Mathematical and Computer Modelling 54 (2011) 1613–1619

Contents lists available at ScienceDirect

Mathematical and Computer Modelling

journal homepage: www.elsevier.com/locate/mcm

On 4-ordered 3-regular graphs[☆]

Ming Tsai [a], Tsung-Han Tsai [a], Jimmy J.M. Tan [a], Lih-Hsing Hsu [b,*]

[a] Department of Computer Science, National Chiao Tung University, Hsinchu, 300, Taiwan ROC

[b] Department of Computer Science and Information Engineering, Providence University, Taichung, 43301, Taiwan ROC

ARTICLE INFO

Article history:
Received 8 October 2010
Received in revised form 13 February 2011
Accepted 27 April 2011

Keywords:
k-ordered
Cubic graphs

ABSTRACT

A graph G is *k-ordered* if for any sequence of k distinct vertices $v_1, v_2, \ldots, v_k$ of G there exists a cycle in G containing these k vertices in the specified order. In 1997, Ng and Schultz posed the question of the existence of 4-ordered 3-regular graphs other than the complete graph K_4 and the complete bipartite graph $K_{3,3}$. In 2008, Meszaros solved the question by proving that the Petersen graph and the Heawood graph are 4-ordered 3-regular graphs. Moreover, the generalized Honeycomb torus $\mathrm{GHT}(3, n, 1)$ is 4-ordered for any even integer n with $n \geq 8$. Up to now, all the known 4-ordered 3-regular graphs are vertex transitive. Among these graphs, there are only two non-bipartite graphs, namely the complete graph K_4 and the Petersen graph. In this paper, we prove that there exists a bipartite non-vertex-transitive 4-ordered 3-regular graph of order n for any sufficiently large even integer n. Moreover, there exists a non-bipartite non-vertex-transitive 4-ordered 3-regular graph of order n for any sufficiently large even integer n.

© 2011 Elsevier Ltd. All rights reserved.

1. Introduction

For the graph definitions and notation, we follow the definitions and notation of [1]. Let $G = (V, E)$ be a graph if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices u and v are *adjacent* if $(u, v) \in E$. A graph is of order n if $|V| = n$. The *degree* of a vertex u in G, denoted by $\deg_G(u)$, is the number of vertices adjacent to u. A graph G is *k-regular* if $\deg_G(x) = k$ for any $x \in V$. A *cubic graph* is a 3-regular graph. A *path* between vertices v_0 and v_k is a sequence of vertices represented by $\langle v_0, v_1, \ldots, v_k \rangle$ with no repeated vertex and (v_i, v_{i+1}) is an edge of G for every i, $0 \leq i \leq k-1$. We also write the path $\langle v_0, v_1, \ldots, v_k \rangle$ as $\langle v_0, \ldots, v_i, Q, v_j, \ldots, v_k \rangle$ where Q is a path from v_i to v_j. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one.

A graph G is *k-ordered* if for any sequence of k distinct vertices $v_1, v_2, \ldots, v_k$ of G there exists a cycle in G containing these k vertices in the specified order. The concept of k-ordered graphs was introduced in 1997 by Ng and Schultz [2]. Previous results focus on the conditions for minimum degree and forbidden subgraphs that imply k-ordered graphs [3–6]. A comprehensive survey of the results can be found in [6].

In [2], Ng and Schultz posed the question of the existence of 4-ordered 3-regular graphs other than K_4 and $K_{3,3}$. In [7], Meszaros solved the question by proving that the Petersen graph and the Heawood graph are 4-ordered 3-regular graphs. Moreover, the generalized Honeycomb torus $\mathrm{GHT}(3, n, 1)$ is 4-ordered if n is an even integer with $n \geq 8$. Up to now, all the known 4-ordered 3-regular graphs are vertex transitive. Among these graphs, there are only two non-bipartite graphs,

☆ This work was supported in part by the National Science Council of the Republic of China under Contract 97-2221-E-126-001-MY3.

* Corresponding author.
E-mail address: lhhsu@cs.pu.edu.tw (L.-H. Hsu).

0895-7177/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.mcm.2011.04.034

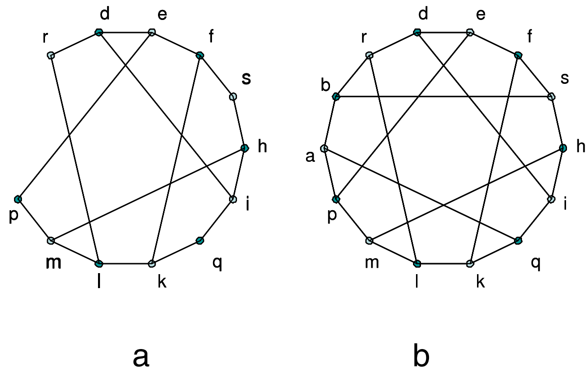

Fig. 1. (a) A semi-cubic cell $C_1 = (H, p, q, r, s)$ and (b) its $OR((H, p, q, r, s))$.

namely the complete graph K_4 and the Petersen graph. In this paper, we prove that there exists a bipartite non-vertex-transitive 4-ordered 3-regular graph of order n for any sufficiently large even integer n. Moreover, there exists a non-bipartite non-vertex-transitive 4-ordered 3-regular graph of order n for any sufficiently large even integer n.

