}$$

The $\circ$ composition of the vector $a = (a_1, a_2, \ldots, a_n)$, corresponding to the fuzzy subset A of U, with the matrix $R = [r_{ij}]$, corresponding to the fuzzy relation R of $U \times V$, where $V = \{v_1, v_2, \ldots, v_m\}$, is denoted by $a \circ R$ and is equal to the vector $c = (c_1, c_2, \ldots, c_m)$, where

$$c_j = \bigvee (a_i \wedge r_{ij}), \tag{4}$$

$\wedge$ denotes the minimum operator, and $\vee$ denotes the maximum operator.
The α composition of a scalar s with a scalar t which is denoted by $s \alpha t$ is defined by

$$s \alpha t = \begin{cases} 1 & \text{if } s \leqslant t, \\ t & \text{otherwise.} \end{cases} \tag{5}$$

The α composition of the vector $x = (x_1, x_2, \ldots, x_n)$ with the scalar s is formed by substituting each element x_i of x with $x_i \alpha s$. The α composition of the matrix R with the vector $x = (x_1, x_2, \ldots, x_n)$ is formed by substituting each column vector r_j of R with $r_j \alpha x_j$ and is denoted by $R \alpha x$. $\bigwedge (R \alpha x)$ denotes the vector whose elements are formed by taking the minimum element of the respective row vector of $R \alpha x$.

3. Measure based on the geometric distance model

Zwick et al. [5] introduced a one-parameter class of distance functions defined as follows:

$$d_r(a, b) = \left[\sum_{i=1}^{n} |a_i - b_i|^r \right]^{1/r}, \tag{6}$$

where a and b are two points in an n-dimensional space, $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$. It is obvious that when $r = 1$, Eq. (6) becomes

$$d_1(a, b) = \sum_{i=1}^{n} |a_i - b_i|. \tag{7}$$

According to [5], when r approaches ∞, Eq. (6) becomes

$$d_\infty(a, b) = \max_i |a_i - b_i|. \tag{8}$$

Let a and b be the vector representations of the fuzzy sets A and B, respectively, where $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$. We can see that one of the measures presented in [3] is based on Eq. (8), where the grade of similarity $L_{A,B}$ of the fuzzy sets A and B is defined by

$$L_{A,B} = 1 - \max_i (|a_i - b_i|). \tag{9}$$

The properties of $L_{A,B}$ have been investigated in [3]. Furthermore, the properties of the similarity measure $S_{A,B}$ of the fuzzy sets A and B are also investigated in [3], where

$$S_{A,B} = 1 - \frac{\sum_{i=1}^{n} |a_i - b_i|}{\sum_{i=1}^{n} (a_i + b_i)}. \tag{10}$$

In the following, we will investigate the properties of the measure $W_{A,B}$ based on Eq. (7), where the grade of similarity $W_{A,B}$ of the fuzzy sets A and B is defined by

$$W_{A,B} = 1 - \frac{\sum_{i=1}^{n} |a_i - b_i|}{n}. \tag{11}$$

According to [3], the fuzzy sets A and B are said to be approximately equal (denoted by $A \sim B$) if and only if given a small nonnegative number ε, $W_{A,B} \leqslant \varepsilon$, where the number ε is said to be a proximity measure of A and B.

The properties of $W_{A,B}$ are shown as follows:

(W1) $W_{A,B} = W_{B,A}$.

(W2) $A = B \Leftrightarrow W_{A,B} = 1$.

(W3) $A \cap B = O \Leftrightarrow W_{A,B} = 0$ is not true. It is obvious that if $A = I$ and $B = O$, then $A \cap B = O$ and $W_{A,B} = 0$. However, if consider $a = (0.8, 0, 0.5)$ and $b = (0, 0.9, 0)$, then we can see that $A \cap B = O$, but $W_{A,B} = 1 - (0.8 + 0.9 + 0)/3 \fallingdotseq 0.43 \neq 0$.

(W4) $W_{A,\bar{A}} = 1 \Leftrightarrow A = M$.

(W5) $W_{A,\bar{A}} = 0 \Leftrightarrow A = I$ or $A = O$.

