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A generalization of the stable matching problem[☆]

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Abstract

It is known that there may not exist any stable matching for a given instance of the stable roommates problem. A stable partition is a structure that generalizes the notion of a stable matching; Tan (1991) proved that every instance of the stable roommates problem contains at least one such structure. In this paper we propose a new algorithm for finding a stable partition, and hence a new algorithm for finding a stable matching if one exists. Our algorithm processes the problem dynamically as long as certain relative preference orders are maintained. Some theoretical results about stable partitions are also presented.

Keywords: Stable roommates problem; Stable matching; Stable partition; Algorithm

1. Introduction

The stable roommates problem has been the subject of much research in recent years. This problem involves matching n people into $n/2$ disjoint pairs to achieve a certain type of stability. Such a matching is called "a complete stable matching". However, it is known [1, 5] that there may exist no complete stable matching for a given instance of the stable roommates problem. Irving [3] proposed an $O(n^2)$ algorithm that finds a complete stable matching or confirms that none exists. Recently Gusfield and Irving [2] listed over a hundred research papers related to this problem. Much of the recent research concerning this problem is explicitly or implicitly subject to one or both of the following restrictions:

(i) The discussion assumes that there exists at least one complete stable matching.
(ii) The preference lists are static. In other words, the entire preference lists are given beforehand.

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In a recent paper [6], Tan established a necessary and sufficient condition for the existence of a complete stable matching. Toward that end, he defined a new structure, called "a stable partition", that is a generalization of the notion of a stable matching, and proved that every instance of the stable roommates problem contains at least one such structure.

In this paper, we treat the problem by relaxing the above two restrictions, and instead of finding a complete stable matching, which may not exist, we look for the more general structure, a stable partition. Our approach in a sense extends that of Itoga [4], which considers the bipartite case (the stable marriage problem), while ours considers the non-bipartite case (the stable roommates problem). Our approach leads to the following results:

(i) Our algorithm processes the problem on line, i.e., the preference lists are allowed to expand dynamically as long as certain relative preference orders are maintained.

(ii) We introduce a new algorithm to find a stable partition, and hence a new algorithm to find a complete stable matching if one exists.

(iii) We give a new proof of the known fact [6] that there exists at least one stable partition for every instance of the stable roommates problem.

(iv) We obtain some theoretical properties of stable partitions that are interesting in their own right.

2. Definitions

In this section, we state the stable roommates problem and recall the definition of a stable partition introduced by Tan [6]. There is a set S of n people. Each person i has a preference list consisting of a subset S_i of $S - \{i\}$ and a rank ordering (most preferred first) of the persons in S_i. For person i, the subset S_i includes all of the persons he is willing to be matched with. A preference relation R is defined to be a pair (S, T), where S is a set of n persons and T is the table of preference lists of these n people. A complete matching M is a partition of the n persons into $n/2$ disjoint pairs of roommates such that for every pair $\{a, b\}$ in M, a is on b's list and b is on a's list. A complete matching M is *unstable* if there are two persons who are not matched together, but each of whom prefers the other to his mate in the matching. A complete matching which is not unstable is called a *stable* matching. The stable roommates problem, as originally stated [1, 5], is to find a complete stable matching. It is known that there may exist no complete stable matching. A stable partition is a structure that generalizes the notion of a complete stable matching; Tan [6] proved that every preference relation contains at least one such structure. We now introduce it.

Let T be a table of preference lists. If person b is on the preference list of person a, then we write $(a \mid b)$ to denote the entry b on a's preference list. We define $r(a \mid b) = k$ to mean that person b occupies position k on a's preference list. The expression $r(a \mid b) < r(a \mid c)$ means that person a prefers b to c.

Let (S, T) be a preference relation, and let A be a subset of S. Denote by $|A|$ the cardinality of set A. A cyclic permutation $\Pi(A) = \langle a_1, a_2, a_3, \ldots, a_k \rangle$ of the persons in A, where $k = |A|$, is called a *semi-party permutation* if one of the following three conditions holds:

(i) $|A| \geqslant 3$, a_{i+1} and a_{i-1} are on a_i's preference list, and $r(a_i \mid a_{i+1}) < r(a_i \mid a_{i-1})$, $i = 1, 2, 3, \ldots, k$ (subscripts modulo k);

(ii) $|A| = 2$, and a_{i-1} is on a_i's preference list, $i = 1, 2$ (subscripts modulo 2);

(iii) $|A| = 1$.

For a specified semi-party permutation $\Pi(A) = \langle a_1, a_2, \ldots, a_k \rangle$ of the persons in A, the entries in the preference lists of A are classified into the following categories.

(I) If $|A| \geqslant 3$, entry $(a_i \mid b)$ is said to be
  - (i) a *superior* entry with respect to $\Pi(A)$, if $r(a_i \mid b) < r(a_i \mid a_{i-1})$;
  - (ii) an *inferior* entry with respect to $\Pi(A)$, if $r(a_i \mid a_{i-1}) \leqslant r(a_i \mid b)$ (note that this inequality is " $\leqslant$ ", not " $<$ ");
  - (iii) a *party* entry with respect to $\Pi(A)$, if $b = a_{i+1}$ or $b = a_{i-1}$ for $i = 1, 2, 3, \ldots, k$ (subscripts modulo k).

(II) If $|A| = 2$, i.e., $k = 2$, $(a_i \mid b)$ is said to be
  - (i) a *superior* entry with respect to $\Pi(A)$, if $r(a_i \mid b) < r(a_i \mid a_{i-1})$;
  - (ii) an *inferior* entry with respect to $\Pi(A)$, if $r(a_i \mid a_{i-1}) < r(a_i \mid b)$ (note that this inequality is " $<$ ", not " $\leqslant$ ");
  - (iii) a *party* entry with respect to $\Pi(A)$, if $b = a_{i-1}$ for $i = 1, 2$ (subscripts modulo 2).

