Bidirectional approximate reasoning based on interval-valued fuzzy sets

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Abstract

This paper proposes a bidirectional approximate reasoning method based on interval-valued fuzzy sets, where fuzzy production rules are used for knowledge representation, and the fuzzy terms appearing in the fuzzy production rules of a rule-based system are represented by interval-valued fuzzy sets. The proposed method is more flexible than the one presented in the paper by Bien and Chun [IEEE Trans. Fuzzy Systems 2 (1994) 177] due to the fact that it allows the fuzzy terms appearing in the fuzzy production rules of a rule-based system to be represented by interval-valued fuzzy sets rather than general fuzzy sets. Furthermore, because the proposed method requires only simple arithmetic operations, and because it allows bidirectional approximate reasoning, it can be executed much faster and more flexible than the single-input-single-output approximate reasoning scheme presented in the paper by Gorzalczany [Fuzzy Sets and Systems 21 (1987) 1]. © 1997 Elsevier Science B.V.

Keywords: Bidirectional approximate reasoning; Fuzzy production rule; Interval-valued fuzzy set; Knowledge base; Rule-based system

1. Introduction

Much knowledge residing in the knowledge base of a rule-based system is fuzzy and imprecise. A powerful rule-based system must have the capability of approximate reasoning [1–7, 9–13]. The following single-input-single-output (SISO) approximate reasoning scheme is discussed by many researchers:

R_1:   IF X is A_1 THEN Y is B_1

R_2:   IF X is A_2 THEN Y is B_2

$\vdots$ (1)

R_p: IF X is A_p THEN Y is B_p

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Fact: X is A_0
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Consequence: Y is B_0

where R_i are fuzzy production rules [15], $1 \leqslant i \leqslant p$; X and Y are linguistic variables [18], A_0, A_1, A_2, $\ldots, A_p, B_1, B_2, \ldots$, and B_p are fuzzy terms, such as "very small", "large", etc. A linguistic variable is a variable whose values are fuzzy terms. For example, let "speed" be a linguistic variable, its values may be fuzzy terms, such as "slow", "moderate", "fast", "very slow", "more or less fast", etc. The fuzzy terms are usually represented by fuzzy sets [17].

In [1], Bien and Chun presented an inference network for bidirectional approximate reasoning based on fuzzy sets; if a fuzzy input is given for the inference network, then the network renders a reasonable fuzzy output after performing approximate reasoning based on an equality measure, and conversely, for a given fuzzy output, the network can yield its corresponding reasonable fuzzy input after performing approximate reasoning. In [16], Turksen proposed the definitions of interval-valued fuzzy sets for the representation of combined concepts based on normal forms. In [12], Gorzalczany presented a method of inference in approximate reasoning based on interval-valued fuzzy sets. In [13], Gorzalczany further presented some properties of the interval-valued fuzzy inference method described in [12].

In this paper, we extend the works of [1, 12] to develop a new method for bidirectional approximate reasoning based on interval-valued fuzzy sets. The proposed method is more flexible than the one presented in [1] due to the fact that it allows the fuzzy terms appearing in the fuzzy production rules of a rule-based system to be represented by interval-valued fuzzy sets rather than general fuzzy sets. Furthermore, because the proposed method requires only simple arithmetic operations, and because it allows bidirectional approximate reasoning, it can be executed much faster and more flexible, than the single-input-single-output approximate reasoning scheme presented in [12].

The rest of this paper is organized as follows. In Section 2, we briefly review some basic definitions of interval-valued fuzzy sets from [12, 13, 16]. In Section 3, a method for measuring the degree of similarity between interval-valued fuzzy sets is presented. In Section 4, we present a method for bidirectional approximate reasoning based on interval-valued fuzzy sets. The conclusions are discussed in Section 5.

2. Interval-valued fuzzy sets

In 1965, Zadeh proposed the theory of fuzzy sets [17]. Roughly speaking, a fuzzy set is a class with fuzzy boundaries. A fuzzy set A of the universe of discourse U, $U = \{u_1, u_2, \ldots, u_n\}$, is a set of ordered pairs, $\{(u_1, f_A(u_1)), (u_2, f_A(u_2)), \ldots, (u_n, f_A(u_n))\}$, where f_A is the membership function of A, $f_A : U \to [0, 1]$, and $f_A(u_i)$ indicates the grade of membership of u_i in A, where $1 \leqslant i \leqslant n$.

In [12, 13], Gorzalczany presented interval-valued fuzzy inference methods based on interval-valued fuzzy sets. If a fuzzy set is represented by an interval-valued membership function, then it is called an interval-valued fuzzy set. An interval-valued fuzzy set A of the universe of discourse U, $U = \{u_1, u_2, \ldots, u_n\}$, can be represented by

$$A = \{(u_1, [a_{11}, a_{12}]), (u_2, [a_{21}, a_{22}]), \ldots, (u_n, [a_{n1}, a_{n2}])\}, \tag{2}$$

where interval $[a_{i1}, a_{i2}]$ indicates that the grade of membership of u_i in the interval-valued fuzzy set A is between a_{i1} and a_{i2}, where $0 \leqslant a_{i1} \leqslant a_{i2} \leqslant 1$ and $1 \leqslant i \leqslant n$.

Let A and B be two interval-valued fuzzy sets,

$$A = \{(u_1, [a_{11}, a_{12}])(u_2, [a_{21}, a_{22}]), \ldots, (u_n, [a_{n1}, a_{n2}])\} = \{(u_i, [a_{i1}, a_{i2}]) \mid 1 \leqslant i \leqslant n\}, \tag{3}$$

$$B = \{(u_1, [b_{11}, b_{12}]), (u_2, [b_{21}, b_{22}]), \ldots, (u_n, [b_{n1}, b_{n2}])\} = \{(u_i, [b_{i1}, b_{i2}]) \mid 1 \leqslant i \leqslant n\}. \tag{4}$$

The union, intersection, and complement operations of the interval-valued fuzzy sets are defined as follows:

$$A \cup B = \{(u_i, [c_{i1}, c_{i2}]) \mid c_{i1} = \mathrm{Max}(a_{i1}, b_{i1}), c_{i2} = \mathrm{Max}(a_{i2}, b_{i2}), \text{ and } 1 \leqslant i \leqslant n\}, \tag{5}$$

$$A \cap B = \{(u_i, [d_{i1}, d_{i2}]) \mid d_{i1} = \mathrm{Min}(a_{i1}, b_{i1}), d_{i2} = \mathrm{Min}(a_{i2}, b_{i2}), \text{ and } 1 \leqslant i \leqslant n\}, \tag{6}$$

$$A' = \{(u_i, [x_{i1}, x_{i2}]) \mid x_{i1} = 1 - a_{i2}, x_{i2} = 1 - a_{i1}, \text{ and } 1 \leqslant i \leqslant n\}. \tag{7}$$

The interval-valued fuzzy sets A and B are called equal (i.e., $A = B$) if and only if $\forall i$, $a_{i1} = b_{i1}$ and $a_{i2} = b_{i2}$ (i.e., $[a_{i1}, a_{i2}] = [b_{i1}, b_{i2}]$), where $1 \leqslant i \leqslant n$.

