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Discrete gauge states and w_∞ charges in $c = 1$ 2D gravity

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Abstract

We give a general formula for gauge states at the discrete momenta in Liouville theory. These discrete gauge states carry the w_∞ charges. As in the case of the 26D (or 10D) string theory, they are decoupled from the correlation functions and can be considered as the symmetry parameters in the old covariant quantization of the theory.

1. Introduction

For the past few years, the toy 2D string model (or $c = 1$ 2D gravity) [1] has been an important laboratory to study non-perturbative information on string theory. In the continuum Liouville approach [2], in addition to the massless tachyon mode, an infinite number of massive discrete momentum physical degrees of freedom was discovered [3,4] and the target space-time w_∞ symmetry and Ward identities were then identified [5,6,7]. This important high energy ($\alpha' \to \infty$) structure turns out to be impossible to be extracted in the 26D (or 10D) string theory due to the high dimensionality of the space-time. However, some interesting progress for these high dimensional string theories has been made by using two types of gauge states (physical zero norm states) in the spectrum [8]. For example, the massive inter-particle broken symmetries and Ward identities for the first few levels were demonstrated although the symmetry algebra is still difficult to identify. In contrast to the BRST quantization used in the 2D case [7,9], these were done in the old covariant quantization of the theory.

In this letter, we will derive the w_∞ structure from the gauge states point of view in the old covariant quantization scheme. This is in parallel with the works of [6] and [7] where the ground ring structure of ghost number zero operators was identified in the BRST quantization. Moreover, the results we obtained will justify the idea of gauge states used in the 26D (or 10D) theories as discussed above. Unlike the discrete Polyakov states, we will find that there is still an infinite number of continuum momentum gauge states in the massive levels of the 2D spectrum and it is very difficult to give a general formula for them just as in the case of 26D theory [8]. However, as far as the dynamics of the theory is concerned, only those gauge states with Polyakov's

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discrete momenta are relevant. This is because all other gauge states are trivially decoupled from the correlation functions due to kinematic reason. Hence, we will only identify all discrete gauge states (DGS) in the spectrum. The higher the momentum is, the more numerous the DGS are found. In particular, we will give an explicit formula for one such set of DGS in terms of Schur polynomials. Finally, we can show that these DGS carry the w_∞ charges and serve as the symmetry parameters of the theory. This is in complete analogy with, e.g., Gupta-Bleuler quantization of QED where the gauge state $\theta(x)$ serves as the $U(1)$ parameter of the theory.

2. Gauge states in 2D gravity

We consider the two dimensional critical string action [2]

$$S = \frac{1}{8\pi}\int d^2\sigma\sqrt{\hat{g}}[g^{\mu\nu}(\partial_\mu X\partial_\nu X + \partial_\mu\phi\partial_\nu\phi) - Q\hat{R}\phi], \tag{2.1}$$

with ϕ being the Liouville field. For $c = 1$ theory Q, which represents the background charge of the Liouville field, is set to be $2\sqrt{2}$ so that the total anomalies cancel that from ghost contribution.

For simplicity here we consider only one of the chiral sectors, while the other sector (denoted by $\bar{z}$) is the same. The stress energy tensor is

$$T_{zz} = -\tfrac{1}{2}(\partial_z X)^2 - \tfrac{1}{2}(\partial_z\phi)^2 - \tfrac{1}{2}Q\partial_z^2\phi. \tag{2.2}$$

If we define the mode expansion of $X^\mu = (\phi, X)$ by

$$\partial_z X^\mu = -\sum_{n=-\infty}^{\infty} z^{-n-1}(\alpha_n^0, i\alpha_n^1), \tag{2.3}$$

with the Minkowski metric $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}$, $Q^\mu = (2\sqrt{2}, 0)$ and the zero mode $\alpha_0^\mu = f^\mu = (\epsilon, p)$, we find the Virasoro generators

$$L_n = (\frac{n+1}{2}Q^\mu + f^\mu)\alpha_{\mu,n} + \tfrac{1}{2}\sum_{k\neq 0} :\alpha_{\mu,-k}\alpha^\mu_{n+k}: \quad n \neq 0,$$

$$L_0 = \tfrac{1}{2}(Q^\mu + f^\mu)f_\mu + \sum_{k=1}^{\infty} :\alpha_{\mu,-k}\alpha_k^\mu: . \tag{2.4}$$