(III) If $|A| = 1$, then $(a_i \mid b)$ is a *superior* entry with respect to $\Pi(A)$ for every person b on a_i's preference list.

In the above definition, if there is no ambiguity, we will omit the phrase "with respect to $\Pi(A)$". For convenience, we will assume that the table of preference lists is *symmetric*, i.e., a is on b's list if and only if b is on a's.

Given a preference relation (S, T), a *stable partition* Π of (S, T) consists of a partition of the set S; $S = \bigcup_{i=1}^{m} A_i$, $A_i \cap A_j = \emptyset$ if $i \neq j$, and a specified semi-party permutation $\Pi(A_i)$ for each A_i, $i = 1, 2, \ldots, m$, such that the following *stable condition* is satisfied:

If $(a \mid b)$ is a superior entry then $(b \mid a)$ is an inferior entry.

Remark. If T is not symmetric, then the above stable condition should be modified as follows: If $(a \mid b)$ is a superior entry then either $(b \mid a)$ is an inferior entry or a is not on b's preference list.

In the context of the above definition, the associated semi-party permutation $\Pi(A_i)$ is called a *party permutation* for A_i, and each A_i is called a *party*. An *odd party* (respectively, *even party*) is a party having odd (respectively, even) cardinality. More precisely, these terms are defined with respect to the given stable partition Π. If there are ambiguities, we will say that A_i is a party in Π (or a Π-party), $(a \mid b)$ is a superior entry in Π (or a Π-superior entry), and so on.

A stable partition Π is specified by its party permutations and will be denoted by $\Pi = \{\Pi(A_1), \Pi(A_2), \Pi(A_3), \ldots, \Pi(A_m)\}$. Persons a and b are said to be a *matching pair* (or *matched*) in Π if $\{a, b\}$ forms a 2-person party in Π. A subset A of the all-person set S is said to form a *party* (respectively, an odd party) if there exists a stable partition Π such that A is a party (respectively, an odd party) in Π.

We give the following example to illustrate the above definitions.

Person		Preference list			
1	.	2	.	4	.
2	.	3	.	1	.
3	Superior	4	Superior	2	Inferior
4	.	1	.	3	.
5	Superior		6		Inferior
6			5		
7	.	8	.	11	.
8	Superior	9	Superior	7	Inferior
9	.	10	.	8	.
10	.	11	.	9	.
11	.	7	.	10	.
12			Superior		

This example depicts a stable partition in which there are four parties, $A_1 = \{1,2,3,4\}$, $A_2 = \{5,6\}$, $A_3 = \{7,8,9,10,11\}$, and $A_4 = \{12\}$, and $\Pi = \{\langle 1,2,3,4\rangle, \langle 5,6\rangle, \langle 7,8,9,10,11\rangle, \langle 12\rangle\}$. To complete the example, we merely have to fill in all the other entries and follow the rule that whenever $(a\,|\,b)$ is a superior entry, then $(b\,|\,a)$ is inferior.

A preference relation may have more than one stable partition. We can identify at least two other stable partitions in the above example:

$$\Pi_1 = \{\langle 1,2\rangle, \langle 3,4\rangle, \langle 5,6\rangle, \langle 7,8,9,10,11\rangle, \langle 12\rangle\}$$

and

$$\Pi_2 = \{\langle 2,3\rangle, \langle 4,1\rangle, \langle 5,6\rangle, \langle 7,8,9,10,11\rangle, \langle 12\rangle\}.$$

As one can see, all three of these stable partitions contain the same odd parties. Tan [6] proved that every preference relation contains at least one stable partition, and that any two stable partitions contain the same odd parties. Therefore the existence of an odd party depends on the preference relation, and not on a particular stable partition. In the following, we cite some results from [6] that are relevant here.

As stated in [6], the notion of a stable partition is a generalization of that of a complete stable matching in the following sense.

Proposition 2.1 (Tan [6]). *A complete stable matching is a stable partition in which every party has cardinality two and vice versa.*

Proof. This follows directly from the definitions. $M = \{\{a_i, b_i\} \mid i = 1 \text{ to } n/2\}$ is a complete stable matching if and only if $\Pi = \{\langle a_i, b_i \rangle \mid i = 1 \text{ to } n/2\}$ is a stable partition. □

We shall need the technique described in the following result later.

Proposition 2.2 (Tan [6]). *A stable partition without any odd party induces a complete stable matching.*

Proof. Suppose that Π is a stable partition without any odd party. Let A be an even party in Π with party permutation $\langle a_1, a_2, a_3, \ldots, a_{2k} \rangle$, $k \geqslant 2$. Then by decomposing party A into k matching pairs $\langle a_1, a_2 \rangle, \langle a_3, a_4 \rangle, \ldots, \langle a_{2k-1}, a_{2k} \rangle$, we have a new stable partition $\Pi' = (\Pi - \{\langle a_1, a_2, \ldots, a_{2k} \rangle\}) \cup \{\langle a_1, a_2 \rangle, \langle a_3, a_4 \rangle, \ldots, \langle a_{2k-1}, a_{2k} \rangle\}$. This is because every superior entry in Π' is a superior entry in Π, and every inferior entry in Π, other than the party entries, is an inferior entry in Π'. By continuing to decompose any even party having cardinality 4 or more, we eventually obtain a stable partition in which every party has cardinality two. □

Proposition 2.3 (Tan [6]). *Given an instance of the stable roommates problem, there exists a complete stable matching if and only if there exists no odd party.*

3. Proposal-rejection alternating sequence

Let $R = (S, T)$ be a preference relation, and let $a \in S$. We define the *deletion* of person a from R, denoted by $R - a$, to be the preference relation (S', T'), where $S' = S - \{a\}$, and T' is the table of preference lists obtained from T by deleting the preference list of person a and the entries $(x \mid a)$, for every $x \in S'$. Suppose that b is a new person, i.e., $b \notin S$. The *addition* of b into R, denoted by $R + b$, is defined to be the reverse operation of deletion. More specifically, each person in S inserts person b into his preference list without changing the relative order of his original list. These enlarged lists together with the preference list of b toward the other persons constitute the table of preference lists of $R + b$.