The following lemma will be used later.

Lemma 1.1 ([7]). *Any 4-ordered 3-regular graph with more than six vertices does not contain a cycle of length 4.*

Let X be a set. An *r-permutation* of X is an ordered selection of r elements of X. A *partition* of X is a collection of disjoint subsets whose union is X. In particular, let X be a set consisting of four elements p, q, r, and s. Obviously, there are exactly twelve 2-permutations of X. Moreover, there are exactly three partitions that divide X into two disjoint subsets Y and Z such that $|Y| = |Z| = 2$.

In the following section, we introduce the concept of 4-ordered cells. With 4-ordered cells, we can construct 4-ordered 3-regular graphs. In Section 3, we present examples of 4-ordered cells with generalized honeycomb tori. In the final section, we give our conclusion and some unsolved problems.

2. 4-ordered cells

A *cell* is a 5-tuple (H, p, q, r, s), where H is a graph, and p, q, r, s are four distinct vertices in H. A cell H is *semi-cubic* if $\deg_H(x) = 3$ for every $x \in V(H) - \{p, q, r, s\}$ and $\deg_H(x) = 2$ for $x \in \{p, q, r, s\}$. The graph $OR((H, p, q, r, s))$ is obtained from H by adding two vertices a, b, and five edges (a, b), (a, p), (a, q), (b, r) and (b, s). We also say that the cell (H, p, q, r, s) is *derived from* $OR((H, p, q, r, s))$ by deleting two adjacent vertices a and b. A semi-cubic cell $C_1 = (H, p, q, r, s)$, for example, is illustrated in Fig. 1(a). The corresponding $OR((H, p, q, r, s))$ is illustrated in Fig. 1(b). We note that the graph in Fig. 1(b) is actually the Heawood graph. Obviously, $OR((H, p, q, r, s))$ is a cubic graph if H is semi-cubic.

A *4-ordered cell* is a cell (H, p, q, r, s) with the following properties.

(1) $OR((H, p, q, r, s))$ is cubic and 4-ordered.
(2) Let x_1, x_2, and x_3 be any three vertices of H. There exists a path P of H joining u to v with $\{u, v\} \subset \{p, q, r, s\}$ and traversing x_1, x_2, and x_3 in the order specified by the indices.
(3) Let x be any vertex of H and $\{u, v\}$ be any two vertices of $\{p, q, r, s\}$. There exists a path P of H joining u to v that traverses x.
(4) Let x_1 and x_2 be two vertices of H. There are at least seven 2-permutations uv of $\{p, q, r, s\}$ such that there exists a path P of H joining u to v that traverses x_1 and x_2 in the order specified by the indices.
(5) For any partition that divides $\{p, q, r, s\}$ into two pairs $\{\{u, v\}, \{w, x\}\}$, there exist two disjoint paths P and Q of H such that P joins u to v and Q joins w to x.
(6) Let x_1 and x_2 be two vertices of H. There are two partitions that divide $\{p, q, r, s\}$ into two subsets $\{\{u, v\}, \{w, x\}\}$ such that there exist two disjoint paths P and Q of H where P joins u to v traversing x_1 and Q joins w to x traversing x_2.

Now, we can check that C_1 in Fig. 1(a) is actually a 4-ordered cell. Since C_1 has twelve vertices, we can prove that C_1 is a 4-ordered cell using a computer. The program can be downloaded and the computer result viewed on the website http://www.cs.pu.edu.tw/lhhsu/FourOrdered/.

Suppose that we delete vertices a and q from the Heawood graph in Fig. 1(b). We will obtain the graph K, shown in Fig. 2, with four vertices b, p, i and k of degree 2. Again, we can check that $C_2 = (K, b, p, i, j)$ is a 4-ordered cell using the computer.

Suppose that we delete vertices a and b from the Petersen graph in Fig. 3(a). We will obtain the cell $C_3 = (L, p, q, r, s)$ shown in Fig. 3(b). Now, we claim that C_3 is not a 4-ordered cell. Let $x_1 = p$ and $x_2 = s$. Assume that u and v are two elements of $\{p, q, r, s\}$ such that there exists a path P of L joining u to v that traverses x_1 and x_2 in the order specified by the indices. Obviously, $u \neq s$ and $v \neq p$. By brute force, we can check that there is no path joining q to r that traverses x_1 and x_2 in the

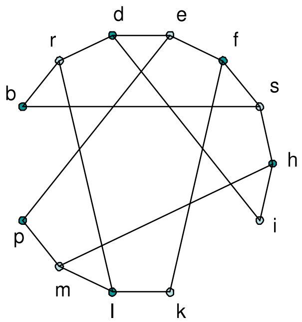

Fig. 2. Another 4-ordered cell C_2.

