J. Math. Anal. Appl. 381 (2011) 897–909

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Numerical ranges of weighted shifts

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ARTICLE INFO

Article history:
Received 23 February 2011
Available online 9 April 2011
Submitted by J.A. Ball

Keywords:
Numerical range
Numerical radius
Numerical contraction
Unilateral weighted shift
Bilateral weighted shift

ABSTRACT

Let A be a unilateral (resp., bilateral) weighted shift with weights w_n, $n \geqslant 0$ (resp., $-\infty < n < \infty$). Eckstein and Rácz showed before that A has its numerical range $W(A)$ contained in the closed unit disc if and only if there is a sequence $\{a_n\}_{n=0}^{\infty}$ (resp., $\{a_n\}_{n=-\infty}^{\infty}$) in $[-1,1]$ such that $|w_n|^2 = (1-a_n)(1+a_{n+1})$ for all n. In terms of such a_n's, we obtain a necessary and sufficient condition for $W(A)$ to be open. If the w_n's are periodic, we show that the a_n's can also be chosen to be periodic. As a result, we give an alternative proof for the openness of $W(A)$ for an A with periodic weights, which was first proven by Stout. More generally, a conjecture of his on the openness of $W(A)$ for A with split periodic weights is also confirmed.

1. Introduction

Let A be a bounded linear operator on a complex Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and its associated norm $\|\cdot\|$. The *numerical range* $W(A)$ of A is, by definition, the subset $\{\langle Ax,x\rangle\colon\ x\in H,\ \|x\|=1\}$ of the complex plane. Its *numerical radius* $w(A)$ is $\sup\{|z|\colon\ z\in W(A)\}$. A is said to be a *numerical contraction* if $W(A)$ is contained in $\overline{\mathbb{D}}$ ($\mathbb{D}\equiv\{z\in\mathbb{C}\colon\ |z|<1\}$) or, equivalently, if $w(A)\leqslant 1$.

The purpose of this paper is to study the numerical ranges of unilateral and bilateral weighted shifts. Recall that the *unilateral* (resp., *bilateral*) *weighted shift* with weights w_n, $n\geqslant 0$ (resp., $-\infty<n<\infty$), is the operator with matrix

$$A=\begin{pmatrix} 0 & & & \\ w_0 & 0 & & \\ & w_1 & 0 & \\ & & \ddots & \ddots \end{pmatrix} \quad \left(\text{resp., } \begin{pmatrix} \ddots & \ddots & & & \\ & w_{-1} & \underline{0} & & \\ & & w_0 & 0 & \\ & & & w_1 & \ddots \\ & & & & \ddots \end{pmatrix}\right)$$

on $\ell^2=\{(x_0,x_1,\ldots)\colon\ \sum_{n=0}^{\infty}|x_n|^2<\infty\}$ (resp., $\ell^2(\mathbb{Z})=\{(\ldots,x_{-1},\underline{x_0},x_1,\ldots)\colon\ \sum_{n=-\infty}^{\infty}|x_n|^2<\infty\}$). Here, in the bilateral case, we underline the $(0,0)$-entry of a matrix and the 0th component of a vector. We will also consider the *finite weighted shift* with weights w_j, $1\leqslant j\leqslant n-1$:

$$A=\begin{pmatrix} 0 & & & \\ w_1 & 0 & & \\ & \ddots & \ddots & \\ & & w_{n-1} & 0 \end{pmatrix} \quad \text{on } \mathbb{C}^n.$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.04.010

Since a weighted shift A (unilateral, bilateral or finite) is unitarily equivalent to $e^{i\theta}A$ for any real θ, its numerical range is an open or closed circular disc centered at the origin. Therefore, the study of their numerical ranges boils down to two things: (1) to determine whether $W(A)$ is open or closed, and (2) to find its radius $w(A)$. In the present paper, we are mainly concerned with the first problem. A prototypical example of the shifts we consider is the *simple unilateral* (resp., *bilateral*) *shift*, that is, the one with weights $1, 1, \ldots$ (resp., $\ldots, 1, \underline{1}, 1, \ldots$). It is known that its numerical range equals the open unit disc (cf. [6, Solution 212(2)]). In Section 2 below, we first prove some preliminary results on the openness (or closedness) of the numerical ranges of the weighted shifts by using only their basic properties. For example, we show that if A is a unilateral (resp., bilateral) weighted shift with weights w_n convergent to a nonzero a and satisfying $|w_n| \geqslant |a|$ for all n, then $W(A)$ is open if and only if $w(A)$ equals $|a|$ (Proposition 2.3). Then, in Section 3, we consider the parametric representation, due to Eckstein and Rácz [3], of the weights of a weighted shift A with $w(A) \leqslant 1$, namely, a unilateral (resp., bilateral) weighted shift A with weights w_n, $n \geqslant 0$ (resp., $-\infty < n < \infty$), is a numerical contraction if and only if there is a sequence $\{a_n\}_{n=0}^{\infty}$ (resp., $\{a_n\}_{n=-\infty}^{\infty}$) in $[-1, 1]$ such that $|w_n|^2 = (1 - a_n)(1 + a_{n+1})$ for all n. In terms of such a_n's, we obtain a necessary and sufficient condition for $W(A)$ to be open (or closed) (Theorem 3.3). If the weights w_n are periodic, we show that the corresponding a_n's can also be chosen to be periodic (Lemma 4.1). As a consequence of these results, we can give an alternative proof for the openness of the numerical ranges of such operators (Proposition 4.2), first proven by Stout [13, Proposition 6]. More generally, a conjecture of his on the openness of $W(A)$ for a weighted shift A with asymptotic split periodic weights is also verified (Theorem 4.3).

Since a weighted shift A with weights w_n is unitarily equivalent, via a unitary operator of the form $\operatorname{diag}(e^{i\theta_n})$, to one with weights $|w_n|$, in our discussions below we may assume from time to time that the weights are all nonnegative. Furthermore, if some of the weights are zero, then A is the direct sum of (finite or unilateral) weighted shifts. Its numerical range is then the largest circular disc among the numerical ranges of the summands. Hence for our purposes we need only consider, in many cases, the numerical ranges of the weighted shifts with strictly positive weights.

For properties of the numerical ranges, the reader may consult [6, Chapter 22] or [5]. A useful reference for weighted shifts is [11].

2. Generalities

We start with a result on the attaining vectors for the numerical radius of a weighted shift. For any operator A, let $\operatorname{Re} A = (A + A^*)/2$ denote its *real part*.

Proposition 2.1. *Let A be a unilateral (resp., bilateral) weighted shift with weights $w_n > 0$ for all n. If $W(A)$ is closed, then*

(a) *the set $\{x \in \ell^2$ (resp., $\ell^2(\mathbb{Z})$): $\langle Ax, x\rangle = w(A)\|x\|^2\}$ is a subspace of dimension one, and*
(b) *there is a unique unit vector $x = (x_n)$ in ℓ^2 (resp., $\ell^2(\mathbb{Z})$) with $x_n > 0$ for all n such that $\langle Ax, x\rangle = w(A)$.*

Proof. We only prove for the unilateral weighted shifts; the proof for the bilateral case is similar.

Let $M = \{x \in \ell^2\colon \langle Ax, x\rangle = w(A)\|x\|^2\}$. For any vector x, we have

$$\begin{aligned}\big\langle \big(w(A)I - \operatorname{Re} A\big)x, x\big\rangle &= w(A)\|x\|^2 - \operatorname{Re}\langle Ax, x\rangle \\ &\geqslant w(A)\|x\|^2 - \big|\langle Ax, x\rangle\big| \geqslant 0,\end{aligned}$$

which implies that $w(A)I - \operatorname{Re} A \geqslant 0$. In particular, if x is in M, then $\langle (w(A)I - \operatorname{Re} A)x, x\rangle = 0$ and hence $(\operatorname{Re} A)x = w(A)x$. Conversely, if $(\operatorname{Re} A)x = w(A)x$, then $\operatorname{Re}\langle Ax, x\rangle = w(A)\|x\|^2$. Since $W(A)$ is closed, we have $W(A) = \{z \in \mathbb{C}\colon |z| \leqslant w(A)\}$. It thus follows from $\langle A(x/\|x\|), x/\|x\|\rangle \in W(A)$ for $x \neq 0$ that $\langle Ax, x\rangle = w(A)\|x\|^2$. Therefore, $M = \ker(w(A)I - \operatorname{Re} A)$ is a subspace.

For any unit vector $x = (x_n)$ in M, we have

$$w(A) = \big|\langle Ax, x\rangle\big| \leqslant \big\langle A|x|, |x|\big\rangle \leqslant w(A),$$

where $|x|$ denotes the (unit) vector $(|x_n|)$. It follows that $\langle A|x|, |x|\rangle = w(A)$. Hence we may assume from the outset that $x_n \geqslant 0$ for all n. If $x_{n_0} = 0$ for some $n_0 \geqslant 1$, then, from $(\operatorname{Re} A)x = w(A)x$, we have $(w_{n_0-1}x_{n_0-1} + w_{n_0}x_{n_0+1})/2 = w(A)x_{n_0} = 0$, which implies that $x_{n_0-1} = x_{n_0+1} = 0$. Repeating these arguments with x_{n_0-1} and x_{n_0+1} replacing x_{n_0} yields $x_n = 0$ for all n. Similarly for $x_0 = 0$. This shows that $x = 0$ contradicting our assumption on x. Hence we must have $x_n > 0$ for all n.

To show that $\dim M = 1$, let $x = (x_n)$ and $y = (y_n)$ be any two vectors in M. There are scalars a and b, not both zero, such that $ax_0 + by_0 = 0$. Then $ax + by$ in M has its 0th component equal to zero. This contradicts what was proven in the preceding paragraph. Hence M is of dimension one as asserted. (a) and (b) follow immediately. □

Two comments are in order. Firstly, the unit vector x in Proposition 2.1(b) can be easily seen to be the normalized vector of (y_n), where $y_0 = 1$, $y_1 = 2w(A)/w_0$ and $y_n = (2w(A)y_{n-1} - w_{n-2}y_{n-2})/w_{n-1}$ for $n \geqslant 2$. Properties of such vectors, even for the more general Jacobi matrices, were investigated in [12, Section X.4]. Secondly, analogous results as (a) and (b) also hold for the finite weighted shifts A. These can be proven either as above or by invoking the Perron–Frobenius theorem [8, Theorem 1, p. 536] since $\operatorname{Re} A$ is nonnegative irreducible.