3. Similarity measures

In [19], Zwick et al. have reviewed 19 similarity measures of fuzzy sets and compared their performance in an experiment. In [14], Ke and Her have presented a similarity function S to measure the degree of similarity between two vectors. The definition of the similarity function S is reviewed as follows.

Definition 3.1. Let $\bar{a}$ and $\bar{b}$ be two vectors in $\mathbb{R}^n$, where $\mathbb{R}$ is a set of real numbers between zero and one, i.e.,

$$\bar{a} = \langle a_1, a_2, \dots, a_n \rangle,$$

$$\bar{b} = \langle b_1, b_2, \dots, b_n \rangle,$$

where $a_i \in [0, 1]$, $b_i \in [0, 1]$, and $1 \leqslant i \leqslant n$. Then, there is a similarity function S,

$$S(\bar{a}, \bar{b}) = \frac{\bar{a} \cdot \bar{b}}{\mathrm{Max}(\bar{a} \cdot \bar{a}, \bar{b} \cdot \bar{b})}, \tag{8}$$

where $S(\bar{a}, \bar{b}) \in [0, 1]$, which can be considered as the similarity measurement between the vectors $\bar{a}$ and $\bar{b}$. The value of $S(\bar{a}, \bar{b})$ indicates the degree of similarity between $\bar{a}$ and $\bar{b}$. The larger the value of $S(\bar{a}, \bar{b})$, the more the similarity between the vectors $\bar{a}$ and $\bar{b}$.

Based on the similarity function S, we can develop a matching function M to measure the degree of matching between interval-valued fuzzy sets. Let U be the universe of discourse, $U = \{u_1, u_2, \dots, u_n\}$, and let A be an interval-valued fuzzy set of U,

$$A = \{(u_1, [a_{11}, a_{12}]), (u_2, [a_{21}, a_{22}]), \dots, (u_n, [a_{n1}, a_{n2}])\} = \{(u_i, [a_{i1}, a_{i2}]) \mid 1 \leqslant i \leqslant n\},$$

then the lower bound and the upper bound of the interval-valued fuzzy set A can be represented by the subscript vector $\bar{A}$ and the superscript vector $\bar{\bar{A}}$, respectively, where

$$\bar{A} = \langle a_{11}, a_{21}, \dots, a_{n1} \rangle, \tag{9}$$

$$\bar{\bar{A}} = \langle a_{12}, a_{22}, \dots, a_{n2} \rangle. \tag{10}$$

In the following, we present the definition of matching function M to measure the degree of similarity between interval-valued fuzzy sets.

Definition 3.2. Let U be the universe of discourse, $U = \{u_1, u_2, \dots, u_n\}$, and let A and B be two interval-valued fuzzy sets of U, where

$$A = \{(u_1, [a_{11}, a_{12}]), (u_2, [a_{21}, a_{22}]), \dots, (u_n, [a_{n1}, a_{n2}])\} = \{(u_i, [a_{i1}, a_{i2}]) \mid 1 \leqslant i \leqslant n\},$$

$$B = \{(u_1, [b_{11}, b_{12}]), (u_2, [b_{21}, b_{22}]), \dots, (u_n, [b_{n1}, b_{n2}])\} = \{(u_i, [b_{i1}, b_{i2}]) \mid 1 \leqslant i \leqslant n\},$$

then the degree of matching $M(A, B)$ between the interval-valued fuzzy sets A and B can be measured as follows. Let

$$S(\bar{A}, \bar{B}) = \alpha, \tag{11}$$

$$S(\bar{\bar{A}}, \bar{\bar{B}}) = \beta, \tag{12}$$

$$M(A, B) = (\alpha + \beta)/2, \tag{13}$$

where $\alpha \in [0, 1]$, $\beta \in [0, 1]$, and $M(A, B) \in [0, 1]$. The larger the value of $M(A, B)$, the more the degree of matching between the interval-valued fuzzy sets A and B.

Example 3.1. Let U be the universe of discourse, $U = \{u_1, u_2, \ldots, u_{14}\}$, and let A and B be two interval-valued fuzzy sets of U, where

$$A = \{(u_1, [0, 0]), (u_2, [0, 0]), (u_3, [0, 0.5]), (u_4, [0.75, 0.8]), (u_5, [0.94, 0.95]),$$

$$(u_6, [1, 1]), (u_7, [0.94, 0.95]), (u_8, [0.75, 0.83]), (u_9, [0, 0.5]), (u_{10}, [0, 0]),$$

$$(u_{11}, [0, 0]), (u_{12}, [0, 0]), (u_{13}, [0, 0]), (u_{14}, [0, 0])\}.$$

$$B = \{(u_1, [0, 0]), (u_2, [0, 0]), (u_3, [0.90, 0.95]), (u_4, [1, 1]), (u_5, [0.90, 0.95]),$$

$$(u_6, [0, 0.8]), (u_7, [0, 0]), (u_8, [0, 0]), (u_9, [0, 0]), (u_{10}, [0, 0]),$$

$$(u_{11}, [0, 0]), (u_{12}, [0, 0]), (u_{13}, [0, 0]), (u_{14}, [0, 0])\}.$$

Let $\bar{A}$ and $\bar{B}$ be the subscript vectors of the interval-valued fuzzy sets A and B, respectively, and let $\bar{\bar{A}}$ and $\bar{\bar{B}}$ be the superscript vectors of the interval-valued fuzzy sets A and B, respectively, where

$$\bar{A} = \langle 0, 0, 0, 0.75, 0.94, 1, 0.94, 0.75, 0, 0, 0, 0, 0, 0 \rangle,$$

$$\bar{B} = \langle 0, 0, 0.90, 1, 0.90, 0, 0, 0, 0, 0, 0, 0, 0, 0 \rangle,$$

$$\bar{\bar{A}} = \langle 0, 0, 0.5, 0.8, 0.95, 1, 0.95, 0.83, 0.5, 0, 0, 0, 0, 0 \rangle,$$

$$\bar{\bar{B}} = \langle 0, 0, 0.95, 1, 0.95, 0.8, 0, 0, 0, 0, 0, 0, 0, 0 \rangle.$$

Then, based on formulas (11)–(13), the degree of matching $M(A, B)$ between the interval-valued fuzzy sets A and B can be measured as follows:

$$S(\bar{A}, \bar{B}) = 0.41,$$

$$S(\bar{\bar{A}}, \bar{\bar{B}}) = 0.65,$$

$$M(A, B) = (0.41 + 0.65)/2 = 0.53.$$

It indicates that the degree of similarity between the interval-valued fuzzy sets A and B is equal to 0.53.