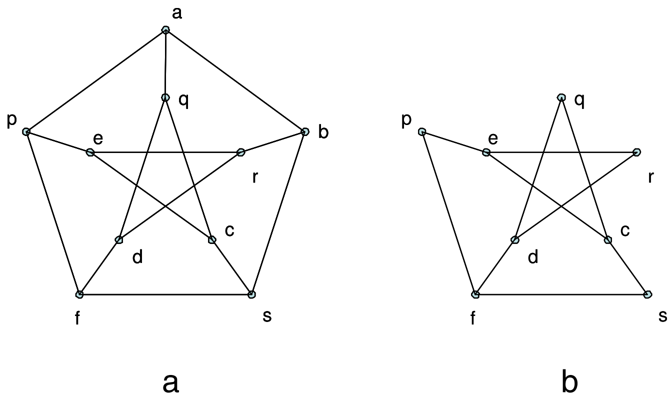

Fig. 3. Example of non-4-ordered cell.

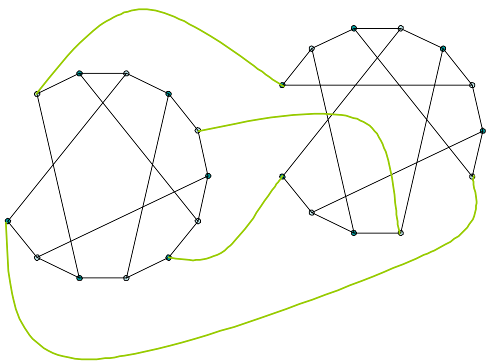

Fig. 4. $OP_f(C_1, C_2)$ where C_1 is the cell in Fig. 1, C_2 is the cell in Fig. 2, $f(p) = i$, $f(q) = p$, $f(r) = b$, and $f(s) = j$.

order specified by the indices. Therefore, uv can only be pq, pr, ps, qs, rq, or rs. Thus, there are at most six 2-permutations uv of $\{p, q, r, s\}$ such that there exists a path P of L joining u to v that traverses x_1 and x_2 in the order specified by the indices. Therefore, C_3 is not a 4-ordered cell.

Cells are used for the construction of various families of graphs [8–10]. In this section, we will use the following operation to combine two cells. Let $C_i = (G_i, p_i, q_i, r_i, s_i)$ be a cell for $i = 1, 2$ and f be a 1–1 correspondence between $\{p_1, q_1, r_1, s_1\}$ and $\{p_2, q_2, r_2, s_2\}$. The graph $OP_f(C_1, C_2)$ is obtained from the disjoint union of G_1 and G_2 by adding the edges $(p_1, f(p_1))$, $(q_1, f(q_1))$, $(r_1, f(r_1))$, and $(s_1, f(s_1))$. See Fig. 4 for illustration. Obviously, all vertices in $OP_f(C_1, C_2)$ are of degree 3 if C_1 and C_2 are semi-cubic.

Theorem 2.1. *Assume that $C_i = (G_i, p_i, q_i, r_i, s_i)$ is a 4-ordered cell for $i = 1, 2$. Then $OP_f(C_1, C_2)$ is 4-ordered 3-regular for any 1–1 correspondence f between $\{p_1, q_1, r_1, s_1\}$ and $\{p_2, q_2, r_2, s_2\}$.*

Proof. The proof is mainly based on the pigeonhole principle. Let x_1, x_2, x_3, and x_4 be any four vertices of $OP_f(C_1, C_2)$. We need to find a cycle of $OP_f(C_1, C_2)$ that traverses x_1, x_2, x_3 and x_4 in the order specified by the indices. Without loss of generality, we have the following four cases.

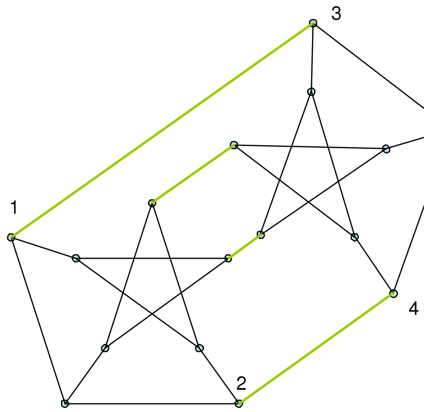

Fig. 5. A graph obtained by combining the two isomorphic cells in Fig. 3.

Case 1: x_1, x_2, x_3, and x_4 are all in G_1. We assume that $OR((G_1, p_1, q_1, r_1, s_1))$ is the graph obtained from G_1 by adding two vertices a_1 and b_1 together with the edges set $\{(p_1, a_1), (q_1, a_1), (r_1, b_1), (s_1, b_1), (a_1, b_1)\}$. By Property (1), there exists a cycle C in $OR((G_1, p_1, q_1, r_1, s_1))$ that traverses x_1, x_2, x_3 and x_4 in the order specified by the indices.