(W6) $A \sim B$ does not necessarily imply that $A \cup C \sim B \cup C$. Consider $a = (0.8, 0.3, 0.5)$, $b = (0.6, 0.7, 0.6)$, and $c = (0.6, 0.5, 0.3)$. It follows that

$$W_{A,B} = 1 - \frac{0.2 + 0.4 + 0.1}{3} \fallingdotseq 0.77,$$

$$a \vee c = (0.8, 0.5, 0.5), \qquad b \vee c = (0.6, 0.7, 0.6),$$

$$W_{A \cup C, B \cup C} = 1 - \frac{0.2 + 0.2 + 0.1}{3} \fallingdotseq 0.83,$$

i.e., $W_{A \cup C, B \cup C} > W_{A,B}$, which means that the proximity measure of $A \cup C$ and $B \cup C$ is greater than that of A and B. Thus, $A \sim B$ does not necessarily imply that $A \cup C \sim B \cup C$.

(W7) $A \sim B$ does not necessarily imply that $A \cap C \sim B \cap C$. Consider $a = (0.8, 0.3, 0.5)$, $b = (0.6, 0.7, 0.6)$, and $c = (0.6, 0.5, 0.3)$. It follows that

$$W_{A,B} = 1 - \frac{0.2 + 0.4 + 0.1}{3} \fallingdotseq 0.77,$$

$$a \wedge c = (0.6, 0.3, 0.3), \qquad b \vee c = (0.6, 0.5, 0.3),$$

$$W_{A \cap C, B \cap C} = 1 - \frac{0 + 0.2 + 0}{3} \fallingdotseq 0.93,$$

i.e., $W_{A\cap C,B\cap C} > W_{A,B}$, which means that the proximity measure of $A\cap C$ and $B\cap C$ is greater than that of A and B. Thus, $A \sim B$ does not necessarily imply that $A\cap C \sim B\cap C$.

(W8) $A \sim B$ does not necessarily imply that $A \circ R \sim B \circ R$. Consider $a = (0.8, 0.3, 0.5)$, $b = (0.6, 0.7, 0.6)$, and

$$R = \begin{bmatrix} 0.1 & 0.4 \\ 0.7 & 0.9 \\ 0.5 & 0.5 \end{bmatrix}.$$

It follows that

$$W_{A,B} = 1 - \frac{0.2 + 0.4 + 0.1}{3} \doteq 0.77,$$

$$a \circ R = (0.5, 0.5), \qquad b \circ R = (0.7, 0.7),$$

$$W_{A\circ R,B\circ R} = 1 - \frac{0.2 + 0.2}{2} = 0.8,$$

i.e., $W_{A\circ R,B\circ R} > W_{A,B}$, which means that the proximity measure of $A \circ R$ and $B \circ R$ is greater than that of A and B. Thus, $A \sim B$ does not necessarily imply that $A \circ R \sim B \circ R$.

(W9) $R \sim S$ does not necessarily imply that $A \circ R \sim A \circ S$.
Consider

$$R = \begin{bmatrix} 1 & 0.1 & 0.7 \\ 0.7 & 0.2 & 0.9 \\ 0.3 & 0.5 & 1 \end{bmatrix}, \qquad S = \begin{bmatrix} 0.9 & 0.5 & 1 \\ 0.7 & 0.8 & 0.5 \\ 0.2 & 0.4 & 0.7 \end{bmatrix}, \qquad a = (0.8, 0.3, 0.5).$$

It follows that

$$W_{R,S} = \frac{\left(1 - \frac{0.1 + 0.4 + 0.3}{3}\right) + \left(1 - \frac{0 + 0.6 + 0.4}{3}\right) + \left(1 - \frac{0.1 + 0.1 + 0.3}{3}\right)}{3}$$

$$= \frac{0.733 + 0.667 + 0.833}{3} \doteq 0.744,$$

$$a \circ R = (0.8, 0.5, 0.7), \qquad a \circ S = (0.8, 0.5, 0.8),$$

$$W_{a\circ R,a\circ S} = 1 - \frac{0 + 0 + 0.1}{3} \doteq 0.97,$$

i.e., $W_{a\circ R,a\circ S} > W_{R,S}$, which means that the proximity measure of $A \circ R$ and $A \circ S$ is greater than that of R and S. Thus, $R \sim S$ does not necessarily imply that $A \circ R \sim A \circ S$.

