Invited Paper

Minimum Redundancy ISI Free FIR Filterbank Transceivers

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ABSTRACT

The discrete multitone modulation system (DMT) has been demonstrated to be a very useful technique for high speed transmission over frequency selective channels such as the digital subscriber loops. The DMT system can be realized using a filterbank transceiver, the synthesis bank as the transmitter and the analysis bank as the receiver. With proper time domain equalization, the channel can usually be modeled as an FIR filter with order L. It is known that if a redundancy of length L is introduced, FIR filterbank transceivers with zero ISI (Inter-Symbol Interference) property can be achieved. For example, in DFT based DMT system, redundancy is introduced by adding a cyclic prefix of length L. In this paper, we will derive the minimum length of redundancy required for FIR filterbank transceivers with ISI Free property. For a give channel, we will show that the minimum length is directly related to the Smith form of an appropriately defined channel matrix.

1. INTRODUCTION

The discrete multitone modulation (DMT) is now a widely used technique for high speed transmission over channels such as digital subscriber loops[1]-.[5] In the DMT scheme, the channel is divided into subbands, each with a different frequency band. The transmission power and bits are judiciously allocated according to the SNR (signal to noise ratio) in each band.[4] This is similar to the water pouring scheme for discrete transmission channels. The DMT scheme is realized by designing a transmitter and a receiver that effectively divides the channel into appropriate subbands.

The DFT based DMT system has been proposed as a practical implementation of DMT system,[5].[2] A certain redundancy known as cyclic prefix is added to allow complete removal of ISI. Very good transmission rate can be accomplished using DFT based DMT systems for channels such as ADSL and HDSL. In the DFT based systems, the transmitter and receiver consists of DFT filters. In,[7] Kasturia et. al. extend the DFT based transceiver to more general vector coding system. The transmitting filter or transmitting vectors are eigen vectors of an appropriately defined channel matrix. When the channel noise is AWGN, the vector coding is shown to be optimal in terms of bit rate maximization subject to a transmission power budget. Bit rate maximization for general noise sources are considered in.[8]

Connection between an M-band filter bank and an M-band transmultiplexer (an M-band filterbank transceiver or DWMT system) was first observed by Vetterli.[12] When the analysis and synthesis bank banks of a perfect reconstruction filter bank are interchanged, the new structure becomes a transmultiplexer or a filterbank transceiver (Fig. 1). The DMWT system in this case has interpolation ratio $N = M$ and it is called *minimally interpolated.* When the transmission channel is ideal, the minimally interpolated M-subband filterbank transceiver is ISI free if the corresponding filter bank has perfect reconstruction.[12] The ISI free property means there is no intra- and inter-subband ISI. The DWMT (discrete wavelet multitone) system[11] is obtained by interchanging perfect reconstruction analysis and synthesis banks. However, when the channel is not ideal, the perfect reconstruction property of the filter bank no longer translates to ISI free property of filterbank transceivers. Performance evaluation conducted in,[13][14] shows that the resulting ISI can seriously degrade the system performance. To reduce the amount of ISI, inter- and intra-subband equalization are performed on the receiver outputs in,[11].[15]

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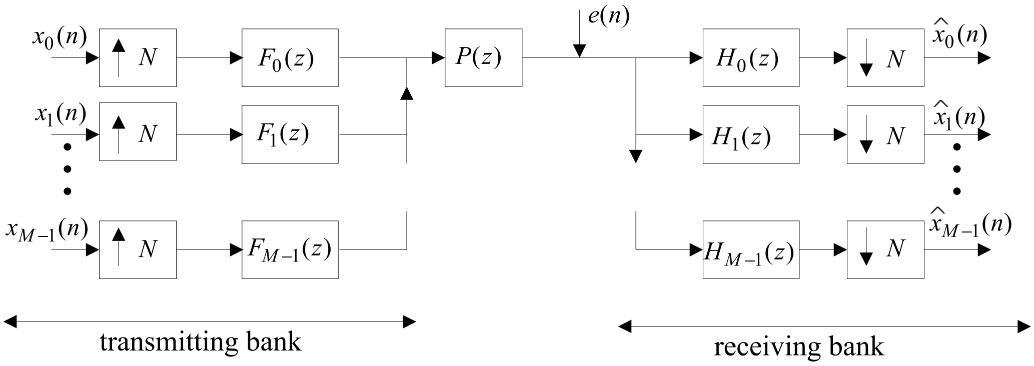

Figure 1. An M-subband filterbank transceiver over a frequency selective channel $P(z)$.

When the interpolation ratio $N > M$, the filterbank transceiver is called *over interpolated*; in average every N output samples of the transmitter contains $K = N - M$ redundant samples. The cyclic prefix in DFT based DMT system and zero padding in vector coding DMT system[7] are examples of such redundant samples. Using over interpolated filterbank transceiver, it is possible to cancel ISI completely with appropriate redundancy K. In a typical system model, the channel is an FIR filter $P(z)$ of order L upon time domain equalization. In the DFT based DMT system, zero ISI is achieved if prefix length $K \geq L$.