Suppose that $\{(p_1, a_1), (q_1, a_1), (r_1, b_1), (s_1, b_1), (a_1, b_1)\} \cap E(C) = \emptyset$. Obviously, C is also a cycle in $OP_f(C_1, C_2)$ that traverses x_1, x_2, x_3, and x_4 in the order specified by the indices. Suppose that $\{(p_1, a_1), (q_1, a_1), (r_1, b_1), (s_1, b_1), (a_1, b_1)\} \cap E(C) \neq \emptyset$. Without loss of generality, we have the following three subcases.

Subcase 1.1: $\langle p_1, a_1, q_1\rangle$ and $\langle r_1, b_1, s_1\rangle$ are subpaths of C. By Property (5), there exist two disjoint paths Q_1 and Q_2 in G_2 such that Q_1 joins $f(p_1)$ to $f(q_1)$ and Q_2 joins $f(r_1)$ to $f(s_1)$. In C, we replace $\langle p_1, a_1, q_1\rangle$ by the path $\langle p_1, f(p_1), Q_1, f(q_1), q_1\rangle$ and replace $\langle r_1, b_1, s_1\rangle$ by the path $\langle r_1, f(r_1), Q_2, f(s_1), s_1\rangle$ to obtain a cycle C' in $OP_f(C_1, C_2)$. Obviously, C' traverses x_1, x_2, x_3, and x_4 in the order specified by the indices.

Subcase 1.2: $\langle p_1, a_1, q_1\rangle$ is a subpath of C but $\langle r_1, b_1, s_1\rangle$ is not a subpath of C. By Property (3), there exists a path Q in G_2 such that Q joins $f(p_1)$ to $f(q_1)$. In C, we replace $\langle p_1, a_1, q_1\rangle$ by the path $\langle p_1, f(p_1), Q, f(q_1), q_1\rangle$ to obtain a cycle C' in $OP_f(C_1, C_2)$. Obviously, C' traverses x_1, x_2, x_3, and x_4 in the order specified by the indices.

Subcase 1.3: $\langle p_1, a_1, b_1, r_1\rangle$ is a subpath of C. By Property (3), there exists a path Q in G_2 such that Q joins $f(p_1)$ to $f(r_1)$. In C, we replace $\langle p_1, a_1, b_1, r_1\rangle$ by the path $\langle p_1, f(p_1), Q, f(r_1), r_1\rangle$ to obtain a cycle C' in $OP_f(C_1, C_2)$. Obviously, C' traverses x_1, x_2, x_3, and x_4 in the order specified by the indices.

Case 2: x_1, x_2, and x_3 are in G_1 and x_4 is in G_2. By Property (2), there exists path P of G_1 joining u to v for some $\{u, v\} \in \{p, q, r, s\}$ that traverses x_1, x_2, and x_3 in the order specified by the indices. By Property (3), there exists a path Q in G_2 such that Q joins $f(v)$ to $f(u)$ that traverses x_4. We set C as $\langle u, P, v, f(v), Q, f(u), u\rangle$. Obviously, C traverses x_1, x_2, x_3, and x_4 in the order specified by the indices.

Case 3: x_1 and x_2 are in G_1; x_3 and x_4 are in G_2. By Property (4), there are at least seven 2-permutations uv among all the twelve 2-permutations from $\{p_1, q_1, r_1, s_1\}$ such that there exists a path P of G_1 joining u to v that traverses x_1 and x_2 in the order specified by the indices. Similarly, there are at least seven 2-permutations $u'v'$ among all the twelve 2-permutations from $\{p_2, q_2, r_2, s_2\}$ such that there exists a path Q of G_2 joining v' to u' that traverses x_3 and x_4 in the order specified by the indices. Suppose that there is no 2-permutation uv such that there exists a path P of G_1 joining u to v that traverses x_1 and x_2 in the order specified by the indices and there exists a path Q of G_2 joining $f(v)$ to $f(u)$ that traverses x_3 and x_4 in the order specified by the indices. Then there are at least 14 different 2-permutations from $\{p_1, q_1, r_1, s_1\}$ which is impossible. Thus, we can find a 2-permutation uv such that there exists a path P of G_1 joining u to v that traverse x_1 and x_2 in the order specified by the indices and there exists a path Q of G_2 joining $f(v)$ to $f(u)$ that traverse x_3 and x_4 in the order specified by the indices. Now, we set C as $\langle u, P, v, f(v), Q, f(u), u\rangle$. Obviously, C traverses x_1, x_2, x_3, and x_4 in the order specified by the indices.

