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Subsonic Solutions of Hydrodynamic Model for Semiconductors

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This paper is concerned with the existence and uniqueness of the steady-state solution of hydrodynamic model for semiconductor devices. Boundary conditions are prescribed by vorticity on inflow boundary as well as by electron density, temperature, and normal component of electron velocity on whole boundary. If the ambient temperature is large, and if both vorticity on inflow boundary and the variation of density on boundary are small, a unique subsonic solution exists. © 1997 by B.G. Teubner Stuttgart–John Wiley & Sons Ltd.

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1. Introduction

This paper is concerned with the existence and uniqueness of the steady-state solution of hydrodynamic model for semiconductor devices. The model is derived from moments of the Boltzmann's equation, taken over group velocity space. When coupled with the charge conservation equation, it describes the behaviour of small semiconductor devices and accounts for special features such as hot electrons and velocity overshoots. In steady-state case [6], the model consists of the following equations:

$$\nabla \cdot (\rho V) = 0, \tag{1.1}$$

$$(V \cdot \nabla) V + \frac{1}{\mathbf{m}\rho} \nabla(\rho T) - \frac{\mathbf{e}}{\mathbf{m}} \nabla \Psi = -\frac{V}{\tau_a}, \tag{1.2}$$

$$-\frac{2}{3\rho} \nabla \cdot (\beta_1 T_0 \rho \nabla T) + V \nabla T + \frac{2}{3} T \nabla \cdot V - \frac{2\mathbf{m} V^2}{3\tau_a} + \frac{\mathbf{m} V^2}{3\tau_b} + \frac{T - T_0}{\tau_b} = 0, \tag{1.3}$$

$$\Delta \Psi = \frac{\mathbf{e}}{\mathbf{d}} (\rho - Z), \tag{1.4}$$

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in a bounded semiconductor domain $\Omega \subset \mathbb{R}^3$. ρ denotes the electron density, V the average electron velocity, T the temperature in energy units, Ψ the electrostatic potential. Z (a positive function) is the prescribed ion background density. $\mathbf{m}$, $\mathbf{e}$, β_1, T_0, $\mathbf{d}$ are given positive constants. $\mathbf{m}$ is the effective electron mass, $\mathbf{e}$ the electron charge, T_0 the constant ambient temperature, $\mathbf{d}$ the dielectric constant. τ_a and τ_b represent momentum and energy relaxation times, respectively:

$$\tau_a = \beta_2 \frac{T_0}{T}, \quad \tau_b = \beta_2 \frac{T_0}{2T} + \beta_3 \frac{T T_0}{T + T_0} \tag{1.5}$$

for positive constants β_2, β_3. Let $\mathbf{n}$ be the unit outward normal vector on $\partial\Omega$. Define $\Gamma_1 := \{x \in \partial\Omega \mid V \cdot \mathbf{n} < 0\}$ to be the inflow boundary, and $\Gamma_2 := \{x \in \partial\Omega \mid V \cdot \mathbf{n} \geqslant 0\}$ the outflow boundary. The boundary conditions for system (1.1)–(1.4) are

$$\rho|_{\partial\Omega} = \rho_D|_{\partial\Omega}, \quad T|_{\partial\Omega} = T_D|_{\partial\Omega}, \tag{1.6}$$

$$V \cdot \mathbf{n}|_{\Gamma_1 \cup \Gamma_2} = V_D, \quad \operatorname{curl} V|_{\Gamma_1} = w_D. \tag{1.7}$$

Because of (1.1), the following condition should hold:

$$\int_{\Gamma_1 \cup \Gamma_2} \rho_D V_D \, \mathrm{d}s = 0.$$

w_D also need to satisfy some compatibility condition on inflow boundary, which will be explained in section 3.

Existence of solutions for a simplified case of the hydrodynamic model, Euler–Poisson equation, in one or two-dimensional cases have been studied by several researchers [1, 5, 8, 13]. Here we consider the existence and uniqueness of a subsonic solution of hydrodynamic model in the three-dimensional case. Boundary conditions are prescribed by vorticity on inflow boundary as well as by electron density, temperature, and normal component of electron velocity on whole boundary. We show that if ambient temperature is large, and if both vorticity on inflow boundary and the variation of density on boundary are small, a unique subsonic solution exists. The strategy to show these results is to write the model in terms of density, vorticity, potential, temperature, and electrostatic potential. That would result in four elliptic systems and one transport equation. One can show that the new differential equations are equivalent to original hydrodynamic model. By fixed point theorem, we prove the new system has a unique solution, so does the hydrodynamic model. To prove the existence of solution for the transport equation, we need to work on a domain with edges because it allows us to reduce the transport equaion to an initial value problem, which is not the case in a smooth domain.

This paper consists of the following sections. In section 2, notations are recalled. In section 3, we discuss the compatibility condition for w_D on the inflow boundary. In section 4, we derive auxiliary linear systems which are equivalent to the system (1.1)–(1.7). In section 5, useful lemmas are presented. Existence and uniqueness of a subsonic solution for the system (1.1)–(1.7) is showed in section 6 (see Theorem 6.1). Proofs of lemmas in section 5 are given in section 7.

2. Notation

For convenience, we need the following variables: $\rho = \exp(\zeta)$, $\operatorname{curl} V = w$, $T/T_0 = E$, and $T_D/T_0 = E_D$. 'ln' denotes the inverse function of 'exp'. So $\rho = \exp(\zeta)$ means $\ln(\rho) = \zeta$. r.h.s. is the abbreviation of right-hand side. Summation convention is used. c is used to denote various constants. $C^{m,\alpha}(\Omega)$ represents the Hölder space. $W^{m,p}$ denotes the Sobolev space and if $p = 2$, then $W^{m,2}(\Omega) = H^m(\Omega)$. For a function φ, $\varphi_{,i} := \partial\varphi/\partial x_i$. $\operatorname{diam}\Omega$ is the diameter of Ω.

In this paper, we consider the model in a non-smooth domain. More precisely, domain Ω is assumed to be simply connected with one edge $L = \bar{\Gamma}_1 \cap \bar{\Gamma}_2$, where $\bar{\Gamma}_i$ are the closure of Γ_i ($i = 1, 2$). For any point $q \in L$, there is a positive dihedral angle $\theta(q)$ between Γ_1 and Γ_2. $(1.7)_1$ can be written as

$$V \cdot \mathbf{n}|_{\Gamma_1} = V_D|_{\Gamma_1} < 0, \quad V \cdot \mathbf{n}|_{\Gamma_2} = V_D|_{\Gamma_2} \geqslant 0. \tag{2.1}$$

In a neighbourhood of Γ_1 we introduce a curvilinear system of orthonormal co-ordinates. By $\{\boldsymbol{\tau}_1(x), \boldsymbol{\tau}_2(x), \mathbf{n}(x)\}$, we denote the orthonormal basis corresponding to the co-ordinate system in such a way that, for $x \in \Gamma_1$, $\{\boldsymbol{\tau}_1(x), \boldsymbol{\tau}_2(x)\}$ are vectors tangent to Γ_1 and $\mathbf{n}(x)$ is the unit outward normal vector to Γ_1. For fixed μ, v, if we look at the following expressions:

$$(\boldsymbol{\tau}_\mu \cdot \nabla)\, \boldsymbol{\tau}_v = \kappa^i_{\mu v} \boldsymbol{\tau}_i + \kappa_{\mu v} \mathbf{n},$$

$$(\boldsymbol{\tau}_v \cdot \nabla)\, \boldsymbol{\tau}_\mu = \kappa^i_{v\mu} \boldsymbol{\tau}_i + \kappa_{v\mu} \mathbf{n},$$

then one can show

$$\kappa_{\mu v} = \kappa_{v\mu} \quad \text{on } \Gamma_1. \tag{2.2}$$

3. Compatibility condition

We now discuss the compatibility condition for w_D on the inflow boundary. First let us assume $w_D \cdot \mathbf{n}|_{\Gamma_1} = 0$ and $T_D|_{\Gamma_1} = \text{constant}$. Taking the curl of (1.2) and using $(1.5)_1$, then

$$(V \cdot \nabla)w + (\nabla \cdot V)w - (w \cdot \nabla)V + \frac{Tw}{\beta_2 T_0} = \frac{-\nabla T \times V}{\beta_2 T_0} + \frac{\nabla\rho \times \nabla T}{\mathbf{m}\rho} \tag{3.1}$$

where $w := \operatorname{curl} V$ and '$\times$' is the cross product. Next we write w, V as follows:

$$w = w_n \mathbf{n} + w_\mu \boldsymbol{\tau}_\mu, \tag{3.2}$$

$$V = V_n \mathbf{n} + V_\mu \boldsymbol{\tau}_\mu, \tag{3.3}$$

where $\mathbf{n}$, $\boldsymbol{\tau}_\mu$ are the normal and tangential vectors. Summation convention is used. By (3.2) and (3.3),

$$\begin{aligned}(V \cdot \nabla)w &= (V_n \mathbf{n} \cdot \nabla)w + (V_\mu \boldsymbol{\tau}_\mu \cdot \nabla)w \\ &= (V_n \mathbf{n} \cdot \nabla)(w_n \mathbf{n} + w_v \boldsymbol{\tau}_v) + (V_\mu \boldsymbol{\tau}_\mu \cdot \nabla)(w_n \mathbf{n} + w_v \boldsymbol{\tau}_v) \\ &= V_n(\mathbf{n} \cdot \nabla w_n)\mathbf{n} + V_n w_n (\mathbf{n} \cdot \nabla)\mathbf{n} + V_n(\mathbf{n} \cdot \nabla w_v)\boldsymbol{\tau}_v + V_n w_v (\mathbf{n} \cdot \nabla)\boldsymbol{\tau}_v \\ &\quad + V_\mu(\boldsymbol{\tau}_\mu \cdot \nabla w_n)\mathbf{n} + V_\mu w_n(\boldsymbol{\tau}_\mu \cdot \nabla)\mathbf{n} + V_\mu(\boldsymbol{\tau}_\mu \cdot \nabla w_v)\boldsymbol{\tau}_v + V_\mu w_v(\boldsymbol{\tau}_\mu \cdot \nabla)\boldsymbol{\tau}_v.\end{aligned}$$

Note $w_n|_{\Gamma_1} = w_D \cdot \mathbf{n}|_{\Gamma_1} = \mathbf{0}$. If we take inner product of $(V \cdot \nabla)w$ and $\mathbf{n}$ on Γ_1, then

$$\langle (V \cdot \nabla) w, \mathbf{n} \rangle |_{\Gamma_1} = V_n(\mathbf{n} \cdot \nabla w_n) + V_n w_v \langle (\mathbf{n} \cdot \nabla) \boldsymbol{\tau}_v, \mathbf{n} \rangle + V_\mu w_v \langle (\boldsymbol{\tau}_\mu \cdot \nabla) \boldsymbol{\tau}_v, \mathbf{n} \rangle |_{\Gamma_1}. \tag{3.4}$$

By a similar argument we can derive, on boundary Γ_1,

$$\langle (w \cdot \nabla) V, \mathbf{n} \rangle |_{\Gamma_1} = w_\mu (\boldsymbol{\tau}_\mu \cdot \nabla V_n) + w_\mu V_n \langle (\boldsymbol{\tau}_\mu \cdot \nabla) \mathbf{n}, \mathbf{n} \rangle + w_\mu V_v \langle (\boldsymbol{\tau}_\mu \cdot \nabla) \boldsymbol{\tau}_v, \mathbf{n} \rangle |_{\Gamma_1}. \tag{3.5}$$