Recall that the *essential numerical range* $W_e(A)$ of an operator A on a separable infinite-dimensional space H is the intersection set $\bigcap\{\overline{W(A+K)}\colon K \text{ compact on } H\}$. Properties of the essential numerical ranges can be found in [7,4]. In particular, it is known that if $W(A)$ is open, then $\overline{W(A)} = W_e(A)$ (cf. [7, Corollary 2]). The *essential numerical radius* $w_e(A)$ of A is $\max\{|z|\colon z \in W_e(A)\}$.

The next proposition gives information on the (essential) numerical range of a weighted shift with convergent weights.

Proposition 2.2. *Let A be a unilateral (resp., bilateral) weighted shift with weights w_n, $n \geqslant 0$ (resp., $-\infty < n < \infty$), satisfying $\lim_{n\to\infty} |w_n| = a$ (resp., $\lim_{n\to\infty} |w_n| = b$ and $\lim_{n\to-\infty} |w_n| = c$). Let $a = \max\{b, c\}$ in the bilateral case. Then*

(a) $W_e(A) = \{z \in \mathbb{C}\colon |z| \leqslant a\}$,
(b) $w(A) \geqslant a$,
(c) *$W(A)$ is closed if and only if a is in $W(A)$, and*
(d) *$W(A) = \{z \in \mathbb{C}\colon |z| < a\}$ if $W(A)$ is open.*

Proof. We may assume that $w_n \geqslant 0$ for all n.

(a) If B is the unilateral (resp., bilateral) weighted shift with weights $a, a, \ldots$ (resp., $\ldots, c, c, \underline{0}, b, b, \ldots$), then $A - B$ is compact. Hence $W_e(A) = W_e(B) = \{z \in \mathbb{C}\colon |z| \leqslant a\}$, where the last equality follows from the fact that $W_e(S) = \overline{W(S)} = \overline{\mathbb{D}}$, S being the simple unilateral shift (cf. [7, Corollary 2]).

(b) Since $\overline{W(A)}$ contains $W_e(A)$, the assertion follows from (a).

(c) This is a consequence of (a) and the fact that $W(A)$ is closed if and only if $W_e(A)$ is contained in $W(A)$ (cf. [7, Corollary 1]).

(d) If $W(A)$ is open, then $\overline{W(A)} = W_e(A)$ by [7, Corollary 2]. Thus $W(A) = \{z \in \mathbb{C}\colon |z| < a\}$ from (a). □

We remark that, under the conditions of Proposition 2.2(c), $W(A)$ may be bigger than $\{z \in \mathbb{C}\colon |z| \leqslant a\}$. One example is the A with weights $w, 1, 1, \ldots$ (resp., $\ldots, 1, 1, \underline{w}, 1, 1, \ldots$), where $w > \sqrt{2}$ (resp., $w > 1$). In this case, $W(A) = \{z \in \mathbb{C}\colon |z| \leqslant w^2/(2\sqrt{w^2-1})\}$ (resp., $\{z \in \mathbb{C}\colon |z| \leqslant (w^2+1)/(2w)\}$), which is bigger than $\overline{\mathbb{D}}$ (cf. [1, pp. 1053–1054], [13, p. 500], [14, Propositions 2 and 3] or Corollary 4.7 and Theorem 4.9 later). Note also that the converse of Proposition 2.2(d) is false, that is, $w(A) = a$ does not guarantee the openness of $W(A)$. Examples will be given after Theorem 3.3. The next proposition says that under the extra condition that $|w_n| \geqslant a$ for all n we do have the equivalence of $w(A) = a$ and the openness of $W(A)$.

Proposition 2.3. *Let A be a unilateral (resp., bilateral) weighted shift with nonzero weights w_n, $n \geqslant 0$ (resp., $-\infty < n < \infty$), satisfying $\lim_{n\to\infty} |w_n| = a > 0$ (resp., $\lim_{n\to\pm\infty} |w_n| = a > 0$) and $|w_n| \geqslant a$ for all $n \geqslant n_0$, where n_0 is some fixed nonnegative integer (resp., some fixed integer). Then $W(A)$ is open if and only if $w(A) = a$.*

Proof. We only prove the unilateral case. In view of Proposition 2.2(d), we need only show that $w(A) = a$ implies the openness of $W(A)$. Assume that $w_n > 0$ for all n and $W(A)$ is closed. Proposition 2.1(b) gives a unit vector $x = (x_n)$ in ℓ^2 with $x_n > 0$ for all n such that $(\operatorname{Re} A)x = ax$. Then $w_0 x_1 = 2ax_0$ and $w_{n-1}x_{n-1} + w_n x_{n+1} = 2ax_n$ for all $n \geqslant 1$. Together with our assumption of $w_n \geqslant a$ for $n \geqslant n_0$, this yields

$$a(x_{n-1} - x_n) \leqslant w_{n-1}x_{n-1} - ax_n = ax_n - w_n x_{n+1} \leqslant a(x_n - x_{n+1})$$

for $n \geqslant n_0 + 1$. Hence the sequence $\{x_n - x_{n+1}\}_{n=n_0}^{\infty}$ is increasing. Since $\lim_n (x_n - x_{n+1}) = a - a = 0$, this implies that $x_{n_0} \leqslant x_{n_0+1} \leqslant \cdots$. That x is in ℓ^2 yields that $x_{n_0} = x_{n_0+1} = \cdots = 0$, which contradicts our assumption. Hence $W(A)$ is open as asserted. □

The preceding proposition generalizes [2, Example 2], where it is assumed that the unilateral $|w_n|$'s decrease to a. Another condition for $W(A)$ to be equal to $\{z \in \mathbb{C}\colon |z| < a\}$ is $\lim_{n\to\infty} |w_n| = a > 0$ (resp., $\lim_{n\to\infty} |w_n| = a > 0$ or $\lim_{n\to-\infty} |w_n| = a > 0$) and $|w_n| \leqslant a$ for all n (cf. [14, Theorem 1]).

The next proposition relates the numerical range $W(A)$ of a weighted shift A to those of its compressions for open $W(A)$.

Proposition 2.4. *Let A be a unilateral (resp., bilateral) weighted shift with nonzero weights w_n, $n \geqslant 0$ (resp., $-\infty < n < \infty$), and, for each $m \geqslant 1$, let A_m be the weighted shift with weights $w_m, w_{m+1}, \ldots$ (resp., $\ldots, w_{-m-1}, w_{-m}, \underline{w_m}, w_{m+1}, \ldots$). If $W(A)$ is open, then $W(A) = W(A_m)$ for all $m \geqslant 1$.*

Proof. We only prove the unilateral case. Since A_m is a compression of A, we obviously have $W(A_m) \subseteq W(A)$ for all $m \geqslant 1$. On the other hand, if B_m denotes the unilateral weighted shift with weights $\underbrace{0, \ldots, 0}_{m}, w_m, w_{m+1}, \ldots,$ then

$$\overline{W(A)} = W_e(A) = W_e(B_m) \subseteq \overline{W(B_m)} = \overline{W(A_m)},$$

where the first equality is the consequence of the openness of $W(A)$ (cf. [7, Corollary 2]). Thus $\overline{W(A)} = \overline{W(A_m)}$. Together with $W(A_m) \subseteq W(A)$ and the openness of $W(A)$, this yields $W(A) = W(A_m)$ for $m \geqslant 1$. □

Here the openness of $W(A)$ is essential. For example, if A is the unilateral (resp., bilateral) weighted shift with weights $w, 1, 1, \ldots$ (resp., $\ldots, 1, 1, \underline{w}, 1, 1, \ldots$), where $w > \sqrt{2}$ (resp., $w > 1$), then $W(A)$ is a closed circular disc centered at the origin, which is bigger that $W(A_1) = \mathbb{D}$ (cf. the remarks after Proposition 2.2). Examples of A with $W(A)$ closed and $W(A) = W(A_m)$ for all $m \geqslant 1$ will be given in Section 3 as applications of the parametric representation in Theorem 3.1.

We now compare the numerical ranges of two weighted shifts with the moduli of the weights of one less than or equal to those of the other. In this case, their numerical ranges have the same containment relation.

Proposition 2.5. *Let A and B be unilateral (resp., bilateral) weighted shifts with weights w_n and u_n, $n \geqslant 0$ (resp., $-\infty < n < \infty$), respectively. If $|w_n| \leqslant |u_n|$ for all n, then the following hold:*

(a) *$W(A) \subseteq W(B)$ and $W_e(A) \subseteq W_e(B)$.*
(b) *Assume further that the w_n's are all nonzero. Then $\overline{W(A)} = \overline{W(B)}$ if and only if either $|w_n| = |u_n|$ for all n (when $W(A)$ is closed) or $W(A) = \operatorname{Int} W(B)$ (when $W(A)$ is open). In the latter case, we have $\overline{W(A)} = W_e(A) = W_e(B) = \overline{W(B)}$.*

For the proof, we need the following lemma relating the numerical range of a general infinite matrix to those of its compressions. If $A = [a_{ij}]_{i,j=0}^{\infty}$ (resp., $[a_{ij}]_{i,j=-\infty}^{\infty}$) and $0 \leqslant m \leqslant n \leqslant \infty$ (resp., $-\infty \leqslant m \leqslant n \leqslant \infty$), we let $A[m, n]$ denote the matrix $[a_{ij}]_{i,j=m}^{n}$. For a subset $\triangle$ of $\mathbb{C}$, $\triangle^\wedge$ denotes its *convex hull*, that is, $\triangle^\wedge$ is the smallest convex set which contains $\triangle$.