Case 4: x_1 and x_3 are in G_1; x_2 and x_4 are in G_2. By Property (6), there are at least two different $\{\{u, v\}, \{w, x\}\}$ among all the three partitions that divide $\{p_1, q_1, r_1, s_1\}$ into two pairs such that there exist two disjoint paths P_1 and P_3 of G_1 where P_1 joins u to v traversing x_1 and P_3 joins w to x traversing x_3. Similarly, there are at least two different $\{\{u', v'\}, \{w', x'\}\}$ among all the three partitions that divide $\{p_2, q_2, r_2, s_2\}$ into two pairs such that there exist two disjoint paths P_2 and P_4 of G_2 where P_2 joins u' to v' traversing x_3 and P_4 joins w' to x' traversing x_4. By the pigeonhole principle, we can find a partition $\{\{u, v\}, \{w, x\}\}$ of $\{p_1, q_1, r_1, s_1\}$ such that $\{u', v'\} \cap \{f(u), f(v)\} = 1$. By interchanging the roles of u and v and interchanging the roles of w and x, we can assume without loss of generality that P_1 joins u to v, P_2 joins $f(v)$ to $f(w)$, P_3 joins w to x, and P_4 joins $f(x)$ to $f(u)$. We set C as $\langle u, P_1, v, f(v), P_2, f(w), w, P_3, x, f(x), P_4, f(u), u\rangle$. Obviously, C traverses x_1, x_2, x_3, and x_4 in the order specified by the indices. $\square$

With Theorem 2.1, we can easily conclude that the graph in Fig. 4 is 4-ordered. However, the graph in Fig. 5, which is obtained by combining two isomorphic cells in Fig. 3, is not 4-ordered. One can check that there is no cycle that traverses the vertices labeled 1, 2, 3, and 4 in that order.

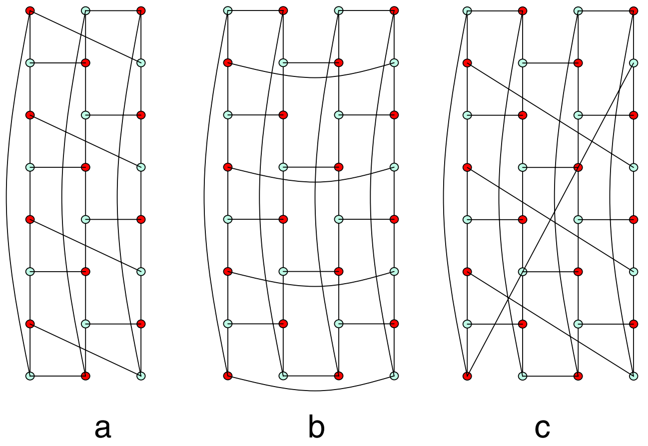

Fig. 6. The generalized honeycomb tori (a) GHT(3, 8, 1), (b) GHT(4, 8, 0), and (c) GHT(4, 8, 2).

3. Generalized honeycomb torus

In this section, we present more examples of 4-ordered cells.

Stojmenovic [11] proposed three classes of honeycomb torus architectures: honeycomb hexagonal torus, honeycomb rectangular torus, and honeycomb rhombic torus. Cho and Hsu [12] found that all these honeycomb torus networks can be characterized in a unified way, and thereby proposed a class of interconnection networks known as the generalized honeycomb torus.

Let n be a positive even integer with $n \geq 4$, m be a positive integer, and d be a nonnegative integer that is less than n and is of the same parity as m. An (m, n, d) generalized honeycomb torus, denoted by GHT(m, n, d), is a graph with the vertex set $\{(i, j) \mid i \in \{0, 1, \ldots, m-1\}, j \in \{0, 1, \ldots, n-1\}\}$. We call m, n, and d the *width*, *height*, and *slope* of GHT(m, n, d), respectively. For a vertex (i, j) of GHT(m, n, d), i and j are called its *first* and *second components*, respectively. Here and in what follows, all arithmetic operations carried out on the first and second components are modulo m and n, respectively. Two vertices (i, j) and (k, l) with $i \leq k$ are adjacent if and only if one of the following three conditions is satisfied:

(1) $(k, l) = (i, j+1)$ or $(k, l) = (i, j-1)$;
(2) $0 \leq i \leq m-2$, $i+j$ is odd, and $(k, l) = (i+1, j)$;
(3) $i = 0$, j is even, and $(k, l) = (m-1, j+d)$.

The generalized honeycomb tori GHT(3, 8, 1), GHT(4, 8, 0), and GHT(4, 8, 2), for example, are shown in Fig. 6. Obviously, any GHT(m, n, d) is 3-regular and vertex transitive. We can color vertices (i, j) white when $i + j$ is even or black otherwise. Thus, any GHT(m, n, d) is bipartite. It is proved in [12] that $\{$GHT$(m, n, d) \mid m$ is even and $d = 0\}$is the set of honeycomb rectangular tori. In [7], an infinite family of 4-ordered 3-regular graphs is proposed. Actually, this family of graphs is $\{$GHT$(3, n, 1) \mid n$ is an even integer with $n \geq 8\}$. Using our terminology, we prove the following lemma in [7].

Lemma 3.1 (*[7]*). *Any* GHT$(3, n, 1)$ *is* 4-*ordered for any even n with* $n \geq 8$.