α-composition. Let $\bigwedge(R \vee x) = f$ and $\bigwedge(R \vee y) = g$, and F, G be the fuzzy sets with membership vectors equal to f and g, respectively.

(W10) $X \sim Y$ does not necessarily imply that $F \sim G$. Consider $x = (0.8, 0.3, 0.5)$, $y = (0.6, 0.7, 0.6)$, and

$$R = \begin{bmatrix} 1 & 0.1 & 0.3 \\ 0.7 & 0.2 & 0.9 \\ 0.3 & 0.5 & 1 \end{bmatrix}.$$

It follows that

$$W_{X,Y} = 1 - \frac{0.2 + 0.4 + 0.1}{3} \doteqdot 0.77,$$

$$R \alpha x = \begin{bmatrix} 0.8 & 1 & 1 \\ 1 & 1 & 0.5 \\ 1 & 0.3 & 0.5 \end{bmatrix}, \qquad R \alpha y = \begin{bmatrix} 0.6 & 1 & 1 \\ 0.6 & 1 & 0.6 \\ 1 & 1 & 0.6 \end{bmatrix},$$

$$f = \bigwedge (R \alpha x) = (0.8, 0.5, 0.3), \qquad g = \bigwedge (R \alpha y) = (0.6, 0.6, 0.6),$$

$$W_{F,G} = 1 - \frac{0.2 + 0.1 + 0.3}{3} = 0.8,$$

i.e., $W_{F,G} > W_{X,Y}$, which means that the proximity measure of F and G is greater than that of X and Y. Thus, $X \sim Y$ does not necessarily imply that $F \sim G$.

Similarly, it can be shown that if R and S are the matrices corresponding to the fuzzy relations R and S, if $\bigwedge(R \alpha x) = f$ and $\bigwedge(S \alpha x) = k$, then

(W11) $R \sim S$ does not necessarily imply that $F \sim K$. Consider

$$R = \begin{bmatrix} 1 & 0.1 & 0.7 \\ 0.7 & 0.2 & 0.9 \\ 0.3 & 0.5 & 1 \end{bmatrix}, \qquad S = \begin{bmatrix} 0.9 & 0.5 & 1 \\ 0.7 & 0.8 & 0.5 \\ 0.2 & 0.4 & 0.7 \end{bmatrix}, \qquad x = (0.8, 0.3, 0.5).$$

It follows that

$$W_{R,S} = \frac{0.733 + 0.667 + 0.833}{3} \doteqdot 0.744,$$

$$R \alpha x = \begin{bmatrix} 0.8 & 1 & 0.5 \\ 1 & 1 & 0.5 \\ 1 & 0.3 & 0.5 \end{bmatrix},$$

$$S \alpha x = \begin{bmatrix} 0.8 & 0.3 & 0.5 \\ 1 & 0.3 & 1 \\ 1 & 0.3 & 0.5 \end{bmatrix},$$

$$\bigwedge (R \alpha x) = (0.5, 0.5, 0.3) = f, \qquad \bigwedge (S \alpha x) = (0.3, 0.3, 0.3) = k,$$

$$W_{F,K} = 1 - \frac{0.2 + 0.2 + 0}{3} \doteqdot 0.87,$$

i.e., $W_{F,K} > W_{R,S}$, which means that the proximity measure of F and K is greater than that of R and S. Thus, $R \sim S$ does not necessarily imply that $F \sim K$.