The vacuum $|0\rangle$ is annihilated by all α_n^μ with $n > 0$. In the old covariant quantization, physical states $|\psi\rangle$ are those satisfy the condition

$$L_n|\psi\rangle = 0 \quad \text{for} \quad n > 0,$$

$$L_0|\psi\rangle = |\psi\rangle. \tag{2.5}$$

One can easily check that the two branches of massless "tachyon"

$$T^\pm(p) = e^{ipX + (\pm|p| - \sqrt{2})\phi} \tag{2.6}$$

are positive norm physical states. In the "material gauge" [6], it was also known that there exist discrete states [3,10] ($J = \{0, \frac{1}{2}, 1, ...\}$ and $M = \{-J, -J+1, ..., J\}$)

$$\psi_{J,M}^{(\pm)} \sim (H_-)^{J-M}\psi_{J,J}^{(\pm)} \sim (H_+)^{J+M}\psi_{J,-J}^{(\pm)}, \tag{2.7}$$

which are also positive norm physical states. In (2.7) $H_{\pm} = \int \frac{dz}{2\pi i} T^{+}(\pm\sqrt{2})$ are the zero modes of the ladder operators of the $SU(2)$ Kac-Moody currents at the self-dual radius in $c = 1$ 2D conformal field theory and $\psi^{(\pm)}_{J,\pm J} = T^{(\pm)}(\pm\sqrt{2}J)$. These exhaust all positive norm physical states. In this letter we are interested in the discrete gauge states (DGS), i.e., the zero norm physical states at the same discrete momenta as those states in (2.7). We thus no longer restrict ourselves in the "material" gauge, and the Liouville field ϕ will play an important role in the following discussions.

In general, there are two types of gauge states,

$$\text{Type I}: \quad |\psi\rangle = L_{-1}|\chi\rangle \quad \text{where} \quad L_m|\chi\rangle = 0 \quad m \geq 0, \tag{2.8}$$

$$\text{Type II}: \quad |\psi\rangle = \left(L_{-2} + \tfrac{3}{2}L_{-1}^2\right)|\tilde{\chi}\rangle \quad \text{where} \quad L_m|\tilde{\chi}\rangle = 0 \quad m > 0 \quad (L_0 + 1)|\tilde{\chi}\rangle = 0. \tag{2.9}$$

They satisfy the physical state conditions (2.5), and have zero norm. It is important to note that (2.9) is a gauge state only when $Q = \sqrt{\frac{25-c}{3}}$, while the states in (2.8) are insensitive to this condition. In this section we will explicitly calculate the gauge states at the two lowest mass levels. At mass level one (i.e. spin one), $f_\mu(f^\mu + Q^\mu) = 0$, only gauge states of type I are found: $f_\mu \alpha^\mu_{-1}|f\rangle$, where $|f\rangle =: e^{ipX+\epsilon\phi} : |0\rangle$. The DGS $G^-_{1,0} =: \partial\phi e^{-2\sqrt{2}\phi} : |0\rangle$ corresponds to the momentum of $\psi^-_{1,0}$. There is no corresponding DGS for $\psi^+_{1,0}$.

At mass level two, $f_\mu(f^\mu + Q^\mu) = -2$, if $e_\mu(f^\mu + Q^\mu) = 0$ then the type I gauge state is

$$|\psi\rangle = [\tfrac{1}{2}(f_\mu e_\nu + e_\mu f_\nu)\alpha^\mu_{-1}\alpha^\nu_{-1} + e_\mu \alpha^\mu_{-2}]|f\rangle, \tag{2.10}$$

while the type II state is

$$|\psi\rangle = \tfrac{1}{2}[(3f_\mu f_\nu + \eta_{\mu\nu})\alpha^\mu_{-1}\alpha^\nu_{-1} + (5f_\mu - Q_\mu)\alpha^\mu_{-2}]|f\rangle. \tag{2.11}$$