Advances to the more general FIR over interpolated system has been made in,[16][17] for ISI cancellation using precoding. It has been shown that FIR transceivers exist for redundancy $K < L$ under very general condition. In particular, for a given number of subbands M and interpolation ratio N, the condition for the existence of FIR transceivers can be given in terms of the zeros of the channel $P(z)$. Let S be the set that contains the zeros of $P(z)$: $S = \{\alpha_0, \alpha_1, \cdots, \alpha_{L-1}\}$, with $P(\alpha_\ell) = 0$. The necessary and sufficient condition for the existence of FIR transceiver is

$$\bigcap_{0 \leq \ell_1 < \ell_2 < \cdots < \ell_M \leq N-1} (S_{\ell_1} \bigcup S_{\ell_2} \bigcup \cdots \bigcup S_{\ell_M}) = \phi,$$

where

$$S_{\ell_k} = \{e^{-j2\pi\ell_k/N}\alpha_0, e^{-j2\pi\ell_k/N}\alpha_1, \cdots, e^{-j2\pi\ell_k/N}\alpha_{L-1}\}.$$

Extensions to the time varying transceivers have been considered,[9].[10]

In this paper we will derive a new necessary and sufficient condition for the existence of FIR transceivers. The new condition is easier to test and understand. Furthermore it provides additional insight to the problem of designing FIR transceivers. For example, given the zeros of the channel $P(z)$, we will be able to determine explicitly the redundancy for which FIR transceivers exist. There are cases that minimum redundancy $K = L$. The condition for such cases will be given. Under very general condition, we can choose redundancy $K = 1$. This condition is the same as that given in.[17] We will also see that, when FIR transceivers exist for a certain redundancy K and number of subbands M, FIR solutions may not exist if the redundancy is increased. One such example will be given.

1.1. Notations and Preliminaries

- Boldfaced lower case letters are used to represent vectors and boldfaced upper case letters are reserved for matrices. The notations $\mathbf{A}^T$ and $\mathbf{A}^\dagger$ represent the transpose of $\mathbf{A}$ and transpose-conjugate of $\mathbf{A}$.
- The notation $\widetilde{\mathbf{A}}(z)$ denotes $\mathbf{A}^\dagger(1/z^*)$. For matrices with real coefficients, $\widetilde{\mathbf{A}}(z) = \mathbf{A}^T(z^{-1})$.
- The function $\mathcal{E}[y]$ denotes the expected value of the random variable y.
- The notation $\mathbf{I}_N$ is used to represent the $N \times N$ identity matrix. The subscript is omitted whenever the size is clear from the context.
- *Unimodular matrices.* An $N \times N$ matrix $\mathbf{A}(z)$ is called unimodular if $\det \mathbf{A}(z) = c$, a constant.[19] A causal unimodular FIR matrix $\mathbf{A}(z)$ the property that $\mathbf{A}^{-1}(z)$ is also causal and FIR.

1.2. Channel Models

Fig. 2(a) shows the block diagram of a filterbank transceiver. The discrete time channel is modeled as an LTI filter $h(n)$ with additive noise $\nu(n)$ as shown in Fig. 2(a). A time domain equalizer (TEQ) $T(z)$ precedes the filterbank receiver. Typically the filter $H(z)$ can be further modeled as a rational transfer function, $H(z) = P(z)/B(z)$. The equalizer $T(z)$ is usually designed to cancel the poles of $H(z)$ and the resulting overall transfer function becomes the FIR filter $P(z)$ as shown in Fig. 2(b). Suppose $P(z)$ is of order L and

$$P(z) = p_0 + p_1 z^{-1} + \cdots + p_L z^{-L}.$$

The equalized impulse response of the channel is thus shorten to L. Each input sample will be spread to a duration of length $L+1$ as a result. The noise $e(n)$ shown in Fig. 2(b) is obtained by feeding the original noise $\nu(n)$ to the equalizer $T(z)$. The equalized channel model in Fig. 2(b) will be used throughout this paper; the channel refers to the equalized channel $P(z)$ and the channel noise refers to the equalized noise $e(n)$ in this paper.

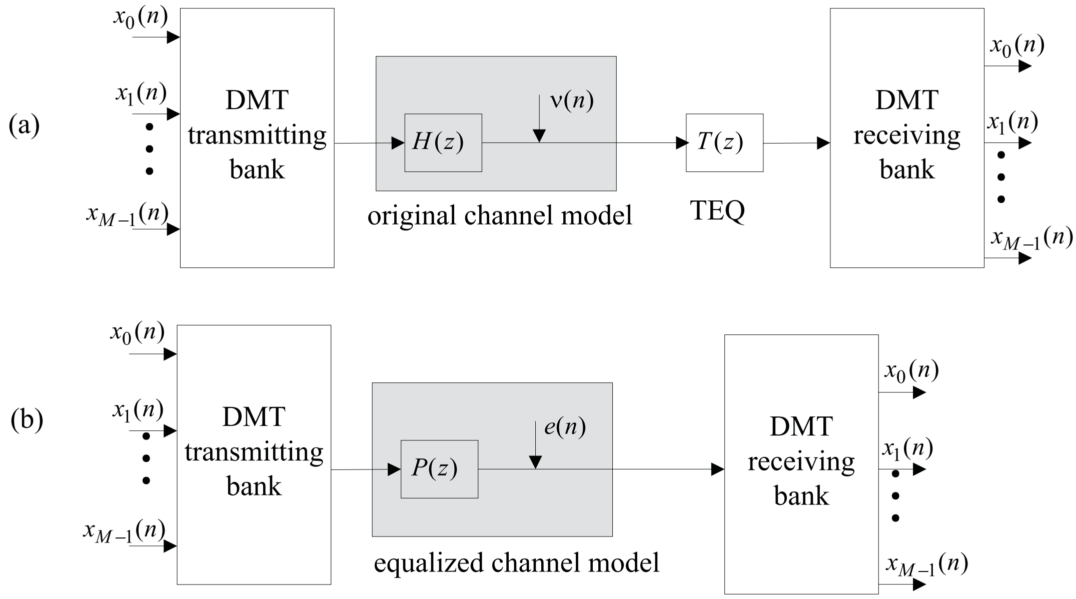

Figure 2. (a) Block diagram of the filterbank transceiver, including a discrete time channel model and an equalizer $T(z)$. (b) Block diagram of the filterbank transceiver with an equalized channel model.