Now, we take inner product of (3.1) and $\mathbf{n}$ on Γ_1. By (2.2), (3.4) and (3.5), and that $T_D|_{\Gamma_1}$ is constant, we obtain, in Γ_1,

$$V_n(\mathbf{n} \cdot \nabla w_n) + V_n w_\mu \langle (\mathbf{n} \cdot \nabla) \boldsymbol{\tau}_\mu, \mathbf{n} \rangle - w_\mu (\boldsymbol{\tau}_\mu \cdot \nabla V_n) - w_\mu V_n \langle (\boldsymbol{\tau}_\mu \cdot \nabla) \mathbf{n}, \mathbf{n} \rangle |_{\Gamma_1} = 0. \tag{3.6}$$

Since $w_n|_{\Gamma_1} = 0$,

$$0 = \nabla \cdot w|_{\Gamma_1} = (\nabla w_n \cdot \mathbf{n} + \nabla w_\mu \cdot \boldsymbol{\tau}_\mu + w_\mu \nabla \cdot \boldsymbol{\tau}_\mu)|_{\Gamma_1}. \tag{3.7}$$

By (3.6) and (3.7) and $V_n|_{\Gamma_1} < 0$ (because $V_n|_{\Gamma_1} = V_D|_{\Gamma_1} < 0$), we obtain

$$V_n \nabla w_\mu \boldsymbol{\tau}_\mu + V_n w_\mu \nabla \cdot \boldsymbol{\tau}_\mu - V_n w_\mu \langle (\mathbf{n} \nabla) \boldsymbol{\tau}_\mu, \mathbf{n} \rangle + w_\mu \boldsymbol{\tau}_\mu \nabla V_n + w_\mu V_n \langle (\boldsymbol{\tau}_\mu \nabla) \mathbf{n}, \mathbf{n} \rangle = 0.$$

In other words, if $w_D \cdot \mathbf{n}|_{\Gamma_1} = 0$ and $T_D|_{\Gamma_1} =$ constant, then $w_\mu (= w_D \cdot \boldsymbol{\tau}_\mu, \mu = 1, 2)$ have to satisfy the following equation on Γ_1:

$$\nabla \cdot (w_\mu V_D \boldsymbol{\tau}_\mu) - w_\mu V_D (\langle (\mathbf{n} \nabla) \boldsymbol{\tau}_\mu, \mathbf{n} \rangle - \langle (\boldsymbol{\tau}_\mu \nabla) \mathbf{n}, \mathbf{n} \rangle) = 0. \tag{3.8}$$

So w_μ ($\mu = 1, 2$) have to depend on V_D and the geometry of the inflow boundary Γ_1. One trivial solution such that w_D satisfies (3.8) is 0. Also note if w_D ($\neq 0$) is a solution of (3.8), then $c \cdot w_D$ is a solution of (3.8) for any constant c.

4. Auxiliary system

In this section, we derive auxiliary linear systems for (1.1)–(1.7). One can easily see that a solution of system (1.1)–(1.7) corresponds to a fixed point of the new linear systems. Proof for the other direction will be given in section 6.

Taking the curl of (1.2), we get

$$(V \cdot \nabla) w + (\nabla \cdot V) w - (w \cdot \nabla) V + \frac{wE}{\beta_2} = \frac{-\nabla E \times V}{\beta_2} + \frac{T_0 \nabla \rho \times \nabla E}{\mathbf{m} \rho} \tag{4.1}$$

where $w = \operatorname{curl} V$, $E = T/T_0$. Next, taking the divergence of (1.2) and using (1.1), (1.3)–(1.4), we obtain

$$\begin{aligned} \Delta \zeta - \frac{\mathbf{m} V \nabla (V \Delta \zeta)}{T_0 E} - \left(\frac{\mathbf{m}}{\beta_2} + \frac{1}{\beta_1} \right) \frac{V \nabla \zeta}{T_0} - \frac{\mathbf{e}^2}{T_0 E \mathbf{d}} \exp(\zeta) \\ = -\frac{\mathbf{e}^2 Z}{T_0 E \mathbf{d}} - \frac{\mathbf{m} V_{i,j} V_{j,i}}{T_0 \mathbf{E}} - \left(\frac{\mathbf{m}}{\beta_2} + \frac{3}{2\beta_1} \right) \frac{V \nabla E}{T_0 E} + \frac{\mathbf{m} V^2}{\beta_1 \beta_2 T_0^2} \\ - \frac{\mathbf{m} V^2}{2\beta_1 T_0^2 E \tau_b} - \frac{3(E-1)}{2\beta_1 T_0 E \tau_b}, \end{aligned} \tag{4.2}$$

where $\rho = \exp(\zeta)$, $V_{i,j} := \partial V_i / \partial x_j$. We then split V in the following way [4]:

$$V = -\nabla\psi + \sigma \tag{4.3}$$

such that

$$\nabla \cdot \sigma = 0 \quad \text{in } \Omega, \ \sigma \cdot \mathbf{n}|_{\Gamma_1 \cup \Gamma_2} = 0. \tag{4.4}$$

So boundary condition $(1.7)_1$ can be written as

$$\nabla\psi \cdot \mathbf{n}|_{\Gamma_1 \cup \Gamma_2} = -V_D|_{\Gamma_1 \cup \Gamma_2}. \tag{4.5}$$

Equation (1.1) then becomes

$$\nabla \cdot (\rho \nabla \psi) = \nabla \cdot (\rho\sigma), \quad x \in \Omega. \tag{4.6}$$

By (4.1)–(4.6), we now define a map $\mathscr{F}(S, U) = (E, V)$ as follows: Given (S, U), solve

$$\Delta\zeta - \frac{\mathbf{m} U \nabla(U\Delta\zeta)}{T_0 S} - \left(\frac{\mathbf{m}}{\beta_2} + \frac{1}{\beta_1}\right)\frac{U\nabla\zeta}{T_0} - \frac{\mathbf{e}^2}{T_0 S \mathbf{d}}\exp(\zeta)$$
$$= -\frac{\mathbf{e}^2 Z}{T_0 S \mathbf{d}} - \frac{\mathbf{m} U_{i,j} U_{j,i}}{T_0 \mathbf{S}} - \left(\frac{\mathbf{m}}{\beta_2} + \frac{3}{2\beta_1}\right)\frac{U\nabla S}{T_0 S} + \frac{\mathbf{m}U^2}{\beta_1\beta_2 T_0^2} - \frac{\mathbf{m}U^2}{2\beta_1 T_0^2 S\tau_b}$$
$$- \frac{3(S-1)}{2\beta_1 T_0 S \tau_b} := \mathscr{F}_{\text{rhs}}(S, U), \tag{4.7}$$

$$\zeta|_{\partial\Omega} = \ln(\rho_D)|_{\partial\Omega}, \tag{4.8}$$

where $\tau_b = \tau_b(S)$. Then, using ζ from (4.7), compute ρ from

$$\rho = \exp(\zeta). \tag{4.9}$$

Then, by ρ above, we solve

$$\Delta E + \frac{\nabla\rho\nabla E}{\rho} - \frac{3U\nabla E}{2\beta_1 T_0} - \frac{3E}{2\beta_1 T_0 \tau_b}$$
$$= \frac{-\mathbf{m}U^2 S}{\beta_1\beta_2 T_0^2} - \frac{U\nabla\rho}{\beta_1 T_0 \rho} S - \frac{3}{2\beta_1 T_0 \tau_b} + \frac{\mathbf{m}U^2}{2\beta_1 T_0^2 \tau_b}, \tag{4.10}$$

$$E|_{\partial\Omega} = T_D / T_0|_{\partial\Omega}, \tag{4.11}$$

where $\tau_b = \tau_b(S)$. Next, by ρ, E from (4.9)–(4.10), we solve the following, for w and P,

$$(U \cdot \nabla)w + (\nabla \cdot U)w - (w \cdot \nabla)U + \frac{wE}{\beta_2} + \nabla P = \frac{-\nabla E \times U}{\beta_2} + \frac{T_0 \nabla\rho \times \nabla E}{\mathbf{m}\rho}, \tag{4.12}$$

$$\nabla \cdot w = 0, \tag{4.13}$$

$$w|_{\Gamma_1} = w_D, \quad \nabla P \cdot \mathbf{n}|_{\Gamma_1} = 0, \quad P|_{\Gamma_1} = 0. \tag{4.14}$$

Variable P and boundary conditions $(4.14)_{2,3}$ are introduced to let w be divergence free. Next, we solve the following system for σ:

$$\operatorname{curl} \sigma = w, \quad x \in \Omega, \tag{4.15}$$

$$\nabla \cdot \sigma = 0, \quad x \in \Omega, \tag{4.16}$$

$$\sigma \cdot \mathbf{n} = 0, \quad x \in \Gamma_1 \cup \Gamma_2. \tag{4.17}$$

Then compute ψ as a solution of the following system:

$$\nabla \cdot (\rho \nabla \psi) = \nabla \cdot (\rho \sigma), \quad x \in \Omega, \tag{4.18}$$

$$\rho \nabla \psi \cdot \mathbf{n} = -\rho_D V_D, \quad x \in \Gamma_1 \cup \Gamma_2, \tag{4.19}$$

$$\int_\Omega \psi \, \mathrm{d}x = 0. \tag{4.20}$$

Finaly, we can compute the velocity V from

$$V = -\nabla \psi + \sigma. \tag{4.21}$$

Existence of a fixed point of the operator $\mathscr{F}$ will be shown in section 6. (4.7)–(4.21) forms auxiliary linear systems for the system (1.1)–(1.7). From the derivation, we see that a solution of system (1.1)–(1.7) corresponds to a fixed point of the linear systems (4.7)–(4.21).

5. Auxiliary lemma

In this section, we present four lemmas (proofs are lengthy and will be given in Section 7). They are used to prove the existence of a fixed point for systems (4.7)–(4.21). Part of results can be found in Reference 14.

Domain Ω considered in this section is a smooth simply-connected domain with edge and $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$ (section 2). Lemma 5.1 is a result for a linear elliptic equation and is for solving (4.10)–(4.11). Lemma 5.2 is an existence theorem of a semilinear elliptic equation in Ω, used to solve (4.7) and (4.8). Lemma 5.3 is to establish an existence theorem for a linear elliptic equation with Neuman boundary condition for computing a solution for (4.18)–(4.20). Lemma 5.4 is an existence theorem for a transport equation in Ω and is for solving (4.12)–(4.14).

Lemma 5.1. *Consider the equation*

$$\begin{cases} a_{ij}(x)\varphi_{,ij} + a_i(x)\varphi_{,i} + a(x)\varphi = f(x), & x \in \Omega, \\ \eta_k \varphi + (1-\eta_k)\nabla\varphi \cdot \mathbf{n} = 0, & on\ \Gamma_k (k = 1, 2), \end{cases} \tag{5.1}$$

where η_k *is either* 0 *or* 1 *and* $\eta_1 + \eta_2 \neq 0$. *If the following conditions hold*:

1. $a_{ij}, a_i, a, f \in C^{m,\alpha}(\bar{\Omega})$, $a(x) \leqslant 0$, $0 < \alpha < 1$, $0 \leqslant m$,
2. $a_{ij} = a_{ji}$, $a_{ij}(x)\xi_i\xi_j \geqslant \lambda|\xi|^2$ *for* $\xi \in \mathbb{R}^2$ *and for some positive constant* λ,
3. $\omega(q) < (\eta_1 + \eta_2)\pi/(m + 2 + \alpha)$ *for all* $q \in \bar{\Gamma}_1 \cap \bar{\Gamma}_2$ (see Remark 1),

then there exists a unique solution $\varphi \in C^{m+2,\alpha}(\bar{\Omega})$ *and*

$$\| \varphi \|_{C^{m+2,\alpha}(\bar{\Omega})} \leqslant c(\lambda, \| a_{ij}, a_i, a \|_{C^{m,\alpha}(\bar{\Omega})}) \| f \|_{C^{m,\alpha}(\bar{\Omega})}. \tag{5.2}$$

Proof. Existence of the solution is the Theorem 1 of [2]. (5.2) is obtained by tracing the proof of Theorem 1 of [2]. □

Remark 1. $\omega(q)$ of condition 3 is obtained as follows: In section 2, we assume domain Ω has one edge $L = \bar{\Gamma}_1 \cap \bar{\Gamma}_2$. For all $q \in L$, we denote by $R_1(q)$ and $R_2(q)$ the two one-sided tangential planes which touch $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ at point q. For a fixed point $q \in L$ we transform to canonical form the second-order term of the system $(5.1)_1$

$$a_{ij}(q)\varphi_{,ij} = 0.$$

Since point q is fixed, this is an equation with constant coefficients. After the transformation, the planes $R_1(q)$ and $R_2(q)$ will be transformed to other planes that intersect at an angle $\omega(q)$.