The DGS corresponding to $\psi^-_{\frac{3}{2},\pm\frac{1}{2}}$ are $G^-_{\frac{3}{2},\pm\frac{1}{2}}$

$$\text{Type I}: \quad G^{-(1)}_{\frac{3}{2},\pm\frac{1}{2}} \sim \left[\begin{pmatrix} \frac{5}{2} & \pm\frac{3}{2} \\ \pm\frac{3}{2} & \frac{1}{2} \end{pmatrix} \alpha^\mu_{-1}\alpha^\nu_{-1} + \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} \end{pmatrix} \alpha^\mu_{-2}\right] |f_\mu = (-\tfrac{5}{2}, \pm\tfrac{1}{2})\rangle, \tag{2.12}$$

$$\text{Type II}: \quad G^{-(2)}_{\frac{3}{2},\pm\frac{1}{2}} \sim \frac{1}{2}\left[\begin{pmatrix} \frac{73}{2} & \pm\frac{15}{2} \\ \pm\frac{15}{2} & \frac{5}{2} \end{pmatrix} \alpha^\mu_{-1}\alpha^\nu_{-1} + \begin{pmatrix} \frac{29}{\sqrt{2}} \\ \pm\frac{5}{\sqrt{2}} \end{pmatrix} \alpha^\mu_{-2}\right] |f_\mu = (-\tfrac{5}{2}, \pm\tfrac{1}{2})\rangle. \tag{2.13}$$

Note that a linear combination of these two states produces a "pure ϕ DGS":

$$G^-_{\frac{3}{2},\pm\frac{1}{2}} \sim [(\partial\phi)^2 - \frac{1}{\sqrt{2}}\partial^2\phi] e^{\pm\frac{i}{\sqrt{2}}X - \frac{5}{\sqrt{2}}\phi}|0\rangle. \tag{2.14}$$

The gauge states corresponding to discrete momenta of $\psi^+_{\frac{3}{2},\pm\frac{1}{2}}$ are degenerate, i.e., the type I and type II gauge states are linearly dependent:

$$G^+_{\frac{3}{2},\pm\frac{1}{2}} \sim \left[\begin{pmatrix} -\frac{1}{2} & \pm\frac{3}{2} \\ \pm\frac{3}{2} & -\frac{5}{2} \end{pmatrix} \alpha^\mu_{-1}\alpha^\nu_{-1} + \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \mp\frac{5}{\sqrt{2}} \end{pmatrix} \alpha^\mu_{-2}\right] |f_\mu = (\tfrac{1}{2}, \pm\tfrac{1}{2})\rangle. \tag{2.15}$$

There is no "pure ϕ DGS" here. In general, the ψ^+ sector has fewer DGS than the ψ^- sector at the same discrete momenta, as a result, the "pure ϕ DGS" only arise at the minus sector. This fact is related to the degeneracy of the DGS in the plus sector. Historically the ψ^+ sector discrete states arise when one considers the "singular gauge" transformation constructed from the difference of the two plus gauge states [3,11].

3. Generating the discrete gauge states

In this section, we will give a general formula for the DGS. In general, there are many DGS for each discrete momentum. The higher the momentum is, the more numerous the DGS are found. We first express the discrete states in (2.7) in terms of Schur polynomials, which are defined as follows:

$$\exp\left(\sum_{k=1}^{\infty} a_k x^k\right) = \sum_{k=0}^{\infty} S_k\left(\{a_k\}\right) x^k, \tag{3.1}$$

where S_k is the Schur polynomial, a function of $\{a_k\} = \{a_i : i \in \mathbb{Z}_k\}$. Performing the operator products in (2.7) , the discrete states $\psi^{\pm}_{J,M}$ can be written as

$$\psi^{\pm}_{J,M} \sim \prod_{i=1}^{J-M} \int \frac{dz_i}{2\pi i} z_i^{-2J} \prod_{j<k}^{J-M} (z_j - z_k)^2 \exp\left[\sum_{i=1}^{J-M} [\,-i\sqrt{2}X(z_i)] + \sqrt{2}(iJX(0) + (-1 \pm J)\phi(0))\right]. \tag{3.2}$$