2. POLYPHASE REPRESENTATION OF FILTERBANK TRANSCEIVERS

Consider Fig. 1, where an M-subband filterbank transceiver is shown. The channel is represented by an FIR filter $P(z)$ with additive noise $e(n)$ as explained in Sec. 1.2. The filters $F_k(z)$ and $H_k(z)$ are called transmitting and receiving filters respectively. The interpolation ratio N is not necessary to be the same as the number of subbands M. Two cases will be studied: (i) When $N = M$, we say the system is minimally interpolated; (ii) When $N > M$, we say it is over interpolated and redundancy is introduced in this case. The case of $N < M$ is of no interest in our application because in this case the input data $x_k(n)$ can never be fully recovered, no matter what the channel is.

Using polyphase decomposition we can decompose the k-th transmitting filter $F_k(z)$ with respect to the integer N,[19]

$$F_k(z) = \sum_{n=0}^{N-1} G_{n,k}(z^N) z^{-n}. \tag{1}$$

Writing the polyphase representation for all the M transmitting filters, we have

$$[F_0(z)\ F_1(z)\ \cdots\ F_{M-1}(z)] = [1\ z^{-1}\ \cdots\ z^{-N+1}] \underbrace{\begin{pmatrix} G_{0,0}(z^N) & G_{0,1}(z^N) & \cdots & G_{0,M-1}(z^N) \\ G_{1,0}(z^N) & G_{1,1}(z^N) & \cdots & G_{1,M-1}(z^N) \\ \vdots & \vdots & \ddots & \vdots \\ G_{N-1,0}(z^N) & G_{N-1,1}(z^N) & \cdots & G_{N-1,M-1}(z^N) \end{pmatrix}}_{\mathbf{G}(z^N)}, \quad (2)$$

where the $N \times M$ matrix $\mathbf{G}(z)$ is the polyphase matrix of the transmitter. Using the noble identity,[19] we can interchange the expander and $\mathbf{G}(z^N)$. The transmitter can be implemented using its polyphase matrix as shown in Fig. 3. In a similar manner, we can decompose the receiving filters as

$$H_k(z) = \sum_{n=0}^{N-1} S_{k,n}(z^N) z^n. \quad (3)$$

Then by invoking the noble identity, the receiver can be redrawn as Fig. 3. The receiving filters $H_k(z)$ are related to the $M \times N$ polyphase matrix $\mathbf{S}(z)$ of the receiver as:

$$\begin{pmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{pmatrix} = \underbrace{\begin{pmatrix} S_{0,0}(z^N) & S_{0,1}(z^N) & \cdots & S_{0,N-1}(z^N) \\ S_{1,0}(z^N) & S_{1,1}(z^N) & \cdots & S_{1,N-1}(z^N) \\ \vdots & \vdots & \ddots & \vdots \\ S_{M-1,0}(z^N) & S_{M-1,1}(z^N) & \cdots & S_{M-1,N-1}(z^N) \end{pmatrix}}_{\mathbf{S}(z^N)} \begin{pmatrix} 1 \\ z \\ \vdots \\ z^{N-1} \end{pmatrix}, \quad (4)$$

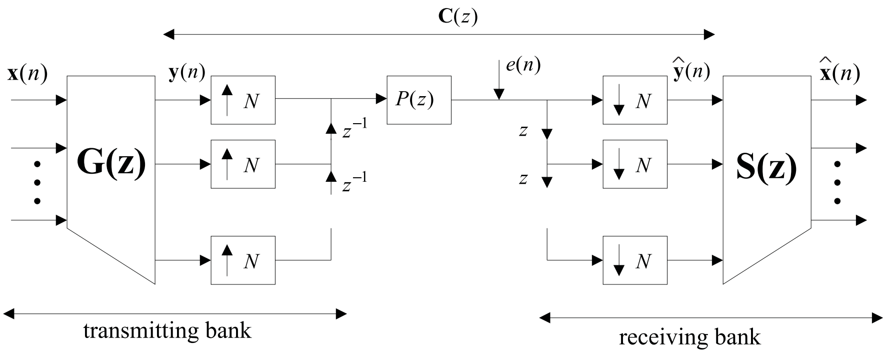

Figure 3. The polyphase representation of the transmitter and receiver in a filterbank transceiver.

Decomposition of the Channel

Using polyphase representation, we can decompose the channel as

$$P(z) = P_0(z^N) + P_1(z^N)z^{-1} + \cdots + P_{N-1}(z^N)z^{-N+1}. \quad (5)$$

In order to further simplify Fig. 3, we need to apply an identity from the multirate theory. It is shown in[19] that the multirate system in Fig. 4 is in fact equivalent to an LTI system with transfer function $A(z)$ given by

$$A(z) = \begin{cases} P_{i-j}(z), & \text{for } i \geq j; \\ z^{-1} P_{N+i-j}(z), & \text{for } i < j, \end{cases}$$

where $P_k(z)$ is defined in (5). We see that the $N \times N$ system from $\mathbf{y}(n)$ to $\hat{\mathbf{y}}(n)$ in Fig. 3 is in fact an LTI system with transfer matrix $\mathbf{C}(z)$ given by,

$$\mathbf{C}(z) = \begin{pmatrix} P_0(z) & z^{-1}P_{N-1}(z) & z^{-1}P_{N-2}(z) & \cdots & z^{-1}P_1(z) \\ P_1(z) & P_0(z) & z^{-1}P_{N-1}(z) & \cdots & z^{-1}P_2(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{N-1}(z) & P_{N-2}(z) & P_{N-3}(z) & \cdots & P_0(z) \end{pmatrix}. \tag{6}$$

Matrices in the above form are known as pseudo circulant matrices.[19] Properties of $\mathbf{C}(z)$ that will be used in later sections are given in the next section.