Lemma 5.2. *Consider the equation*

$$\begin{cases} a_{ij}(x)\varphi_{,ij} + a_i(x)\varphi_{,i} - a(x)g(\varphi) = -f(x), & x \in \Omega, \\ \varphi|_{\partial\Omega} = \varphi_D|_{\partial\Omega}. \end{cases} \tag{5.3}$$

If the following conditions hold:

1. $a_{ij}, a_i, a, f \in C^{m,\alpha}(\bar{\Omega})$, $0 < a(x)$, $0 < \alpha < 1$, $0 \leqslant m$,
2. $a_{ij} = a_{ji}$, $a_{ij}(x)\xi_i\xi_j \geqslant \lambda|\xi|^2$ *for* $\xi \in \mathbb{R}^2$ *and for some positive constant* λ,
3. $g \in C^{m+1}(\mathbb{R})$, $g' > 0$, *there are constants* $\underline{\varphi}$, $\bar{\varphi}$ *satisfying* $g(\underline{\varphi}) \leqslant f/a(x) \leqslant g(\bar{\varphi})$,
4. $\varphi_D \in C^{m+2,\alpha}(\bar{\Omega})$, $\varphi_1 \leqslant \varphi_D(x) \leqslant \varphi_2$ *for all* $x \in \partial\Omega$,
5. $\omega(q) < 2\pi/(m+2+\alpha)$ *for all* $q \in \bar{\Gamma}_1 \cap \bar{\Gamma}_2$ (see Remark 1),

then there exists a unique solution $\varphi \in C^{m+2,\alpha}(\bar{\Omega})$ *satisfying*

$$\min(\varphi_1, \underline{\varphi}) \leqslant \varphi(x) \leqslant \max(\varphi_2, \bar{\varphi}) \ \forall x \in \Omega,$$

$$\| \varphi \|_{C^{m+2,\alpha}} \leqslant c(\lambda \| a_{ij}, a_i, a \|_{C^{m,\alpha}}) \mathscr{P}(\| \varphi_D \|_{C^{m+2,\alpha}}, \| f \|_{C^{m,\alpha}}, \| g \|_{C^{m+1}(\varpi_1)} + \| g^{-1} \|_{C^0(\varpi_2)}),$$

where $\mathscr{P}$ *is a polynomial with* $\mathscr{P}(0,0,0) = 0$ *and* ϖ_1, ϖ_2 *denote ranges of* $\varphi, f/a$ *over* Ω.

Lemma 5.3. *Consider the equation*

$$\begin{cases} \nabla \cdot (a(x)\nabla\varphi) = f & \text{in } \Omega, \\ a(x)\nabla\varphi \cdot \mathbf{n} = g & \text{on } \Gamma_1 \cup \Gamma_2, \\ \displaystyle\int_\Omega \varphi(x)\,\mathrm{d}x = 0. \end{cases} \tag{5.4}$$

If the following conditions are satisfied:

1. $a \in C^{m+1,\alpha}(\bar{\Omega})$, $0 \leqslant m$, $0 < \lambda < a(x)$ *for some positive constant* λ,
2. $f \in W^{m,p}(\Omega)$, $g \in W^{m+1-1/p,p}(\Gamma_i)$, $\int_\Omega f\,\mathrm{d}x + \sum_{i=1}^2 \int_\Gamma g\,\mathrm{d}s = 0$, $p \geqslant 2$,
3. $\theta(q) < \pi/(m+2-2/p)$ *for all* $q \in \bar{\Gamma}_1 \cap \bar{\Gamma}_2$ (see section 2 for $\theta(q)$),

then there is a unique solution $\varphi \in W^{m+2,p}(\Omega)$ *satisfying*

$$\|\varphi\|_{W^{m+2,p}(\Omega)} \leqslant c(\|a\|_{C^{m+1,\alpha}(\bar{\Omega})}, 1/\lambda)\left(\|f\|_{W^{m,p}(\Omega)} + \sum_{i=1}^{2} \|g\|_{W^{m+1-1/p,p}(\bar{\Gamma}_i)}\right). \tag{5.5}$$

In next lemma, $\{b, U, f, g\}$ are vector functions and Λ is a matrix function.

Lemma 5.4. *Consider the system*

$$(b\cdot\nabla)U + \Lambda U = f \quad \text{in } \Omega,$$

$$U = g \quad \text{on } \Gamma_1.$$

If the following conditions are satisfied:

1. $b \in W^{\ell+1,p}(\Omega)$, $b\cdot\mathbf{n}|_{\Gamma_1} < -\lambda < 0$, $b\cdot\mathbf{n}|_{\Gamma_2} \geqslant 0$, $3 < p < 4$, $2 \leqslant \ell$,
2. $\Lambda, f \in W^{\ell,p}(\Omega)$, $g \in W^{\ell,p}(\Gamma_1)$, Λ *is a positive-definite matrix*,
3. $\Lambda(x) > \Lambda_m I$ *in* Ω, $\Lambda_m - \kappa(p, \|b\|_{W^{\ell+1,p}(\Omega)}, 1/\lambda) = \Lambda_d > 0$,

where κ *is a continuous positive function of its orguments, I is identity matrix, and* λ, Λ_m, Λ_d *are some positive constants, then there exists a unique solution* $U \in W^{\ell,p}(\Omega)$ *and, for* $0 \leqslant s \leqslant \ell$,

$$\|U\|^p_{W^{s,p}(\Omega)} \leqslant c(\|\Lambda\|_{W^{\ell,p}(\Omega)}, \|b\|_{W^{\ell+1,p}(\Omega)}, 1/\lambda, 1/\Lambda_d)(\|f\|^p_{W^{s,p}(\Omega)} + \|g\|^p_{W^{s,p}(\Gamma_1)}). \tag{5.6}$$

6. Existence and uniqueness of a subsonic solution

In this section, we prove existence and uniqueness of a subsonic solution of (1.1)–(1.7). To do this, we first define a set $\mathscr{A} \times \mathscr{D}$ (see Remark 2), and show that operator $\mathscr{F}$ defined by (4.7)–(4.21) is a map from $\mathscr{A} \times \mathscr{D}$ to itself (see Lemmas 6.1–6.5). Next, we show that the map $\mathscr{F}$ is continuous in some weaker space (see Lemma 6.6). Then by fixed point theorem, we conclude that a fixed point of (4.7)–(4.21) exists. Moreover, we see that if the ambient temperature is large and if the variation of density on boundary is small, a unique fixed point exists. Then we show a fixed point of system (4.7)–(4.21) corresponds to a solution of (1.1)–(1.7) (see Theorem 6.1). Now let us make the following assumptions:

(A1) $3 < p < 4$ and $0 < \alpha < 1$ such that $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\bar{\Omega})$ (continuous imbedding),

(A2) $\Omega \subset R^3$ is a smooth simply-connected domain with edge $L := \bar{\Gamma}_1 \cap \bar{\Gamma}_2$; $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$; $0 < \theta(q) < \pi/(5 - 2/p)$ and $\theta(q) \neq \pi/j$ for all $q \in L$ and $j \in N$ (positive integer),

(A3) $Z \in C^{1,\alpha}(\bar{\Omega})$, $0 < 2\rho_1 \leqslant Z(x) \leqslant \rho_2/2$ for all $x \in \Omega$,

(A4) $\rho_D \in C^{3,\alpha}(\bar{\Omega})$, $0 < \rho_1 \leqslant \rho_D(x) \leqslant \rho_2$ for all $x \in \Omega$,

(A5) $V_D \in W^{3-1/p,p}(\Gamma_i)$, $\sum_{i=1}^{2} \int_{\Gamma_i} \rho_D V_D \, ds = 0$, $V_D|_{\Gamma_1} < -v_d < 0 \leqslant V_D|_{\Gamma_2}$,

(A6) $w_D \in W^{2,p}(\Gamma_1)$, $w_D \cdot \mathbf{n}|_{\Gamma_1} = 0$ and $w_\mu := w_D \cdot \boldsymbol{\tau}_\mu|_{\Gamma_1}$ ($\mu = 1, 2$) satisfy (3.8),

(A.7) $T_D \in C^{3,\alpha}(\bar{\Omega})$, $0 < E_1 \leqslant T_D(x)/T_0$ for all $x \in \Omega$, $T_D|_{\Gamma_1}$ and T_0 are positive constants.

Remark 2. For constants $\mathbf{c}_m(>1)$ and U_m, we define $\mathscr{A}$ and $\mathscr{D}$ as follows:

$$\mathscr{A} = \{S \in C^{2,\alpha}(\bar{\Omega}) \mid S|_{\partial\Omega} = T_D/T_0|_{\partial\Omega},\ \|\nabla S\|_{C^{1,\alpha}(\bar{\Omega})} \leqslant E_m := E_1/\mathbf{c}_m\},$$

$$\mathscr{D} = \{U \in W^{3,p}(\Omega) \mid U \cdot \mathbf{n}|_{\Gamma_1 \cup \Gamma_2} = V_D,\ \|U\|_{W^{3,p}(\Omega)} \leqslant U_m\}.$$

To show that operator $\mathscr{F}$ is a map from $\mathscr{A} \times \mathscr{D}$ to itsel, constant E_m, U_m cannot be chosen arbitrary. As we will see below, $E_m, U_m, \sum_{i=1}^{2} \|V_D\|_{W^{3-1/p,p}(\Gamma_i)}$, and E_1 are related to each other. We state our main result below:

Theorem 6.1. *Under* (A1)–(A7), *there exist two continuous functions* κ_1, κ_2 *such that if*

1. *The following hold for* $E_m, U_m, \sum_i \|V_D\|_{W^{3-1/p,p}(\Gamma_i)}$, *and* E_1:
   (a) *There is a constant* ρ_3 *satisfying* $0 < \rho_3 \leqslant S(x)\ \forall x \in \Omega,\ S \in \mathscr{A}$,
   (b) $0 < \rho_1 \leqslant -T_0 S\mathrm{d}/\mathbf{e}^2 \mathscr{F}_{\mathrm{rhs}}(S, U) \leqslant \rho_2, \quad \forall (S, U) \in \mathscr{A} \times \mathscr{D}$ (see (4.7) for $\mathscr{F}_{\mathrm{rhs}}(S, U)$),
   (c) $\kappa_1(1/\rho_1, \|\ln \rho_D\|_{C^{3,\alpha}}, \|\rho_D\|_{C^{3,\alpha}}, \|Z\|_{C^{1,\alpha}}) \sum_{i=1}^{2} \|V_D\|_{W^{3-1/p,p}(\Gamma_i)} < U_m$,
   (d) *There is a constant* $\mathbf{k}_s$ *satisfying* $0 < \mathbf{k}_s < E_1(1 - \operatorname{diam} \Omega/\mathbf{c}_m)/\beta_2 - \kappa_2(U_m, 1/v_d)$,
2. $1/T_0 + \|\nabla \ln(\rho_D)\|_{C^{2,\alpha}(\bar{\Omega})} + \|w_D\|_{W^{2,p}(\Gamma_1)}$ *is small, then a unique subsonic solution of system* (1.1)–(1.7) *exists. In other words, there exists* (T, V, ρ, Ψ) *uniquely such that* (1) $\rho, T \in C^{3,\alpha}(\bar{\Omega})$, $V, \Psi \in W^{3,p}(\Omega)$; (2) (T, V, ρ, Ψ) *satisfies* (1.1)–(1.7), *and* (3) $|V|^2(x) \leqslant T(x)/\mathbf{m}$ *for all* $x \in \Omega$.