4. Measure based on the set-theoretic approach

Let A and B be fuzzy sets of the universe of discourse U, and let μ_A and μ_B be the membership functions of the fuzzy sets A and B, respectively. Defining the following operations between fuzzy subsets [5],

$$\forall x \in U, \quad \mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x)), \tag{12}$$

$$\forall x \in U, \quad \mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x)), \tag{13}$$

$$\forall x \in U, \quad \mu_{\bar{A}}(x) = 1 - \mu_A(x). \tag{14}$$

The scalar cardinality (power) of a fuzzy subset A of U is defined as

$$|A| = \sum_{x \in U} \mu_A(x). \tag{15}$$

When the universe of discourse U is an infinite set, then the power of A is defined by

$$|A| = \int_{-\infty}^{\infty} \mu_A(x)\,\mathrm{d}x. \tag{16}$$

Zwick et al. [5] pointed out that the following indexes have been proposed in the literature as dissimilarity measures between fuzzy subsets:

$$S_1(A, B) = 1 - \frac{|A \cap B|}{|A \cup B|} \tag{17}$$

$$S_4(A, B) = 1 - \sup_{x \in U} \mu_{A \cap B}(x). \tag{18}$$

Pappis et al. [3] have investigated the properties of similarity measure based on Eq. (17), where the grade of similarity $M_{A,B}$ of the fuzzy sets A and B is defined by

$$M_{A,B} = \frac{|A \cap B|}{|A \cup B|} = \frac{\sum_{i=1}^{n} (a_i \wedge b_i)}{\sum_{i=1}^{n} (a_i \vee b_i)}. \tag{19}$$

In the following, we will investigate the properties of similarity measure $T_{A,B}$ based on Eq. (18). Let U be the universe of discourse, $U = \{u_1, u_2, \ldots, u_n\}$. The grade of similarity $T_{A,B}$ of the fuzzy sets A and B is defined by

$$T_{A,B} = \sup_{x \in U} \mu_{A \cap B}(x) = \max(\mu_{A \cap B}(u_1), \mu_{A \cap B}(u_2), \ldots, \mu_{A \cap B}(u_n)). \tag{20}$$

The definitions of approximately equal fuzzy sets and the proximity measure in the case of $T_{A,B}$ are similar to those of Section 3.

The properties of $T_{A,B}$ are shown as follows.

(T1) $T_{A,B} = T_{B,A}$.

(T2) $A = B \Leftrightarrow T_{A,B} = 1$ is not true. Consider $a = (0.5, 1, 0.5)$ and $b = (0.5, 1, 0.5)$. We can see that $T_{A,B} = \max(0.5, 1, 0.5) = 1$. However, if consider $a = (0.5, 0.5, 0.5)$ and $b = (0.5, 0.5, 0.5)$, we can see that

$$T_{A,B} = \max(0.5, 0.5, 0.5) = 0.5 \neq 1.$$

Furthermore, if $a = (1, 0{\cdot}3, 0.5)$ and $b = (1, 0.6, 0.7)$, then we can see that $T_{A,B} = 1$, but $A \neq B$.

(T3) $A \cap B = O \Leftrightarrow T_{A,B} = 0$.

(T4) $T_{A,\bar{A}} = 1 \Leftrightarrow A = M$ is not true due to the fact that $T_{A,\bar{A}} = 1$ is impossible.

(T5) $T_{A,\bar{A}} = 0 \Leftrightarrow A = I$ or $A = O$ is not true. It is obvious that if $A = I$ or $A = O$, then $T_{A,\bar{A}} = 0$. However, if consider $a = (1,0,1)$, then we can see that $\bar{a} = (0,1,0)$. In this case, we can get $T_{A,\bar{A}} = 0$, but $A \neq I$ or $A \neq O$.

(T6) $A \sim B$ does not necessarily imply that $A \cup C \sim B \cup C$. Consider $a = (0.8, 0.3, 0.5)$, $b = (0.6, 0.7, 0.6)$, and $c = (0.6, 0.9, 0.3)$. It follows that

$$T_{A,B} = \max(0.6, 0.3, 0.5) = 0.6,$$

$$a \vee c = (0.8, 0.9, 0.5), \qquad b \vee c = (0.6, 0.9, 0.6),$$

$$T_{A \cup C, B \cup C} = \max(0.6, 0.9, 0.5) = 0.9,$$

i.e., $T_{A \cup C, B \cup C} > T_{A,B}$, which means that the proximity measure of $A \sim C$ and $B \sim C$ is greater than that of A and B.

(T7) $A \sim B \Rightarrow A \cap C \sim B \cap C$.

(T8) $A \sim B \Rightarrow A \circ R \sim B \circ R$.