Proof. We prove this lemma by induction. Using computer programming, we can check that GHT(3, 8, 1) is 4-ordered. Assume that GHT$(3, n-2, 1)$ is 4-ordered and n is any positive even integer with $n \geq 9$. Let x_1, x_2, x_3, and x_4 be any four vertices of GHT$(3, n, 1)$. We want to find a cycle in GHT$(3, n, 1)$ that traverses the vertices x_1, x_2, x_3, and x_4 in the order specified by the indices. Since $n \geq 10$, there exists an integer $j \in \mathcal{Z}_n$ such that $\{x_1, x_2, x_3, x_4\} \cap \{(r, s) \mid r \in \{0, 1, 2\}, s \in \{j, j+1\}\} = \emptyset$. In GHT$(3, n, 1)$, we delete all the vertices in $\{(r, s) \mid r \in \{0, 1, 2\}, s \in \{j, j+1\}\}$ and join $(i, j-1)$ with $(i, j+2)$ for $i = 0, 1, 2$. See Fig. 7 for illustration. Obviously, the resultant graph is isomorphic to GHT$(3, n-2, 1)$. By assumption, there exists a cycle C' in GHT$(3, n-2, 1)$ that traverses the vertices x_1, x_2, x_3, and x_4 in the order specified by the indices. Now, we replace all the edges of the form joining $(i, j-1)$ to $(i, j+2)$ in C' with the path $\langle (i, j-1), (i, j), (i, j+1), (i, j+2) \rangle$ to obtain a cycle C in GHT$(3, n, 1)$ that traverses the vertices x_1, x_2, x_3, and x_4 in the order specified by the indices. The lemma is proved. □

Remarks. In the above proof, we have seen that the desired path pattern of GHT$(3, n, 1)$ can be obtained from the path pattern of GHT$(3, n-2, 1)$ by inserting two rows. We call this operation *row insertion*.

Theorem 3.1. *Assume that m is an odd integer with* $m \geq 3$ *and n is an even integer with* $n \geq 4$. *The generalized honeycomb torus* GHT$(m, n, 1)$ *is* 4-*ordered if and only if* $n \neq 4$.

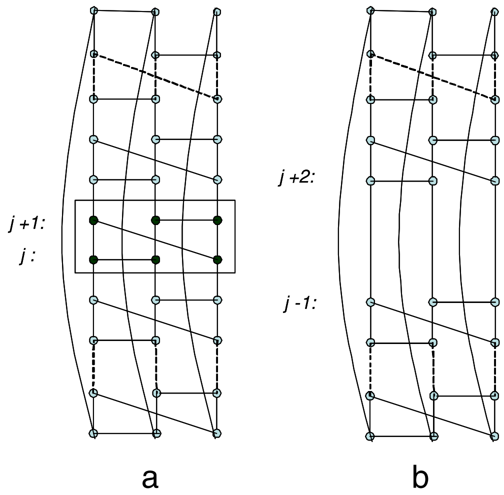

Fig. 7. Illustration for the proof of Lemma 3.1: (a) GHT$(3, n, 1)$ and (b) delete all the vertices in $\{(r, s) \mid r \in \{0, 1, 2\}, s \in \{j, j + 1\}\}$ and join $(i, j - 1)$ with $(i, j + 2)$ for $i = 0, 1, 2$.

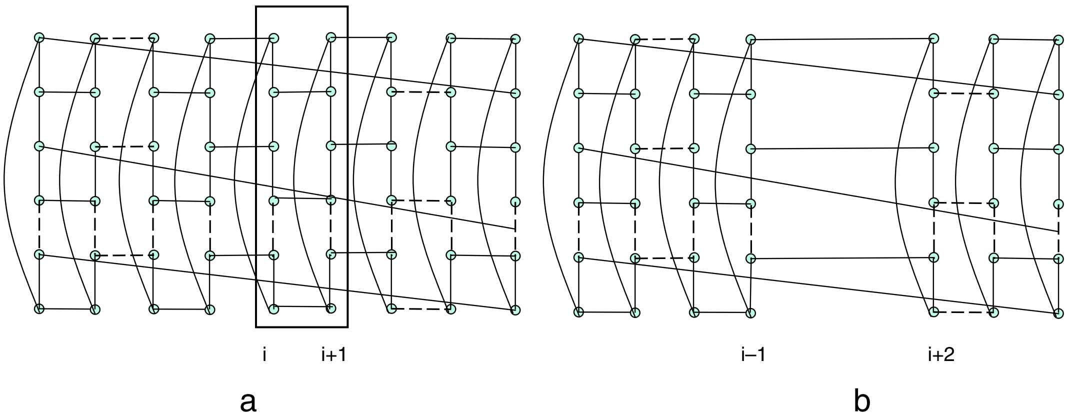

Fig. 8. Illustration for the proof of Lemma 3.1: (a) GHT$(m, n, 1)$ and (b) delete all the vertices in $\{(r, s) \mid r \in \{i, i + 1\}, s \in \{0, 1, \ldots, n - 1\}\}$ and join $(i - 1, j)$ with $(i + 2, j)$ for $j \in \{0, 1, \ldots, n - 1\}$ if $(i - 1, j)$ is adjacent to (i, j).