Note that if $E_m, U_m, \sum_i \|V_D\|_{W^{3-1/p,p}(\Gamma_i)}$ are small, and if E_1 is large, then the four constants satisfy 1. (*a*), (*b*), (*c*), (*d*). To show that operator $\mathscr{F}$ is a map from $\mathscr{A} \times \mathscr{D}$ to itself, let us first consider the solvability of (4.7)–(4.9).

Lemma 6.1. *Under* (A1)–(A7), *there exist constants* E_m, U_m *such that as* $1/T_0$ *is small enough,* (4.7)–(4.9) *has a unique solution* $\rho = \exp(\zeta) \in C^{3,\alpha}(\bar{\Omega})$ *for all* $(S, U) \in \mathscr{A} \times \mathscr{D}$. *Moreover,*

$$0 < \rho_1 \leqslant \rho(x), \quad \forall x \in \Omega, \tag{6.1}$$

$$\|\rho\|_{C^{3,\alpha}(\bar{\Omega})} \leqslant k_3\left(\|\ln(\rho_D)\|_{C^{3,\alpha}}, \|\rho_D, \ln(Z), Z\|_{C^0}, \frac{\|Z\|_{C^{1,\alpha}}}{T_0 E_1}, \frac{U_m}{T_0 E_1}, \frac{E_m}{T_0 E_1}\right), \tag{6.2}$$

$$\left\|\frac{\nabla \rho}{\rho}\right\|_{C^{2,\alpha}(\bar{\Omega})} \leqslant k_4\left(U_m, E_m, \frac{1}{E_1}, \|\ln(\rho_D)\|_{C^{3,\alpha}}, \|Z, \ln Z\|_{C^{1,\alpha}}\right) \times \left(\frac{1}{T_0} + \|\nabla \ln(\rho_D)\|_{C^{2,\alpha}}\right), \tag{6.3}$$

where k_3, k_4 *are continuous functions of its arguments.*

Proof. Let us take constants E_m, U_m such that the following hold:

1. There exists a constant ρ_3 satisfying $0 < \rho_3 \leqslant S(x)\ \forall x \in \Omega, S \in \mathscr{A}$, (6.4)
2. $0 < \rho_1 \leqslant (-T_0 S\mathrm{d}/\mathbf{e}^2)\mathscr{F}_{\mathrm{rhs}}(S, U) \leqslant \rho_2, \forall (S, U) \in \mathscr{A} \times \mathscr{D}$
   (see (4.7) for $\mathscr{F}_{\mathrm{rhs}}$). (6.5)

Next, we look at the second-order terms of (4.7). Setting $A_{ij} = \delta_{ij} - \mathbf{m}U_iU_j/T_0S$, then the eigenvalues of (A_{ij}) are 1, 1, $1 - \mathbf{m}|U|^2/T_0S$. If $1/T_0$ is small enough, then

$$1 - \mathbf{m}|U|^2(x)/(T_0S) \geqslant \underline{\rho}_s > 0 \text{ for } (, U) \in \mathscr{A} \times \mathscr{D} \text{ and some constant } \underline{\rho}_s. \quad (6.6)$$

Also if $1/T_0$ is small enough, by (A1) and (A2) and Remark 1, $\omega(q) < 2\pi/(3+\alpha)$ for $q \in \bar{\Gamma}_1 \cup \bar{\Gamma}_2$. By Lemma 5.2, (4.7) and (4.8) has a unique solution $\zeta \in C^{3,\alpha}(\bar{\Omega})$ and ζ satisfies

$$\ln(\rho_1) \leqslant \zeta(x) \leqslant \ln(\rho_2), \quad (6.7)$$

$$\|\zeta\|_{C^{3,\alpha}(\bar{\Omega})} \leqslant c_1\left(\|\ln(\rho_D)\|_{C^{3,\alpha}}, \|\rho_D, \ln(Z), Z\|_{C^0}, \frac{\|Z\|_{C^{1,\alpha}}}{T_0E_1}, \frac{U_m}{T_0E_1}, \frac{E_m}{T_0E_1}\right). \quad (6.8)$$

(6.7) and (6.8) imply (6.1) and (6.2), respectively. Since $\nabla\rho/\rho = \nabla\zeta$, we obtain (6.3) by (6.8) and Lemma 5.2. □

Next, we consider the solvability of (4.10) and (4.11).

Lemma 6.2. *Under the same assumptions as Lemma* 6.1, (4.10) *and* (4.11) *has a unique solution* $E \in C^{3,\alpha}(\bar{\Omega})$ *for all* $(S, U) \in \mathscr{A} \times \mathscr{D}$ *and* ρ *from* (4.9). *Moreover,*

$$\|\nabla E\|_{C^{2,\alpha}(\bar{\Omega})} \leqslant k_5(\|\nabla\rho/\rho\|_{C^{1,\alpha}}, U_m, \|E_D\|_{C^{1,\alpha}})(\|\nabla E_D\|_{C^{2,\alpha}} + 1/T_0), \quad (6.9)$$

where k_5 *is continuous function of its arguments.*

Proof. By (A1)–(A2) and Lemma 5.1, (4.10) and (4.11) has a unique solution $E \in C^{3,\alpha}(\bar{\Omega})$ and

$$\|E - E_D\|_{C^{3,\alpha}(\bar{\Omega})} \leqslant c_3(\|\nabla\rho/\rho\|_{C^{2,\alpha}}, U_m, \|E_D\|_{C^{1,\alpha}})(\|\nabla E_D\|_{C^{2,\alpha}} + 1/T_0),$$

which implies (6.9). □

By $U \in \mathscr{D}$ and ρ, E from above two lemmas, we are ready to solve (4.12)–(4.14). Let us also define an operator $M(\tilde{w}) = w$ as follows: Given $\tilde{w}$, solve the following for P;

$$\Delta P = -\nabla E \cdot \tilde{w}/\beta_2 - \nabla \cdot (\nabla E \times U/\beta_2), \quad x \in \Omega, \quad (6.10)$$

$$\nabla P \cdot \mathbf{n}|_{\Gamma_1} = 0. \quad P|_{\Gamma_2} = 0 \quad (6.11)$$

By P from (6.10) and (6.11), compute from

$$(U\cdot\nabla)w + (\nabla\cdot U)w - (w\cdot\nabla)U + \frac{Ew}{\beta_2} = -\nabla P - \frac{\nabla E \times U}{\beta_2} + \frac{T_0\nabla\rho \times \nabla E}{\mathbf{m}\rho}, \quad (6.12)$$

$$w|_{\Gamma_1} = w_D. \quad (6.13)$$

Lemma 6.3. *Besides the assumptions of* Lemma 6.1, *if* $\nabla\cdot U/2 + Ew/\beta_2 > 0$ *in* Ω, *then a solution of* (4.12)–(4.14) *is a fixed point of* (6.10)–(6.13) *and vice versa.*

Proof. This lemma only shows the equivalence between a solution of (4.12)–(4.14) and a fixed point of (6.10)–(6.13). Existence of a fixed point of (6.10)–(6.13) is in next lemma.

Taking the divergence of (4.12), we get (6.10). So one side is done. The other side is equivalent to show that $\nabla \cdot w = 0$ in Ω. Taking inner product of (6.12) and $\mathbf{n}$ on Γ_1, we obtain (by (A6)–(A7), (6.11), and using a similar argument in section 3)

$$U_n(\mathbf{n} \cdot \nabla w_n) + U_n w_\mu \langle (\mathbf{n} \cdot \nabla)\tau_\mu, \mathbf{n} \rangle - w_\mu(\tau_\mu \cdot \nabla U_n) - w_\mu U_n \langle (\tau_\mu \cdot \nabla)\mathbf{n}, \mathbf{n} \rangle |_{\Gamma_1} = 0,$$

where $U_n := U \cdot \mathbf{n}$. By (A6) (i.e. (3.8)), we see $\nabla \cdot w|_{\Gamma_1} = 0$. Taking the divergence of (6.12), we obtain, by (6.10),

$$U\nabla(\nabla \cdot w) + (\nabla \cdot U + E/\beta_2)\nabla \cdot w = 0 \quad \text{in } \Omega, \tag{6.14}$$

$$\nabla \cdot w|_{\Gamma_1} = 0. \tag{6.15}$$

Multiplying (6.14) by $\nabla \cdot w$ and doing integration by parts, we see that (6.14)–(6.15) implies $\nabla \cdot w = 0$ in Ω because $\nabla \cdot U/2 + Ew/\beta_2 > 0$. So a fixed point of (6.10)–(6.13) is a solution of system (4.12)–(4.14). □

We now prove the solvability of the system (4.12)–(4.14).

Lemma 6.4. *Under* (A1)–(A7), *there is a continuous function* κ_2 *such that if*

$$0 < \mathbf{k}_s < (1 - \operatorname{diam}\Omega/\mathbf{c}_m)E_1/\beta_2 - \kappa_2(U_m\ 1/v_d) \text{ for some constant } \mathbf{k}_s \tag{6.16}$$

(E_m, U^m are chosen as Lemma 6.1 and $\mathbf{c}_m$ is defined in Remark 2) *and if* $1/T_0$ *is small, then* (4.12)–(4.14) *has a unique solution* $w \in W^{2,p}(\Omega)$ *and*

$$\|w\|_{W^{2,p}(\Omega)} \leqslant k_6\left(U_m, \frac{1}{v_d}, \frac{1}{\mathbf{k}_d}, \|E_D, \ln\rho_D\|_{C^{3,\alpha}(\bar{\Omega})}, \|\ln(Z), Z\|_{C^{1,\alpha}}, \frac{1}{E_1}, \|\nabla T_D\|_{C^{2,\alpha}(\bar{\Omega})}\right)$$
$$\times(1/T_0 + \|\nabla\ln(\rho_D)\|_{C^{2,\alpha}(\bar{\Omega})} + \|w_D\|_{W^{2,p}(\Gamma_1)}), \tag{6.17}$$

where k_6 *is a continuous function of its arguments.*

Proof. By Lemma 6.3, it is equivalent to proving the existence of a fixed point of system (6.10)–(6.13). Given $\tilde{w} \in W^{2,p}(\Omega)$, by (A1)–(A2) and Lemma 5.1 (6.10) and (6.11) is uniquely solvable, and

$$\|P\|_{C^{3,\alpha}} \leqslant c_4\|\nabla E\|_{C^{2,\alpha}}(\|\tilde{w}\|_{W^{2,p}} + \|U\|_{W^{3,p}}). \tag{6.18}$$

By Lemma 5.4, (A5) and (6.16), we see (6.12) and (6.13) has a unique solution $w \in W^{2,p}(\Omega)$ and

$$\|w\|_{W^{2,p}(\Omega)} \leqslant c_5(\|E\|_{W^{2,p}}, U_m, 1/v_d, 1/\mathbf{k}_s)(\|\text{r.h.s. of (6.12)}\|_{W^{2,p}(\bar{\Omega})} + \|w_D\|_{W^{2,p}(\Gamma_1)}). \tag{6.19}$$

So we can define a map $M: W^{2,p}(\Omega) \to W^{2,p}(\Omega)$ by $M(\tilde{w}) = w$.