(T9) $R \sim S$ does not necessarily imply that $A \circ R \sim A \circ S$. Consider $a = (0.8, 0.3, 0.5)$,

$$R = \begin{bmatrix} 1 & 0.1 & 0.7 \\ 0.7 & 0.2 & 0.9 \\ 0.3 & 0.5 & 1 \end{bmatrix}, \qquad S = \begin{bmatrix} 0.9 & 0.5 & 1 \\ 0.7 & 0.8 & 0.5 \\ 0.2 & 0.4 & 0.7 \end{bmatrix}.$$

It follows that

$$T_{R,S} = \frac{\max(0.9, 0.1, 0.7) + \max(0.7, 0.2, 0.5) + \max(0.2, 0.4, 0.7)}{3}$$

$$\fallingdotseq 0.77,$$

$$a \circ R = (0.8, 0.5, 0.7), \qquad a \circ S = (0.8, 0.5, 0.8),$$

$$T_{a \circ R, a \circ S} = \max(0.8, 0.5, 0.7) = 0.8,$$

i.e., $T_{a \circ R, a \circ S} > T_{R,S}$, which means that the proximity measure of $A \circ R$ and $A \circ S$ is greater than that of R and S.

(T10) $X \sim Y \Rightarrow F \sim G$.

(T11) $R \sim S \Rightarrow F \sim K$.

5. Measure based on the matching function *S* [1, 6]

In [1], we have presented a matching function S to calculate the degree of similarity between fuzzy sets A and B. Let a and b be the vector representations of the fuzzy sets A and B, respectively. Then,

$$S(a,b) = \frac{a \cdot b}{\max(a \cdot a, b \cdot b)}, \tag{21}$$

where $S(a,b) \in [0,1]$. The larger the value of $S(a,b)$, the more the similarity between the fuzzy sets A and B. In the following, we further investigate the properties of the matching function S. Let $P_{A,B}$ denote $S(a,b)$, i.e.,

$$P_{A,B} = S(a,b) = \frac{a \cdot b}{\max(a \cdot a, b \cdot b)}. \tag{22}$$

The definitions of approximately equal fuzzy sets and the proximity measure in the case of $P_{A,B}$ are similar to those of Section 3.

The properties of $P_{A,B}$ are shown as follows:

(P1) $P_{A,B} = P_{B,A}$.

(P2) $A = B \Leftrightarrow P_{A,B} = 1$.

(P3) $A \cap B = O \Leftrightarrow P_{A,B} = 0$.

(P4) $P_{A,\bar{A}} = 1 \Leftrightarrow A = M$.

(P5) $P_{A,\bar{A}} = 0 \Leftrightarrow A = I$ or $A = O$.

(P6) $A \sim B$ does not necessarily imply that $A \cup C \sim B \cup C$. Consider $a = (0.8, 0.3, 0.5)$, $b = (0.6, 0.7, 0.6)$, and $c = (0.6, 0.9, 0.3)$. It follows that

$$P_{A,B} = \frac{0.8*0.6 + 0.3*0.7 + 0.5*0.6}{\max(0.8*0.8 + 0.3*0.3 + 0.5*0.5, 0.6*0.6 + 0.7*0.7 + 0.6*0.6)}$$

$$\fallingdotseq 0.82,$$

$$a \vee c = (0.8, 0.9, 0.5), \qquad b \vee c = (0.6, 0.9, 0.6),$$

$$P_{A \cup C, B \cup C} = \frac{0.8*0.6 + 0.9*0.9 + 0.5*0.6}{\max(0.8*0.8 + 0.9*0.9 + 0.5*0.5, 0.6*0.6 + 0.9*0.9 + 0.6*0.6)}$$

$$\fallingdotseq 0.94,$$

i.e., $P_{A \cup C, B \cup C} > P_{A,B}$, which means that the proximity measure of $A \cup C$ and $B \cup C$ is greater than that of A and B.

(P7) $A \sim B \Rightarrow A \cap C \sim B \cap C$.