Proof. Obviously, there are 4-cycles in GHT$(m, 4, 1)$. By Lemma 1.1, GHT$(m, 4, 1)$ is not 4-ordered. Using computer programming, we can check that GHT$(m, n, 1)$ is 4-ordered for $m \in \{3, 5, 7\}$ and $n \in \{6, 8\}$. Using row insertion, we can prove that GHT$(m, n, 1)$ is 4-ordered for $m \in \{3, 5, 7\}$ and $n \geq 10$.

Now, assume that GHT$(m-2, n, 1)$ is 4-ordered and $m \geq 9$. Let x_1, x_2, x_3, and x_4 be any four vertices of GHT$(m, n, 1)$. We want to find a cycle in GHT$(m, n, 1)$ that traverses the vertices x_1, x_2, x_3, and x_4 in the order specified by the indices. Since $m \geq 9$, there exists an integer i such that $\{x_1, x_2, x_3, x_4\} \cap \{(r, s) \mid r \in \{i, i + 1 \ (\mathbf{mod}\ m)\}, s \in \{0, 1, \ldots, n - 1\}\} = \emptyset$. In GHT$(m, n, 1)$, we delete all the vertices in $\{(r, s) \mid r \in \{i, i + 1 \ (\mathbf{mod}\ m)\}, s \in \{0, 1, \ldots, n - 1\}\}$ and join $(i - 1 \ (\mathbf{mod}\ m), j)$ with $(i + 2 \ (\mathbf{mod}\ m), j)$ for $j \in \{0, 1, \ldots, n - 1\}$ if $(i - 1 \ (\mathbf{mod}\ m), j)$ is adjacent to (i, j). See Fig. 8 for illustration. Obviously, the resultant graph is isomorphic to GHT$(m - 2, n, 1)$. By assumption, there exists a cycle C' in GHT$(m - 2, n, 1)$ that traverses the vertices x_1, x_2, x_3, and x_4 in the order specified by the indices. Now, we replace all the edges of the form joining $(i - 1 \ (\mathbf{mod}\ m), j)$ to $(i + 2 \ (\mathbf{mod}\ m), j)$ in C' with the path $\langle (i - 1 \ (\mathbf{mod}\ m), j), (i, j), (i, j + 1), (i + 1 \ (\mathbf{mod}\ m), j + 1), (i + 1 \ (\mathbf{mod}\ m), j), (i + 2 \ (\mathbf{mod}\ m)) \rangle$ to obtain a cycle C in GHT$(m, n, 1)$ that traverses the vertices x_1, x_2, x_3, and x_4 in the order specified by the indices.

The theorem is proved. □

Remarks. In the above proof, we have seen that the desired path pattern of GHT$(m, n, 1)$ can be obtained from the path pattern of GHT$(m - 2, n, 1)$ by inserting two columns. We call this operation *column insertion*.

Using similar techniques to those above, we obtain the following theorem.

Theorem 3.2. *Assume that m is a positive even integer with $m \geq 2$ and n is an even integer with $n \geq 4$. The generalized honeycomb torus* GHT$(m, n, 0)$ *is 4-ordered if and only if $n \neq 4$.*

Next, we will discuss some path patterns for generalized honeycomb tori.

Let (a, b) be any edge of $\mathrm{GHT}(m, n, d)$. We use $\mathrm{XHT}^{a,b}(m, n, d)$ to denote the subgraph $\mathrm{GHT}(m, n, d) - \{a, b\}$. Obviously, there are four vertices p, q, r, and s in $\mathrm{XHT}^{a,b}(m, n, d)$ of degree 2 and all the other vertices are of degree 3.

Using computer programming, we can check that for any edge (a, b) of $\mathrm{GHT}(m, n, 1)$ and any three vertices x_1, x_2, and x_3 of $\mathrm{XHT}^{a,b}(m, n, 1)$ with $m \in \{3, 5, 7\}$ and $n \in \{6, 8\}$ there exists a path P of $\mathrm{XHT}^{a,b}(m, n, 1)$ joining u to v with $\{u, v\} \in \{p, q, r, s\}$ that traverses x_1, x_2, and x_3 in the order specified by the indices. Applying row insertion and column insertion, we can obtain the following lemma.

Lemma 3.2. *Assume that m is odd with $m \geq 3$ and n is even with $n \geq 6$. Let (a, b) be any edge of $\mathrm{GHT}(m, n, 1)$ and x_1, x_2, and x_3 be three vertices of $\mathrm{XHT}^{a,b}(m, n, 1)$. There exists a path P of $\mathrm{XHT}^{a,b}(m, n, 1)$ joining u to v with $\{u, v\} \in \{p, q, r, s\}$ that traverses x_1, x_2, and x_3 in the order specified by the indices.*

Using similar techniques, we have the following lemmas.