Lemma 2.6. *If $A = [a_{ij}]_{i,j=0}^{\infty}$ (resp., $[a_{ij}]_{i,j=-\infty}^{\infty}$) is an operator on ℓ^2 (resp., $\ell^2(\mathbb{Z})$), then*

(a) *$\overline{W(A)} = \overline{\bigcup_{n=0}^{\infty} W(A[0,n])}$ (resp., $\overline{\bigcup_{n=0}^{\infty} W(A[-n,n])}$), and*
(b) *$W_e(A) = \bigcap_{n=0}^{\infty} \overline{W(A[n,\infty])}$ (resp., $[(\bigcap_{n=0}^{-\infty} \overline{W(A[-\infty,n])}) \cup (\bigcap_{n=0}^{\infty} \overline{W(A[n,\infty])})]^\wedge$).*

Proof. We leave out the easy proof of (a) and prove only (b). Assume first that $A = [a_{ij}]_{i,j=0}^{\infty}$. For each $n \geqslant 0$, let z_n be a point in $\overline{W(A[n,\infty])}$ and let $A_n = z_n I_n \oplus A[n,\infty]$ ($A_0 = A[0,\infty]$). Since A_n is a finite-rank perturbation of A, we have

$$W_e(A) \subseteq \overline{W(A_n)} = \big(\{z_n\} \cup \overline{W\big(A[n,\infty]\big)}\big)^\wedge = \overline{W\big(A[n,\infty]\big)}$$

for $n \geqslant 0$. Hence $W_e(A) \subseteq \bigcap_{n=0}^{\infty} \overline{W(A[n,\infty])}$. For the converse containment, let z be any point in $\bigcap_{n=0}^{\infty} \overline{W(A[n,\infty])}$. Then, for each $n \geqslant 0$, there is a unit vector x_n in ℓ^2 such that $|\langle A[n,\infty]x_n, x_n\rangle - z| < 1/(n+1)$. Letting $x_n' = 0_n \oplus x_n$, where 0_n denotes the vector with n zero components, we have $|\langle Ax_n', x_n'\rangle - z| < 1/(n+1)$. Since the x_n''s are unit vectors which converge to 0 in the weak topology, we infer that $z \in W_e(A)$ (cf. [4, corollary of Theorem 5.1]). This proves that $W_e(A) = \bigcap_{n=0}^{\infty} \overline{W(A[n,\infty])}$.

Next assume that $A = [a_{ij}]_{i,j=-\infty}^{\infty}$. Let z_0 be any point in $W_e(A[-\infty,-1])$ and let $B = A[-\infty,-1] \oplus [z_0] \oplus A[1,\infty]$. Since $A - B$ is of finite rank, we have

$$\begin{aligned} W_e(A) = W_e(B) &= \big[W_e\big(A[-\infty,-1]\big) \cup W_e\big(A[1,\infty]\big)\big]^\wedge \\ &= \left[\left(\bigcap_{n=-1}^{-\infty} \overline{W\big(A[-\infty,n]\big)}\right) \cup \left(\bigcap_{n=1}^{\infty} \overline{W\big(A[n,\infty]\big)}\right)\right]^\wedge, \end{aligned}$$

where the last equality is an easy consequence of what was just proven for $W_e([a_{ij}]_{i,j=0}^{\infty})$. Our assertion for $W_e(A)$ follows immediately. □

Proof of Proposition 2.5. We only prove the unilateral case and assume that $w_n, u_n \geqslant 0$ for all n.

(a) Let $z = \langle Ax, x\rangle$ be any point in $W(A)$, where $x = (x_n)$ is a unit vector in ℓ^2. Then

$$|z| = \left|\sum_{n=0}^{\infty} w_n x_n \bar{x}_{n+1}\right| \leqslant \sum_{n=0}^{\infty} w_n |x_n||x_{n+1}| \leqslant \sum_{n=0}^{\infty} u_n |x_n||x_{n+1}| = \big\langle B|x|, |x|\big\rangle,$$

where $|x| = (|x_n|)$ is also a unit vector. Since $W(B)$ is a circular disc centered at the origin, this implies that $|z|$, and hence z, is in $W(B)$. Therefore, $W(A) \subseteq W(B)$.

For the essential numerical range, let A_n and B_n be the unilateral weighted shifts with weights $w_n, w_{n+1}, \ldots$ and $u_n, u_{n+1}, \ldots$, respectively, for each $n \geqslant 0$. Then

$$W_e(A) = \bigcap_n \overline{W(A_n)} \subseteq \bigcap_n \overline{W(B_n)} = W_e(B)$$

by Lemma 2.6(b).

(b) Assume that $\overline{W(A)} = \overline{W(B)}$, $W(A)$ is closed, and $w_{n_0} < u_{n_0}$ for some $n_0 \geqslant 0$. Proposition 2.1(b) says that $w(A) = \langle Ax, x\rangle$ for some unit vector $x = (x_n)$ with $x_n > 0$ for all n. Then

$$w(A) = \sum_{n=0}^{\infty} w_n x_n x_{n+1} < \sum_{n=0}^{\infty} u_n x_n x_{n+1} = \langle Bx, x\rangle \leqslant w(B),$$

which is a contradiction. Hence in this case we must have $w_n = u_n$ for all n. On the other hand, if $\overline{W(A)} = \overline{W(B)}$ and $W(A)$ is open, then $W(A) = \operatorname{Int} W(B)$ and $\overline{W(A)} = W_e(A)$ by [7, Corollary 2]. Hence

$$\overline{W(A)} = W_e(A) \subseteq W_e(B) \subseteq \overline{W(B)} = \overline{W(A)}$$

and thus the equalities hold throughout.

The converse implication is trivial. □

Note that in Proposition 2.5(b) the assumption of nonzero w_n's is essential. For example, the unilateral weighted shifts A and B with weights $2, 0, 1, 1, \ldots$ and $2, 0, \sqrt{2}, 1, 1, \ldots$, respectively, are such that $W(A) = W(B) = \overline{\mathbb{D}}$ (cf. [14, Proposition 2]).

3. Parametric representations

We now consider an additional tool for the study of numerical ranges of the weighted shifts. The following theorem gives a refinement of the parametric representation for numerical contractions among weighted shifts. It is due to Eckstein and Rácz [3, Theorem 2.5]. Since the proof in [3] depends on a result from an unpublished article, we present here a more detailed operator-theoretic proof for completeness.

Theorem 3.1.

(a) *Let A be a unilateral (resp., bilateral) weighted shift with weights w_n, $n \geqslant 0$ (resp., $-\infty < n < \infty$). Then $w(A) \leqslant 1$ if and only if there is a sequence $\{a_n\}_{n=0}^{\infty}$ (resp., $\{a_n\}_{n=-\infty}^{\infty}$) in $[-1, 1]$ such that $|w_n|^2 = (1 - a_n)(1 + a_{n+1})$ for all n.*
Moreover, in this case, if $w_n \neq 0$ for all n, then the set $\alpha_A \equiv \{a_0\colon\ |w_n|^2 = (1 - a_n)(1 + a_{n+1})$ for some $\{a_n\}_{n=0}^{\infty}$ (resp., $\{a_n\}_{n=-\infty}^{\infty}$) in $[-1, 1)\}$ equals $[-1, a]$ for some a in $[-1, 1)$ (resp., $[a, b]$ for some $a \leqslant b$ in $(-1, 1)$), and for each $a_0 \in \alpha_A$ such a_n's are uniquely determined.

(b) *Let A be the n-by-n weighted shift*

$$\begin{pmatrix} 0 & & & \\ w_1 & 0 & & \\ & \ddots & \ddots & \\ & & w_{n-1} & 0 \end{pmatrix}$$

with $w_j \neq 0$ for all j. Then $w(A) < 1$ (resp., $w(A) = 1$) if and only if $|w_j|^2 = (1 - a_j)(1 + a_{j+1})$ for some $\{a_j\}_{j=1}^{n}$ with $a_1 = -1$ and $-1 < a_j < 1$ for $1 < j \leqslant n$ (resp., $a_1 = -1$, $-1 < a_j < 1$ for $1 < j < n$ and $a_n = 1$).

Proof. (a) Assume first that A is a unilateral weighted shift with nonnegative weights w_n. Since A is the direct sum of finite weighted shifts and a unilateral shift with nonzero weights (either may be absent), we may consider these latter two types separately.

Assume that $w_n \neq 0$ for all n and $w(A) \leqslant 1$. Since the latter is equivalent to $\operatorname{Re} A \leqslant I$, we may apply the Gram–Schmidt process to the columns of $(I - \operatorname{Re} A)^{1/2}$ to obtain its QR-decomposition: $(I - \operatorname{Re} A)^{1/2} = QR$, where Q is an isometry and R is upper triangular. Note that since $I - \operatorname{Re} A$ is a real matrix, so are $(I - \operatorname{Re} A)^{1/2}$, Q and R (cf. [6, Problem 121]). Then $I - \operatorname{Re} A = R^*Q^*QR = R^*R$ is a Cholesky decomposition of $I - \operatorname{Re} A$:

$$\begin{pmatrix} 1 & -w_0/2 & & \\ -w_0/2 & 1 & -w_1/2 & \\ & -w_1/2 & 1 & \ddots \\ & & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} r_{00} & & & \\ r_{01} & r_{11} & & \\ r_{02} & r_{12} & r_{22} & \\ \vdots & \vdots & & \ddots \end{pmatrix} \begin{pmatrix} r_{00} & r_{01} & r_{02} & \cdots \\ & r_{11} & r_{12} & \cdots \\ & & r_{22} & \\ & & & \ddots \end{pmatrix}. \tag{1}$$

In particular, we have $r_{00}^2 = 1$. If, for some $n_0 \geqslant 1$, $r_{n_0 n_0} = 0$ and $r_{nn} \neq 0$ for $0 \leqslant n < n_0$, then, carrying out the matrix multiplication in (1), we deduce that $r_{mn} = 0$ for $0 \leqslant m < n_0$ and $n \geqslant m + 2$, and $-w_{n_0}/2 = \sum_{m=0}^{n_0} r_{m,n_0+1} r_{m n_0} =$