4. A bidirectional approximate reasoning method based on interval-valued fuzzy sets

Let us consider the following generalized modus ponens (GMP):

Rule: IF X is A THEN Y is B

Fact: X is A^* $\tag{14}$

Consequence: Y is B^*

where X and Y are linguistic variables, A^* and A are interval-valued fuzzy sets of the universe of discourse U, $U = \{u_1, u_2, \ldots, u_n\}$, and B^* and B are interval-valued fuzzy sets of the universe of discourse V, $V = \{v_1, v_2, \ldots, v_m\}$. Assume that the interval-valued fuzzy sets A^*, A, and B have the following forms:

$$A^* = \{(u_1, [x_{11}, x_{12}]), (u_2, [x_{21}, x_{22}]), \ldots, (u_n, [x_{n1}, x_{n2}])\},$$

$$A = \{(u_1, [y_{11}, y_{12}]), (u_2, [y_{21}, y_{22}]), \ldots, (u_n, [y_{n1}, y_{n2}])\},$$

$$B = \{(v_1, [z_{11}, z_{12}]), (v_2, [z_{21}, z_{22}]), \ldots, (v_m, [z_{m1}, z_{m2}])\},$$

where $0 \leqslant x_{i1} \leqslant x_{i2} \leqslant 1, 0 \leqslant y_{i1} \leqslant y_{i2} \leqslant 1, 1 \leqslant i \leqslant n, 1 \leqslant z_{j1} \leqslant z_{j2} \leqslant 1$, and $1 \leqslant j \leqslant m$. Let $\overline{A^*}$ and $\overline{A}$ be the subscript vectors of the interval-valued fuzzy sets A^* and A, respectively, and let $\overline{\overline{A^*}}$ and $\overline{\overline{A}}$ be the superscript vectors of the interval-valued fuzzy sets A^* and A, respectively, where

$$\overline{A^*} = \langle x_{11}, x_{21}, \ldots, x_{n1} \rangle,$$

$$\overline{A} = \langle y_{11}, y_{21}, \ldots, y_{n1} \rangle,$$

$$\overline{\overline{A^*}} = \langle x_{12}, x_{22}, \ldots, x_{n2} \rangle,$$

$$\overline{\overline{A}} = \langle y_{12}, y_{22}, \ldots, y_{n2} \rangle.$$

Then based on formulas (11)–(13), the degree of matching between the interval-valued fuzzy sets A^* and A can be measured. Let $M(A^*, A) = k$, where $k \in [0, 1]$. The deduced consequence of the rule is "Y is B^*", where the membership function of the interval-valued fuzzy set B^* is as follows:

$$B^* = \{(v_1, [w_{11}, w_{12}]), (v_2, [w_{21}, w_{22}]), \ldots, (v_m, [w_{m1}, w_{m2}])\}, \tag{15}$$

where $w_{i1} = k^* z_{i1}$, $w_{i2} = k^* z_{i2}$, and $1 \leqslant i \leqslant m$.

It is obvious that if A^* and A are identical interval-valued fuzzy sets (i.e., $A^* = A$), then $M(A^*, A) = 1$ and B^* is equal to B.

Let us consider the following single-input-single-output (SISO) approximate reasoning scheme:

R_1:   IF X is A_1 THEN Y is B_1

R_2:   IF X is A_2 THEN Y is B_2

$\qquad \vdots$ $\hspace{9cm}$ (16)

R_p: IF X is A_p THEN Y is B_p

Fact: X is A_0

———————————————————————

Consequence: Y is B_0

where $A_0, A_1, A_2, \ldots, A_p$ are interval-valued fuzzy sets of the universe of discourse U, $U = \{u_1, u_2, \ldots, u_n\}$, and $B_1, B_2, \ldots$ and B_p are interval-valued fuzzy sets of the universe of discourse V, $V = \{v_1, v_2, \ldots, v_m\}$. Assume that

$$A_i = \{(u_1, [x_{i1}, x_{i1}^*]), (u_2, [x_{i2}, x_{i2}^*]), \ldots, (u_n, [x_{in}, x_{in}^*])\},$$

$$B_j = \{(v_1, [y_{j1}, y_{j1}^*]), (v_2, [y_{j2}, y_{j2}^*]), \ldots, (v_m, [y_{jm}, y_{jm}^*])\},$$

where $0 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant p$. Based on formulas (9) and (10), the interval-valued fuzzy sets A_i can be represented by the subscript vectors $\overline{A_i}$ and the superscript vectors $\overline{\overline{A_i}}$, $0 \leqslant i \leqslant p$, where

$$\overline{A_0} = \langle x_{01}, x_{02}, \ldots, x_{0n} \rangle,$$

$$\overline{A_1} = \langle x_{11}, x_{12}, \ldots, x_{1n} \rangle,$$

$$\overline{A_2} = \langle x_{21}, x_{22}, \dots, x_{2n} \rangle,$$

$$\vdots \tag{17}$$

$$\overline{A_p} = \langle x_{p1}, x_{p2}, \dots, x_{pn} \rangle,$$

$$\overline{\overline{A_0}} = \langle x_{01}^*, x_{02}^*, \dots, x_{0n}^* \rangle,$$

$$\overline{\overline{A_1}} = \langle x_{11}^*, x_{12}^*, \dots, x_{1n}^* \rangle,$$

$$\overline{\overline{A_2}} = \langle x_{21}^*, x_{22}^*, \dots, x_{2n}^* \rangle,$$

$$\vdots$$

$$\overline{\overline{A_p}} = \langle x_{p1}^*, x_{p2}^*, \dots, x_{pn}^* \rangle.$$

Based on the previous discussions, we can get the following results:

$M(A_0, A_1) = k_1 \Rightarrow$ the deduced consequence of rule R_1 is "Y is B_1^*", where

$$B_1^* = \{(v_1, [k_1{}^*y_{11}, k_1{}^*y_{11}^*]), (v_2, [k_1{}^*y_{12}, k_1{}^*y_{12}^*]), \dots, (v_m, [k_1{}^*y_{1m}, k_1{}^*y_{1m}^*])\},$$