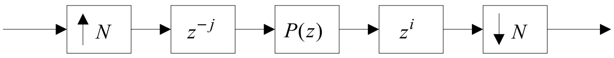

Figure 4. The polyphase identity.

With the channel matrix $\mathbf{C}(z)$, we can redraw Fig. 3 as Fig. 5. As we will see later, the polyphase representation in Fig. 5 will facilitate a systematic study of filterbank transceivers. Many useful theoretical and practical results can be drawn from such representation.

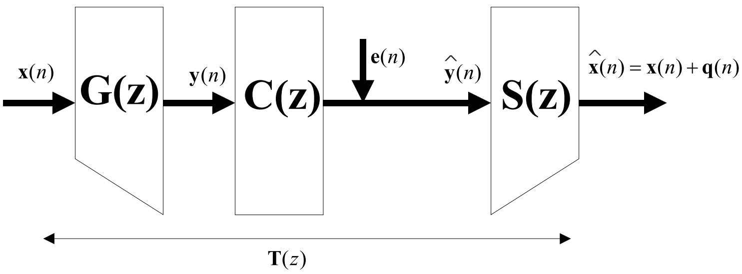

Figure 5. The polyphase representation of a filterbank transceiver.

Zero ISI Condition. From the polyphase decomposition in Fig. 5, we see that even though multirate building blocks are used in a filterbank transceiver, it is in fact an LTI M-input M-output system. The transfer matrix $\mathbf{T}(z)$ of the overall system can be expressed as

$$\mathbf{T}(z) = \mathbf{S}(z)\mathbf{C}(z)\mathbf{G}(z). \tag{7}$$

The overall system is free from inter-subband ISI if $\mathbf{T}(z)$ is a diagonal matrix. It is free from intra-subband ISI when the diagonal elements of $\mathbf{T}(z)$ are merely delays. If it is free from both inter- and intra-subband ISI, we say the filterbank transceiver is ISI free; in the absence of channel noise the outputs of an ISI free filterbank transceiver are identical to the inputs except delays and scalers. Without much loss of generality, we can use the ISI free condition,

$$\mathbf{S}(z)\mathbf{C}(z)\mathbf{G}(z) = \mathbf{I}. \tag{8}$$

3. PROPERTIES OF THE CHANNEL MATRIX

This section aims to provide a collection of the properties of the channel matrix (give in (6)) that will be useful for our subsequent discussion. Some of these properties are known and can be found in text books, e.g.,.[19] Some have not been shown explicitly before and will be derived. The first one gives the diagonalization of pseudo circulant matrices using DFT matrices.[19] The second property relates $\det \mathbf{C}(z)$ to the underlying filter $P(z)$. The third one is the property of the Smith form of $\mathbf{C}(z)$. The fourth property gives a simplified form of $\mathbf{C}(z)$ when $L < N$.

1. A pseudo circulant matrix $\mathbf{C}(z)$ of the form in (6) assumes the decomposition,[19]

$$\mathbf{C}(z^N) = \mathbf{D}(z)\mathbf{W}\mathbf{\Sigma}(z)\mathbf{W}^\dagger\mathbf{D}(z^{-1}), \tag{9}$$

where

$$\mathbf{D}(z) = \operatorname{diag}\begin{pmatrix}1 & z^{-1} & \cdots & z^{-N+1}\end{pmatrix},$$
$$\mathbf{\Sigma}(z) = \operatorname{diag}\begin{pmatrix}P(z) & P(zW^{-1}) & \cdots & P(zW^{-N+1})\end{pmatrix}.$$

The matrix $\mathbf{W}$ is the $N \times N$ DFT matrix given by,

$$[\mathbf{W}]_{kn} = \frac{1}{\sqrt{N}} W^{kn} \quad \text{with} \quad W = e^{-j2\pi/N} \quad \text{for } 0 \le k, n \le N-1.$$

Note that the k-th diagonal entry of $\mathbf{\Sigma}(z)$ is $P(zW^{-k})$. The zeros of $P(zW^{-k})$ can be obtained by rotating the zeros of $P(z)$ by $\frac{2k\pi}{N}$.

2. When the channel $P(z)$ is a causal FIR filter, $\det \mathbf{C}(z)$ is also a causal FIR filter. Furthermore, suppose $P(z)$ has zeros at α_ℓ, for $\ell = 1, 2, \cdots, L$, and

$$P(z) = p_0 \prod_{\ell=1}^{L} (1 - \alpha_\ell z^{-1}).$$

Then $\det \mathbf{C}(z)$ has zeros at α_ℓ^N, in particular,

$$\det \mathbf{C}(z) = p_0^N \prod_{\ell=1}^{L} (1 - \alpha_\ell^N z^{-1}). \tag{10}$$

Proof: Using (9), we can obtain $\det \mathbf{C}(z)$ as,

$$\det \mathbf{C}(z) = \det(\mathbf{\Sigma}(z)) = \prod_{k=0}^{N-1} P(z^{1/N} W^{-k}).$$

We observe that

$$\prod_{k=0}^{N-1} \left(1 - \alpha_\ell \left(z^{1/N} W^k\right)^{-1}\right) = \prod_{k=0}^{N-1} \left(1 - \left(\alpha_\ell z^{-1/N}\right) W^{-k}\right) = (1 - \alpha_\ell^N z^{-1}).$$

The property in (10) follows immediately from the above equation.

Suppose $P(z)$ has q zeros on the same circle centered at the origin and these zeros are separated by angles that are multiples of $2\pi/N$. To be more specific, suppose these q zeros are $\alpha_\ell = re^{j\theta_\ell}$, for $\ell = 1, 2, \cdots, q$ and $\theta_i - \theta_\ell =$ an integer multiple of $2\pi/N$. Then

$$\alpha_1^N = \alpha_2^N = \cdots = \alpha_q^N.$$

In this case, $\det \mathbf{C}(z)$ has zeros at α_1^N of multiplicity q.