$r_{n_0,n_0+1}r_{n_0n_0}=0$, which contradicts our assumption. Hence we must have $r_{nn}\neq 0$ for all n and thus $r_{mn}=0$ for $0\leqslant m\leqslant n-2$, $r_{n,n+1}^2+r_{n+1,n+1}^2=1$ and $r_{n,n+1}r_{nn}=-w_n/2$ for $n\geqslant 0$. If $a_n=1-2r_{nn}^2$ for $n\geqslant 0$, then $a_0=-1$, $-1<a_n<1$ for $n\geqslant 1$ and

$$w_n^2=4r_{n,n+1}^2r_{nn}^2=4\big(1-r_{n+1,n+1}^2\big)r_{nn}^2=(1-a_n)(1+a_{n+1}),\quad n\geqslant 0.$$

Conversely, if the a_n's in $[-1,1]$ satisfy $w_n^2=(1-a_n)(1+a_{n+1})$ for all n, then, defining b_n's inductively by $b_0=-1$ and $(1-b_n)(1+b_{n+1})=w_n^2$ for $n\geqslant 0$, we have $-1\leqslant b_n\leqslant a_n<1$ for all n. If

$$r_{mn}=\begin{cases}\sqrt{(1-b_n)/2} & \text{for } m=n\geqslant 0,\\ -\sqrt{(1+b_n)/2} & \text{for } m=n-1\geqslant 0,\\ 0 & \text{for } 0\leqslant m<n-1,\end{cases}$$

then $r_{00}^2=1$, $r_{n,n+1}^2+r_{n+1,n+1}^2=1$ and $r_{n,n+1}r_{nn}=-w_n/2$ for $n\geqslant 0$. Hence (1) holds. This shows that $\operatorname{Re}A\leqslant I$ and thus $w(A)\leqslant 1$.

Now consider the set α_A under the assumption of $w_n>0$ for all n. We have $-1\in\alpha_A$ from above. If the sequence $\{a_0^{(m)}\}_{m=1}^\infty$ is in α_A with $\lim_m a_0^{(m)}=a_0$ and with the corresponding $\{a_n^{(m)}\}_{n=0}^\infty$ satisfying $w_n^2=(1-a_n^{(m)})(1+a_{n+1}^{(m)})$ for all m and n, then we easily infer that $a_n\equiv\lim_m a_n^{(m)}$ exists for each $n\geqslant 1$ and a_0 is in α_A with the corresponding sequence $\{a_n\}_{n=0}^\infty$ in $[-1,1)$. This shows that α_A is a closed subset of $[-1,1)$. Moreover, if $a_0<b_0$ are both in α_A with the corresponding $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ and if $a_0<c_0<b_0$, then the sequence $\{c_n\}_{n=0}^\infty$ defined inductively by $w_n^2=(1-c_n)(1+c_{n+1})$ obviously satisfies $-1\leqslant a_n<c_n<b_n<1$ for all n. We conclude that α_A is a closed subinterval of $[-1,1)$ and hence of the form $[-1,a]$ for some a in $[-1,1)$.

We now move to the proof of (b) and will be back to (a) again for the bilateral case.

(b) Assume that $w_j>0$ for all j. If $w(A)<1$, then $I_n-\operatorname{Re}A$ is positive definite. We may proceed as in (a) to obtain the Cholesky decomposition of $I_n-\operatorname{Re}A$:

$$\begin{pmatrix}1 & -w_1/2 & & \\ -w_1/2 & 1 & \ddots & \\ & \ddots & \ddots & -w_{n-1}/2\\ & & -w_{n-1}/2 & 1\end{pmatrix}=\begin{pmatrix}r_{11} & & & \\ r_{12} & r_{22} & & \\ \vdots & \ddots & \ddots & \\ r_{1n} & \cdots & r_{n-1,n} & r_{nn}\end{pmatrix}\begin{pmatrix}r_{11} & r_{12} & \cdots & r_{1n}\\ & r_{22} & \ddots & \vdots\\ & & \ddots & r_{n-1,n}\\ & & & r_{nn}\end{pmatrix}\tag{2}$$

with $r_{jj}\neq 0$ for all j. Letting $a_j=1-2r_{jj}^2$ for $1\leqslant j\leqslant n$, we have $a_1=-1$, $-1<a_j<1$ for $1<j\leqslant n$ and $w_j^2=(1-a_j)(1+a_{j+1})$ for all j as before. On the other hand, if $w(A)=1$, then $I_n-\operatorname{Re}A$ is noninvertible and hence, in (2), some r_{jj} must be equal to 0. If $r_{j_0j_0}=0$ and $r_{jj}\neq 0$ for $1\leqslant j<j_0$, then, as in (a), we would have $-w_{j_0}/2=\sum_{k=0}^{j_0}r_{k,j_0+1}r_{kj_0}=r_{j_0,j_0+1}r_{j_0j_0}=0$, a contradiction. Thus $r_{jj}\neq 0$ for all j, $1\leqslant j<n$, and hence $r_{nn}=0$. This gives $a_1=-1$, $-1<a_j<1$ for $1<j<n$, and $a_n=1$.

Conversely, if the a_j's satisfying $w_j^2=(1-a_j)(1+a_{j+1})$ for all j exist, then letting

$$r_{ij}=\begin{cases}\sqrt{(1-a_j)/2} & \text{for } i=j,\\ -\sqrt{(1+a_j)/2} & \text{for } i=j-1,\\ 0 & \text{for } i<j-1,\end{cases}$$

we obtain the equality in (2). Hence $I_n-\operatorname{Re}A$ is either positive definite or positive semidefinite with nonzero kernel depending on whether $r_{nn}>0$ or $=0$, and thus $w(A)<1$ or $=1$ accordingly.

Finally, we come back to (a).

(a) Consider a bilateral weighted shift A with weights $w_n>0$. For each integer m, let A_m be the unilateral weighted shift with weights $w_m,w_{m+1},\ldots$. If $w(A)\leqslant 1$, then $w(A_m)\leqslant 1$ for all m. By what was proven before, for each m there is a sequence $\{a_n^{(m)}\}_{n=0}^\infty$ in $[-1,1)$ such that $w_{m+n}^2=(1-a_n^{(m)})(1+a_{n+1}^{(m)})$ for all m and n, and a_0^m is such that

$$\big[-1,a_0^{(m)}\big]=\big\{b_0^{(m)}\colon\ w_{m+n}^2=\big(1-b_n^{(m)}\big)\big(1+b_{n+1}^{(m)}\big)\text{ for some }\big\{b_n^{(m)}\big\}_{n=0}^\infty\text{ in }[-1,1)\big\}.$$

Since $a_n^{(m-n)}$ is in $[-1,a_0^{(m)}]$ for all integers m and all $n\geqslant 0$, we have $a_0^{(m)}\geqslant a_1^{(m-1)}$ and $a_0^{(m-1)}\geqslant a_1^{(m-2)}$. From

$$\big(1-a_0^{(m-1)}\big)\big(1+a_1^{(m-1)}\big)=w_{m-1}^2=\big(1-a_1^{(m-2)}\big)\big(1+a_2^{(m-2)}\big),$$

we deduce that $a_1^{(m-1)}\geqslant a_2^{(m-2)}$. Similarly, we obtain, by induction, that the sequence $\{a_j^{(m-j)}\}_{j=0}^\infty$ is decreasing. If $a_m=\lim_j a_j^{(m-j)}$ for each m, then a_m is in $[-1,1)$ and satisfies $w_m^2=(1-a_m)(1+a_{m+1})$ for all m. Conversely, the existence of such a_n's would imply, by the proven unilateral case, that $w(A_m)\leqslant 1$ for all m. Hence $w(A)=\lim_{m\to-\infty}w(A_m)\leqslant 1$ by Lemma 2.6(a). Finally, if $w_n\neq 0$ for all n, then $\alpha_A=[a,b]$ for some $a\leqslant b$ in $(-1,1)$ can be proven as in the unilateral case. $\square$

The next lemma will be needed in Section 4. It implies, in particular, that if A is the simple unilateral (resp., bilateral) shift, then $\alpha_A = [-1, 0]$ (resp., $\alpha_A = \{0\}$).

Lemma 3.2. *If A is the unilateral (resp., bilateral) weighted shift with weights $w, w, \ldots$ (resp., $\ldots, w, \underline{w}, w, \ldots$), where $0 < |w| \leqslant 1$, then $\alpha_A = [-1, \sqrt{1-|w|^2}\,]$ (resp., $\alpha_A = \{\sqrt{1-|w|^2}\,\}$).*

Proof. We first consider the unilateral case. Let $\{a_n\}_{n=0}^\infty$ be such that $-1 \leqslant a_n < 1$ and $|w|^2 = (1-a_n)(1+a_{n+1})$ for all n. If $a_0^2 > 1-|w|^2$, then we derive from $1+a_1 = |w|^2/(1-a_0)$ that $1+a_1 > 1+a_0$ and hence $a_1^2 > a_0^2 > 1-|w|^2$. Repeating this argument, we obtain the increasingness of the a_n's. If $a = \lim_n a_n$, then

$$|w|^2 = \lim_n (1-a_n)(1+a_{n+1}) = (1-a)(1+a) = 1-a^2,$$

which yields that $a^2 = 1-|w|^2 < a_0^2$, a contradiction. Hence we must have $a_0^2 \leqslant 1-|w|^2$ or $\alpha_A \subseteq [-1, \sqrt{1-|w|^2}\,]$. On the other hand, letting $a_n = \sqrt{1-|w|^2}$ for all n gives $|w|^2 = (1-a_n)(1+a_{n+1})$, which shows that $\sqrt{1-|w|^2}$ is in α_A. Hence $\alpha_A = [-1, \sqrt{1-|w|^2}\,]$ as asserted.

Similarly, for the bilateral case, if $\{a_n\}_{n=-\infty}^\infty$ is such that $-1 < a_n < 1$ and $|w|^2 = (1-a_n)(1+a_{n+1})$ for all n, and if $a_0 < \sqrt{1-|w|^2}$, then, as above, the sequence $\{a_n\}_{n=0}^{-\infty}$ is decreasing and we would obtain the contradictory $\lim_{n\to-\infty} a_n = \sqrt{1-|w|^2}$. Thus $\alpha_A = \{\sqrt{1-|w|^2}\}$ as required. $\square$

As was noted in Section 1, if A is a weighted shift, then $W(A)$ is either open or closed. In terms of the parameters in Theorem 3.1, we can now give a complete characterization of those A's with $W(A)$ open (or closed).