$M(A_0, A_2) = k_2 \Rightarrow$ the deduced consequence of rule R_2 is "Y is B_2^*", where

$$B_2^* = \{(v_1, [k_2{}^*y_{21}, k_2{}^*y_{21}^*]), (v_2, [k_2{}^*y_{22}, k_2{}^*y_{22}^*]), \dots, (v_m, [k_2{}^*y_{2m}, k_2{}^*y_{2m}^*])\},$$

$$\vdots$$

$M(A_0, A_p) = k_p \Rightarrow$ the deduced consequence of rule R_p is "Y is B_p^*", where

$$B_p^* = \{(v_1, [k_p{}^*y_{p1}, k_p{}^*y_{p1}^*]), (v_2, [k_p{}^*y_{p2}, k_p{}^*y_{p2}^*]), \dots, (v_m, [k_p{}^*y_{pm}, k_p{}^*y_{pm}^*])\},$$

where $k_i \in [0, 1]$, $1 \leqslant i \leqslant p$, and the deduced consequence of the SISO approximate reasoning scheme is "Y is B_0", where

$$B_0 = B_1^* \cup B_2^* \cup \cdots \cup B_p^*, \tag{18}$$

and "$\cup$" is the union operator of the interval-valued fuzzy sets. That is,

$$B_0 = \{(v_1, [z_1, z_1^*]), (v_2, [z_2, z_2^*]), \dots, (v_m, [z_p, z_m^*])\}, \tag{19}$$

where

$$z_1 = \mathrm{Max}(k_1{}^*y_{11}, k_2{}^*y_{21}, \dots, k_p{}^*y_{p1})$$

$$z_1^* = \mathrm{Max}(k_1{}^*y_{11}^*, k_2{}^*y_{21}^*, \dots, k_p{}^*y_{p1}^*)$$

$$z_2 = \mathrm{Max}(k_1{}^*y_{12}, k_2{}^*y_{22}, \dots, k_p{}^*y_{p2})$$

$$z_2^* = \mathrm{Max}(k_1{}^*y_{12}^*, k_2{}^*y_{22}^*, \dots, k_p{}^*y_{p2}^*)$$

$$\vdots \tag{20}$$

$$z_m = \mathrm{Max}(k_1{}^*y_{1m}, k_2{}^*y_{2m}, \dots, k_p{}^*y_{pm})$$

$$z_m^* = \mathrm{Max}(k_1{}^*y_{1m}^*, k_2{}^*y_{2m}^*, \dots, k_p{}^*y_{pm}^*),$$

$0 \leqslant z_i \leqslant z_i^* \leqslant 1$, and $1 \leqslant i \leqslant m$. If k_i is the largest value among the values $k_1, k_2, \dots,$ and k_p, then the interval-valued fuzzy set B_0 is the most similar to the interval-valued fuzzy set B_i, where $1 \leqslant i \leqslant p$.

Example 4.1. Let us consider the following single-input-single-output interval-valued approximate reasoning scheme:

R_1:   IF X is A_1 THEN Y is B_1

R_2:   IF X is A_2 THEN Y is B_2

R_3:   IF X is A_3 THEN Y is B_3

R_4:   IF X is A_4 THEN Y is B_4

R_5:   IF X is A_5 THEN Y is B_5

Fact: X is A_0

Consequence: Y is B_0

where A_0, A_1, $A_2, \ldots,$ and A_5 are interval-valued fuzzy sets of the universe of discourse U, and B_0, B_1, $B_2, \ldots,$ and B_5 are interval-valued fuzzy sets of the universe of discourse V. These interval-valued fuzzy sets are shown as follows:

$A_0 = \{(u_1, [0, 0]), (u_2, [0, 0]), (u_3, [0.90, 0.95]), (u_4, [1, 1]), (u_5, [0.90, 0.95]),$

$\quad (u_6, [0, 0.8]), (u_7, [0, 0]), (u_8, [0, 0]), (u_9, [0, 0]), (u_{10}, [0, 0]),$

$\quad (u_{11}, [0, 0]), (u_{12}, [0, 0]), (u_{13}, [0, 0]), (u_{14}, [0, 0])\},$

$A_1 = \{(u_1, [1, 1]), (u_2, [1, 1]), (u_3, [0.82, 0.95]), (u_4, [0, 0.7]), (u_5, [0, 0]),$

$\quad (u_6, [0, 0]), (u_7, [0, 0]), (u_8, [0, 0]), (u_9, [0, 0]), (u_{10}, [0, 0]),$

$\quad (u_{11}, [0, 0]), (u_{12}, [0, 0]), (u_{13}, [0, 0]), (u_{14}, [0, 0])\},$

$A_2 = \{(u_1, [0, 0]), (u_2, [0, 0]), (u_3, [0, 0.5]), (u_4, [0.75, 0.8]), (u_5, [0.94, 0.95]),$

$\quad (u_6, [1, 1]), (u_7, [0.94, 0.95]), (u_8, [0.75, 0.83]), (u_9, [0, 0.5]), (u_{10}, [0, 0]),$

$\quad (u_{11}, [0, 0]), (u_{12}, [0, 0]), (u_{13}, [0, 0]), (u_{14}, [0, 0])\},$

$A_3 = \{(u_1, [0, 0]), (u_2, [0, 0]), (u_3, [0, 0]), (u_4, [0, 0]), (u_5, [0, 0]),$

$\quad (u_6, [0, 0]), (u_7, [0, 0.6]), (u_8, [0.87, 0.92]), (u_9, [1, 1]), (u_{10}, [0.87, 0.92]),$

$\quad (u_{11}, [0, 0.6]), (u_{12}, [0, 0]), (u_{13}, [0, 0]), (u_{14}, [0, 0])\},$

$A_4 = \{(u_1, [0, 0]), (u_2, [0, 0]), (u_3, [0, 0]), (u_4, [0, 0]), (u_5, [0, 0]),$

$\quad (u_6, [0, 0]), (u_7, [0, 0]), (u_8, [0, 0]), (u_9, [0, 0.6]), (u_{10}, [0.87, 0.92]),$

$\quad (u_{11}, [1, 1]), (u_{12}, [0.87, 0.92]), (u_{13}, [0, 0.6]), (u_{14}, [0, 0])\},$

$A_5 = \{(u_1, [0, 0]), (u_2, [0, 0]), (u_3, [0, 0]), (u_4, [0, 0]), (u_5, [0, 0]),$

$\quad (u_6, [0, 0]), (u_7, [0, 0]), (u_8, [0, 0]), (u_9, [0, 0]), (u_{10}, [0, 0]),$

$\quad (u_{11}, [0, 0]), (u_{12}, [0, 0.6]), (u_{13}, [0.87, 0.92]), (u_{14}, [1, 1])\},$

$B_1 = \{(v_1, [1, 1]), (v_2, [0.94, 0.96]), (v_3, [0, 0.65]), (v_4, [0, 0]), (v_5, [0, 0]),$

$\quad (v_6, [0, 0]), (v_7, [0, 0]), (v_8, [0, 0]), (v_9, [0, 0]), (v_{10}, [0, 0]),$

$\quad (v_{11}, [0, 0]), (v_{12}, [0, 0]), (v_{13}, [0, 0]), (v_{14}, [0, 0])\},$

$B_2 = \{(v_1, [0, 0]), (v_2, [0, 0.6]), (v_3, [0.87, 0.92]), (v_4, [1, 1]), (v_5, [0.87, 0.92]),$