3. *Smith form decomposition.* An $N \times N$ polynomial matrix $\mathbf{A}(z)$ in z^{-1} and can be represented using the Smith form decomposition[19]

$$\mathbf{A}(z) = \mathbf{U}(z)\mathbf{\Gamma}(z)\mathbf{V}(z), \tag{11}$$

where all three matrices in the decomposition are matrix polynomials in the variable z^{-1}. The matrices $\mathbf{U}(z)$ and $\mathbf{V}(z)$ are unimodular matrices, and $\mathbf{\Gamma}(z)$ is a diagonal matrix,

$$\mathbf{\Gamma}(z) = \begin{pmatrix} \gamma_0(z) & 0 & \cdots & 0 \\ 0 & \gamma_1(z) & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & \gamma_{N-1}(z) \end{pmatrix}.$$

Moreover the unimodular matrices $\mathbf{U}(z)$ and $\mathbf{V}(z)$ can be so chosen that the polynomial $\gamma_k(z)$ is a factor of $\gamma_{k+1}(z)$, i.e.,

$$\gamma_k(z)|\gamma_{k+1}(z), \quad 0 \le k \le N-2.$$

The matrix $\mathbf{\Gamma}(z)$, called the Smith form of $\mathbf{A}(z)$, is unique. Although $\mathbf{\Gamma}(z)$ is unique, the unimodular matrices $\mathbf{U}(z)$ and $\mathbf{V}(z)$ are not. The decomposition can be obtained using elementary row operations and elementary column operations.

Smith form of $\mathbf{C}(z)$.

Let the Smith form decomposition of $\mathbf{C}(z)$ be,

$$\mathbf{C}(z) = \mathbf{U}(z)\mathbf{\Gamma}(z)\mathbf{V}(z).$$

Note that $\det \mathbf{C}(z) = \det \mathbf{\Gamma}(z)$, which is equal to the product $\Pi_{k=0}^{N-1}\gamma_k(z)$. Also $\det \mathbf{C}(z)$ by (10) is an FIR filter with order L. This implies that $\mathbf{\Gamma}(z)$ has at most L non unity diagonal terms. Recall that the polynomials $\gamma_k(z)$, for $k = 0, 1, \cdots, N-1$, in the diagonal of $\mathbf{\Gamma}(z)$ have the property that $\gamma_k(z)$ divides $\gamma_{k+1}(z)$. It follows that at least the first $N-L$ diagonal terms are equal to 1, i.e.,

$$\mathbf{\Gamma}(z) = \begin{pmatrix} 1 & 0 & \cdots & & & & \\ 0 & 1 & & & & & \\ \vdots & & \ddots & & & & \\ 0 & & & 1 & 0 & & \\ 0 & & & & \gamma_{N-\rho}(z) & & \\ \vdots & & & & & \ddots & \\ 0 & 0 & \dots & & & & \gamma_{N-1}(z) \end{pmatrix}, \tag{12}$$

with the number of non unity terms $\rho \le L$.

4. Usually the interpolation ratio is chosen to be larger than the order L of $P(z)$. In this case, the N polyphases of $P(z)$ are constants and the last $N-L-1$ polyphases are zero. The matrix $\mathbf{C}(z)$ can be partitioned as an $N \times (N-L)$ constant matrix $\mathbf{C}_0$ and an $N \times L$ FIR causal matrix $\mathbf{C}_1(z)$ that is of order 1,

$$\mathbf{C}(z) = [\underbrace{\mathbf{C}_0}_{N\times(N-L)} \; \vdots \; \underbrace{\mathbf{C}_1(z)}_{N\times L}], \tag{13}$$

where

$$\mathbf{C}_0 = \begin{pmatrix} p_0 & 0 & \cdots & 0 & & 0 \\ p_1 & p_0 & & \vdots & & 0 \\ \vdots & & \ddots & & & \vdots \\ p_L & p_{L-1} & & & & 0 \\ 0 & p_L & & & & 0 \\ \vdots & \vdots & \ddots & & & \vdots \\ 0 & 0 & & p_L & \cdots & p_0 \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & & 0 & \cdots & p_L \end{pmatrix} \quad \text{and} \quad \mathbf{C}_1(z) = \begin{pmatrix} z^{-1}p_L & z^{-1}p_{L-1} & \cdots & z^{-1}p_1 \\ 0 & z^{-1}p_L & \cdots & z^{-1}p_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & z^{-1}p_L \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ p_0 & 0 & \cdots & 0 \\ p_1 & p_0 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{L-1} & p_{L-2} & & p_0 \end{pmatrix} \tag{14}$$

4. OVER-INTERPOLATED FILTERBANK TRANSCEIVERS

We say a filterbank transceiver is over-interpolated if the interpolation ratio N is greater than the number of subbands M. In this case there are more samples at the output of the transmitter than the input. There are

$K = N - M$ redundant samples in every N samples of the transmitter output. By introducing proper redundancy to the transmitter output, the channel can be equalized perfectly to achieve ISI free property using FIR transceivers. For example, in the DFT based DMT system, redundancy is introduced by adding cyclic prefix. The transmitting and receiving filters are FIR of length N and M, respectively. In this section, we will consider general FIR transceivers. For a given interpolation ratio N, we will derive the minimum redundancy for the existence of FIR transceivers. In particular, the minimum redundancy will be related to the Smith form decomposition of the channel matrix $\mathbf{C}(z)$.