Theorem 3.3. *Let A be a unilateral (resp., bilateral) weighted shift with nonzero weights w_n, $n \geqslant 0$ (resp., $-\infty < n < \infty$). Assume that $w(A) = 1$ and let a_n, $n \geqslant 0$ (resp., $-\infty < n < \infty$), be such that $-1 \leqslant a_n < 1$ (resp., $-1 < a_n < 1$) and $|w_n|^2 = (1-a_n)(1+a_{n+1})$ for all n. Then $W(A)$ is closed if and only if $a_0 = -1$ and $\sum_{n=0}^\infty \prod_{k=0}^n (1-a_k)/(1+a_{k+1}) < \infty$ (resp., $\sum_{n=0}^\infty [(\prod_{k=0}^n (1-a_k)/(1+a_{k+1})) + (\prod_{k=0}^n (1+a_{-k})/(1-a_{-k-1}))] < \infty$).*

Proof. We only prove the unilateral case and assume that $w_n > 0$ for all n. Then $-1 \leqslant a_0 < 1$ and $-1 < a_n < 1$ for all $n \geqslant 1$.

Assume first that $W(A)$ is closed, that is, $W(A) = \overline{\mathbb{D}}$, and $a_0 > -1$. Let

$$u_n = \begin{cases} \sqrt{(1-a_n)(1+a_{n+1})} & \text{for } n \geqslant 0, \\ \sqrt{1+a_0} & \text{for } n = -1, \\ 1 & \text{for } n \leqslant -2, \end{cases}$$

and let B be the bilateral weighted shift with weights $u_n > 0$, $-\infty < n < \infty$. Then, obviously, $W(A) \subseteq W(B)$, and $w(B) \leqslant 1$ by Theorem 3.1(a). Hence we have $W(B) = \overline{\mathbb{D}}$. By Proposition 2.1(b), there is a unit vector $x = (x_n)$ in $\ell^2(\mathbb{Z})$ with $x_n > 0$ for all n such that $\langle Bx, x\rangle = w(B) = 1$. In particular, we have $(\operatorname{Re} B)x = x$. A simple calculation yields that $(x_{n-1}+x_{n+1})/2 = x_n$ or $x_{n+1} - x_n = x_n - x_{n-1} \equiv d$ for all $n \leqslant -2$. Hence $x_n = x_{-1} + (n+1)d$ for $n \leqslant -2$. Since $\sum_n x_n^2 = 1$, this implies that $d = 0$ and thus $x_{-1} = x_{-2} = \cdots = 0$, which contradicts our assumption. Therefore, we must have $a_0 = -1$. Moreover, by Theorem 3.1(a), there is a unit vector $y = (y_n)$ in ℓ^2 with $y_n > 0$ for all n such that $\langle Ay, y\rangle = w(A) = 1$. This yields that

$$\begin{aligned} 0 &= 1 - \sum_{n=0}^\infty w_n y_n y_{n+1} \\ &= \sum_{n=0}^\infty y_n^2 - \sum_{n=0}^\infty w_n y_n y_{n+1} \\ &= \sum_{n=0}^\infty \left(\frac{1}{2}(1-a_n)y_n^2 + \frac{1}{2}(1+a_{n+1})y_{n+1}^2\right) - \sum_{n=0}^\infty \sqrt{1-a_n}\sqrt{1+a_{n+1}}\,y_n y_{n+1} \\ &= \sum_{n=0}^\infty \left(\sqrt{\frac{1}{2}(1-a_n)}\,y_n - \sqrt{\frac{1}{2}(1+a_{n+1})}\,y_{n+1}\right)^2. \end{aligned} \tag{3}$$

Hence $y_{n+1} = \sqrt{(1-a_n)/(1+a_{n+1})}\,y_n$ for all $n \geqslant 0$ and thus $y_{n+1} = y_0 \prod_{k=0}^n \sqrt{(1-a_k)/(1+a_{k+1})}$, $n \geqslant 0$. Thus $\sum_{n=0}^\infty \prod_{k=0}^n (1-a_k)/(1+a_{k+1}) < \infty$ as asserted.

Conversely, if $a_0 = -1$ and $\alpha \equiv \sum_{n=1}^\infty \prod_{k=0}^n (1-a_k)/(1+a_{k+1}) < \infty$, then, letting $y_0 = 1/\sqrt{1+\alpha}$ and $y_{n+1} = \sqrt{(1-a_n)/(1+a_{n+1})}\,y_n$ for $n \geqslant 0$, we obtain that $\sum_{n=0}^\infty y_n^2 = 1$ and $\sum_{n=0}^\infty w_n y_n y_{n+1} = 1$ by (3). Hence $y = (y_n)$ is a unit vector in ℓ^2 with $\langle Ay, y\rangle = 1$. This shows that $W(A) = \overline{\mathbb{D}}$ is closed. $\square$

We now use this theorem to give an example of a unilateral weighted shift A with positive weights w_n, $n \geqslant 0$, such that $W(A) = W_e(A) = W(A_m)$ for all $m \geqslant 1$, where A_m is the weighted shift with weights $w_m, w_{m+1}, \dots$ (cf. [13, Note (4), p. 502]). Indeed, let $a_0 = -1$, $a_n = 1/(n+1)$ for $n \geqslant 1$, $w_n = \sqrt{(1-a_n)(1+a_{n+1})}$ for $n \geqslant 0$, and A be the weighted shift with weights w_n, $n \geqslant 0$. Then $w_0 = \sqrt{3}$, $0 < w_n = \sqrt{n(n+3)/((n+1)(n+2))}$ for $n \geqslant 1$, $\lim_n w_n = 1$, and $w(A) = w(A_m) = 1$ for all $m \geqslant 1$ (by Proposition 2.2(b) and Theorem 3.1(a)). Moreover, since

$$\sum_{n=m}^{\infty}\prod_{k=m}^{n}\frac{1-a_k}{1+a_{k+1}} \leqslant \sum_{n=m}^{\infty}\prod_{k=m}^{n}\frac{1}{(1+a_{k+1})^2} = (m+2)^2\sum_{n=m}^{\infty}\frac{1}{(n+3)^2} < \infty$$

for any $m \geqslant 0$, Theorem 3.3 says that $W(A)$ and $W(A_m)$ are closed for all $m \geqslant 1$. Hence $W(A) = W(A_m) = \overline{\mathbb{D}} = W_e(A)$ by Proposition 2.2(a).

Similarly, it can be shown that if

$$a_n = \begin{cases} 1/(n+1) & \text{for } n \geqslant 1, \\ 0 & \text{for } n = 0, \\ 1/(n-1) & \text{for } n \leqslant -1, \end{cases}$$

and $w_n = \sqrt{(1-a_n)(1+a_{n+1})}$ for all n, then the bilateral weighted shift A with weights w_n is such that $W(A) = W_e(A) = W(A_m) = \overline{\mathbb{D}}$ for all integers m, where A_m is the unilateral weighted shift with weights $w_m, w_{m+1}, \dots$.

4. Periodic weights

If A is a (unilateral or bilateral) weighted shift with positive periodic weights, then it was shown in [13, Proposition 6] that $W(A)$ is open (cf. also [14, Theorem 4]). In the following, we prove this via the openness criterion in Section 3. We start by showing that if the weights w_n are periodic, then the parameters a_n can be chosen to be periodic too.

Lemma 4.1. *Let A be a unilateral (resp., bilateral) weighted shift with weights w_n, $n \geqslant 0$ (resp., $-\infty < n < \infty$). Assume that $w(A) \leqslant 1$. Then the $|w_n|$'s are periodic with period p ($\geqslant 1$) if and only if there is a periodic sequence $\{a_n\}_{n=0}^{\infty}$ (resp., $\{a_n\}_{n=-\infty}^{\infty}$) with period p in $[-1,1]$ such that $|w_n|^2 = (1-a_n)(1+a_{n+1})$ for all n. In this case, if the w_n's are all nonzero, then the a_n's are in $(-1,1)$.*

Proof. We only prove the unilateral case and assume that $w_{n+p} = w_n \geqslant 0$ for all $n \geqslant 0$. Let $\{b_n\}_{n=0}^{\infty}$ be a sequence in $[-1,1]$ satisfying $w_n^2 = (1-b_n)(1+b_{n+1})$ for all n. If $b_0 \leqslant b_p$, then we deduce from

$$(1-b_p)(1+b_{p+1}) = w_p^2 = w_0^2 = (1-b_0)(1+b_1)$$

that $b_1 \leqslant b_{p+1}$. Inductively, we obtain $b_n \leqslant b_{n+p}$ for all n or, in other words, $b_l \leqslant b_{l+p} \leqslant \cdots \leqslant b_{l+mp} \leqslant \cdots \leqslant 1$ for all l, $0 \leqslant l < p$, and all $m \geqslant 0$. If $c_l = \lim_m b_{l+mp}$ for $0 \leqslant l < p$ and $c_p = c_0$, then

$$w_l^2 = \lim_m w_{l+mp}^2 = \lim_m (1-b_{l+mp})(1+b_{l+mp+1}) = (1-c_l)(1+c_{l+1}).$$

We define the sequence $\{a_n\}_{n=0}^{\infty}$ by $a_{l+mp} = c_l$ for $0 \leqslant l < p$ and $m \geqslant 0$. It is easily seen that $\{a_n\}_{n=0}^{\infty}$ is periodic with period p and satisfies $-1 \leqslant a_n \leqslant 1$ and $w_n^2 = (1-a_n)(1+a_{n+1})$ for all n. Similarly, if $b_0 > b_p$, then the sequence $\{b_{l+mp}\}_{m=0}^{\infty}$ is decreasing and we can also obtain the required periodic $\{a_n\}_{n=0}^{\infty}$ as above.