Next, we claim M is a contractive map. If w_1^*, w_2^* are given and if $M(w_1^*) = w_1$, $M(w_2^*) = w_2$, then by (6.10) and (6.11).

$$\| P_1 - P_2 \|_{C^{3,\alpha}} \leqslant c_6 \| \nabla E \|_{C^{2,\alpha}} \| w_1^* - w_2^* \|_{W^{2,p}}. \tag{6.20}$$

By (6.12), (6.13) and (6.20),

$$\| w_1 - w_2 \|_{W^{2,p}} \leqslant c_7(\| E \|_{W^{2,p}}, U_m, 1/v_d, 1/\mathbf{k}_s) \| \nabla E \|_{C^{2,\alpha}} \| w_1^* - w_2^* \|_{W^{2,p}}.$$

Because $1/T_0$ is small, M is a contractive map.

Therefore, the fixed point of (6.10)–(6.13) exists uniquely. If w is the fixed point of (6.10)–(6.13), by (6.19), we see w satisfies (6.17). □

Next, we prove operator $\mathscr{F}$ is a map from $\mathscr{A} \times \mathscr{D}$ to itself.

Lemma 6.5. *Under* (A1)–(A7), *there exist two continuous functions* κ_1, κ_2 *such that if*

1. E_m, U_m (chosen as Lemma 6.1) $\sum_{i=1}^{2} \| V_D \|_{W^{3-1/p,p}(\Gamma_i)}$ *and* E_1 *satisfy*
   (a) $\kappa_1(1/\rho_1, \| \ln \rho_D \|_{C^{3,\alpha}}, \| \rho_D \|_{C^{3,\alpha}}, \| Z \|_{C^{1,\alpha}}) \sum_{i=1}^{2} \| V_D \|_{W^{3-1/p,p}(\Gamma_i)} < U_m$,
   (b) (6.16) *hold*,
2. $1/T_0 + \| \nabla \ln(\rho_D) \|_{C^{2,\alpha}(\bar\Omega)} + \| w_D \|_{W^{2,p}(\Gamma_1)}$ *is small*,
3. $\| \nabla T_D \|_{C^{2,\alpha}(\bar\Omega)}$ *is bounded*,

the operator $\mathscr{F}$ *of* (4.7)–(4.21) *is a map from* $\mathscr{A} \times \mathscr{D}$ *to itself.*

Proof. For any $(S, U) \in \mathscr{A} \times \mathscr{D}$, if $1/T_0$ is small by Lemma 6.1, (4.7)–(4.9) has a unique solution $\rho \in C^{3,\alpha}(\bar\Omega)$ and (6.1)–(6.3) hold. If $1/T_0$ is small and if $\| \nabla T_D \|_{C^{2,\alpha}(\bar\Omega)}$ is bounded, by Lemma 6.2, (4.10)–(4.11) has a unique solution and

$$\| \nabla E \|_{C^{2,\alpha}(\bar\Omega)} \leqslant E_m, \quad E > 0, \tag{6.21}$$

i.e. $E \in \mathscr{A}$. By Lemma 6.4, system (4.12)–(4.14) has a unique solution $w \in W^{2,p}(\Omega)$ and w satisfies (6.17). With w from (4.12)–(4.14), by Theorem 10.3 of [15] and (A2), the system (4.15)–(4.17) is uniquely solvable, $\sigma \in W^{3,p}(\Omega)$, and

$$\| \sigma \|_{W^{3,p}(\Omega)} \leqslant c_8 \| w \|_{W^{3,p}(\Omega)}. \tag{6.22}$$

Next, we consider (4.18)–(4.20). By (A5), Lemmas 5.3, 6.1, and let $1/T_0$ small, we obtain

$$\| \psi \|_{W^{4,p}(\Omega)} \leqslant c_9(1/\rho_1, \| \rho \|_{C^{3,\alpha}})(\| \nabla \rho \sigma \|_{W^{2,p}(\Omega)} + \sum_i \| \rho_D V_D \|_{W^{3-1/p,p}(\Gamma_i)}). \tag{6.23}$$

Let $V = -\nabla \psi + \sigma$. By (6.22)–(6.23), if $1/T_0$ is small enough,

$$\| V \|_{W^{3,p}} \leqslant k_1(1/\rho_1, \| \ln \rho_D \|_{C^{3,\alpha}}, \| \rho_D \|_{C^{3,\alpha}}, \| Z \|_{C^{1,\alpha}})(\| w \|_{W^{2,p}} + \sum_i \| V_D \|_{W^{3-1/p,p}(\Gamma_i)}).$$

By (6.17) and above assumptions, as long as $1/T_0 + \| \nabla \ln(\rho_D) \|_{C^{2,\alpha}(\bar{\Omega})} + \| w_D \|_{W^{2,p}(\Gamma_1)}$ is small enough,

$$\| V \|_{W^{3,p}} \leqslant U_m. \tag{6.24}$$

So $V \in \mathscr{D}$. By (6.21) and (6.24), we conclude that $\mathscr{F}$ maps $\mathscr{A} \times \mathscr{D}$ to itself. □

Next we show $\mathscr{F}$ is a continuous map in $C^{1,\alpha} \times W^{2,p}$. If it is true, by Schauder fixed point theorem we know the operator $\mathscr{F}$ has a fixed point in $\mathscr{A} \times \mathscr{D}$.

Lemma 6.6. *By assumptions of* Lemma 6.5, *$\mathscr{F}$ is a continuous map in $C^{1,\alpha} \times W^{2,p}$. $\mathscr{F}$ is a contractive map if* $1/T_0 + \| \nabla \ln(\rho_D) \|_{C^{2,\alpha}(\bar{\Omega})}$ *is even smaller.*

Proof. This is shown by a straightforward way, so we only sketch the proof. First let us given a notation:

$$\mathbf{c}_{11} = 1/T_0 + \| \nabla \ln(\rho_D) \|_{C^{2,\alpha}(\bar{\Omega})},$$

$\mathbf{c}_{12}$ is a constant which depending on $\| \rho_D, E_D \|_{C^{3,\alpha}(\bar{\Omega})}$, $U_m, 1/\rho_1, 1/E_1, \| Z \|_{C^{1,\alpha}(\bar{\Omega})}$, $1/v_d$, $1/\mathbf{k}_s$.

Given (S_a, U_a), (S_b, U_b), by solving (4.7)–(4.21), we obtain $(\zeta_a, \rho_a, E_a, w_a, \sigma_a, \psi_a, V_a)$ and $(\zeta_b, \rho_b, E_b, w_b, \sigma_b, \psi_b, V_b)$. $\mathscr{F}(S_a, U_a) = (E_a, V_a)$, $\mathscr{F}(S_b, U_b) = (E_b, V_b)$. By (4.7)–(4.9), we can derive

$$\| \zeta_a - \zeta_b \|_{C^{2,\alpha}} \leqslant \frac{\mathbf{c}_{12}}{T_0} [\| S_a - S_b \|_{C^{1,\alpha}} + \| U_a - U_b \|_{C^{1,\alpha}}]. \tag{6.25}$$

By (4.10)–(4.11) and (6.25), we obtain

$$\| E_a - E_b \|_{C^{2,\alpha}} \leqslant \frac{\mathbf{c}_{12}}{T_0} [\| S_a - S_b \|_{C^{1,\alpha}} + \| U_a - U_b \|_{C^{1,\alpha}}]. \tag{6.26}$$

By Lemmas 5.4, 6.1, 6.2, 6.4, and (6.25)–(6.26), we have

$$\| w_a - w_b \|_{W^{1,p}} \leqslant \mathbf{c}_{11} \mathbf{c}_{12} [\| S_a - S_b \|_{C^{1,\alpha}} + \| U_a - U_b \|_{W^{2,p}}]. \tag{6.27}$$

By Reference 15 and (4.15)–(4.17),

$$\| \sigma_a - \sigma_b \|_{W^{2,p}} \leqslant c_{13} \| w_a - w_b \|_{W^{1,p}}. \tag{6.28}$$

By Eq. (4.18) and Lemma 5.3, we get the estimate

$$\| \nabla(\psi_a - \psi_b) \|_{W^{2,p}} \leqslant c_{14}(1/\rho_1, \rho_D) [\| \sigma_a - \sigma_b \|_{C^{1,\alpha}} + \| \rho_a - \rho_b \|_{C^{2,\alpha}}]. \tag{6.29}$$

By (6.27)–(6.29),

$$\| V_a - V_b \|_{W^{2,p}} \leqslant \mathbf{c}_{11} \mathbf{c}_{12} [\| S_a - S_b \|_{C^{1,\alpha}} + \| U_a - U_b \|_{W^{2,p}}]. \tag{6.30}$$

By (6.29) and (6.30), we see that $\mathscr{F}$ is a continuous map in $C^{1,\alpha} \times W^{2,p}$. If $\mathbf{c}_{11}$ is even smaller, $\mathscr{F}$ is a contractive map. □

Finally, we prove our main result.

Proof of Theorem 6.1. By Lemmas 6.5, 6.6, we know that $\mathscr{F}$ maps $\mathscr{A}\times\mathscr{D}$ to itself and that $\mathscr{F}$ is continuous in $C^{1,\alpha}\times W^{2,p}$. Since $\mathscr{A}\times\mathscr{D}$ is a compact, convex subset of $C^{1,\alpha}\times W^{2,p}$, by Schauder fixed point theorem, a fixed point exists. If $1/T_0 + \|\nabla \ln(\rho_D\|_{C^{2,\alpha}(\bar{\Omega})}$ is small enough, fixed point of $\mathscr{F}$ exists uniquelybecause $\mathscr{F}$ is contractive map.

Next, we show a solution of (1.1)–(1.7) corresponds to a fixed point of (4.7)–(4.21) and vice versa. Suppose (ρ, V, T, Ψ) is a solution of the system (1.1)–(1.7), by tracing the derivation of (4.7)–(4.21), it is easy to see that $(T/T_0, V)$ is a fixed point of the system (4.7)–(4.21). On the other hand, let us assume that (E, V) is a fixed point of (4.7)–(4.21). Define $T = ET_0 > 0$. By (4.9) and (4.21), we obtain $(\rho, \mathrm{V}, \mathrm{T})$. (ρ, V, T) satisfy equations (1.1), (1.3), $(1.6)_2$, $(1.7)_1$, $(1.7)_2$ by (4.18), (4.10), (4.11), (4.19), $(4.14)_1$ respectively. By (4.15) and (4.21), (4.12) can be written as

$$\operatorname{curl}\left[(V\cdot\nabla)V + \frac{1}{\mathbf{m}\rho}\nabla(\rho T) + \frac{TV}{\beta_2 T_0}\right] = -\nabla P. \tag{6.31}$$

By $(4.14)_{2,3}$, we see (6.31) implies $P = 0$ in (4.12). Since domain is simply connected, (6.31) implies there is a function Ψ such that

$$(V\cdot\nabla)V + \frac{1}{\mathbf{m}\rho}\nabla(\rho T) - \frac{\mathbf{e}}{\mathbf{m}}\nabla\Psi + \frac{TV}{\beta_2 T_0} = 0, \tag{6.32}$$

i.e. (1.2). Next taking divergence of equaton (6.32), and comparing with (4.7), we obtain equations (1.4) and $(1.6)_1$. Therefore, we conclude a fixed point $(T/T_0, V)$ of (4.7)–(4.21) also corresponds to a solution (ρ, V, T, Ψ) of (1.1)–(1.7). Uniqueness of the fixed point of (4.7)–(4.21) is equivalent to uniqueness of the solution of (1.1)–(1.7).