(P8) $A \sim B$ does not necessarily imply that $A \circ R \sim B \circ R$. Consider $a = (0.4, 0.6, 0.6)$, $b = (0.5, 0.7, 0.8)$, and

$$R = \begin{bmatrix} 0.2 & 1.0 \\ 0.6 & 0.6 \\ 1.0 & 0.3 \end{bmatrix}.$$

It follows that

$$P_{A,B} = \frac{0.4*0.5 + 0.6*0.7 + 0.6*0.8}{\max(0.4*0.4 + 0.6*0.6 + 0.6*0.6, 0.5*0.5 + 0.7*0.7 + 0.8*0.8)}$$

$$\fallingdotseq 0.80,$$

$$a \circ R = (0.6, 0.6), \qquad b \circ R = (0.8, 0.6),$$

$$P_{A \circ R, B \circ R} = \frac{0.6*0.8 + 0.6*0.6}{\max(0.6*0.6 + 0.6*0.6, 0.8*0.8 + 0.6*0.6)}$$

$$\fallingdotseq 0.84,$$

i.e., $P_{A \circ R, B \circ R} > P_{A,B}$, which means that the proximity measure of $A \circ R$ and $B \circ R$ is greater than that of A and B.

(P9) $R \sim S$ does not necessarily imply that $A \circ R \sim A \circ S$. Consider $a = (0.8, 0.3, 0.5)$,

$$R = \begin{bmatrix} 1 & 0.1 & 0.7 \\ 0.7 & 0.2 & 0.9 \\ 0.3 & 0.5 & 1 \end{bmatrix}, \qquad S = \begin{bmatrix} 0.9 & 0.5 & 1 \\ 0.7 & 0.8 & 0.5 \\ 0.2 & 0.4 & 0.7 \end{bmatrix}.$$

We can see that the degree of similarity between (1, 0.1, 0.7) and (0.9, 0.5, 1) can be evaluated and is equal to

$$\frac{1*0.9 + 0.1*0.5 + 0.7*1}{\max(1*1 + 0.1*0.1 + 0.7*0.7, 0.9*0.9 + 0.5*0.5 + 1*1)} \doteq 0.801.$$

The degree of similarity between (0.7, 0.2, 0.9) and (0.7, 0.8, 0.5) can be evaluated and is equal to

$$\frac{0.7*0.7 + 0.2*0.8 + 0.9*0.5}{\max(0.7*0.7 + 0.2*0.2 + 0.9*0.9, 0.7*0.7 + 0.8*0.8 + 0.5*0.5)} \doteq 0.797.$$

The degree of similarity between (0.3, 0.5, 1) and (0.2, 0.4, 0.7) can be evaluated and is equal to

$$\frac{0.3*0.2 + 0.5*0.4 + 0.1*0.7}{\max(0.3*0.3 + 0.5*0.5 + 1*1, 0.2*0.2 + 0.4*0.4 + 0.7*0.7)} \doteq 0.716.$$

Thus, we can get

$$P_{R,S} = \frac{0.801 + 0.797 + 0.716}{3} \doteq 0.77,$$

$$a \circ R = (0.8, 0.5, 0.7), \qquad a \circ S = (0.8, 0.5, 0.8),$$

$$P_{A \circ R, A \circ S} = \frac{0.8*0.8 + 0.5*0.5 + 0.7*0.8}{\max(0.8*0.8 + 0.5*0.5 + 0.7*0.7, 0.8*0.8 + 0.5*0.5 + 0.8*0.8)} \doteq 0.95,$$

i.e., $P_{A \circ R, A \circ S} > P_{R,S}$, which means that the proximity measure of $A \circ R$ and $A \circ S$ is greater than that of R and S.

(P10) $X \sim Y$ does not necessarily imply that $F \sim G$. Consider $x = (0.8, 0.3, 0.5)$, $y = (0.6, 0.7, 0.6)$, and

$$R = \begin{bmatrix} 1 & 0.1 & 0.3 \\ 0.7 & 0.2 & 0.9 \\ 0.3 & 0.5 & 1 \end{bmatrix}.$$