Let us consider the following situation. Suppose that we have already found a stable partition Π for preference relation $R = (S, T)$ (note that for $|S| = 1$ or 2 a stable partition is immediately at hand), and one additional person a is then added to the relation. The question then arises whether there is a stable partition for the enlarged preference relation $R + a$. If the answer is yes, another natural question is how to find this partition. Is it necessary to start all over again, or is there a way of augmenting the current stable partition to incorporate the new person? Tan [6] proved that every preference relation contains at least one stable partition. Therefore, the answer to the first question is affirmative. However, the proof in [6] is quite long and complicated. In this paper we propose a new algorithm and a

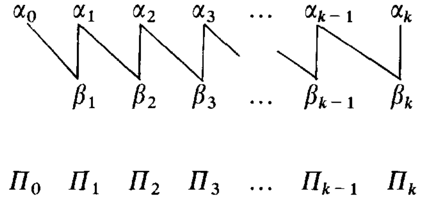


Fig. 1.

new proof that resolve both questions. The basic idea behind our algorithm is that of "adding one person at a time". For ease of exposition, let us define the following terms.

Let $R = (S, T)$ be a preference relation and let $\alpha_0 \in S$. Given a stable partition Π_0 of $R - \alpha_0$, a sequence of persons $\alpha_0, \beta_1, \alpha_1, \beta_2, \alpha_2, \ldots, \beta_k, \alpha_k$, $k \geqslant 0$, is called a *proposal-rejection alternating sequence* starting from α_0 (or simply *an alternating sequence*), if there is a sequence of stable partitions Π_i, $i = 0, 1, 2, \ldots, k$, for preference relation $R - \alpha_i$ (see Fig. 1) such that

(i) $(\alpha_i \mid \beta_{i+1})$ is the first entry on α_i's list (starting from the most preferred person) such that $(\beta_{i+1} \mid \alpha_i)$ is a Π_i-superior entry;
(ii) $\langle \beta_{i+1}, \alpha_{i+1} \rangle$ is a two-person party in Π_i;
(iii) $\Pi_{i+1} = (\Pi_i - \{\langle \beta_{i+1}, \alpha_{i+1} \rangle\}) \cup \{\langle \alpha_i, \beta_{i+1} \rangle\}$.

The motivation for this definition is as follows. Consider person α_k and stable partition Π_k for $R - \alpha_k$ (see Fig. 1). One may think of α_k as being out of the relation initially. To incorporate this person into a stable partition, let α_k propose to others successively in the order of his preference list, until either there is someone x who finds that $(x \mid \alpha_k)$ is Π_k-superior and accepts α_k, or everyone rejects α_k. There are three cases, as follows.

(i) α_k is rejected by every person on his list, i.e., $(x \mid \alpha_k)$ is Π_k-inferior for every x on α_k's list. Then α_k by himself forms an odd party, and $\Pi_k \cup \{\langle \alpha_k \rangle\}$ is a stable partition for R.

In case (ii) and (iii), there is someone who accepts α_k. Let x be the first one on α_k's list who finds that $(x \mid \alpha_k)$ is Π_k-superior.

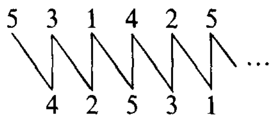
(ii) If x is currently in a Π_k-odd party, say $\langle a_1, a_2, \ldots, a_{2m}, x \rangle$, then this odd party is decomposed and $\Pi_{k+1} = (\Pi_k - \{\langle a_1, a_2, \ldots, a_{2m}, x \rangle\}) \cup \{\langle a_1, a_2 \rangle, \langle a_3, a_4 \rangle, \ldots, \langle a_{2m-1}, a_{2m} \rangle, \langle x, a_k \rangle\}$ is a stable partition for R.

(iii) Suppose that x is currently in a Π_k-even party. In view of Proposition 2.2, we may assume that x is currently in a Π_k-even party of size 2, say $\langle x, y \rangle$. (In fact, we may assume from the beginning that all the even parties encountered have size exactly 2.) Let $\beta_{k+1} = x$ and $\alpha_{k+1} = y$. Then $\Pi_{k+1} = (\Pi_k - \{\langle \beta_{k+1}, \alpha_{k+1} \rangle\}) \cup \{\langle \alpha_k, \beta_{k+1} \rangle\}$ is a stable partition for $R - \alpha_{k+1}$, and $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k, \beta_{k+1}, \alpha_{k+1}$ is a longer alternating sequence. One may then repeat the process for person α_{k+1}.

Example.

Person	Preference list
1	2 5
2	3 1
3	4 2
4	5 3
5	1 4

Given the preference table shown above, if person 5 is deleted from the table, $\Pi_0 = \{\langle 1,2\rangle \langle 3,4\rangle\}$ is a stable partition for the remaining persons. Then the following diagram is an alternating sequence starting from person 5.



Let us discuss some properties of the alternating sequence.