We can write

$$\prod_{j<k}^{J-M} (z_j - z_k)^2 = \sum_f \begin{vmatrix} 1 & z_{f_1} & \cdots & z_{f_1}^{J-M-1} \\ z_{f_2} & z_{f_2}^2 & \cdots & z_{f_2}^{J-M-1} \\ \vdots & \vdots & \ddots & \vdots \\ z_{f_{J-M}}^{J-M-1} & z_{f_{J-M}}^{J-M} & \cdots & z_{f_{J-M}}^{2J-2M-2} \end{vmatrix} \tag{3.3}$$

and Taylor expand $X(z_i)$ around $z_i = 0$:

$$e^{-i\sqrt{2}X(z_i)} = e^{-i\sqrt{2}X(0)} \left[\sum_{k=0}^{\infty} S_k\left(\{\frac{-i\sqrt{2}}{k!}\partial^k X(0)\}\right) z_i^k\right]. \tag{3.4}$$

In (3.3) the sum is over all permutations $f = (f_1, ..., f_{J-M})$ of $(1, 2, ..., J-M)$. Putting (3.3) and (3.4) into (3.2), and using the symmetry of the integrand over the index i, we have

$$\psi^{\pm}_{J,M} \sim \begin{vmatrix} S_{2J-1} & S_{2J-2} & \cdots & S_{J+M} \\ S_{2J-2} & S_{2J-3} & \cdots & S_{J+M-1} \\ \vdots & \vdots & \ddots & \vdots \\ S_{J+M} & S_{J+M-1} & \cdots & S_{2M+1} \end{vmatrix} \exp\left[\sqrt{2}(iMX(0) + (-1 \pm J)\phi(0))\right], \tag{3.5}$$

with $S_k = S_k\left(\{\frac{-i\sqrt{2}}{k!}\partial^k X(0)\}\right)$ and $S_k = 0$ if $k < 0$. We will denote the rank $(J-M)$ determinant in (3.5) as $\Delta(J, M, -i\sqrt{2}X)$. As a by-product, comparing the two definitions in (2.7) we can use (3.5) to deduce a mathematical identity relating the determinants of rank $(J-M)$ and $(J+M)$,

$$\Delta(J, M, -i\sqrt{2}X) = (-1)^{J+M+1}\Delta(J, -M, i\sqrt{2}X). \tag{3.6}$$

We now begin to study the DGS. One first notes that the DGS in (2.14) can be generated by $\int \frac{dz}{2\pi i} e^{-\sqrt{2}\phi(z)} \psi^-_{\frac{1}{2},\pm\frac{1}{2}}(0)$. In general it is also possible to write down explicitly one of the many gauge states for each discrete momentum in the ψ^- sector as follows:

$$G^-_{J,M} \sim \left[\int \frac{dz}{2\pi i} e^{-\sqrt{2}\phi}(z)\right] \psi^-_{J-1,M}(0) \sim S_{2J-1}(\{\frac{-\sqrt{2}}{k!}\partial^k \phi\})\Delta(J-1, M, -i\sqrt{2}X) e^{\sqrt{2}[iMX + (-1-J)\phi]}. \tag{3.7}$$

Using (2.2) it can be verified explicitly after some lengthy algebra that they are primary, and are of dimension 1. For $M = J - 1$ (3.7) are "pure ϕ" states, but orthogonal to the "pure X" discrete physical states at the same momenta, and are therefore gauge states. For general M the polynomial prefactor in (3.7) factorizes into "pure ϕ" and "pure X" parts, and are still orthogonal to the physical states at the same momenta. They are, therefore, also gauge states. This is also suggested by the following result [6]:

$$\int \frac{dz}{2\pi i}\psi^-_{J_1,M_1}(z)\psi^-_{J_2,M_2}(0) \sim 0, \tag{3.8}$$

where the r.h.s. is meant to be a DGS. We thus have explicitly obtained a DGS for each ψ^- discrete momentum. We stress that there are still other DGS in this sector, for example, the states

$$G'^-_{J,M} \sim \left[\int \frac{dz}{2\pi i} e^{-\sqrt{2}\phi(z)}\right]^{J-M} \psi^-_{M,M}(0) \tag{3.9}$$

can be shown to be of dimension 1. Since they are "pure ϕ" states, they are also DGS. This expression reminds us of (2.7). However, there is no $SU(2)$ structure in the ϕ direction, and the usual techniques of $c = 1$ 2D conformal field theory cannot be applied. The "pure ϕ DGS" are only found in the minus sector.