$\quad (v_6, [0, 0.6]), (v_7, [0, 0]), (v_8, [0, 0]), (v_9, [0, 0]), (v_{10}, [0, 0]),$

$\quad (v_{11}, [0, 0]), (v_{12}, [0, 0]), (v_{13}, [0, 0]), (v_{14}, [0, 0])\},$

$B_3 = \{(v_1, [0, 0]), (v_2, [0, 0]), (v_3, [0, 0]), (v_4, [0, 0.5]), (v_5, [0.74, 0.82]),$

$\quad (v_6, [0.94, 0.95]), (v_7, [1, 1]), (v_8, [0.94, 0.95]), (v_9, [0.74, 0.82]), (v_{10}, [0, 0.5]),$

$\quad (v_{11}, [0, 0]), (v_{12}, [0, 0]), (v_{13}, [0, 0]), (v_{14}, [0, 0])\},$

$B_4 = \{(v_1, [0, 0]), (v_2, [0, 0]), (v_3, [0, 0]), (v_4, [0, 0]), (v_5, [0, 0]),$

$\quad (v_6, [0, 0]), (v_7, [0, 0.5]), (v_8, [0.74, 0.82]), (v_9, [0.94, 0.95]), (v_{10}, [1, 1]),$

$\quad (v_{11}, [0.94, 0.95]), (v_{12}, [0.74, 0.82]), (v_{13}, [0, 0.5]), (v_{14}, [0, 0])\},$

$B_5 = \{(v_1, [0, 0]), (v_2, [0, 0]), (v_3, [0, 0]), (v_4, [0, 0]), (v_5, [0, 0]),$

$\quad (v_6, [0, 0]), (v_7, [0, 0]), (v_8, [0, 0]), (v_9, [0, 0]), (v_{10}, [0, 0]),$

$\quad (v_{11}, [0, 0.6]), (v_{12}, [0.87, 0.92]), (v_{13}, [1, 1]), (v_{14}, [1, 1])\}.$

The membership function curves of these interval-valued fuzzy sets are shown in Fig. 1. Based on formulas (9) and (10), the interval-valued fuzzy sets A_i can be represented by the subscript vectors $\overline{A_i}$ and the superscript
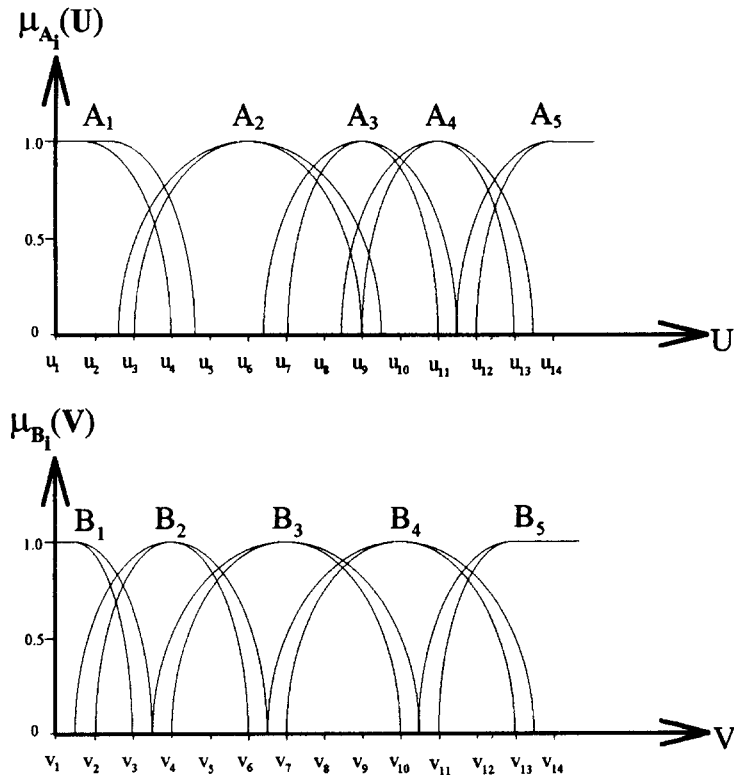

Fig. 1. The membership functions of interval-valued fuzzy sets.

vectors $\overline{\overline{A_i}}$, $0 \leqslant i \leqslant 5$, where

$\overline{A_0} = \langle 0, 0, 0.90, 1, 0.90, 0, 0, 0, 0, 0, 0, 0, 0, 0 \rangle$,

$\overline{A_1} = \langle 1, 1, 0.82, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \rangle$,

$\overline{A_2} = \langle 0, 0, 0, 0.75, 0.94, 1, 0.94, 0.75, 0, 0, 0, 0, 0, 0 \rangle$,

$\overline{A_3} = \langle 0, 0, 0, 0, 0, 0, 0, 0.87, 1, 0.87, 0, 0, 0, 0 \rangle$,

$\overline{A_4} = \langle 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.87, 1, 0.87, 0, 0 \rangle$,

$\overline{A_5} = \langle 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.87, 1 \rangle$,

$\overline{\overline{A_0}} = \langle 0, 0, 0.95, 1, 0.95, 0.8, 0, 0, 0, 0, 0, 0, 0, 0 \rangle$,

$\overline{\overline{A_1}} = \langle 1, 1, 0.95, 0.7, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \rangle$,

$\overline{\overline{A_2}} = \langle 0, 0, 0.5, 0.8, 0.95, 1, 0.95, 0.83, 0.5, 0, 0, 0, 0, 0 \rangle$,

$\overline{\overline{A_3}} = \langle 0, 0, 0, 0, 0, 0, 0.6, 0.92, 1, 0.92, 0.6, 0, 0, 0 \rangle$,

$\overline{\overline{A_4}} = \langle 0, 0, 0, 0, 0, 0, 0, 0, 0.6, 0.92, 1, 0.92, 0.6, 0 \rangle$,

$\overline{\overline{A_5}} = \langle 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.6, 0.92, 1 \rangle$,

and the interval-valued fuzzy sets B_1, B_2, B_3, B_4, B_5 can also be represented by the subscript vectors $\overline{B_1}$, $\overline{B_2}$, $\overline{B_3}$, $\overline{B_4}$, $\overline{B_5}$ and the superscript vectors $\overline{\overline{B_1}}$, $\overline{\overline{B_2}}$, $\overline{\overline{B_3}}$, $\overline{\overline{B_4}}$, $\overline{\overline{B_5}}$, respectively, where

$\overline{B_1} = \langle 1, 0.94, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \rangle$,

$\overline{B_2} = \langle 0, 0, 0.87, 1, 0.87, 0, 0, 0, 0, 0, 0, 0, 0, 0 \rangle$,

$\overline{B_3} = \langle 0, 0, 0, 0, 0.74, 0.94, 1, 0.94, 0.74, 0, 0, 0, 0, 0 \rangle$,