With the number of subbands M and interpolation ratio N, the transmitter $\mathbf{G}(z)$ and $\mathbf{S}(z)$ are respectively of dimension $N \times M$ and $M \times N$. The channel matrix $\mathbf{C}(z)$ is of dimension $N \times N$. The following theorem gives the condition for the existence of FIR transceivers in terms of the Smith form of the channel matrix.

LEMMA 4.1. *The smallest rank of $\mathbf{C}(z)$ is $N-\rho$, where ρ is the number of non-identity terms on the diagonal of the Smith form $\mathbf{\Gamma}(z)$ as given in (12).*

Proof:

The determinants of the two unimodular matrices $\mathbf{U}(z)$ and $\mathbf{V}(z)$ in the Smith form decomposition are constants. The rank of $\mathbf{C}(z)$ is the same the rank of the Smith form $\mathbf{\Gamma}(z)$. We can consider the rank of $\mathbf{\Gamma}(z)$. Observing (12), we can see that the smallest rank of $\mathbf{\Gamma}(z)$ is $N-\rho$. This happens when $z = z_0$ is a zero of $\gamma_{N-\rho}(z)$.

$\Delta\Delta\Delta$

THEOREM 4.2. *Consider the DMT system in Fig. 1. Let the Smith form $\mathbf{\Gamma}(z)$ of the matrix $\mathbf{C}(z)$ be as given in (12). Then there exist FIR $\mathbf{G}(z)$ and $\mathbf{S}(z)$ such that the transceiver is ISI free if and only if one of the following conditions is true*

(i) *the added redundancy $K \geq \rho$, where ρ is the number of non-identity terms on the diagonal of the smith form $\mathbf{\Gamma}(z)$ of $\mathbf{C}(z)$;*

(ii) *$rank(\mathbf{C}(z)) \geq M$, for all z.*

The minimum redundancy for FIR transceiver solutions is ρ. Furthermore, the solutions for FIR ISI free transceivers are not unique whenever one solution exists.

proof:

We first show the sufficiency and necessity of condition (i) and then the equivalence of these two conditions.

Sufficiency of condition (i)

Consider the following choice of FIR transmitter $\mathbf{G}(z)$,

$$\mathbf{G}(z) = \mathbf{V}^{-1}(z) \begin{pmatrix} \mathbf{I} \\ \mathbf{0} \end{pmatrix}.$$

Then,

$$\mathbf{C}(z)\mathbf{G}(z) = \mathbf{U}(z)\mathbf{\Gamma}(z)\mathbf{V}(z)\mathbf{V}^{-1}(z) \begin{pmatrix} \mathbf{I} \\ \mathbf{0} \end{pmatrix} = \mathbf{U}(z) \begin{pmatrix} \mathbf{I} \\ \mathbf{0} \end{pmatrix}.$$

So if we choose the receiver as

$$\mathbf{S}(z) = \begin{pmatrix} \mathbf{I} & \mathbf{0} \end{pmatrix} \mathbf{U}^{-1}(z),$$

the transceiver is FIR and ISI free. The unimodular matrices $\mathbf{U}(z)$ and $\mathbf{V}(z)$ are not unique in the smith form decomposition, so $\mathbf{G}(z)$ and $\mathbf{S}(z)$ are not unique.

Necessity of condition (i)

Suppose the added redundancy $K < \rho$ and there exist FIR $\mathbf{G}(z)$ and $\mathbf{S}(z)$ such that the system is ISI free, i.e., $\mathbf{S}(z)\mathbf{C}(z)\mathbf{G}(z) = \mathbf{I}_M$. Using $\mathbf{C}(z) = \mathbf{U}(z)\mathbf{\Gamma}(z)\mathbf{V}(z)$, we have

$$(\mathbf{S}(z)\mathbf{U}(z))\,\mathbf{\Gamma}(z)\,(\mathbf{V}(z)\mathbf{G}(z)) = \mathbf{I}_M. \tag{15}$$

As $\gamma_k(z)$ divides $\gamma_{k+1}(z)$, the zeros of $\gamma_k(z)$ are also zeros of $\gamma_{k+1}(z)$. The last ρ non identity terms have a common factor $\gamma_{N-\rho}(z)$. If $\gamma_{N-\rho}(z_0)=0$, then

$$\gamma_{N-\rho}(z_0)=\gamma_{N-\rho+1}(z_0)=\cdots=\gamma_{N-1}(z_0)=0.$$

It follows that $\mathbf{\Gamma}(z_0)$ has rank $N-\rho$ and the left hand side of (15) at $z=z_0$ has at most rank $N-\rho$. But the right hand side of (15) has rank always equal to $M=N-K$, which is greater than $N-\rho$ when $K<\rho$. So we have a contradiction in this case.

Equivalence of conditions (i) and (ii)

From Lemma 4.1, we know the smallest rank of $\mathbf{C}(z)$ is $N-\rho$, so

$$rank(\mathbf{\Gamma}(z))\geq N-\rho, \quad \text{for all } z.$$

As the added redundancy $K=N-M$, condition (i) is the same as $N-M\geq\rho$ or, equivalently,

$$N-\rho\geq M.$$

When condition (i) holds, we have

$$rank(\mathbf{\Gamma}(z))\geq N-\rho\geq M, \quad \text{for all } z.$$

Conversely, when condition (ii) holds, the rank of $\mathbf{\Gamma}(z)$ is no less than M, including the smallest rank $N-\rho$; therefore, we have $N-\rho\geq M$.