It follows that

$$P_{X,Y} = \frac{0.8*0.6 + 0.3*0.7 + 0.5*0.6}{\max(0.8*0.8 + 0.3*0.3 + 0.5*0.5, 0.6*0.6 + 0.7*0.7 + 0.6*0.6)} \doteq 0.82,$$

$$f = \bigwedge(R \alpha x) = (0.8, 0.5, 0.3), \qquad g = \bigwedge(R \alpha y) = (0.6, 0.6, 0.6),$$

$$P_{F,G} = \frac{0.8*0.6 + 0.5*0.6 + 0.3*0.6}{\max(0.8*0.8 + 0.5*0.5 + 0.3*0.3, 0.6*0.6 + 0.6*0.6 + 0.6*0.6)} \doteq 0.89,$$

i.e., $P_{F,G} > P_{X,Y}$, which means that the proximity measure of F and G is greater than that of X and Y.

(P11) $R \sim S$ does not necessarily imply that $F \sim K$. Consider $a = (0.4, 0.9, 0.8)$,

$$R = \begin{bmatrix} 1 & 0.1 & 0.7 \\ 0.7 & 0.2 & 0.9 \\ 0.3 & 0.5 & 1 \end{bmatrix}, \qquad S = \begin{bmatrix} 0.9 & 0.5 & 1 \\ 0.7 & 0.8 & 0.5 \\ 0.2 & 0.4 & 0.7 \end{bmatrix}.$$

Table 1
Properties of similarity measures

Property	M [3]	L [3]	S [3]	W	T	P
$X_{A,B} = X_{B,A}$	Y	Y	Y	Y	Y	Y
$A = B \Leftrightarrow X_{A,B} = 1$	Y	Y	Y	Y	N	Y
$A \cap B = O \Leftrightarrow X_{A,B} = 0$	Y	N	Y	N	Y	Y
$X_{A,\bar{A}} = 1 \Leftrightarrow A = M$	Y	Y	Y	Y	N	Y
$X_{A,\bar{A}} = 0 \Leftrightarrow A = I$ or $A = O$	Y	N	Y	Y	N	Y
$A \sim B \Rightarrow A \cup C \sim B \cup C$	N	Y	Y	N	N	N
$A \sim B \Rightarrow A \cap C \sim B \cap C$	N	Y	N	N	Y	Y
$A \sim B \Rightarrow A \circ R \sim B \circ R$	N	Y	N	N	Y	N
$R \sim S \Rightarrow A \circ R \sim A \circ S$	N	Y	N	N	N	N
$X \sim Y \Rightarrow \bigwedge(R\alpha x) \sim \bigwedge(R\alpha y)$	N	N	N	N	Y	N
$R \sim S \Rightarrow \bigwedge(R\alpha x) \sim \bigwedge(S\alpha x)$	N	N	N	N	Y	N

Y = Yes, N = No.

From (P9), we know that

$$P_{R,S} \doteqdot 0.77.$$

Furthermore, we can get the following results:

$$R\alpha a = \begin{bmatrix} 0.4 & 1 & 1 \\ 0.4 & 1 & 0.8 \\ 1 & 1 & 0.8 \end{bmatrix}, \qquad S\alpha a = \begin{bmatrix} 0.4 & 1 & 0.8 \\ 0.4 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$\bigwedge(R\alpha a) = (0.4, 0.4, 0.8) = f, \qquad \bigwedge(S\alpha a) = (0.4, 0.4, 1) = k,$$

$$P_{F,K} = \frac{0.4 * 0.4 + 0.4 * 0.4 + 0.8 * 1}{\max(0.4 * 0.4 + 0.4 * 0.4 + 0.8 * 0.8, 0.4 * 0.4 + 0.4 * 0.4 + 1 * 1)}$$

$$\doteqdot 0.85,$$

i.e., $P_{F,K} > P_{R,S}$, which means that the proximity measure of F and K is greater than that of R and S.

6. A comparison of properties

Table 1 summarizes the properties of the six measures of similarity of fuzzy values, where three of them are presented in [3], and the others are investigated in this paper. It can be shown that several of these properties are common to all measures, and some properties did not hold for all of them.

7. Conclusion

We have extended the work of [3] to make a comparison of measures of similarity of fuzzy values. It is shown that several properties are common to these measures and several properties do not hold for all of them. This investigation can provide us some useful information to choose a suitable similarity measure in applications of fuzzy sets.

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