$\overline{B_4} = \langle 0, 0, 0, 0, 0, 0, 0, 0.74, 0.94, 1, 0.94, 0.74, 0, 0 \rangle$,

$\overline{B_5} = \langle 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.87, 1, 1 \rangle$,

$\overline{\overline{B_1}} = \langle 1, 0.96, 0.65, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \rangle$,

$\overline{\overline{B_2}} = \langle 0, 0.6, 0.92, 1, 0.92, 0.6, 0, 0, 0, 0, 0, 0, 0, 0 \rangle$,

$\overline{\overline{B_3}} = \langle 0, 0, 0, 0.5, 0.82, 0.95, 1, 0.95, 0.82, 0.5, 0, 0, 0, 0 \rangle$,

$\overline{\overline{B_4}} = \langle 0, 0, 0, 0, 0, 0, 0.5, 0.82, 0.95, 1, 0.95, 0.82, 0.5, 0 \rangle$,

$\overline{\overline{B_5}} = \langle 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.6, 0.92, 1, 1 \rangle$.

Assume that given the fact "X is A_0", where

$A_0 = \{(u_1, [0, 0]), (u_2, [0, 0]), (u_3, [0.90, 0.95]), (u_4, [1, 1]), (u_5, [0.90, 0.95]),$

$\qquad (u_6, [0, 0.8]), (u_7, [0, 0]), (u_8, [0, 0]), (u_9, [0, 0]), (u_{10}, [0, 0]),$

$\qquad (u_{11}, [0, 0]), (u_{12}, [0, 0]), (u_{13}, [0, 0]), (u_{14}, [0, 0])\}$,

then

(i) Because $k_1 = M(A_0, A_1) = 0.47$, we can get

$B_1^* = \{(v_1, [0.47, 0.47]), (v_2, [0.44, 0.45]), (v_3, [0, 0.3]), (v_4, [0, 0]), (v_5, [0, 0]),$

$\quad (v_6, [0, 0]), (v_7, [0, 0]), (v_8, [0, 0]), (v_9, [0, 0]), (v_{10}, [0, 0]),$

$\quad (v_{11}, [0, 0]), (v_{12}, [0, 0]), (v_{13}, [0, 0]), (v_{14}, [0, 0])\}.$

(ii) Because $k_2 = M(A_0, A_2) = 0.53$, we can get

$B_2^* = \{(v_1, [0, 0]), (v_2, [0, 0.32]), (v_3, [0.46, 0.49]), (v_4, [0.53, 0.53]), (v_5, [0.46, 0.49]),$

$\quad (v_6, [0, 0.32]), (v_7, [0, 0]), (v_8, [0, 0]), (v_9, [0, 0]), (v_{10}, [0, 0]),$

$\quad (v_{11}, [0, 0]), (v_{12}, [0, 0]), (v_{13}, [0, 0]), (v_{14}, [0, 0])\}.$

(iii) Because $k_3 = M(A_0, A_3) = 0$, we can get

$B_3^* = \{(v_i, [0, 0]) \mid 1 \leqslant i \leqslant 14\}.$

(iv) Because $k_4 = M(A_0, A_4) = 0$, we can get

$B_4^* = \{(v_i, [0, 0]) \mid 1 \leqslant i \leqslant 14\}.$

(v) Because $k_5 = M(A_0, A_5) = 0$, we can get

$B_5^* = \{(v_i, [0, 0]) \mid 1 \leqslant i \leqslant 14\}.$

Finally, we can get the deduced consequence "Y is B_0" of the SISO interval-valued approximate reasoning scheme, where,

$B_0 = B_1^* \cup B_2^* \cup B_3^* \cup B_4^* \cup B_5^*$

$\quad = \{(v_1, [0.47, 0.47]), (v_2, [0.44, 0.45]), (v_3, [0.46, 0.49]), (v_4, [0.53, 0.53]), (v_5, [0.46, 0.46]),$

$\qquad (v_6, [0, 0.32]), (v_7, [0, 0]), (v_8, [0, 0]), (v_9, [0, 0]), (v_{10}, [0, 0]),$

$\qquad (v_{11}, [0, 0]), (v_2, [0, 0]), (v_{13}, [0, 0]), (v_{14}, [0, 0])\}.$

The reasoning result is shown in Fig. 2. Because $M(A_0, A_2)$ has the largest value among the values of $M(A_0, A_1)$, $M(A_0, A_2)$, $M(A_0, A_3)$, $M(A_0, A_4)$ and $M(A_0, A_5)$, we can see that the interval-valued fuzzy set B_0 is the most similar to the interval-valued fuzzy set B_2.

Conversely, let us consider the following SISO approximate reasoning scheme:

R_1:   IF X is A_1 THEN Y is B_1

R_2:   IF X is A_2 THEN Y is B_2

$\qquad \vdots$ (21)

R_p: IF X is A_p THEN Y is B_p

Fact: Y is B_0

_____

Consequence: X is A_0

where

$A_i = \{(u_1, [x_{i1}, x_{i1}^*]), (u_2, [x_{i2}, x_{i2}^*]), \dots, (u_n, [x_{in}, x_{in}^*])\},$

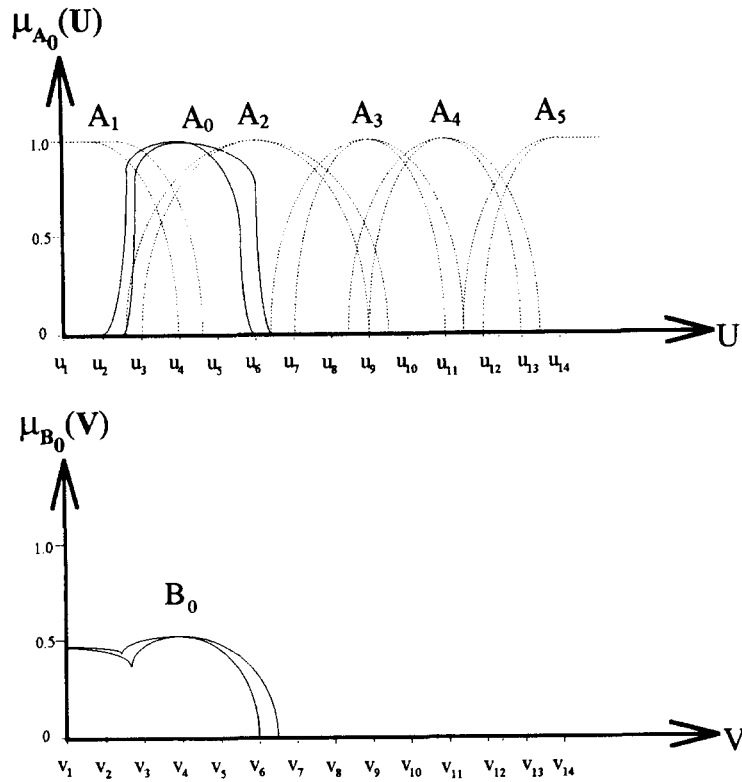$B_j = \{(v_1, [y_{j1}, y_{j1}^*]), (v_2, [y_{j2}, y_{j2}^*]), \dots, (v_m, [y_{jm}, y_{jm}^*])\},$