Proposition 3.1. *Let $R = (S, T)$ be a preference relation, and let $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k$ be an alternating sequence with associated stable partitions $\Pi_0, \Pi_1, \ldots, \Pi_k$, $k \geqslant 0$. Then*

(i) *each α_i has a worse partner in Π_{i+1} than he has in Π_{i-1} i.e., $r(\alpha_i \mid \beta_i) < r(\alpha_i \mid \beta_{i+1})$, $\forall i = 1, 2, \ldots, k-1$;*

(ii) *each β_i has a better partner in Π_i than he has in Π_{i-1}, i.e., $r(\beta_i \mid \alpha_{i-1}) < r(\beta_i \mid \alpha_i)$, $\forall i = 1, 2, \ldots, k$.*

Proof. By the definition of an alternating sequence, (ii) is trivial; this is because β_i is matched with α_i in Π_{i-1} and $(\beta_i \mid \alpha_{i-1})$ is Π_{i-1}-superior. To prove part (i), we observe that if $r(\alpha_i \mid x) < r(\alpha_i \mid \beta_i)$, then $(\alpha_i \mid x)$ is Π_{i-1}-superior. So $(x \mid \alpha_i)$ is Π_{i-1}-inferior and it is still Π_i-inferior, since $\Pi_i = (\Pi_{i-1} - \{\langle \alpha_i, \beta_i \rangle\}) \cup \{\langle \alpha_{i-1}, \beta_i \rangle\}$. Considering the stable partition Π_i for preference relation $R - \alpha_i$, when α_i proposes to the other persons in the order of his list, anyone before β_i in α_i's list will not accept his proposal. Therefore the first person who accepts α_i's proposal, β_{i+1}, must be behind β_i. This proves (i). □

The above result (i) implies that anyone who is rejected by another person can participate in the proposing process by continuing down his list of choices.

Proposition 3.2. *Let $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k$ be an alternating sequence. If $\beta_j \neq \alpha_i$ for $i < j$, then $\beta_j \neq \alpha_i$ for all i and j.*

Proof. Suppose not. Take j_0 and i_0 such that $j_0 < i_0$, $\beta_{j_0} = \alpha_{i_0}$, $\beta_j \neq \beta_{j_0}$ for all $j_0 < j \leqslant i_0$, and $\alpha_i \neq \alpha_{i_0}$ for all $j_0 \leqslant i < i_0$. By the definition of an alternating sequence, we know that $j_0 \neq i_0$, α_{j_0-1} and β_{j_0} are matched together in Π_{j_0}, and they are still together in Π_{i_0-1}. However, β_{j_0} $(=\alpha_{i_0})$ is left unmatched in Π_{i_0}. So $\beta_{i_0} = \alpha_{j_0-1}$, which is a contradiction, and the result follows. □

Given an alternating sequence $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k$, if $\beta_j \neq \alpha_i$ for all $i < j$, then by Propositions 3.1 and 3.2, each α_i gets an increasingly worse partner as the alternating sequence continues, while each β_j gets an increasingly better partner. Each person has at most $n-1$ entries in his list. Therefore within $O(n^2)$ steps, one of the following three cases occurs.

(i) α_k proposes to the others but no one accepts his proposal.
(ii) There is a person who accepts α_k's proposal, and the first such person is currently in a Π_k-odd party.
(iii) The first person who accepts α_k's proposal, say β_{k+1}, is in a Π_k-even party and $\beta_{k+1} = \alpha_i$ for some $0 \leqslant i \leqslant k-1$.

As discussed before, in cases (i) and (ii), the alternating sequence terminates and a stable partition for preference relation $R = (S, T)$ has been found. Before discussing case (iii), we need some further definitions.

Definition. Let $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k, \beta_{k+1}, \alpha_{k+1}$ be an alternating sequence. If $\beta_{k+1} = \alpha_i$ for some $0 \leqslant i \leqslant k-1$, then we say that the alternating sequence has a *return* at β_{k+1}. Let i_0 be the largest index, $0 \leqslant i_0 \leqslant k-1$, such that $\beta_{k+1} = \alpha_{i_0}$. Then we say the sequence at β_{k+1} returns to i_0 (or returns to α_{i_0}, if there is no ambiguity). The subsequence $\alpha_{i_0}, \beta_{i_0+1}, \alpha_{i_0+1}, \ldots, \beta_k, \alpha_k, \beta_{k+1}$ is said to be the return sequence corresponding to β_{k+1}, or simply a return sequence.

When considering the subsequence starting from α_{i_0}, we may assume that the first return sequence is that starting from α_0.

Definition. Let $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k, \beta_{k+1}$ be a return sequence. The length of this return sequence is defined to be $2k+1$.

Now let us return to case (iii), in which the alternating sequence has a return at β_{k+1}. Following the method of extending the alternating sequence described before, we have the following result.

Theorem 3.3. *Let $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k$ be an alternating sequence. Suppose that there is a return at β_{k+1}; then the alternating sequence can be extended so that at a certain step, say at β_{k+m+1}, it returns to α_k. Moreover, the corresponding return sequence has the following two properties:*

(i) $\alpha_k, \beta_{k+1}, \alpha_{k+1}, \ldots, \beta_{k+m}, \alpha_{k+m}$ are $2m+1$ *distinct persons.*

(ii) $A = \langle \alpha_{k+m}, \beta_{k+m}, \alpha_{k+m-1}, \beta_{k+m-1}, \ldots, \alpha_{k+1}, \beta_{k+1}, \alpha_k \rangle$ *forms an odd party. (This is the reverse order of the return sequence.) More precisely,* $(\Pi_0 - \{\langle \beta_i, \alpha_i \rangle \mid i = k+1$ *to* $k+m\}) \cup \{A\}$ *is a stable partition for the whole preference relation* R.

We shall prove the above theorem in the next section. In the remainder of this section, we investigate some properties of return sequences and give a new proof of the fact that there exists a stable partition for every preference relation.