$\Delta\Delta\Delta$

Remark. When $\rho=1$, we only need to use redundancy $K=1$, which is the lowest redundancy possible for any non ideal channel when the transceiver is FIR. The case $\rho=1$ occurs when $\det\mathbf{C}(z)$ has distinct roots. From (10), we see that $\det\mathbf{C}(z)$ has roots at α_ℓ^N, where α_ℓ for $\ell=1,2,\cdots,L$ are the L roots of the channel $P(z)$. The roots of $\det\mathbf{C}(z)$ are distinct if and only if α_ℓ have the property that

$$\frac{\alpha_k}{\alpha_\ell}\neq e^{j2m\pi/N}, \quad \text{where } m \text{ is any integer in the range } 1\leq m\leq N-1. \tag{16}$$

That means, if two roots α_k and α_ℓ are of the same magnitude, their phase difference can not be a multiple of $2\pi/N$. This condition is the same as that given in.[16] For practical channels, the probability that the roots of $P(z)$ satisfy (16) is almost one. Therefore redundancy of $K=1$ is sufficient for the existence of FIR ISI free transceivers for most practical cases.

THEOREM 4.3. *Consider the $N\times N$ channel matrix $\mathbf{C}(z)$ given in (6). The smallest rank of $\mathbf{C}(z))$ is equal to $N-\mu$, where μ is the largest number of distinct zeros that have the same magnitude and their difference in angles are integer multiples of $2\pi/N$.*

Proof:

Consider the decomposition in (9). Because $\mathbf{W}$ and $\mathbf{D}(z)$ are unitary matrices, the rank of $\mathbf{C}(z)$ is the same as the rank of $\mathbf{\Sigma}(z)$. Recall that

$$\mathbf{\Sigma}(z)=\text{diag}\begin{pmatrix}P(z) & P(zW^{-1}) & \cdots & P(zW^{-(N-1)})\end{pmatrix}.$$

The number of terms on the diagonal of $\mathbf{\Sigma}(z)$ that have common zeros determines the smallest rank of $\mathbf{\Sigma}(z)$. Suppose the largest number of terms that have common zeros is μ. Then the smallest rank of $\mathbf{\Sigma}(z)$ is $N-\mu$. Let the μ terms be

$$P(zW^{-\ell_1}),P(zW^{-\ell_2}),\cdots,P(zW^{-\ell_\mu})$$

and the common zero is $z=z_0$. It follows that $P(z)$ has distinct zeros at $z_0W^{-\ell_1},z_0W^{-\ell_2},\cdots z_0W^{-\ell_m u}$. These zeros have the same magnitude $|z_0|$ and they differ in the angle by integer multiples of $2\pi/N$. So μ is also the largest number of distinct zeros of $P(z)$ that have the same magnitude and differ in angles by integer multiples of $2\pi/N$.

$\Delta\Delta\Delta$

Combining Lemma 4.1 and Theorem 4.3, we can conclude the following:

LEMMA 4.4. *The number of non-identity terms ρ on the diagonal of $\mathbf{\Gamma}(z)$ is equal to μ, the largest number of distinct zeros that have the same magnitude and their differences in angles are integer multiples of $2\pi/N$.*

By combining Theorem 4.1 and Theorem 4.2, we can relate the existence of FIR transceivers to the zeros of the channel $P(z)$.

THEOREM 4.5. *Consider the DMT system in Fig. 1 with added redundancy $K = N - M$. Then FIR transceivers exist if and only if $P(z)$ does not have more than K distinct zeros that have the same magnitude and differ in angles by integer multiples of $2\pi/N$.*

Remarks:

1. *When is ρ equal to L?* The minimum redundancy required for FIR transceivers falls into the range $1 \le \rho \le L$. The minimum redundancy $\rho = L$ if and only if there are L non identity terms in the Smith form $\mathbf{\Gamma}(z)$. This happens only when $P(z)$ has distinct zeros and these zeros lie on the same circle with angles difference that are integer multiples of $2\pi/N$,

2. The number ρ is the largest number of *distinct* zeros of $P(z)$ that have the same magnitude but differs in angles by integer multiples of $2\pi/N$. Zeros of multiplicity greater than one count as one. For example, consider $P(z) = 1 + 2z^{-1} + z^{-2}$. The channel has double zeros at $z = -1$. The number of zeros on the unit circle is 2 but the number of distinct zeros is one. In this case $\rho = 1$ for all N. For instance $M = 1$ and $K = 1$; we have $N = 2$. The polyphases of $P(z)$ with respect to $N = 2$ are $P_0(z) = 1 + z^{-1}$ and $P_1(z) = 2$. The channel matrix $\mathbf{C}(z)$ is given by

$$\mathbf{C}(z) = \begin{pmatrix} 1 + z^{-1} & 2z^{-1} \\ 2 & 1 + z^{-1} \end{pmatrix}$$

   The Smith form $\mathbf{\Gamma}(z)$ of $\mathbf{C}(z)$ is

$$\mathbf{\Gamma}(z) = \mathbf{C}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 - 2z^{-1} + z^{-2} \end{pmatrix}$$

3. When $P(z)$ has distinct zeros, the number ρ can be determined from the multiplicities of the zeros of $\det(\mathbf{C}(z))$. Suppose $\det(\mathbf{C}(z))$ has q distinct roots $\beta_0, \beta_1, \cdots, \beta_{q-1}$ with multiplicities, respectively, $\rho_0, \rho_1, \cdots, \rho_{q-1}$. Then it can be verified that ρ is equal to the maximum of the multiplicities, i.e.,

$$\rho = \max\{\rho_0, \rho_1, \cdots, \rho_{q-1}\}.$$

4. Suppose solutions of FIR transceivers exist for a given K. FIR solutions do not necessarily exist if we increase redundancy from K to $K + 1$. For example consider the channel $P(z) = 1 + z^{-3}$ with $L = 3$. The channel has zeros at $e^{j\pi/3}$, $e^{-j\pi/3}$ and -1. Let $M = 1$ and $K = 1$. In this case $N = 2$. The polyphases of $P(z)$ with respect to $N = 2$ are $P_0(z) = 1$ and $P_1(z) = z^{-1}$. The channel matrix is given by

$$\mathbf{C}(z) = \begin{pmatrix} 1 & z^{-2} \\ z^{-1} & 1 \end{pmatrix}$$

   One can verify that the Smith form of $\mathbf{C}(z)$ is

$$\mathbf{\Gamma} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - z^{-3} \end{pmatrix}.$$