Because of the fixed point $(T/T_0, V)\in\mathscr{A}\times\mathscr{D}$, solution (ρ, V, T, Ψ) of (1.1)–(1.7) satisfy $\rho, T\in C^{3,\alpha}(\bar{\Omega})$, $V, \Psi\in W^{3,p}(\Omega)$. Because of (6.6), we see that solutin (ρ, V, T, Ψ) satisfy $|V|^2(x)\leqslant T(x)/\mathbf{m}$ for all $x\in\Omega$, i.e. the solution (ρ, V, T, Ψ) is a subsonic solution. So we complete the proof.

7. Proof of auxillary lemmas

In this section, we prove Lemmas 5.2–5.4. Lemma 5.2 is proved by employing the Leray–Schauder fixed point theorem [7] and Lemma 5.1, Lemma 5.3 is proved by results in [11] and method of continuity [7]. Lemma 5.4 is proved based on Lemmas 7.1–7.3. Lemma 7.1 is an extension theorem, and its proof is similar to that of Theorem 7.25 in [7]. By energy method, Lemma 7.2 gives a priori estimate for a transport equation in a bounded smooth domain. Then an existence result of a transport equation in a bounded smooth domain is shown in Lemma 7.3. Finally, by Lemma 7.3, we prove Lemma 5.4.

Proof of Lemma 5.2. Set $0<\gamma\leqslant\alpha$, $K_m = \max(|\underline{\varphi}|, |\bar{\varphi}|)$, $K_M = \sup_{\partial\Omega}|\varphi_D| + K_m$,

$$y_{K_M}(x) := \begin{cases} K_M & \text{if } |y(x)| > K_M, \\ y(x) & \text{if } |y(x)| \leqslant K_M. \end{cases}$$

Next, define a map $M: C^{0,\gamma}(\bar{\Omega}) \times [0,1] \to C^{0,\gamma}(\bar{\Omega})$ by $M(y,t) = z$, where z is the solution of

$$\mathscr{L}(z) := a_{ij} z_{,ij} + a_i z_{,i} = t(a(x) g(y_{K_M}) - f(x)) \quad \text{in } \Omega,$$

$$z|_{\partial\Omega} = t\varphi_D|_{\partial\Omega}.$$

Observe that if y_n convergences to y in $C^{0,\gamma}$, then y_{n,K_M} also converges to y_{K_M} in $C^{0,\gamma}$. By Lemma 5.1, we set that

$$\| z \|_{C^{2,\gamma}} \leqslant tc(\lambda, \| a_{ij}, a_i \|_{C^{0,\alpha}})(\| \varphi_D \|_{C^{2,\gamma}} + \| a(x) g(y_{K_M}) \|_{C^{0,\gamma}} + \| f(x) \|_{C^{0,\gamma}}).$$

So M is a continuous and compact operator. If z_t is a fixed point of the following system:

$$\mathscr{L}(z_t) = t(a(x) g(y_{K_M}) - f(x)),$$

$$z_t|_{\partial\Omega} = t\varphi_D|_{\partial\Omega},$$

then $z_t \in C^{2,\gamma}(\bar{\Omega})$ [2]. By De Giorgi–Nash theorem [7], there is a positive number ℓ such that $\| z_t \|_{C^{0,\ell}}$ is bounded, and the bound is independent of y, t. Note ℓ is independent of γ, so we may assume $\gamma = \ell$. Therefore by Leray–Schauder fixed point theorem [7], a fixed point, φ, of $M(\cdot, 1)$ exists.

Suppose the fixed point φ satisfies $|\varphi(x)| \leqslant K_M$ in Ω, then φ is a solution of (5.3). Next we prove $|\varphi(x)| \leqslant K_M$ in Ω.

(a) **Claim.** *If φ is a fixed point of $M(\cdot, 1)$ then $|\varphi(x)| \leqslant K_M$ in Ω.*

Proof. The set $\Omega_+ \subseteq \Omega$ of points at which $\varphi(x) > K_m$ holds is open in Ω, and the boundary of Ω_+ consists of points x at which either $\varphi(x) = K_m$ or the point is contained on $\partial\Omega$. We assume that Ω_+ is non-empty. Let $x^* \in \Omega_+$. We denote the maximal connected component of Ω_+ containing x^* by Ω_+^*. Define

$$\varphi_{K_m, K_M}(x) := \begin{cases} K_M & \text{if } \varphi(x) > K_M, \\ \varphi(x) & \text{if } K_m \leqslant \varphi(x) \leqslant K_M, \\ K_m & \text{if } \varphi(x) < K_m. \end{cases}$$

Then, $\varphi|_{\Omega_+^*}$ satisfies

$$\mathscr{L}(\varphi) = a(x) g(\varphi_{K_m, K_M}) - f(x) \quad \text{in } \Omega_+^*,$$

$$\varphi|_{\partial\Omega_+^*} = \varphi_D|_{\partial\Omega} \text{ or } K_m.$$

Note $a(x) g(K_m) \geqslant f(x)$, so $\mathscr{L}(\varphi) \geqslant 0$. We obtain $\sup_{\Omega_+^*} \varphi(x) \leqslant \sup_{\partial\Omega_+^*} \varphi(x)$ by maximal principle [7]. Therefore $\sup_{\Omega_+^*} \varphi(x) \leqslant K_M$, which implies $\varphi(x) \leqslant K_M$ in Ω. A similar argument can be used to prove the other side, i.e. $\varphi(x) \geqslant -K_M$ in Ω. Therefore, we conclude that $\| \varphi \|_{L^\infty(\Omega)} \leqslant K_M$.

(b) By *Claim* (a), the fixed point φ is a solution of (5.3) and is in $C^{2,\ell}(\bar{\Omega})$. So $g(\varphi_{K_M}) = g(\varphi) \in C^{0,\alpha}(\bar{\Omega})$ by Theorem 7.26 of [7]. By iteration we see $\varphi \in C^{m+2,\alpha}(\bar{\Omega})$.

Uniqueness of solution is obtained by the maximal principle [7]. The upper and lower bounds of $\varphi(x)$ are obtained as follows: Define $I := \min(\varphi_1, \underline{\varphi})$, so

$$a_{ij}(x)(\varphi - I)_{,ij} + a_i(\varphi - I)_{,i} - a(x)(g(\varphi) - g(I)) = -f(x) + a(x)g(I),$$
$$\varphi - I|_{\partial\Omega} \geqslant 0.$$

Since $-f(x) + a(x)g(I) \leqslant 0$ for all $x \in \Omega$, $\min(\varphi_1, \underline{\varphi}) \leqslant \varphi(x)$ by the maximal principle. By a similar argument, we can prove $\varphi(x) \leqslant \max(\varphi_2, \bar{\varphi})$ for all $x \in \Omega$.

By (5.2), we see that solution φ satisfies

$$\|\varphi\|_{C^{m+2,\alpha}} \leqslant c(\lambda, \|a_{ij}, a_i, a\|_{C^{m,\alpha}})(\|\varphi_D\|_{C^{m+2,\alpha}} + \|f\|_{C^{m,\alpha}} + \|g(\varphi)\|_{C^{m+1}}). \tag{7.1}$$

By interpolation inequality [7, p. 176] and (7.1), one can derive

$$\|\varphi\|_{C^{m+2,\alpha}} \leqslant c(\lambda, \|a_{ij}, a_i, a\|_{C^{m,\alpha}})\mathscr{P}(\|\varphi_D\|_{C^{m+2,\alpha}} \|f\|_{C^{m,\alpha}}, \|g\|_{C^{m+1}(\varpi_1)} + \|g^{-1}\|_{C^0(\varpi_2)}). \qquad \square$$

Proof of Lemma 5.3. This lemma will be proved by method of continuity [7]. We first consider the case $g = 0$. Define

$$\mathscr{B} = \left\{\varphi \in W^{m+2,p}(\Omega) \,\middle|\, \int_\Omega \varphi \,\mathrm{d}x = 0, \nabla\varphi \cdot \mathbf{n}|_{\Gamma_1 \cup \Gamma_2} = 0\right\},$$

$$\mathscr{V} = \left\{f \in W^{m,p}(\Omega) \,\middle|\, \int_\Omega f(x)\,\mathrm{d}x = 0\right\}.$$

Then $\mathscr{B}$ and $\mathscr{V}$ are Banach spaces. Let $\mathscr{L}_0\varphi := \Delta\varphi$ and $\mathscr{L}_1\varphi := \nabla \cdot (a(x)\nabla\varphi)$. Then $\mathscr{L}_0$ and $\mathscr{L}_1$ are bounded linear operators from $\mathscr{B}$ to $\mathscr{V}$. Define $\mathscr{L}_t$, $t \in [0, 1]$, as follows:

$$\mathscr{L}_t\varphi := (1 - t)\Delta\varphi + t\nabla \cdot (a(x)\nabla\varphi).$$

By [11], $\mathscr{L}_0$ is a one-to-one and onto map. Suppose $\|\varphi\|_{W^{m+2,p}} \leqslant c\|\mathscr{L}_t\varphi\|_{W^{m,p}}$, by method of continuity, $\mathscr{L}_1$ is also one-to-one and onto. Then the theorem holds true.

To show that $\|\varphi\|_{W^{m+2,p}} \leqslant c\|\mathscr{L}_t\varphi\|_{W^{m,p}}$ for all $\varphi \in \mathscr{B}$, we note

$$\Delta\varphi = \frac{\mathscr{L}_t\varphi}{(1 - t) + ta(x)} - \frac{t\nabla a(x)}{(1 - t) + ta(x)}\nabla\varphi.$$

By [11] and interpolation theorem [7],

$$\|\varphi\|_{W^{m+2,p}} \leqslant c(\|a\|_{C^{m+1,\alpha}}, 1/\lambda)(\|\mathscr{L}_t\varphi\|_{W^{m,p}} + \|\varphi\|_{L^p}). \tag{7.2}$$

Next, we want to show $\|\varphi\|_{L^p} \leqslant c\|\mathscr{L}_t\varphi\|_{W^{m,p}}$ for all $\varphi \in \mathscr{B}$. If not, then there exists a sequence $\{\varphi_{n,t_n}\} \subset \mathscr{B}$ such that $\|\varphi_{n,t_n}\|_{L^p} = 1$ and $\|\mathscr{L}_{t_n}\varphi_{n,t_n}\|_{W^{m,p}} \to 0$. By (7.2),

$$\|\varphi_{n,t_n}\|_{W^{m+2,p}} \leqslant c(\|a\|_{C^{m+1,\alpha}}, 1/\lambda)(\|\mathscr{L}_{t_n}\varphi_{n,t_n}\|_{W^{m,p}} + \|\varphi_{n,t_n}\|_{L^p}),$$

i.e. $\|\varphi_{n,t_n}\|_{W^{m+2,p}}$ is bounded. Since $t_n \in [0, 1]$, we assume $t_n \to t^*$. Because $W^{m+2,p}$ is a reflexive Banach space [3], there exists a subsequence $\{\varphi'_{n,t_n}\}$ of $\{\varphi_{n,t_n}\}$ such that

$\varphi'_{n,t_n} \to \varphi^* \in W^{m+2,p}(\Omega)$ weakly. One can see $\|\varphi^*\|_{L^p} = 1$. However,

$$\mathscr{L}_{t^*}\varphi^* = 0 \quad \text{in } \Omega,$$

$$\nabla\varphi^* \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2,$$

$$\int_\Omega \varphi^*(x)\,\mathrm{d}x = 0. \tag{7.3}$$

Equation (7.3) implies $\varphi^* = 0$ [9], which is a contradiction. So $\|\varphi\|_{L^p} \leqslant c\|\mathscr{L}_t\varphi\|_{W^{m,p}}$. Therefore, by (7.2), we have

$$\|\varphi\|_{W^{m+2,p}} \leqslant c(\|a\|_{C^{m+1,\alpha}}, 1/\lambda)\|\mathscr{L}_t\varphi\|_{W^{m,p}}, \quad \forall\varphi \in \mathscr{B}. \tag{7.4}$$

So $\mathscr{L}_1 : \mathscr{B} \to \mathscr{V}$ is an one-to-one and onto map, that is, solution of (5.4) exists uniquely for $g = 0$. By (7.4), (5.5) holds for the case $g = 0$.