That the periodicity of the a_n's implies that of the w_n's is trivial. So is our assertion for nonzero w_n's. □

The next proposition was proven in [13, Proposition 6] and [14, Theorem 4].

Proposition 4.2. *If A is a unilateral (resp., bilateral) weighted shift with nonzero periodic weights, then $W(A)$ is open.*

Proof. We only prove the unilateral case. Assume that A has weights w_n, $n \geqslant 0$, with $w_{n+p} = w_n > 0$ ($p \geqslant 1$) for all n, and $w(A) = 1$. By Lemma 4.1, there are a_n's in $(-1,1)$ such that $a_{n+p} = a_n$ and $w_n^2 = (1-a_n)(1+a_{n+1})$ for all n. Then

$$\sum_{n=0}^{\infty}\prod_{k=0}^{n}\frac{1-a_k}{1+a_{k+1}} \geqslant \sum_{m=1}^{\infty}\prod_{k=0}^{mp-1}\frac{1-a_k}{1+a_{k+1}} = \sum_{m=1}^{\infty} m\prod_{l=0}^{p-1}\frac{1-a_l}{1+a_{l+1}} = \infty.$$

Thus Theorem 3.3 yields the openness of $W(A)$. □

In [13, p. 499], Stout introduced the class of *split periodic shifts*, namely, the bilateral weighted shifts A with weights of the form $\dots, a_l, \dots, a_1, a_l, \dots, a_1, b_1, \dots, b_m, c_1, \dots, c_n, c_1, \dots, c_n, \dots$ and conjectured that if $w(A) = \max\{w(A'), w(C')\}$, where

$$A' = \begin{pmatrix} 0 & & & a_l \\ a_1 & 0 & & \\ & \ddots & \ddots & \\ & & a_{l-1} & 0 \end{pmatrix} \quad \text{and} \quad C' = \begin{pmatrix} 0 & & & c_n \\ c_1 & 0 & & \\ & \ddots & \ddots & \\ & & c_{n-1} & 0 \end{pmatrix},$$

then $W(A)$ is open. Note that, if true, then this is a generalization of the preceding proposition by way of a result of Ridge's [10, Theorem 1]. In the following, we confirm this conjecture by proving an even more general result.

For the ease of exposition, we use the following notations. For $p \geqslant 1$ and scalars $w_0, \ldots, w_{p-1}$, let $A(w_0, \ldots, w_{p-1})$ (resp., $B(w_0, \ldots, w_{p-1})$) denote the unilateral (resp., bilateral) weighted shift with periodic weights $w_0, \ldots, w_{p-1}, w_0, \ldots, w_{p-1}, \ldots$ (resp., $\ldots, w_0, \ldots, w_{p-1}, \underline{w_0}, \ldots, w_{p-1}, \ldots$), and $C(w_0, \ldots, w_{p-1})$ denote the p-by-p weighted cyclic matrix

$$\begin{pmatrix} 0 & & & w_{p-1} \\ w_0 & 0 & & \\ & \ddots & \ddots & \\ & & w_{p-2} & 0 \end{pmatrix}$$

if $p \geqslant 2$, and the 1-by-1 matrix $[w_0]$ if $p = 1$.

Theorem 4.3.

(a) *Let A be a unilateral weighted shift with nonzero weights w_n, $n \geqslant 0$, such that, for some $a_0, \ldots, a_{p-1} > 0$ and some integer $n_0 \geqslant 0$, we have $|w_{n_0+kp+j}| \geqslant a_j$ for all $k \geqslant 0$ and all j, $0 \leqslant j < p$, and $\lim_{k\to\infty} |w_{n_0+kp+j}| = a_j$ for all j. Then $W(A)$ is open if and only if $w(A) = w(C(a_0, \ldots, a_{p-1}))$.*

(b) *Let B be a bilateral weighted shift with nonzero weights w_n, $-\infty < n < \infty$, such that for some $a_0, \ldots, a_{p-1}, b_0, \ldots, b_{q-1} > 0$ and some integer $n_0 \geqslant 0$, we have $|w_{n_0+kp+j}| \geqslant a_j$ for all $k \geqslant 0$ and all j, $0 \leqslant j < p$, $\lim_{k\to\infty} |w_{n_0+kp+j}| = a_j$ for all j, $|w_{kq-j-1}| \geqslant b_j$ for $k \leqslant 0$ and all j, $0 \leqslant j < q$, and $\lim_{k\to\infty} |w_{kq-j-1}| = b_j$ for all j. Then $W(B)$ is open if and only if $w(A) = \max\{w(C(a_0, \ldots, a_{p-1})), w(C(b_0, \ldots, b_{q-1}))\}$.*

For the proof, we need the following lemmas.

Lemma 4.4. *Let A and B be unilateral (resp., bilateral) weighted shifts with nonzero periodic weights w_n and u_n, $n \geqslant 0$, respectively, of period p. Then*

(a) *$w(A) = w(C(w_0, \ldots, w_{p-1}))$,*
(b) *$w(A) \leqslant w(B)$ if $|w_n| \leqslant |u_n|$ for all n, and*
(c) *$w(A) < w(B)$ if $|w_n| \leqslant |u_n|$ for all n and $|w_{n_0}| < |u_{n_0}|$ for some n_0.*

Proof. (a) was proven in [10, Theorem 1]. (b) and (c) follow from (a) and [9, Corollary 3.6]. (Note that (b) also follows from Proposition 2.5(a).) □

Lemma 4.5. *Let A be a unilateral (resp., bilateral) weighted shift with nonzero periodic weights w_n, $n \geqslant 0$ (resp., $-\infty < n < \infty$), of period p.*

(a) *Assume that $w(A) \leqslant 1$ and $\{a_n\}_{n=0}^{\infty}$ (resp., $\{a_n\}_{n=-\infty}^{\infty}$) is a periodic sequence of period p with $-1 < a_n < 1$ and $|w_n|^2 = (1 - a_n)(1 + a_{n+1})$ for all n. Then $w(A) = 1$ if and only if $\prod_{n=0}^{p-1}(1 - a_n)/(1 + a_{n+1}) = 1$.*

(b) *If $w(A) = 1$, $P = \{\{\alpha_n\}_{n=0}^{p-1}\colon -1 < \alpha_n < 1$ and $|w_n|^2 = (1 - \alpha_n)(1 + \alpha_{n+1})$ for $0 \leqslant n < p$ $(\alpha_p \equiv \alpha_0)\}$, and $M_j = \sup\{\alpha_j\colon \{\alpha_n\}_{n=0}^{p-1} \in P\}$ for $0 \leqslant j < p$, then $\{M_n\}_{n=0}^{p-1}$ is in P and, for any sequence $\{b_n\}_{n=0}^{\infty}$ (resp., $\{b_n\}_{n=-\infty}^{\infty}$) satisfying $-1 < b_n < 1$ and $|w_n|^2 = (1 - b_n)(1 + b_{n+1})$ for all n, we have $b_n \leqslant M_{n (\mathrm{mod}\, p)}$ for all n.*

Proof. We assume that A is a unilateral weighted shift with periodic weights $w_n > 0$.

(a) Suppose that $w(A) = 1$ and $\prod_{n=0}^{p-1}(1 - a_n)/(1 + a_{n+1}) < 1$. Let $\epsilon > 0$ be such that $\prod_{n=0}^{p-1}[((1 - a_n)/(1 + a_{n+1})) + \epsilon] < 1$, $b_0, \ldots, b_{p-1}$ be such that $b_n > ((1 - a_n)/(1 + a_{n+1})) + \epsilon$ for $0 \leqslant n < p$ and $\prod_{n=0}^{p-1} b_n = 1$, and ϵ_0 be such that $0 < \epsilon_0 < \min\{1 + a_0, (1 + a_1)\epsilon\}$ and $\epsilon_n \equiv \epsilon_0/(b_0 \cdots b_{n-1}) < \min\{1 + a_n, (1 + a_{n+1})\epsilon\}$ for $1 \leqslant n < p$. Then

$$\frac{\epsilon_n}{\epsilon_{n+1}} = b_n > \frac{1 - a_n}{1 + a_{n+1}} + \epsilon > \frac{1 - a_n + \epsilon_n}{1 + a_{n+1}}$$

or

$$u_n^2 \equiv (1 - a_n + \epsilon_n)(1 + a_{n+1} - \epsilon_{n+1}) > (1 - a_n)(1 + a_{n+1}) = w_n^2$$

for $0 \leqslant n < p$ ($\epsilon_p \equiv \epsilon_0$). Let $B = A(u_0, \ldots, u_{p-1})$. Since $u_n > w_n$ for all n, $0 \leqslant n < p$, we have $w(A) < w(B)$ by Lemma 4.4(c). Note that $w(B) \leqslant 1$ by Theorem 3.1(a) because its weights u_n are associated with the sequence $\{a_0 - \epsilon_0, \ldots, a_{p-1} - \epsilon_{p-1}, a_0 - \epsilon_0, \ldots, a_{p-1} - \epsilon_{p-1}, \ldots\}$ in $(-1, 1)$. These two together yield $w(A) < 1$ in contradiction to our assumption. In a similar fashion, if $\prod_{n=0}^{p-1}(1 - a_n)/(1 + a_{n+1}) > 1$, then we can choose $\epsilon'_n > 0$, $0 \leqslant n < p$, such that $a_n + \epsilon'_n$ is in $(-1, 1)$ and $\epsilon'_n/\epsilon'_{n+1} < (1 - a_n - \epsilon'_n)/(1 + a_{n+1})$ for all n, which would lead to $w(A) < 1$ as above, a contradiction again. We thus conclude that $\prod_{n=0}^{p-1}(1 - a_n)/(1 + a_{n+1}) = 1$.