Fig. 2. The reasoning result of Example 4.1.

where $1 \leqslant i \leqslant p$ and $0 \leqslant j \leqslant p$. Based on formulas (9) and (10), the interval-valued fuzzy sets B_j can be represented by the subscript vectors $\overline{B_j}$ and the superscript vectors $\overline{\overline{B_j}}$, $0 \leqslant j \leqslant p$, where

$$\overline{B_0} = \langle y_{01}, y_{02}, \dots, y_{0m} \rangle,$$

$$\overline{B_1} = \langle y_{11}, y_{12}, \dots, y_{1m} \rangle,$$

$$\overline{B_2} = \langle y_{21}, y_{22}, \dots, y_{2m} \rangle,$$

$$\vdots$$

$$\overline{B_p} = \langle y_{p1}, y_{p2}, \dots, y_{pm} \rangle, \tag{22}$$

$$\overline{\overline{B_0}} = \langle y_{01}^*, y_{02}^*, \dots, y_{0m}^* \rangle,$$

$$\overline{\overline{B_1}} = \langle y_{11}^*, y_{12}^*, \dots, y_{1m}^* \rangle,$$

$$\overline{\overline{B_2}} = \langle y_{21}^*, y_{22}^*, \dots, y_{2m}^* \rangle,$$

$$\vdots$$

$$\overline{\overline{B_p}} = \langle y_{p1}^*, y_{p2}^*, \dots, y_{pm}^* \rangle.$$

Based on the previous discussions, we can get the following results:

$M(B_0, B_1) = s_1 \Rightarrow$ the deduced consequence of rule R_1 is "X is A_1^*", where

$$A_1^* = \{(u_1, [s_1{}^*x_{11}, s_1{}^* x_{11}^*]), (u_2, [s_1{}^*x_{12}, s_1{}^*x_{12}^*]), \ldots, (u_n, [s_1{}^*x_{1n}, s_1{}^*x_{1n}^*])\},$$

$M(B_0, B_2) = s_2 \Rightarrow$ the deduced consequence of rule R_2 is "X is A_2^*", where

$$A_2^* = \{(u_1, [s_2{}^*x_{21}, s_2{}^*x_{21}^*]), (u_2, [s_2{}^*x_{22}, s_2{}^*x_{22}^*]), \ldots, (u_n, [s_2{}^*x_{2n}, s_2{}^*x_{2n}^*])\},$$

$$\vdots$$

$M(B_0, B_p) = s_p \Rightarrow$ the deduced consequence of rule R_p is "X is A_p^*", where

$$A_p^* = \{(u_1, [s_p{}^* x_{p1}, s_p{}^* x_{p1}^*]), (u_2, [s_p{}^*x_{p2}, s_p{}^*x_{p2}^*]), \ldots, (u_n, [s_p{}^*x_{pn}, s_p{}^* x_{pn}^*])\},$$

where $s_i \in [0, 1]$ and $1 \leq i \leq p$, and the deduced consequence of the SISO approximate reasoning scheme is "Y is A_0", where

$$A_0 = A_1^* \cup A_2^* \cup \cdots \cup A_p^*, \tag{23}$$

and "$\cup$" is the union operator of the interval-valued fuzzy sets. That is,

$$A_0 = \{(u_1, [w_1, w_1^*]), (u_2, [w_2, w_2^*]), \ldots, (u_n, [w_n, w_n^*])\}, \tag{24}$$

where

$$w_1 = \text{Max}(s_1{}^*x_{11}, s_2{}^*x_{21}, \ldots, s_p{}^*x_{p1}),$$

$$w_1^* = \text{Max}(s_1{}^*x_{11}^*, s_2{}^*x_{21}^*, \ldots, s_p{}^*x_{p1}^*),$$

$$w_2 = \text{Max}(s_1{}^*x_{12}, s_2{}^*x_{22}, \ldots, s_p{}^*x_{p2}),$$

$$w_2^* = \text{Max}(s_1{}^*x_{12}^*, s_2{}^*x_{22}^*, \ldots, s_p{}^*x_{p2}^*), \tag{25}$$

$$\vdots$$

$$w_n = \text{Max}(s_1{}^*x_{1n}, s_2{}^*x_{2n}, \ldots, s_p^* x_{pn}),$$

$$w_n^* = \text{Max}(s_1{}^*x_{1n}^*, s_2{}^*x_{2n}^*, \ldots, s_p{}^*x_{pn}^*),$$

$0 \leq w_i \leq w_i^* \leq 1$, and $1 \leq i \leq n$. If s_i is the largest value among the values $s_1, s_2, \ldots,$ and s_p, then the interval-valued fuzzy set A_0 is the most similar to the interval-valued fuzzy set A_i, where $1 \leq i \leq p$.

Example 4.2. Consider the following single-input-single-output (SISO) approximate reasoning scheme:

R_1:   IF X is A_1 THEN Y is B_1

R_2:   IF X is A_2 THEN Y is B_2

R_3:   IF X is A_3 THEN Y is B_3

R_4:   IF X is A_4 THEN Y is B_4

R_5:   IF X is A_5 THEN Y is B_5

Fact: Y is B_0

Consequence: X is A_0

where X and Y are linguistic variables, A_0, A_1, A_2, A_3, A_4, and A_5 are interval-valued fuzzy sets of the universe of discourse U, $U = \{u_1, u_2, \ldots, u_{14}\}$, B_1, B_2, B_3, B_4, and B_5 are interval-valued fuzzy sets of the universe of discourse V, $V = \{v_1, v_2, \ldots, v_{14}\}$. These interval-valued fuzzy sets are the same as those shown in Example 4.1, where the membership functions of these interval-valued fuzzy sets are shown in Fig. 1. Assume that given the fact "Y is B_0", where

$$B_0 = \{(v_1, [0, 0]), (v_2, [0, 0]), (v_3, [0, 0]), (v_4, [0, 0.25]), (v_5, [0.55, 0.67]),$$

$$(v_6, [0.88, 0.90]), (v_7, [1, 1]), (v_8, [0.88, 0.90]), (v_9, [0.55, 0.67]), (v_{10}, [0, 0.25]),$$