For the plus sector, the operator products of the discrete states defined in (2.7) form a w_∞ algebra [6],

$$\int \frac{dz}{2\pi i}\psi^+_{J_1,M_1}(z)\psi^+_{J_2,M_2}(0) = (J_2M_1 - J_1M_2)\psi^+_{J_1+J_2-1,M_1+M_2}(0). \tag{3.10}$$

(Again, the r.h.s. is up to a DGS.) We can subtract two positive norm discrete states to obtain a pure gauge state:

$$\begin{aligned} G^+_{J,M} &= (J+M+1)^{-1}\int \frac{dz}{2\pi i}\left[\psi^+_{1,-1}(z)\psi^+_{J,M+1}(0) + \psi^+_{J,M+1}(z)\psi^+_{1,-1}(0)\right] \\ &\sim (J-M)!\Delta(J,M,-i\sqrt{2}X)\exp\left[\sqrt{2}\left(iMX + (J-1)\phi\right)\right] \\ &+ (-1)^{2J}\sum_{j=1}^{J-M}(J-M-1)! \\ &\times \int \frac{dz}{2\pi i}\mathcal{D}(J,M,-i\sqrt{2}X(z),j)\exp\left[\sqrt{2}\left(i(M+1)X(z) + (J-1)\phi(z) - X(0)\right)\right], \end{aligned} \tag{3.11}$$

where $\mathcal{D}(J,M,-i\sqrt{2}X(z),j)$ is defined as

$$\mathcal{D}(J,M,-i\sqrt{2}X(z),j) = \begin{vmatrix} S_{2J-1} & S_{2J-2} & \cdots & \cdots & S_{J+M} \\ S_{2J-2} & S_{2J-3} & \cdots & \cdots & S_{J+M-1} \\ \vdots & \vdots & \ddots & & \vdots \\ (-z)^{j-1-2J} & (-z)^{j-2J} & & & (-z)^{j-J-M-2} \\ \vdots & \vdots & & \ddots & \vdots \\ S_{J+M} & S_{J+M-1} & \cdots & \cdots & S_{2M+1} \end{vmatrix}, \tag{3.12}$$

which is the same as $\Delta(J,M,-i\sqrt{2}X(z))$ except that the j^{th} row is replaced by $\{(-z)^{j-1-2J}, (-z)^{j-2J}\ldots\}$. As an example, with (3.11) one easily obtains the state $G^+_{\frac{3}{2},\pm\frac{1}{2}}$ of (2.15).

4. w_∞ charges and conclusion

It was shown [6] that the operator products of the states $\psi^+_{J,M}$ defined in (2.7) satisfy the w_∞ algebra in (3.10). By construction (3.11) one can easily see that the plus sector DGS $G^+_{J,M}$ carry the w_∞ charges and can be considered as the symmetry parameters of the theory. In fact, the operator products of the DGS $G^+_{J,M}$ of (3.11) form the same w_∞ algebra

$$\int \frac{dz}{2\pi i} G^+_{J_1,M_1}(z) G^+_{J_2,M_2}(0) = (J_2 M_1 - J_1 M_2) G^+_{J_1+J_2-1,M_1+M_2}(0), \tag{4.1}$$

where the r.h.s. is defined up to another DGS.

In summary, we have shown that the space-time w_∞ symmetry parameters of 2D string theory come from the solution of equations (2.8) and (2.9). This argument is valid also in the case of 26D (or 10D) string theory although it would be very difficult to exhaust all the solutions of the gauge states [8]. This difficulty is, of course, related to the high dimensionality of the space-time. The DGS we introduced in the old covariant quantization in this paper seems to be related to the ghost sectors and the ground ring strucrture [7] in the BRST quantization of the theory. Many issues remain to be studied.

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