Proposition 3.4. *Let* $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k, \beta_{k+1}, \alpha_{k+1}$ *be an alternating sequence. Suppose that at* β_{k+1} *the first return occurs and it returns to* α_0. *Then the following hold*:
(i) $\beta_{k+1} = \alpha_0$, *and* $\beta_j \neq \alpha_i$ *for all* $1 \leqslant j \leqslant k$ *and* $0 \leqslant i \leqslant k$;
(ii) $\alpha_{k+1} = \beta_1$;
(iii) *consider stable partition* Π_{k+1} *for preference relation* $R - \alpha_{k+1}$; *when* α_{k+1} *proposes to others successively in the order of his list, there must be someone who accepts him, and the first such person is one of the* α_i*'s, i.e.,* $\beta_{k+2} = \alpha_i$ *for some* $1 \leqslant i \leqslant k$. *Moreover,* $r(\alpha_{k+1} \mid \beta_{k+2}) \leqslant r(\alpha_{k+1} \mid \alpha_1)$.

Proof. (i) This follows directly from the definition and Proposition 3.2.

(ii) Since α_0 and β_1 are matched together in Π_1, they remain together in Π_k, but not in Π_{k+1}, so $\alpha_{k+1} = \beta_1$.

(iii) Observe that α_{k+1} $(= \beta_1)$ and α_1 are matched together in Π_0. α_1 gets an increasingly worse partner from Π_0 to Π_{k+1}. When α_{k+1} $(= \beta_1)$ makes a proposal, α_1 certainly takes α_{k+1} as a Π_{k+1}-superior entry. So, in α_{k+1}'s list, there must be someone before α_1 (or equal to α_1) who accepts α_{k+1}. Let the first person who accepts α_{k+1}'s proposal be x; thus $(x \mid \alpha_{k+1})$ is Π_{k+1}-superior. If $x \neq \alpha_1$, then $r(\alpha_{k+1} \mid x) < r(\alpha_{k+1} \mid \alpha_1)$. So $(\alpha_{k+1} \mid x)$ is Π_0-superior, and $(x \mid \alpha_{k+1})$ is Π_0-inferior. Person x takes α_{k+1} as inferior in Π_0, but as superior in Π_{k+1}, therefore x has a worse partner in Π_{k+1} than he has in Π_0. By Propositions 3.1 and 3.2, $x = \alpha_i$ for some $1 \leqslant i \leqslant k$, and this completes the proof. □

Proposition 3.5. *Let* $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k, \ldots$ *be an alternating sequence. If there is a return at* β_{k+1}, *then there is another return at* β_{k+2}. *Furthermore, suppose that the return sequence corresponding to* β_{k+1} *(respectively,* β_{k+2}*) starts from* i_1 *(respectively,* i_2*); then* $i_2 > i_1$. *(So the length of the return sequence corresponding to* β_{k+2} *is less than or equal to that of the return sequence corresponding to* β_{k+1}.*)*

Proof. By Proposition 3.4(iii), the result is true if at β_{k+1} the sequence has its first return. Then, by the characteristics of the alternating sequence starting from α_{i_2} and a simple induction, the result follows. □

Corollary 3.6. *Let* $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k, \ldots$ *be an alternating sequence. If there is a return at* β_{k+1} *then the sequence can be extended infinitely. Moreover, there is a return at every* β_j, *for* $j \geqslant k+1$.

With the above definitions and discussion, we are able to give a new proof of the fact that every preference relation contains at least one stable partition and to explore some new properties of the stable partition. The proof is inductive and will lead to an algorithm for finding a stable partition, but we shall not worry about the efficiency of the algorithm until later.

Let $R = (S, T)$ be a preference relation. Suppose that we have already found a stable partition Π_0 for preference relation $R - \alpha_0$. Starting from α_0, an alternating sequence is generated. As discussed before, if the sequence $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k$ terminates at some point α_k, either because that there is no one to accept α_k, or because the first person who accepts α_k is in a Π_k-odd party, then a stable partition for preference relation R is found. Suppose that the sequence extends infinitely; by the finiteness of the problem, eventually $\alpha_{k_1} = \alpha_{k_2}$ and $\Pi_{k_1} = \Pi_{k_2}$ for some $k_1 < k_2$. Then the sequence cycles, and so does the associated sequence of stable partitions. Without loss of generality, we may assume that $\alpha_0 = \alpha_m$ and $\Pi_0 = \Pi_m$.

Theorem 3.7. *Let $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_m, \alpha_m$ be an alternating sequence with associated sequence of stable partitions $\Pi_0, \Pi_1, \ldots, \Pi_m$. Suppose that $\alpha_0 = \alpha_m$ and $\Pi_0 = \Pi_m$. Then*
(i) *there is an element β_{k+1}, $1 < k + 1 < m$, returning to α_0; and*
(ii) *$\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k$ are $2k + 1$ distinct persons, $A = \langle \alpha_k, \beta_k, \alpha_{k-1}, \beta_{k-1}, \ldots, \alpha_1, \beta_1, \alpha_0 \rangle$ forms an odd party, and $\Pi = (\Pi_0 - \{\langle \alpha_i, \beta_i \rangle \mid i = 1 \text{ to } k\}) \cup \{A\}$ is a stable partition for preference relation R.*

Proof. (i) It is obvious that the alternating sequence cycles with pattern $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_m, \alpha_m = \alpha_0$, and that the sequence must have a return at some point, say at β_j. Suppose that the return sequence corresponding to β_j starts at element α_i. Then we claim that the return sequence corresponding to the next element β_{j+1} starts at α_{i+1}. For otherwise, the length of the return sequence strictly decreases, contradicting the fact that the alternating sequence cycles. Since the sequence cycles, every α_i is the starting point of a return sequence. So (i) follows.