   The number of non-identity terms ρ on the diagonal of $\mathbf{\Gamma}(z)$ is 1. So the choice $K = 1$ ensures the existence of FIR transceivers. Now suppose we increase redundancy to $K = 2$ and N becomes 3. The polyphases of $P(z)$ with respect to $N = 3$ are $P_0(z) = 1 + z^{-1}$, $P_1(z) = 0$ and $P_2(z) = 0$. The channel matrix $\mathbf{C}(z)$ is given by

$$\mathbf{C}(z) = \begin{pmatrix} 1 + z^{-1} & 0 & 0 \\ 0 & 1 + z^{-1} & 0 \\ 0 & 0 & 1 + z^{-1} \end{pmatrix}$$

   It is already in Smith form; the Smith form $\mathbf{\Gamma}(z) = \mathbf{C}(z)$. The number of non-identity terms ρ on the diagonal of $\mathbf{\Gamma}(z)$ is 3. FIR transceiver solutions do not exist in this case.

REFERENCES

1. J. W. Lechleider, "High Bit Rate Digital Subscriber Lines: A Review of HDSL Progress," *IEEE Journal of Selected Areas in Communications,* vol. 9, no. 6, pp. 769-784, Aug. 1991.
2. P. S. Chow, J. C. Tu, and J. M. Cioffi, "Performance Evaluation of a Multichannel Transceiver System for ADSL and VHDSL Services," IEEE J. Select. Areas Commun., vol. 9, no. 6, pp. 909-919, Aug. 1991.
3. A. N. Akansu, et. al., "Orthogonal Transmultiplexers in Communication: A Review," *IEEE Trans. Signal Processing*, vol. 46, pp. 979-995, April 1998.
4. I. Kalet, "The Multitone Channel," *IEEE Trans. Commun.*, vol. 37, no. 2, pp. 119-224, Feb. 1989.
5. I. Kalet, "Multitone Modulation," in A. N. Akansu and M. J. T. Smith, Eds., *Subband and Wavelet Transforms: Design and Applications*, Boston, MA: Kluwer, 1995.
6. G. W. Wornell, "Emerging Applications of Multirate Signal Processing and Wavelets in Digital Communications," *Proc. IEEE*, vol. 84, no. 4, April 1996.
7. S. Kasturia, J. T. Aslanis, and J. M. Cioffi, "Vector Coding for Partial Response Channels," *IEEE Trans. Inform. Theory*, vol. 36, pp. 741-762, July 1990.
8. A. Scaglione, S. Barbarossa and G. B. Giannakis, "FilterBank Transceivers Optimizing Information Rate in Block Transmissions over Dispersive Channels," *IEEE Trans. Signal Processing,* vol. 45, no. 4, April 1999.
9. A. Scaglione, G. B. Giannakis, and S. Barbarossa "Redundant FilterBank Precoders and Equalizers Part I: Unification and Optimal Designs," *IEEE Trans. Signal Processing,* vol. 47, no. 7, July 1999.
10. A. Scaglione, G. B. Giannakis, and S. Barbarossa "Redundant FilterBank Precoders and Equalizers Part II: Blind Channel Estimation, Synthronization, and Direct Equalization," *IEEE Trans. Signal Processing,* vol. 47, no. 7, July 1999.
11. S. D. Sandberg and M. A. Tzannes, "Overlapped Discrete Multitone Modulation for High Speed Copper Wire Communications," *IEEE Journal of Selected Areas in Communications,* vol. 13, pp. 1571-1585, Dec. 1995.
12. M. Vetterli, "Perfect transmultiplexers," Proc. IEEE Int. Conf. Acoust. Speech and Signal Proc., pp. 2567-2570, Tokyo, Japan, April 1986.
13. A. D. Rizos, J. G. Proakis, and T. Q. Nguyen, "Comparison of DFT and Cosine Modulated Filter Banks in Multicarrier Modulation," IEEE Global Telecommunications Conference, vol. 2, pp. 687-691, 1994.
14. S. Govardhanagiri, T. Karp, P. Heller and T. Nguyen, "Performance Analysis of Multicarrier Modulations Systems using Cosine Modulated Filter Banks," Proc. Acoustics, Speech, and Signal Processing, vol. 3, pp. 1405 -1408, March 1999.
15. N. J. Fliege and G. Rosel, "Equalizer and Crosstalk Compensation Filters for DFT Polyphase Transmultiplexer Filter Banks," Proc. IEEE Int. Symp. Circuits and Systems, vol. 3, pp. 173-176, London, 1994.
16. Xiang-Gen Xia, "A New Precoding for ISI Cancellation using Multirate Filterbanks," IEEE International Symposium on Circuits and Systems, vol. 4, pp. 2409-2412, 1997.
17. Xiang-Gen Xia, "New Precoding For Intersymbol Interference Cancellation Using Nonmaximally Decimated Multirate FilterBanks with Ideal FIR Equalizers ," IEEE Transactions on Signal Processing, vol. 45, pp. 2431 -2441, 1997.
18. Yuan-Pei Lin and See-May Phoong, "Perfect Discrete Wavelet Multitone Modulation for Fading Channels," Proc. The 6th IEEE International Workshop on Intelligent Signal Processing and Communication Systems, Nov. 1998.
19. P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Englewood Cliffs,Prentice-Hall, 1993.