We now consider the case $g \neq 0$. By [11], one can find $\mathscr{G} \in W^{m+2,p}(\Omega)$ such that

$$\nabla\mathscr{G} \cdot \mathbf{n}|_{\Gamma_i} = g/a, \; i = 1, 2, \quad \|\mathscr{G}\|_{W^{m+2,p}(\Omega)} \leqslant c\left(\sum \|g/a\|_{W^{m+1-1/p,p}(\Gamma_i)}\right). \tag{7.5}$$

Consider the following

$$\nabla \cdot (a(x)\nabla\hat{\varphi}) = f - \nabla \cdot (a(x)\nabla\mathscr{G}), \quad \text{in } \Omega,$$

$$a(x)\nabla\hat{\varphi} \cdot \mathbf{n} = 0, \quad \text{on } \Gamma_1 \cup \Gamma_2,$$

$$\int_\Omega \hat{\varphi}(x)\,\mathrm{d}x = 0. \tag{7.6}$$

By the result of previous case $g = 0$, (7.6) has a unique solution $\hat{\varphi} \in W^{m+2,p}(\Omega)$ and by (7.4) and (7.5),

$$\|\hat{\varphi}\|_{W^{m+2,p}} \leqslant c(\|a\|_{C^{m+1,\alpha}}, 1/\lambda)\left(\|f\|_{W^{m,p}} + \sum \|g\|_{W^{m+1-1/p,p}(\Gamma_i)}\|\right). \tag{7.7}$$

Let us define

$$\varphi = \hat{\varphi} + \mathscr{G} - \frac{1}{|\Omega|}\int_\Omega \mathscr{G}\,\mathrm{d}x,$$

where $|\Omega|$ is the volume of Ω. Then $\varphi \in W^{m+2,p}(\Omega)$ is the unique solution of (5.4), and it is easy to check, by (7.7), (5.5) holds. Thus the conclusion of this lemma follows. □

Remark 3. Next, we give an extension theorem for a domain with edge. Let $\Omega \subset \mathbb{R}^3$ be a $C^{k-1,1}$ domain with an edge L (see Remark 1). Then we can find a bounded smooth domain Ω' such that (1) $\Omega \subset \Omega'$; (2) $\Gamma_1 \subset \partial\Omega'$; (3) for all $q \in L = \bar{\Gamma}_1 \cap \bar{\Gamma}_2$, there exists a neighborhood $N(q)$ of q such that $N(q) \cap \Omega$ is smoothly ($C^{k-1,1}$) homomorphic to the intersection between a unit ball and a quadrant $\mathbb{R}^3/4 := \{(x_1, x_2, x_3) | x_1 > 0, x_3 > 0\}$; (4) $N(q) \cap \Omega'$ is smoothly ($C^{k-1,1}$) homomorphic to the intersection of a unit ball and the half-space $\mathbb{R}^3_+ := \{(x_1, x_2, x_3) | x_1 > 0\}$.

Lemma 7.1. *For any* Ω *and* Ω' *in Remark 3, there exists a bounded linear operator* $\mathscr{E}: W^{k,p}(\Omega) \to W^{k,p}(\Omega')$ *such that* $\mathscr{E}(U)(x) = U(x)$ *for* $x \in \Omega$ *and*

$$\| \mathscr{E}(U) \|_{W^{k,p}(\Omega')} \leqslant c \| U \|_{W^{k,p}(\Omega)}, \quad \forall U \in W^{k,p}(\Omega),$$

$$c = c(k, \Omega, \Omega').$$

Proof (see Theorem 7.25 [7]). As in (7.56) of [7], we define an extension in half-space as follows:

$$\mathscr{E}_0 U(x) = \begin{cases} U(x), & x_3 > 0, \\ \sum_{i=1}^{k} c_i U(x', -x_3/i), & x_3 < 0, \end{cases}$$

where $x = (x', x_3)$ and $c_1, x_2, \ldots, c_k$ are constants determined by the system

$$\sum_{i=1}^{k} c_i(-1/i)^m = 1, \quad m = 0, 1, \ldots, (k-1).$$

If $U \in C^\infty(\mathbb{R}^3/4) \cap W^{k,p}(\mathbb{R}^3/4)$, then

$$\mathscr{E}_0 U \in C^{k-1,1}(\mathbb{R}^3_+) \cap W^{k,p}(\mathbb{R}^3_+) \quad \text{and} \quad \| \mathscr{E}_0 U \|_{W^{k,p}(\mathbb{R}^3_+)} \leqslant c \| U \|_{W^{k,p}(\mathbb{R}^3/4)}.$$

By approximation, one see that the domain of $\mathscr{E}_0$ can be extended to $W^{k,p}(\mathbb{R}^3/4)$, i.e.

$$\mathscr{E}_0 : W^{k,p}(\mathbb{R}^3/4) \to W^{k,p}(\mathbb{R}^3_+) \quad \text{and} \quad \| \mathscr{E}_0 U \|_{W^{k,p}(\mathbb{R}^3_+)} \leqslant c \| U \|_{W^{k,p}(\mathbb{R}^3/4)}.$$

Then, by partition of unity and following the argument of Theorem 7.25 [7], we can show this lemma true. □

Next, we derive *a priori* estimate for a transport equation. Domain Ω considered in the next two lemmas are smooth domains, Γ_- is a closed subset of $\partial\Omega$, $\{b, U, f, g\}$ are vector functions, and Λ is a matrix function.

Lemma 7.2. *Consider the system*

$$(b \cdot \nabla) U + \Lambda U = f, \quad \Omega,$$

$$U = g, \quad \Gamma_-, \tag{7.8}$$

where $\Gamma_- := \{x \in \partial\Omega \mid b \cdot \mathbf{n} < 0\}$. *If the following conditions are satisfied*

1. Ω *is a bounded smooth domain,* $H^1(\Omega) \hookrightarrow L^p(\Omega)$, $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$,
2. $b \in W^{\ell+1,p}(\Omega)$, $2 \leqslant \ell$, $b \cdot \mathbf{n} < -\lambda < 0$ *on* Γ_-,
3. $f, \Lambda \in W^{\ell,p}(\Omega)$, $g \in W^{\ell,p}(\Gamma_-)$, $U \in H^{\ell+1}(\Omega)$,
4. $\Lambda(x) > \Lambda_m I$ *in* Ω, $\Lambda_m - \kappa_1(p, \| b \|_{W^{\ell+1,p}(\Omega)}, 1/\lambda) = \Lambda_d > 0$,

where $\lambda, \Lambda_m, \Lambda_d$ *are constants,* I *is identity matrix, and* κ_1 *is a continuous function of its arguments, then the following estimate holds,* $0 \leqslant s < \ell$:

$$\| U \|^p_{W^{s,p}(\Omega)} \leqslant c(p, \| \Lambda \|_{W^{\ell,p}}, \| b \|_{W^{\ell+1,p}}, 1/\lambda, 1/\Lambda_d)(\| f \|^p_{W^{s,p}(\Omega)} + \| f \|^p_{W^{s,p}(\Gamma_-)}). \tag{7.9}$$

Proof. By means of a partition of unity, a local co-ordinate change, and Lemma 1.1.1 in [10], it is sufficient to prove (7.9) in a half-space. Estimate in a half-plane is obtained by differentiating the equation $(7.8)_1$ to estimate tangential derivatives and then, using $b\cdot\mathbf{n}\neq 0$ on Γ_- to solve for the normal variables in terms of the tangential ones. For convenience, we assume assumptions of this lemma hold in a half place and $\partial\Omega=\Gamma_-$. We use the following notations: $b=(b_1,b')=(b_1,b_2,b_3)$, $\Omega=\{x\,|\,x=(x_1,x')=(x_1,x_2,x_3), x_1\geqslant 0\}$,

$$\partial^s_{x'}U=\sum_{|\gamma|\leqslant s}\frac{\partial^{\gamma_2}}{\partial x_2^{\gamma_2}}\frac{\partial^{\gamma_3}}{\partial x_3^{\gamma_3}}U,\quad \partial^s_{x_1,x'}U=\sum_{|\gamma|\leqslant s}\frac{\partial^{\gamma_2}}{\partial x_1^{\gamma_2}}\frac{\partial^{\gamma_3}}{\partial x'^{\gamma_3}}U,\quad |\gamma|=\gamma_2+\gamma_3,\ \gamma_i\geqslant 0.$$

Consider the following in a half-space;

$$\mathscr{L}U:=(b\cdot\nabla)U+\Lambda U=f,$$
$$U|_{x_1=0}=g. \tag{7.10}$$

We prove the inequality (7.9) by method of induction.

Case $s=0$: Multiply $(7.10)_1$ by $|U|^{p-2}U$ and integrate over Ω:

$$\int_\Omega (b\cdot\nabla)U|U|^{p-2}U\,\mathrm{d}x+\int_\Omega\Lambda|U|^p\,\mathrm{d}x=\int_\Omega f|U|^{p-2}U\,\mathrm{d}x. \tag{7.11}$$

Note

$$\int_\Omega (b\cdot\nabla)U|U|^{p-2}U\,\mathrm{d}x\geqslant 1/p\int_{\Gamma_1} b\cdot\mathbf{n}|U|^p\,\mathrm{d}x'-1/p\int_\Omega(\nabla\cdot b)|U|^p\,\mathrm{d}x. \tag{7.12}$$

By (7.11) and (7.12) and Hölder's inequality,

$$\int_\Omega(\Lambda-(\nabla\cdot b)/p-(p-1)/p)|U|^p\,\mathrm{d}x\leqslant 1/p\,\|f\|^p_{L^p}+1/p\int_{\Gamma_-}|b\cdot\mathbf{n}\|g|^p\,\mathrm{d}x'. \tag{7.13}$$

Case $s=1$: Differentiate $(7.10)_1$ with respect to $\partial_{x'}$, we have

$$(b\cdot\nabla)\partial_{x'}U+\Lambda\partial_{x'}U=\partial_{x'}f-(\partial_{x'}b\cdot\nabla)U-(\partial_{x'}\Lambda)U. \tag{7.14}$$

Applying (7.13) to (7.14), we obtain

$$\begin{aligned}\int_\Omega(\Lambda-(\nabla\cdot b)/p-(p-1)/p)|\partial_{x'}U|^p\,\mathrm{d}x\\ \leqslant 1/p\int_{\Gamma_-}|b\cdot\mathbf{n}||\partial_{x'}g|^p\,\mathrm{d}x'+c(p)\,\|\partial_{x'}f\|^p_{L^p}\\ +c(p)\,\|(\partial_{x'}b\cdot\nabla)U\|^p_{L^p}+c(p)\,\|(\partial_{x'}\Lambda)U\|^p_{L^p}.\end{aligned} \tag{7.15}$$