(ii) Consider the alternating sequence $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k, \beta_{k+1}, \alpha_{k+1}, \ldots$. The first return occurs at β_{k+1}, and it returns to α_0. Then, as proved in part (i), at each β_{k+j} it returns to α_{j-1}, so $\beta_{k+j} = \alpha_{j-1}$ for all $j \geqslant 1$. By Proposition 3.4, we know that $\alpha_{k+j} = \beta_j$ (see Fig. 2).

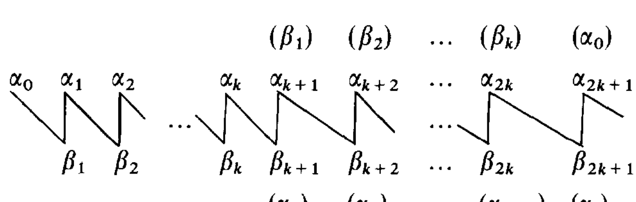


Fig. 2.

Note that $\alpha_0 \neq \alpha_i$ and $\alpha_0 \neq \beta_i$ for $1 \leqslant i \leqslant k$. Therefore, by the definition of a return sequence and by the fact that the sequence cycles, it is easy to verify the following facts:
(A) $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k$ are $2k+1$ distinct persons;
(B) $\langle \beta_i, \alpha_i \rangle$ is a two-person party in Π_0, for $i = 1$ to k;
(C) $\langle \alpha_{i-1}, \beta_i \rangle$ is a two-person party in Π_k for $i = 1$ to k;
(D) when restricted to persons in $S - \{\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k\}$, all the Π_i's are the same, for $i \geqslant 0$;
(E) $A = \langle \alpha_k, \beta_k, \alpha_{k-1}, \beta_{k-1}, \ldots, \alpha_1, \beta_1, \alpha_0 \rangle$ forms a semi-party permutation.

We now need to show that A is an odd party in Π, when $\Pi = (\Pi_0 - \{\langle \beta_i, \alpha_i \rangle \mid i = 1 \text{ to } k\}) \cup \{A\}$. To do this, we only have to show that Π is stable. Suppose not. Since Π_0 is stable, then any instability must involve some α_i or β_j. Because the alternating sequence cycles, without loss of generality, we may assume that person α_0 causes instability. So there is a person x such that both $(\alpha_0 \mid x)$ and $(x \mid \alpha_0)$ are Π-superior. Note that $(\alpha_0 \mid x)$ is Π-superior if and only if $r(\alpha_0 \mid x) < r(\alpha_0 \mid \beta_1)$. In light of the stable partitions Π_0 and Π_1 and the definition of an alternating sequence, x can only be some α_i or β_i, for some $i = 1$ to k. We claim that x cannot be any of the β_i's, nor any of the α_i's, and this will give a contradiction. First we consider the stable partition Π_1 in which α_0 is matched with β_1 and each β_i is matched with α_i for $i = 2$ to k. If $(\alpha_0 \mid \beta_i)$ is Π-superior, i.e., Π_1-superior, then $(\beta_i \mid \alpha_0)$ is Π_1-inferior, i.e., Π-inferior. So $x \neq \beta_i$ for $i = 2$ to k, and obviously $x \neq \beta_1$. Second, let us consider the stable partition Π_k: α_0 is matched with β_1, and each α_i is matched with β_{i+1}, for $i = 1$ to $k-1$. If $(\alpha_0 \mid \alpha_i)$ is Π-superior, i.e., Π_k-superior, then $(\alpha_i \mid \alpha_0)$ is Π_k-inferior, i.e. Π-inferior. Thus $x \neq \alpha_i$, for $i = 1$ to $k-1$, and clearly $x \neq \alpha_k$. This shows that Π is stable and the theorem follows. □

From the previous discussion and Theorem 3.7, by adding one person at a time and by a simple induction, we establish the following fact, first proved in [6].

Corollary 3.8. *There exists a stable partition for every preference relation.*

We also have the following new observation.

Corollary 3.9. *Given a preference relation, adding a new person into the relation results in the number of odd parties either increasing by* 1 *or decreasing by* 1. *Furthermore,*
(i) *when the number of odd parties increases by* 1, *all the original odd parties remain in the new relation, while a new odd party is formed;*
(ii) *when it decreases by* 1, *one of the existing odd parties is eliminated, and all the rest remain in the new relation.*

It can be shown that deleting one person from the relation has the same effect on the number of odd parties. An immediate consequence of these results is the following theorem, first proved in [6].

Theorem 3.10. *For a given preference relation, any two stable partitions have the same number of odd parties.*

Proof. Suppose not. Let Π_1 and Π_2 be two stable partitions having m_1 and m_2 odd parties, respectively, where $m_1 < m_2$. Deleting one person from each odd party in Π_1 results in a stable partition Π'_1 without any odd party. Deleting the same set of persons from Π_2 results in a stable partition Π'_2 with at least $m_2 - m_1$ odd parties; this is because deleting one person reduces the number of odd parties by at most 1. So, without loss of generality, we may assume that $m_1 = 0$ and $m_2 > 0$. By Proposition 2.2, we may also assume that each even party in Π_1 has cardinality two. Hence Π_1 is a complete stable matching.

Let S be the set of persons whose partners in Π_1 are superior entries in Π_2, and let I be the set of persons whose partners in Π_1 are inferior entries in Π_2. For the stability of Π_2, every person in S has a Π_1-partner in I. So $|S| \leqslant |I|$. Consider a party A in stable partition Π_2, and let $\langle a_1, a_2, \ldots, a_k \rangle$ be the associated party permutation of A. For the stability of Π_1, no two consecutive persons a_i and a_{i+1} (subscripts modulo k) can be in I, otherwise a_i and a_{i+1} block the matching Π_1.