To prove the converse, suppose that $w(A) < 1$. Since $w(A/w(A)) = 1$, there is, by Lemma 4.1, a periodic sequence $\{b_n\}_{n=0}^{\infty}$ of period p in $(-1, 1)$ such that $(w_n/w(A))^2 = (1 - b_n)(1 + b_{n+1})$ for all n. Assume first that $a_0 \leqslant b_0$. Then

$$1 + a_1 = \frac{w_0^2}{1 - a_0} < \frac{w_0^2}{w(A)^2} \frac{1}{1 - b_0} = 1 + b_1,$$

which implies that $a_1 < b_1$. Inductively, we obtain $a_n < b_n$ for all n. Let $\delta_n = b_n - a_n > 0$. Then

$$\begin{aligned}(1 - a_n - \delta_n)(1 + a_{n+1} + \delta_{n+1}) = (1 - b_n)(1 + b_{n+1}) &= \frac{w_n^2}{w(A)^2} \\ &> w_n^2 = (1 - a_n)(1 + a_{n+1}),\end{aligned}$$

from which it follows that $(1 - a_n - \delta_n)\delta_{n+1} > (1 + a_{n+1})\delta_n$ for $0 \leqslant n < p$ and thus

$$\prod_{n=0}^{p-1} \frac{1 - a_n}{1 + a_{n+1}} > \prod_{n=0}^{p-1} \frac{1 - a_n - \delta_n}{1 + a_{n+1}} > \prod_{n=0}^{p-1} \frac{\delta_n}{\delta_{n+1}} = 1$$

since $\delta_p = b_p - a_p = b_0 - a_0 = \delta_0$. Similarly, if $a_0 > b_0$, then we obtain $a_n > b_n$ for all n and hence $\prod_{n=0}^{p-1}(1 - a_n)/(1 + a_{n+1}) < 1$ as above. These show that if $\prod_{n=0}^{p-1}(1 - a_n)/(1 + a_{n+1}) = 1$, then $w(A) = 1$.

(b) To prove that $\{M_n\}_{n=0}^{p-1}$ is in P, let $\{\alpha_n^{(m)}\}_{n=0}^{p-1}$, $m \geqslant 1$, be in P such that $M_0 = \lim_m \alpha_0^{(m)}$. From $w_n^2 = (1 - \alpha_n^{(m)})(1 + \alpha_{n+1}^{(m)})$ for $0 \leqslant n < p$ and $m \geqslant 1$, we infer that $N_n \equiv \lim_m \alpha_n^{(m)}$ exists for all n, $1 \leqslant n < p$, $N_p = M_0$ and $\{M_0, N_1, \ldots, N_{p-1}\}$ is in P. In particular, we have $N_n \leqslant M_n$ for $1 \leqslant n < p$. If $N_1 < M_1$, then there exists some $\{\beta_n\}_{n=0}^{p-1}$ in P such that $N_1 < \beta_1 \leqslant M_1$. We infer inductively from

$$w_n^2 = (1 - N_n)(1 + N_{n+1}) = (1 - \beta_n)(1 + \beta_{n+1}), \quad 1 \leqslant n < p$$

($\beta_p = \beta_0$) that $N_n < \beta_n$ for all n, $1 \leqslant n \leqslant p$, and thus, in particular, $M_0 = N_p < \beta_p = \beta_0$, which is a contradiction. Therefore, we must have $N_1 = M_1$. Similarly, we can prove inductively that $N_n = M_n$ for all n, $2 \leqslant n < p$, and hence $\{M_n\}_{n=0}^{p-1} = \{M_0, N_1, \ldots, N_{p-1}\}$ is in P as asserted.

Let $\{b_n\}_{n=0}^{\infty}$ be a sequence in $(-1, 1)$ satisfying $w_n^2 = (1 - b_n)(1 + b_{n+1})$ for all n. We need check that $b_{kp+j} \leqslant M_j$ for $k \geqslant 0$ and $0 \leqslant j < p$. If $b_0 > b_p$, then let

$$u_n = \begin{cases} \sqrt{(1 - b_n)(1 + b_{n+1})} & \text{if } 0 \leqslant n < p - 1, \\ \sqrt{(1 - b_{p-1})(1 + b_0)} & \text{if } n = p - 1, \end{cases}$$

and let $B = A(u_0, \ldots, u_{p-1})$. Since $w_n = u_n$ for all n, $0 \leqslant n < p - 1$, and $w_{p-1} < u_{p-1}$, we infer from Lemma 4.4(c) that $w(A) < w(B)$. On the other hand, we also have $w(B) \leqslant 1$ by Theorem 3.1(a). These result in the contradictory $w(A) < 1$. Hence we must have $b_0 \leqslant b_p$. It follows from $w_n^2 = (1 - b_n)(1 + b_{n+1})$ for all n and the periodicity of the w_n's that $b_{kp+j} \leqslant b_{(k+1)p+j}$ for all $k \geqslant 0$ and $0 \leqslant j < p$. Let $\alpha_j = \lim_k b_{kp+j}$ for each j, $0 \leqslant j < p$, and $\alpha_p = \alpha_0$. Then

$$w_j^2 = \lim_k w_{kp+j}^2 = \lim_k (1 - b_{kp+j})(1 + b_{kp+j+1}) = (1 - \alpha_j)(1 + \alpha_{j+1})$$

for $0 \leqslant j < p$. This shows that $\{\alpha_j\}_{j=0}^{p-1}$ is in P and therefore $b_{kp+j} \leqslant \alpha_j \leqslant M_j$ for all k and j as asserted. This completes the proof. $\square$

We are now ready to prove Theorem 4.3.

Proof of Theorem 4.3. We prove only (a); (b) can be done analogously, which we omit. Assume that $w_n > 0$ for all n and $W(A)$ is open. Since A and the unilateral weighted shift A' with weights $w_0, \ldots, w_{n_0-1}, a_0, \ldots, a_{p-1}, a_0, \ldots, a_{p-1}, \ldots$ differ by a compact operator, we have $W_e(A) = W_e(A')$. Hence

$$\begin{aligned}w(A) = w_e(A) = w_e(A') &= w_e\big(A(a_0, \ldots, a_{p-1})\big) \\ &\leqslant w\big(A(a_0, \ldots, a_{p-1})\big) = w\big(C(a_0, \ldots, a_{p-1})\big),\end{aligned}$$

where the first equality is because $\overline{W(A)} = W_e(A)$ by the openness of $W(A)$ (cf. [7, Corollary 2]), and the last equality is by Lemma 4.4(a). On the other hand, we also have $w(A) \geqslant w(A')$ by Lemma 4.4(b) and $w(A') \geqslant w(A(a_0, \dots, a_{p-1})) = w(C(a_0, \dots, a_{p-1}))$ by Lemma 4.4(a). Thus we conclude that $w(A) = w(C(a_0, \dots, a_{p-1}))$ as asserted.

To prove the converse, we may assume that $w(A) = w(C(a_0, \dots, a_{p-1})) = w(A(a_0, \dots, a_{p-1})) = 1$. Let $\{b_n\}_{n=0}^{\infty}$ be a sequence with $b_0 = -1$, $-1 < b_n < 1$ for $n \geqslant 1$ and $w_n^2 = (1 - b_n)(1 + b_{n+1})$ for all n by Theorem 3.1(a). We need check that

$$\sum_{n=0}^{\infty} \prod_{k=0}^{n} \frac{1 - b_k}{1 + b_{k+1}} = \infty \tag{4}$$

holds. The openness of $W(A)$ will then follow from Theorem 3.3. For this, let $M_j = \sup\{\alpha_j$: there is some $\{\alpha_n\}_{n=0}^{p-1}$ in $(-1, 1)$ such that $a_n^2 = (1 - \alpha_n)(1 + \alpha_{n+1})$ for $0 \leqslant n < p$ $(\alpha_p \equiv \alpha_0)\}$ for $0 \leqslant j < p$. In the following, we show that $b_{n_0+lp+j} \leqslant M_j$ for all $l \geqslant 0$ and all j, $0 \leqslant j < p$. Indeed, if this is the case, then

$$\prod_{j=0}^{p-1} \frac{1 - b_{n_0+lp+j}}{1 + b_{n_0+lp+j+1}} \geqslant \prod_{j=0}^{p-1} \frac{1 - M_j}{1 + M_{j+1}} = 1$$

for all $l \geqslant 0$ by Lemma 4.5(b) and (a), from which we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \prod_{k=0}^{n} \frac{1 - b_k}{1 + b_{k+1}} &\geqslant \sum_{l=0}^{\infty} \left(\prod_{k=0}^{n_0-1} \frac{1 - b_k}{1 + b_{k+1}} \right) \left(\prod_{j=0}^{p-1} \frac{1 - b_{n_0+lp+j}}{1 + b_{n_0+lp+j+1}} \right) \\ &\geqslant \sum_{l=0}^{\infty} \prod_{k=0}^{n_0-1} \frac{1 - b_k}{1 + b_{k+1}} = \infty, \end{aligned}$$

and thus (4) holds.

For convenience, we only show that $b_{n_0} \leqslant M_0$; that $b_{n_0+lp+j} \leqslant M_j$ for general values of l and j can be done analogously. Assume otherwise that $b_{n_0} > M_0$. Then, in particular, $n_0 \geqslant 1$. Let $\{c_n\}_{n=0}^{\infty}$ be a sequence with $c_0 = b_{n_0}$ and $a_k^2 = (1 - c_{mp+k})(1 + c_{mp+k+1})$ for all $m \geqslant 0$ and all k, $0 \leqslant k < p$. Then $-1 < c_0 < 1$. We infer from

$$(1 - b_{n_0})(1 + b_{n_0+1}) = w_{n_0}^2 \geqslant a_0^2 = (1 - c_0)(1 + c_1) = (1 - b_{n_0})(1 + c_1)$$

that $b_{n_0+1} \geqslant c_1$ and thus $-1 < c_1 < 1$. In a similar fashion, we obtain inductively that $-1 < c_n < 1$ for all n. Thus $\{c_n\}_{n=0}^{\infty}$ is a sequence in $(-1, 1)$ associated with $A(a_0, \dots, a_{p-1})$ as in Theorem 3.1(a). On the other hand, we also have $a_k^2 = (1 - M_k)(1 + M_{k+1})$ for $0 \leqslant k < p$ $(M_p \equiv M_0)$ by the first assertion of Lemma 4.5(b). From

$$a_0^2 = (1 - M_0)(1 + M_1) = (1 - c_0)(1 + c_1) = (1 - b_{n_0})(1 + c_1)$$

and our assumption that $b_{n_0} > M_0$, we obtain $c_1 > M_1$, which is in contradiction to the second assertion in Lemma 4.5(b). Hence we have $b_{n_0} \leqslant M_0$ as asserted. This completes the proof of (a). □

Note that the preceding theorem generalizes both Propositions 2.3 and 4.2.