By $(7.10)_1$,

$$\|\partial_{x_1}U\|_{L^p}\leqslant c/\lambda(\|f\|_{L^p}+\|b'\|_{L^\infty}\|\partial_{x'}U\|_{L^p}+\|\Lambda\|_{L^\infty}\|U\|_{L^p}). \tag{7.16}$$

By (7.15) and (7.16), we obtain

$$(\Lambda_m - \kappa_1(p, 1/\lambda, \|b\|_{W^{1,\infty}(\Omega)}))\|\partial_{x'}U\|_{L^p}^p \leqslant \int_{\Gamma_-} |b\cdot\mathbf{n}||\partial_{x'}g|^p\,\mathrm{d}x'$$

$$+ c(p, 1/\lambda, \|\nabla b\|_{L^\infty})\|f\|_{W^{1,p}}^p + c(p, \|\Lambda\|_{W^{1,\infty}}, 1/\lambda, \|\nabla b\|_{L^\infty})\|U\|_{L^p}^p. \tag{7.17}$$

By (7.16) and (7.17),

$$\Lambda_d\|\partial_x U\|_{L^p}^p \leqslant c(p, \|\Lambda\|_{W^{1,\infty}}, 1/\lambda, \|b\|_{W^{1,\infty}})\left(\|f\|_{W^{1,p}}^p + \|U\|_{L^p}^p + \int_{\Gamma_-}|\partial_{x'}g|^p\,\mathrm{d}s\right). \tag{7.18}$$

By (7.13), (7.18),

$$\|U\|_{W^{1,p}}^p \leqslant c(p, \|\Lambda\|_{W^{1,\infty}}, 1/\lambda, \|b\|_{W^{1,\infty}}, 1/\Lambda_d)(\|f\|_{W^{1,p}}^p + \|g\|_{W^{1,p}(\Gamma_-)}^p). \tag{7.19}$$

So we prove (7.9) for cases $s = 0, 1$.

Suppose (7.9) holds for $k = 0, \ldots, s-1$, we plan to show (7.9) holds for $k = s, s \leqslant \ell$. Differentiate $(7.10)_1$ with respect to $\partial_{x'}^s$, then

$$(b\cdot\nabla)\partial_{x'}^s U + \Lambda\partial_{x'}^s U = \partial_{x'}^s f + [\mathscr{L}, \partial_{x'}^s]U,$$

where

$$[\mathscr{L}, \partial_{x'}^s]U := (b\cdot\nabla)\partial_{x'}^s U + \Lambda\partial_{x'}^s U - \partial_{x'}^s((b\cdot\nabla)U + \Lambda U).$$

By (7.13),

$$\int_\Omega (\Lambda - (\nabla\cdot b)/p - (p-1)/p)|\partial_{x'}^s U|^p\,\mathrm{d}x$$

$$\leqslant 1/p\int_{\Gamma_-}|b\cdot\mathbf{n}||\partial_{x'}^s g|^p\,\mathrm{d}x' + c(p)\|\partial_{x'}^s f\|_{L^p}^p + c(p)\|[\mathscr{L}, \partial_{x'}^s]U\|_{L^p}^p,$$

which implies

$$(\Lambda_m - c(p, 1/\lambda, \|b\|_{W^{\ell+1,p}}))\|\partial_{x'}^s U\|_{L^p}^p$$

$$\leqslant c(p)\int_{\Gamma_-}|b\cdot\mathbf{n}||\partial_{x'}^s g|^p\,\mathrm{d}x' + c(p)\|\partial_{x'}^s f\|_{L^p}^p$$

$$+ c(p, \|b\|_{W^{\ell+1,p}}, \|\Lambda\|_{W^{\ell,p}})\|U\|_{W^{s-1,p}}^p + c(p, \|b\|_{W^{\ell+1,p}})\|\partial_{x_1x'}^s U\|_{L^p}^p. \tag{7.20}$$

By $(7.10)_1$, we have

$$\partial_x^{s-1}\partial_{x_1}U = \partial_{x'}^{s-1}(f - b'\partial_{x'}U - \Lambda U)/b_1. \tag{7.21}$$

So

$$\|\partial_{x'}^{s-1}\partial_{x_1}U\|_{L^p} \leqslant c(1/\lambda, \|b\|_{W^{\ell+1,p}}, \|\Lambda\|_{W^{\ell,p}})(\|U\|_{W^{s-1,p}} + \|f\|_{W^{s-1,p}})$$

$$+ c(1/\lambda, \|b\|_{W^{\ell+1,p}})\|\partial_{x'}^s U\|_{L^p}.$$

Moreover, one can show

$$\|\partial^s_{x_1,x'}U\|_{L^p} \leqslant c(1/\lambda, \|b\|_{W^{\ell+1,p}}, \|\Lambda\|_{W^{\ell,p}})(\|U\|_{W^{s-1,p}} + \|f\|_{W^{s-1,p}}) + c(1/\lambda, \|b\|_{W^{\ell+1,p}})\|\partial^s_{x'}U\|_{L^p}. \quad (7.22)$$

By (7.20), (7.22), and assumption of method of induction,

$$\Lambda^d\|\partial^s_{x'}U\|^p_{L^p} \leqslant c(p, \|b\|_{W^{\ell+1,p}}, \|\Lambda\|_{W^{\ell,p}}, 1/\lambda, 1/\Lambda_d)(\|f\|^p_{W^{s,p}} + \|g\|^p_{W^{s,p}(\Gamma_-)}). \quad (7.23)$$

By (7.22) and (7.23),

$$\|U\|^p_{W^{s,p}} \leqslant c(p, \|\Lambda\|_{W^{\ell,p}}, \|b\|_{W^{\ell+1,p}}, 1/\lambda, 1/\Lambda_d)(\|f\|^p_{W^{s,p}} + \|g\|^p_{W^{s,p}(\Gamma_-)}).$$

So we prove (7.9). □

Lemma 7.3. *Besides assumptions* 1, 2, 4 *of* Lemma 7.2, *if*
1. $f, \Lambda \in W^{\ell,p}(\Omega)$, $g \in W^{\ell,p}(\Gamma_-)$,

then the system (7.8) *has a unique solution* $U \in W^{\ell,p}(\Omega)$ *and*

$$\|U\|^p_{W^{\ell,p}(\Omega)} \leqslant c(p, \|\Lambda\|_{W^{\ell,p}}, \|b\|_{W^{\ell+1,p}}, 1/\lambda, 1/\Lambda_d)(\|f\|^p_{W^{s,p}(\Omega)} + \|g\|^p_{W^{s,p}(\Gamma_-)}). \quad (7.24)$$

Proof. By density theorem [7], we can choose the sequences $\{b_k, \Lambda_k, f_k\} \subset C^\infty(\bar{\Omega})$, $\{g_k\} \subset C^\infty(\Gamma_-)$ such that

$$b_k \to b \text{ in } W^{\ell+1,p}(\Omega),\ \Lambda_k, f_k \to \Lambda, f \text{ in } W^{\ell,p}(\Omega),\ g_k \to g \text{ in } W^{\ell,p}(\Gamma_-) \text{ as } k \to \infty.$$

Let k be large enough such that $\{b_k, \Lambda_k, f_k\}$ satisfy the assumptions of Lemma 7.3. Then for each k, the system

$$(b_k \cdot \nabla)U_k + \Lambda_k U_k = f_k \quad \text{in } \Omega,$$
$$U_k = g_k \quad \text{on } \Gamma_-,$$

has a unique weak solution $U_k \in H^{\ell+1}(\Omega)$ [12]. By Lemma 7.2, the following holds:

$$\|U_k\|^p_{W^{\ell,p}} \leqslant c(p, \|\Lambda_k\|_{W^{\ell,p}}, \|b_k\|_{W^{\ell+1,p}}, 1/\lambda, 1/\Lambda_d)(\|f_k\|^p_{W^{\ell,p}} + \|g_k\|^p_{W^{\ell,p}(\Gamma_-)}).$$

Since $\{\|b_k\|_{W^{\ell+1,p}}, \|\Lambda_k\|_{W^{\ell,p}}, \|f_k\|_{W^{\ell,p}}, \|g_k\|_{W^{\ell,p}}\}$ are bounded, $\|U_k\|_{W^{\ell,p}(\Omega)}$ is bounded. Because $W^{\ell,p}$ is a reflexive Banach space [3], there is a subsequence $\{U'_k\}$ of $\{U_k\}$ such that $U'_k \to U^*$ weakly in $W^{\ell,p}$, which implies $\|U^*\|_{W^{\ell,p}} \leqslant \liminf_{k\to\infty}\|U_k\|_{W^{\ell,p}}$. Passing k to ∞, we see that U^* is a classical solution of (7.8) and satisfies (7.24). □

Based on the results of Lemmas 7.1–7.3, we now prove Lemma 5.4.

Proof of Lemma 5.4. First, we extend the domain Ω to a smooth simply connected domain Ω' (see Remark 3) such that $\Gamma_1 \subset \Gamma'_1 \subset \partial\Omega'$. For function b, we extend it to b^* in Ω', by Lemma 7.1, such that $\Gamma'_1 = \{x \in \partial\Omega' \mid b^*(x)\cdot\mathbf{n} < 0\}$,

$$\|b^*\|_{W^{\ell+1,p}(\Omega')} \leqslant c\|b\|_{W^{\ell+1,p}(\Omega)}, \quad b^*\cdot\mathbf{n}|_{\Gamma'_1}(x) < -\lambda < 0,$$

We also extend f, Λ to f^*, Λ^* in Ω' and g to g^* on $\partial\Omega'$ such that

$$\| f^* \|_{W^{\ell,p}(\Omega')} \leqslant c \| f \|_{W^{\ell,p}(\Omega)}, \quad \| g^* \|_{W^{\ell,p}(\Gamma_1')} \leqslant c \| g \|_{W^{\ell,p}(\Gamma_1)},$$

$$\| \Lambda^* \|_{W^{\ell,p}(\Omega')} \leqslant c \| \Lambda \|_{W^{\ell,p}(\Omega)}, \quad \Lambda^*(x) > \Lambda_m > 0, \forall x \in \Omega.$$

Furthermore, b^*, f^*, Λ^*, g^* satisfy the assumptions of Lemma 7.3. Consider the following

$$(b^* \cdot \nabla) U^* + \Lambda^* U^* = f^* \quad \text{in } \Omega',$$

$$U^* = g^* \quad \text{on } \Gamma_1'.$$

Then, by Lemma 7.3, there exists a solution $U^* \in W^{\ell,p}(\Omega')$ in the above equation and

$$\| U^* \|^p_{W^{\ell,p}(\Omega')} \leqslant c(p, \| \Lambda^* \|_{W^{\ell,p}}, \| b^* \|_{W^{\ell+1,p}}, 1/\lambda, 1/\Lambda_d)(\| f^* \|^p_{W^{\ell,p}(\Omega')} + \| g^* \|^p_{W^{\ell,p}(\Gamma_1')}).$$

Then $U = U^*|_\Omega$ is the required solution. Note that $\| U^* \|^p_{W^{\ell,p}(\Omega')} \leqslant \| U^* \|^p_{W^{\ell,p}(\Omega')}$. Next by adjusting the coefficients of the above inequality, we obtain estimate (5.6) for $s = \ell$ case. Other cases of estimate (5.6) can then be obtained by tracing the proof of Lemma 7.2. Uniqueness can be proved simply by energy method. □

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