Therefore, if A is an odd party in Π_2, then

$$|A \cap S| > |A \cap I|.$$

And if A is an even party in Π_2, then

$$|A \cap S| \geqslant |A \cap I|.$$

Since stable partition Π_2 contains at least one odd party, we have

$$|S| = \sum_{\substack{A_i \text{ is a party} \\ \text{in } \Pi_2}} |A_i \cap S| > \sum_{\substack{A_i \text{ is a party} \\ \text{in } \Pi_2}} |A_i \cap I| = |I|.$$

This is a contradiction, and the theorem follows. □

The above result also indicates that, using the alternating sequence approach, no matter which of the stable partitions for a given preference relation is used as the starting point, introducing a new person always leads to the same outcome, either the introduction of a new odd party or the elimination of an existing one. By Corollary 3.8 and Theorem 3.10, we may conclude the known fact [6] that there exists a complete stable matching if and only if there does not exist any odd party. One application of Corollary 3.9 is as follows. Suppose we know that a given preference relation does not contain a complete stable matching, and we wish to know the minimum number of persons that must be added to (deleted from) the relation so that the resulting instance contains a complete stable matching. By Corollary 3.9, this minimum number is the number of odd parties. This is because adding (deleting) one person into (from) the relation reduces the number of odd parties by at most 1, and no complete stable matching exists as long as there are odd parties. On the other hand, suppose the number of odd parties is m. It is then a simple matter to add (delete) m persons into (from) the relation so as to decompose all the odd parties.

4. Locating an odd party

Theorem 3.7 in the previous section does not provide us with an efficient way to locate the odd party it describes. In this section, we will discuss how to identify the odd party algorithmically and examine the time complexity of the algorithm involved. As a result, we will establish Theorem 3.3. Below we provide some more definitions and describe further properties of return sequences.

Let S: $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k, \beta_{k+1}$ be a return sequence, and let x be a person involved in this sequence, i.e., $x = \alpha_i$ or β_i for some i. During the course of this sequence starting from α_0 and ending at β_{k+1}, sometimes person x has a matched partner and sometimes he does not. We define $\text{Worst}_S(x)$ to be the worst person in x's list with whom x has been matched during the course of sequence S.

Proposition 4.1. *Let* S: $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k, \beta_{k+1}$ *be a return sequence and let* $\Pi_0, \Pi_1, \ldots, \Pi_k, \Pi_{k+1}$ *be the corresponding stable partitions. Then*
(i) *in stable partition* Π_{k+1}, *each* α_i *is matched with* $\text{Worst}_S(\alpha_i)$, $i = 1$ *to* k;
(ii) α_0 *is matched with* $\text{Worst}_S(\alpha_0)$ *in stable partition* Π_1;
(iii) *in stable partition* Π_0, *each* β_1 *is matched with* $\text{Worst}_S(\beta_i)$, $i = 1$ *to* k.

Proof. By Propositions 3.1 and 3.2, each α_i receives an increasingly worse partner, while each β_i receives an increasingly better partner in the process of S, $i = 1$ to k. So (i) and (iii) follow. For part (ii), since S is a return sequence with $\beta_{k+1} = \alpha_0, \alpha_0$ has only been matched with β_1 and α_k in Π_0 and Π_{k+1}, respectively, and $r(\alpha_0 \mid \alpha_k) < r(\alpha_0 \mid \beta_1)$. So $\text{Worst}_S(\alpha_0) = \beta_1$. □

Consider an altering sequence $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k, \ldots$. Suppose that there is a return at β_{k+1}. By Proposition 3.4, there is another return at β_{k+2}. The return at β_{k+2} is said to be *the next return* (subsequent to the return at β_{k+1}), and the corresponding return sequence is called the *next return sequence.*

Proposition 4.2. *Let* $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k, \beta_{k+1}$ *be a return sequence* S_1, *and let* S_2 *be the next return sequence. Then*
(i) *the set of persons involved in the next return sequence* S_2 *is contained in the set of persons involved in* S_1;
(ii) $\text{Worst}_{S_2}(x)$ *is no worse than* $\text{Worst}_{S_1}(x)$, *for every person* x *in* S_2, *i.e.,* $r(x \mid \text{Worst}_{S_2}(x)) \leqslant r(x \mid \text{Worst}_{S_1}(x))$.

Proof. (i) By Proposition 3.4, $\alpha_{k+1} = \beta_1$ and $\beta_{k+2} = \alpha_i$ for some $1 \leqslant i \leqslant k$, hence the result is trivial.

(ii) Let $\Pi_0, \Pi_1, \ldots, \Pi_k, \Pi_{k+1}$ be the corresponding stable partitions. Then the next return sequence S_2 starts from α_i, for some $1 \leqslant i \leqslant k$. So $\alpha_i, \beta_{i+1}, \alpha_{i+1}, \ldots, \beta_k, \alpha_k, \beta_{k+1}, \alpha_{k+1}, \beta_{k+2}$ is the next return sequence with corresponding stable partitions $\Pi_i, \Pi_{i+1}, \ldots, \Pi_{k+1}, \Pi_{k+2}$, where $\alpha_{k+1} = \beta_1$ and $\beta_{k+2} = \alpha_i$.