$$(v_{11}, [0, 0]), (v_{12}, [0, 0]), (v_{13}, [0, 0]), (v_{14}, [0, 0])\}.$$

then

(i) Because $k_1 = M(B_0, B_1) = 0$, we can get

$$A_1^* = \{(u_i, [0, 0]) \mid 1 \leqslant i \leqslant 14\}.$$

(ii) Because $k_2 = M(B_0, B_2) = 0.27$, we can get

$$A_2^* = \{(u_1, [0, 0]), (u_2, [0, 0]), (u_3, [0, 0.14]), (u_4, [0.20, 0.22]), (u_5, [0.25, 0.26]),$$

$$(u_6, [0.27, 0.27]), (u_7, [0.25, 0.26]), (u_8, [0.20, 0.22]), (u_9, [0, 0.14]), (u_{10}, [0, 0]),$$

$$(u_{11}, [0, 0]), (u_{12}, [0, 0]), (u_{13}, [0, 0]), (u_{14}, [0, 0])\}.$$

(iii) Because $k_3 = M(B_0, B_3) = 0.89$, we can get

$$A_3^* = \{(u_1, [0, 0]), (u_2, [0, 0]), (u_3, [0, 0]), (u_4, [0, 0]), (u_5, [0, 0]),$$

$$(u_6, [0, 0]), (u_7, [0, 0.53]), (u_8, [0.77, 0.82]), (u_9, [0.89, 0.89]), (u_{10}, [0.77, 0.82]),$$

$$(u_{11}, [0, 0.53]), (u_{12}, [0, 0]), (u_{13}, [0, 0]), (u_{14}, [0, 0])\}.$$

(iv) Because $k_4 = M(B_0, B_4) = 0.38$, we can get

$$A_4^* = \{(u_1, [0, 0]), (u_2, [0, 0]), (u_3, [0, 0]), (u_4, [0, 0]), (u_5, [0, 0]),$$

$$(u_6, [0, 0]), (u_7, [0, 0]), (u_8, [0, 0]), (u_9, [0, 0.23]), (u_{10}, [0.33, 0.35]),$$

$$(u_{11}, [0.38, 0.38]), (u_{12}, [0.33, 0.35]), (u_{13}, [0, 0.23]), (u_{14}, [0, 0])\}.$$

(v) Because $k_5 = M(B_0, B_5) = 0$, we can get

$$A_5^* = \{(u_i, [0, 0]) \mid 1 \leqslant i \leqslant 14\}.$$

Finally, we can get the deduced consequence "X is A_0" of the SISO interval-valued approximate reasoning scheme, where

$$A_0 = A_1^* \cup A_2^* \cup A_3^* \cup A_4^* \cup A_5^*$$

$$= \{(u_1, [0, 0]), (u_2, [0, 0]), (u_3, [0, 0.14]), (u_4, [0.20, 0.22]), (u_5, [0.25, 0.26]),$$

$$(u_6, [0.27, 0.27]), (u_7, [0.25, 0.53]), (u_8, [0.77, 0.82]), (u_9, [0.89, 0.89]), (u_{10}, [0.77, 0.82]),$$

$$(u_{11}, [0.38, 0.53]), (u_{12}, [0.33, 0.35]), (u_{13}, [0, 0.23]), (u_{14}, [0, 0])\}.$$

$\mu_{B_0}(V)$

B_0

1.0

0.5

0

V_1 V_2 V_3 V_4 V_5 V_6 V_7 V_8 V_9 V_{10} V_{11} V_{12} V_{13} V_{14}

$\mu_{A_0}(U)$

A_0

1.0

0.5

0

u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8 u_9 u_{10} u_{11} u_{12} u_{13} u_{14}
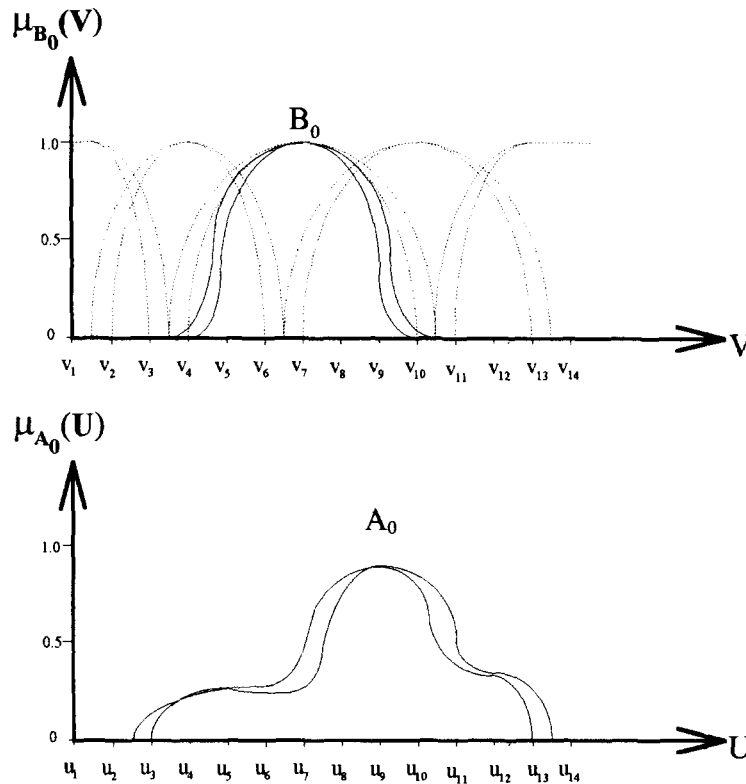
Fig. 3. The reasoning result of Example 4.2.

The reasoning result is shown in Fig. 3. Because $M(B_0, B_3)$ has the largest value among the values of $M(B_0, B_1)$, $M(B_0, B_2)$, $M(B_0, B_3)$, $M(B_0, B_4)$ and $M(B_0, B_5)$, we can see that the interval-valued fuzzy set A_0 is the most similar to the interval-valued fuzzy set A_3.

5. Conclusions

In [1], Bien and Chun have presented an inference network for bidirectional approximate reasoning based on an equality measure, where the fuzzy input and fuzzy output data are represented by fuzzy sets. In this paper, we have extended the work of [1] to propose a new method for bidirectional approximate reasoning based on interval-valued fuzzy sets. If an interval-valued fuzzy input is given for the rule-based system, then the system renders a reasonable interval-valued fuzzy output after performing approximate reasoning based on a similarity measure, and conversely, for a given interval-valued fuzzy output, the system can yield its corresponding reasonable interval-valued fuzzy input after performing approximate reasoning. The proposed method is more flexible than the one presented in [1] due to the fact that it allows the fuzzy terms appearing in the fuzzy production rules of a rule-based system to be represented by interval-valued fuzzy sets rather than general fuzzy sets. Furthermore, because the proposed method requires only simple arithmetic operations, and because it allows bidirectional approximate reasoning, it can be executed much faster and more flexible than the single-input-single-output approximate reasoning scheme presented in [12]. The proposed method allows the rule-based systems to perform bidirectional approximate reasoning in a more flexible and more simple manner.

Acknowledgements

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