Lemma 3.3. *Assume that m is odd with $m \geq 3$ and n is even with $n \geq 6$. Let (a, b) be any edge of $\mathrm{GHT}(m, n, 1)$ and x be any vertex of $\mathrm{XHT}^{a,b}(m, n, 1)$. There exists a path P of $\mathrm{XHT}^{a,b}(m, n, 1)$ joining u to v with $\{u, v\} \in \{p, q, r, s\}$ that traverses x.*

Lemma 3.4. *Assume that m is odd with $m \geq 3$ and n is even with $n \geq 6$. Let (a, b) be any edge of $\mathrm{GHT}(m, n, 1)$ and x_1 and x_2 be two vertices of $\mathrm{XHT}^{a,b}(m, n, 1)$. There are at least seven 2-permutations uv of $\{p, q, r, s\}$ such that there exists a path P of $\mathrm{XHT}^{a,b}(m, n, 1)$ joining u to v that traverses x_1 and x_2 in the order specified by the indices.*

Lemma 3.5. *Assume that m is odd with $m \geq 3$ and n is even with $n \geq 6$. Let (a, b) be any edge of $\mathrm{GHT}(m, n, 1)$. For any partition that divides $\{p, q, r, s\}$ into $\{\{u, v\}, \{w, x\}\}$, there exist two disjoint paths P and Q of $\mathrm{XHT}^{a,b}(m, n, 1)$ such that P joins u to v and Q joins w to x.*

Lemma 3.6. *Assume that m is odd with $m \geq 3$ and n is even with $n \geq 6$. Let (a, b) be any edge of $\mathrm{GHT}(m, n, 1)$ and x_1 and x_3 be two vertices of $\mathrm{XHT}^{a,b}(m, n, 1)$. There are at least two different partitions that divide $\{p, q, r, s\}$ into $\{\{u, v\}, \{w, x\}\}$ such that there exist two disjoint paths P and Q of $\mathrm{XHT}^{a,b}(m, n, 1)$ where P joins u to v traversing x_1 and Q joins w to x traversing x_3.*

Combining the discussion above, we obtain the following theorem.

Theorem 3.3. *Every $\mathrm{XHT}^{a,b}(m, n, 1)$ is a 4-ordered cell if m is odd with $m \geq 3$, n is even with $n \geq 6$ and (a, b) is any edge of $\mathrm{GHT}(m, n, 1)$. Similarly, every $\mathrm{XHT}^{a,b}(m, n, 0)$ is a 4-ordered cell if m is even with $m \geq 4$, n is even with $n \geq 6$ and (a, b) is any edge of $\mathrm{GHT}(m, n, 0)$.*

4. Main result and concluding remarks

Using Theorem 3.3, we can apply Theorem 2.1 combining a 4-ordered cell $\mathrm{XHT}^{a,b}(m, n, 1)$ and a 4-ordered cell $\mathrm{XHT}^{a,b}(i, j, 0)$ in a different manner to obtain a 4-ordered 3-regular graph.

Thus, we obtain the following theorem.

Theorem 4.1. *There exists a bipartite non-vertex-transitive 4-ordered 3-regular graph of order n for any sufficiently large even integer n. Moreover, there exists a non-bipartite non-vertex-transitive 4-ordered 3-regular graph of order n for any sufficiently large even integer n.*

Finally, we will discuss some currently unsolved problems. We have seen that some generalized honeycomb tori are 4-ordered and some are not. Thus, we would like to classify all generalized honeycomb tori that are 4-ordered. Similarly, we would like to classify all generalized Petersen graphs that are 4-ordered.

References

[1] L.H. Hsu, C.K. Lin, Graph Theory and Interconnection Networks, CRC Press, 2008.
[2] L. Ng, M. Schultz, *k*-Ordered hamiltonian graphs, J. Graph Theory 24 (1997) 45–57.
[3] G. Chen, R.J. Gould, F. Pfender, New conditions for ordered *k*-Hamiltonian graphs, Ars Combin. 70 (2003) 245–255.
[4] J.R. Faundree, R.J. Faundree, Forbidden subgraphs that imply *k*-ordered and *k*-ordered hamiltonian, Discrete Math. 243 (2002) 91–108.
[5] J.R. Faundree, R.J. Gould, F. Pfender, A. Wolf, On *k*-ordered bipartite graphs, Electron. J. Combin. 10 (2003) R11.
[6] R.J. Faundree, Survey of results on *k*-ordered graphs, Discrete Math. 229 (2001) 73–87.
[7] K. Meszaros, On 3-regular 4-ordered graphs, Discrete Math. 308 (2008) 2149–2155.
[8] V. Chvátal, Flip-flops in hypohamiltonian graphs, Canad. Math. Bull. 16 (1) (1973) 33–41.
[9] J.B. Collier, E.F. Schmeichel, New flip-flop constructions for hypohamiltonian graphs, Discrete Math. 18 (1977) 265–273.
[10] C.H. Tsai, T.K. Li, Two construction schemes for cubic Hamiltonian 1-node-Hamiltonian graphs, Math. Comput. Modelling 48 (2008) 656–661.
[11] I. Stojmenovic, Honeycomb networks: topological properties and communication algorithms, IEEE Trans. Parallel Distributed Systems 8 (10) (1997) 1036–1042.
[12] H. Cho, L. Hsu, Generalized honeycomb torus, Inform. Process. Lett. 86 (4) (2003) 185–190.