The only stable partition that appears in S_2 but not in S_1 is Π_{k+2}, and the only difference between Π_{k+2} and Π_{k+1} is the changes in the matching status among α_{k+1} $(=\beta_1)$, β_{k+2} $(=\alpha_i)$, and α_{k+2} $(=\beta_{i+1})$. Person α_{k+2} is out of the relation in Π_{k+2} and is not matched with anyone. By Proposition 3.1, person β_{k+2} has a better partner in Π_{k+2} than he has in Π_{k+1}. So the result holds for $x = \alpha_{k+2}$ and for $x = \beta_{k+2}$. Person α_{k+1} $(=\beta_1)$ is matched with α_i (respectively, α_1) in Π_{k+2} (respectively, in Π_0). By Proposition 3.4, $r(\alpha_{k+1} \mid \alpha_i) \leqslant r(\alpha_{k+1} \mid \alpha_1)$, so the result also holds for $x = \alpha_{k+1}$. □

Consider an alternating sequence. Suppose that a return occurs; by Corollary 3.6, the alternating sequence can be extended infinitely. Nevertheless, we have the following properties.

Proposition 4.3. *Let $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k, \beta_{k+1}$ be the first return subsequence S_1. As the alternating sequence extends infinitely, the following hold:*

(i) *the worst possible partner that a person x can have, throughout the whole process of the alternating sequence, is* $\mathrm{Worst}_{S_1}(x)$;

(ii) *no two β_i and β_j can be matched together in any step of the process, for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant k$ and $i \neq j$.*

Proof. (i) The result follows from Proposition 3.5 and from inductively applying Proposition 4.2(ii).

(ii) By the definition of a return sequence and by Proposition 3.2, we have $\beta_{k+1} = \alpha_0$ and $\beta_j \neq \alpha_i$ for all $0 \leqslant i \leqslant k$, $1 \leqslant j \leqslant k$. In the initial stable partition Π_0, it is obvious that a "β" person is matched with an "α" person. Now suppose, to the contrary, that for a certain i and j, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant k$ and $i \neq j$, β_i and β_j are matched together in a certain step of the alternating sequence. By Proposition 4.1, each β_i is matched with $\mathrm{Worst}_S(\beta_i)$ in Π_0. Then both β_i and β_j would prefer each other to their partner in the initial stable partition Π_0, which is a contradiction. □

Proof of Theorem 3.3. Let $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k$ be an alternating sequence with a return at β_{k+1}. Then α_k must be the starting point of a return sequence as the alternating sequence goes on.

Suppose not. Without loss of generality, we may assume that α_0 is the starting point of the return at β_{k+1}. Then at β_{k+2}, the sequence returns to α_i, for some $1 \leqslant i \leqslant k$. As the alternating sequence goes on, let m be the largest index, with $m < k$, such that α_m is the starting point of a return sequence. Therefore, at a certain step β_{q+1}, the alternating sequence $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_m, \alpha_m, \ldots$ returns to α_m, and at β_{q+2}, it returns to one of $\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_q$. Note that α_{q+1} is matched with β_{q+2} in Π_{q+2}. However,

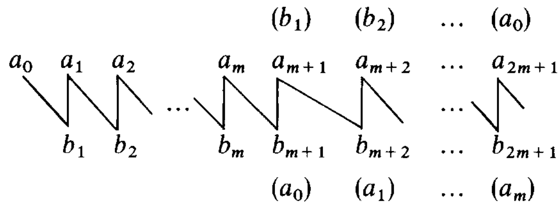


Fig. 3.

by Proposition 3.4 we know that $\alpha_{q+1} = \beta_{m+1}$ and $\beta_{q+2} \in \{\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_q\} \subset \{\beta_1, \beta_2, \ldots, \beta_m\}$. This contradicts that fact that no two β_i and β_j, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant k$, can be matched together in any step of the process. So α_k must be the starting point of a return sequence.

Consider an alternating sequence $\alpha_0, \beta_1, \alpha_1, \ldots, \beta_k, \alpha_k, \ldots$. Suppose that a return occurs, and the first return occurs at β_{k+1}. Then α_k must be the starting point of a return sequence, say $\alpha_k, \beta_{k+1}, \alpha_{k+1}, \ldots, \beta_{k+m}, \alpha_{k+m}, \beta_{k+m+1}$ ($= \alpha_k$). So the sequence can be extended infinitely, there is a return at every β_j, $j \geqslant k+1$, and every α_i is the starting point of a return, $i \geqslant k$. If α_{k+i} is replaced by a_i and β_{k+i} by b_i, $i \geqslant 0$, it is clear that the alternating sequence cycles with the pattern S shown in Fig. 3.

Furthermore, $\Pi_0 = \Pi_{2m+1}$. By Theorem 3.7(ii), $\langle a_m, b_m, a_{m-1}, b_{m-1}, \ldots, a_1, b_1, a_0 \rangle$ forms an odd party, and Theorem 3.3 follows. □

Remark. In the context of the last paragraph, it is not difficult to give an example in which any return sequence starting from an element before α_k does not constitute an odd party.

Given a preference relation R and given a stable partition of R, to add a new participant α_0 into R and find a new stable partition incorporating this new person, we generate an alternating sequence starting from α_0. Within $O(n^2)$ steps, either the alternating sequence terminates with a larger stable partition for $R + \alpha_0$, or a return occurs. Suppose that the first return occurs at β_{k+1}. Then by extending the alternating sequence from α_k, we will locate a new odd party within at most $O(n^2)$ steps as stated in Theorem 3.3, and obtain a stable partition for $R + \alpha_0$.

5. Conclusions

In this paper, we propose a new algorithm for finding a stable partition for a given instance of the stable roommates problem, and therefore a new algorithm for finding a stable matching if one exists. Our algorithm processes the problem dynamically, by allowing new participants to join the relation. In its present form, the algorithm considers the addition of only one person at a time, but we believe that the idea on which the algorithm is based can be extended to handle the case of inserting a set of persons at a time. The new participants would be processed in batch form, which might enhance the efficiency of the algorithm. This is an issue worthy of further study.

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