We conclude this paper by determining the numerical ranges of the unilateral (resp., bilateral) weighted shifts whose weights are such that all but one have equal moduli.

Theorem 4.6. *Let A be the unilateral weighted shift with weights $1, \dots, 1, c, 1, 1, \dots$, where c appears in the mth position ($m \geqslant 0$). Then*

(a) *$W(A) = \mathbb{D}$ if and only if $|c| \leqslant \sqrt{(m+2)/(m+1)}$, and*
(b) *if $|c| > \sqrt{(m+2)/(m+1)}$, then $W(A) = \{z \in \mathbb{C}\colon |z| \leqslant r\}$, where r satisfies the $m+1$ equations $r^{-2} = 2(1 + a_1)$, $r^{-2} = (1 - a_n)(1 + a_{n+1})$ for $1 \leqslant n < m$, and $|c|^2 r^{-2} = (1 - a_m)(1 + \sqrt{1 - r^{-2}})$ for some $a_1, \dots, a_m$ in $(-1, 1)$ (if $m = 0$, this is interpreted as "r satisfies $|c|^2 r^{-2} = 2(1 + \sqrt{1 - r^{-2}})$").*

The following two corollaries give the cases of $m = 0$ and $m = 1$. The former was obtained before in [1, pp. 1053–1054] (cf. also [13, p. 500] and [14, Proposition 2]), whose proof we omit.

Corollary 4.7. *Let A be the unilateral weighted shift with weights $c, 1, 1, \dots$. Then*

(a) *$W(A) = \mathbb{D}$ if and only if $|c| \leqslant \sqrt{2}$, and*
(b) *if $|c| > \sqrt{2}$, then $W(A) = \{z \in \mathbb{C}\colon |z| \leqslant |c|^2/(2\sqrt{|c|^2 - 1})\}$.*

Corollary 4.8. *Let A be the unilateral weighted shift with weights $1, c, 1, 1, \ldots$. Then*

(a) $W(A) = \mathbb{D}$ *if and only if* $|c| \leqslant \sqrt{3/2}$, *and*
(b) *if* $|c| > \sqrt{3/2}$, *then* $W(A) = \{z \in \mathbb{C}\colon |z| \leqslant (2|c|^2\sqrt{|c|^4 + 2|c|^2 - 3} - 2|c|^4 - 2|c|^2 + 4)^{-1/2}\}$.

Proof. We need only prove (b). Assuming that $|c| > \sqrt{3/2}$, let $s = 1/r$ and $t = \sqrt{1 - s^2}$. We have to solve the equations $s^2 = 2(1 + a_1)$ and $|c|^2 s^2 = (1 - a_1)(1 + \sqrt{1 - s^2})$ for s and a_1. These can be written as $1 - t^2 = 2(1 + a_1)$ and $|c|^2(1 - t^2) = (1 - a_1)(1 + t)$. Their solutions are easily seen to be $t = -|c|^2 + \sqrt{|c|^4 + 2|c|^2 - 3}$ and $s = (2|c|^2\sqrt{|c|^4 + 2|c|^2 - 3} - 2|c|^4 - 2|c|^2 + 4)^{1/2}$. Our assertion then follows. □

We now prove Theorem 4.6.

Proof of Theorem 4.6(a). We may assume that $c > 0$. Letting $n_0 = m + 1$ and $p = 1$ in Theorem 4.3(a), we have that $W(A)$ is open if and only if $w(A) = 1$. Thus to complete the proof, we need show that $w(A) = 1$ if and only if $c \leqslant \sqrt{(m+2)/(m+1)}$. Indeed, if $w(A) = 1$, then there is a sequence $\{a_n\}_{n=0}^{\infty}$ with $a_0 = -1, -1 < a_n < 1$ for $n \geqslant 1$, $1 = (1 - a_n)(1 + a_{n+1})$ for $n \neq m$, and $c^2 = (1 - a_m)(1 + a_{m+1})$ by Theorem 3.1(a). We derive from

$$1 = 2(1 + a_1) = (1 - a_1)(1 + a_2) = \cdots = (1 - a_{m-1})(1 + a_m)$$

that $a_1 = -1/2, a_2 = -1/3, \ldots, a_m = -1/(m+1)$ and from $c^2 = (1 - a_m)(1 + a_{m+1})$ that $a_{m+1} = [c^2(m+1) - (m+2)]/(m+2)$. Since $1 = (1 - a_n)(1 + a_{n+1})$ for all $n \geqslant m + 1$, the sequence $\{a_n\}_{n=m+1}^{\infty}$ is associated with the simple unilateral shift as in Theorem 3.1(a). In particular, we have $a_{m+1} \leqslant 0$ by Lemma 3.2. It follows that $c \leqslant \sqrt{(m+2)/(m+1)}$. Conversely, if $c \leqslant \sqrt{(m+2)/(m+1)}$, then, letting

$$a_n = \begin{cases} -1/(n+1) & \text{if } 0 \leqslant n \leqslant m, \\ (c^2(m+1) - (m+2))/(m+2) & \text{if } n = m+1, \\ a_{n-1}/(1 - a_{n-1}) & \text{if } n \geqslant m+2, \end{cases}$$

we have $-1 \leqslant a_n \leqslant 0$ for all n, $1 = (1 - a_n)(1 + a_{n+1})$ for $n \neq m$, and $c^2 = (1 - a_m)(1 + a_{m+1})$. Thus $w(A) \leqslant 1$ by Theorem 3.1(a). Since the simple unilateral shift S is a compression of A, we also have $w(A) \geqslant w(S) = 1$. This shows that $w(A) = 1$ as required.

(b) If $c > \sqrt{(m+2)/(m+1)}$, then obviously $W(A) = \{z \in \mathbb{C}\colon |z| \leqslant r\}$ for some $r \geqslant 1$ from (a). We now show that r is of the asserted form. Let $\{a_n\}_{n=0}^{\infty}$ be such that $a_0 = -1$, $-1 < a_n < 1$ for $n \geqslant 1$, $r^{-2} = (1 - a_n)(1 + a_{n+1})$ for $n \neq m$ and $c^2 r^{-2} = (1 - a_m)(1 + a_{m+1})$. By Lemma 3.2, we have $a_{m+1} \leqslant \sqrt{1 - r^{-2}}$. If $a_{m+1} < \sqrt{1 - r^{-2}}$, then let

$$b_n = \begin{cases} a_n & \text{if } 0 \leqslant n \leqslant m, \\ \sqrt{1 - r^{-2}} & \text{if } n \geqslant m+1. \end{cases}$$

Let $c' > c$ be such that $c'^2 r^{-2} = (1 - b_m)(1 + b_{m+1})$ and let A' be the unilateral weighted shift with weights $1, \ldots, 1, c', 1, 1, \ldots$, where c' is in the mth position. Then the relations $r^{-2} = (1 - b_n)(1 + b_{n+1})$ for $n \neq m$ and $c'^2 r^{-2} = (1 - b_m)(1 + b_{m+1})$ yield that $w(A'/r) \leqslant 1$ or $w(A') \leqslant r$ by Theorem 3.1(a). On the other hand, since $W(A)$ is closed, there is, by Proposition 2.1(b), a unit vector x in ℓ^2 with strictly positive components such that $\langle Ax, x\rangle = w(A)$. Thus

$$r = w(A) = \langle Ax, x\rangle < \langle A'x, x\rangle \leqslant w(A') \leqslant r,$$

which is a contradiction. Hence we must have $a_{m+1} = \sqrt{1 - r^{-2}}$ and, therefore, r satisfies the asserted $m + 1$ equations. □

The bilateral case of Theorem 4.6 was essentially obtained in [13, p. 500] and [14, Proposition 3]. We give an alternative proof here based on Proposition 2.3 and Theorem 4.6.

Theorem 4.9. *Let A be the bilateral weighted shift with weights $\ldots, 1, 1, c, 1, 1, \ldots$, where c appears in the mth position ($-\infty < m < \infty$). Then*

(a) $W(A) = \mathbb{D}$ *if and only if* $|c| \leqslant 1$, *and*
(b) *if* $|c| > 1$, *then* $W(A) = \{z \in \mathbb{C}\colon |z| \leqslant (|c|^2 + 1)/(2|c|)\}$.

Proof. (a) is an easy consequence of Proposition 2.3. To prove (b), assume that $|c| > 1$. Then $W(A) = \{z \in \mathbb{C}\colon |z| \leqslant r\}$ for some $r \geqslant 1$. Since $w(A/r) = 1$, by Theorem 3.1(a) there is a sequence $\{a_n\}_{n=-\infty}^{\infty}$ in $(-1, 1)$ such that $r^{-2} = (1 - a_n)(1 + a_{n+1})$ for $n \neq m$ and $|c|^2 r^{-2} = (1 - a_m)(1 + a_{m+1})$. Then both $\{a_n\}_{n=m+1}^{\infty}$ and $\{-a_n\}_{n=m}^{-\infty}$ are associated with the unilateral weighted shift with weights $1/r, 1/r, \ldots$. We infer from Lemma 3.2 that $a_{m+1}, -a_m \leqslant \sqrt{1 - r^{-2}}$. Then using the closedness of $W(A)$, we may argue as in the proof of Theorem 4.6(b) that $a_{m+1} = -a_m = \sqrt{1 - r^{-2}}$. Thus $|c|^2 r^{-2} = (1 + \sqrt{1 - r^{-2}})^2$ and it follows that $r = (|c|^2 + 1)/(2|c|)$. This completes the proof. □

Acknowledgments

The researches of the two authors were partially supported by the National Science Council of the Republic of China under NSC-99-2115-M-009-013-MY2 and NSC-99-2115-M-009-002-MY2, respectively. They were also supported by the MOE-ATU.

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