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The paths embedding of the arrangement graphs with prescribed vertices in given position

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Abstract Let n and k be positive integers with $n-k\geq 2$. The arrangement graph $A_{n,k}$ is recognized as an attractive interconnection networks. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be three different vertices of $A_{n,k}$. Let l be any integer with $d_{A_{n,k}}(\mathbf{x},\mathbf{y})\leq l\leq \frac{n!}{(n-k)!}-1-d_{A_{n,k}}(\mathbf{y},\mathbf{z})$. We shall prove the following existance properties of Hamiltonian path: (1) for $n-k\geq 3$ or $(n,k)=(3,1)$, there exists a Hamiltonian path $R(\mathbf{x},\mathbf{y},\mathbf{z};l)$ from $\mathbf{x}$ to $\mathbf{z}$ such that $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x},\mathbf{y})=l$; (2) for $n-k=2$ and $n\geq 5$, there exists a Hamiltonian path $R(\mathbf{x},\mathbf{y},\mathbf{z};l)$ except for the case that $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are adjacent to each other.

Keywords Arrangement graph · Panpositionable Hamiltonian · Panconnected · Interconnection network

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1 Introduction

In this paper, a network is represented as an acyclic undirected graph. We follow the graph definitions and notation used in Bondy and Murty (1976). Let $G = (V, E)$ be a graph where V is a finite set and E is a subset of $\{(a, b) \mid (a, b)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. We also use $V(G)$ and $E(G)$ to denote the vertex set and edge set of a graph G, respectively. Two vertices u and v are *adjacent* if $(u, v) \in E$. A vertex v is a *neighbor* of u if v is adjacent to u. We use $Nbd_G(u)$ to denote the neighborhood set $\{v \mid (u, v) \in E(G)\}$. The *degree* of a vertex u in G, denoted by $\deg_G(u)$, is $|Nbd_G(u)|$. A graph is *k-regular* if $\deg_G(u) = k$ for every vertex u in G. A *path* is a sequence of adjacent vertices, written as $\langle v_0, v_1, \ldots, v_m\rangle$, in which all the vertices $v_0, v_1, \ldots, v_m$ are distinct except that possibly $v_0 = v_m$. We also write the path as $\langle v_0, P, v_m\rangle$, where $P = \langle v_0, v_1, \ldots, v_m\rangle$. The *length* of a path P, denoted by $l(P)$, is the number of edges in P. Let u and v be two vertices of G. The *distance* between u and v denoted by $d_G(u, v)$ is the length of the shortest path of G joining u and v. The *diameter* of a graph G, denoted by $D(G)$, is $\max\{d_G(u, v) \mid u, v \in V(G)\}$. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. A *Hamiltonian cycle* is a cycle of length $|V(G)|$. A *Hamiltonian path* is a path of length $|V(G)| - 1$.

The interconnection network has been an important research area for parallel and distributed computer systems. The graph embedding problem is to ask if the guest graph is a subgraph of a host graph. An important benefit of the graph embeddings is that we can apply existing algorithm for guest graphs to host graphs. Therefore, the graph embedding problem is a central issue in evaluating a network and has attracted a burst of studies in recent years. Cycle networks and path networks are suitable for designing simple algorithms with low communication costs. The cycle embedding problem, which deals with all possible lengths of the cycles in a given graph, is investigated in a lot of interconnection networks (Day and Tripathi 1992; Germa et al. 1998; Hwang and Chen 2000; Li et al. 2003; Ma et al. 2007). The path embedding problem, which deals with all possible lengths of the paths between given two vertices in a given graph, is investigated in a lot of interconnection networks (Chang et al. 2004; Fan et al. 2007; Li et al. 2003; Ma and Xu 2006; Xu and Xu 2007).

The hypercube and the star graph (Akers et al. 1986; Akers and Krishnamurthy 1989) are important families of interconnection networks. The hypercube possesses many good properties and is implemented in many multiprocessor systems. Akers et al. (1986) proposed the star graph, which has smaller degree, diameter, and average distance than the hypercube while reserving symmetry properties and desirable fault-tolerant characteristics. As a result, the star graph has been recognized as an alternative to the hypercube. However, the hypercube and the star are less flexible in adjusting their sizes. The arrangement graph was proposed by Day and Tripathi (1992) as a generalization of the star graph. It is more flexible in its size than the star graph.

For the path embedding problem on the arrangement graphs, in Teng et al. (2008), it is proved that between any two distinct vertices $\mathbf{x}$ and $\mathbf{y}$ of the arrangement graph $A_{n,k}$, there exists a path $P_l(\mathbf{x}, \mathbf{y})$ of length l with $d_{A_{n,k}}(\mathbf{x}, \mathbf{y}) \leq l \leq |V(A_{n,k})| - 1 =$

$\frac{n!}{(n-k)!} - 1$. It would be interesting to extend $P_l(\mathbf{x}, \mathbf{y})$ by including all the vertices in $V(A_{n,k}) - V(P_l(\mathbf{x}, \mathbf{y}))$ and terminating at a desired vertex $\mathbf{z}$.

Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be three different vertices of $A_{n,k}$. Let l be any integer with $d_{A_{n,k}}(\mathbf{x}, \mathbf{y}) \le l \le \frac{n!}{(n-k)!} - 1 - d_{A_{n,k}}(\mathbf{y}, \mathbf{z})$. For $n - k \ge 3$ or $(n, k) = (3, 1)$, we shall prove that there exists a Hamiltonian path $R(\mathbf{x}, \mathbf{y}, \mathbf{z}; l)$ from $\mathbf{x}$ to $\mathbf{z}$ such that $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x}, \mathbf{y}) = l$. For $n - k = 2$ and $n \ge 4$, we shall prove that there exists a Hamiltonian path $R(\mathbf{x}, \mathbf{y}, \mathbf{z}; l)$ from $\mathbf{x}$ to $\mathbf{z}$ such that $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x}, \mathbf{y}) = l$ except for the case that $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are adjacent to each other.

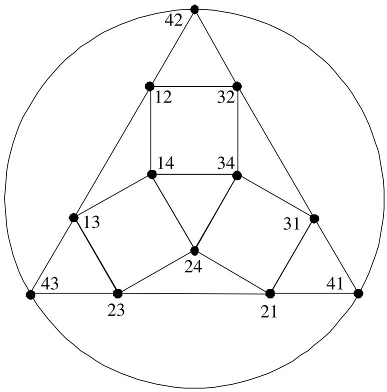
In the following section, we introduce the definition of the arrangement graphs. In Sect. 3, we introduce another property, called 2RP, for arrangement graphs $A_{n,k}$. Then we apply the 2RP-property to prove the aforementioned property. We prove that any $A_{n,k}$ satisfies the 2RP-property if $n - k \ge 2$ and $(n, k) \notin \{(3, 1), (4, 2)\}$ in Sect. 4.

2 Arrangement graphs

Assume that n and k are two positive integers. Let $\langle n \rangle$ and $\langle k \rangle$ denote the sets $\{1, 2, \ldots, n\}$ and $\{1, 2, \ldots, k\}$, respectively. The vertex set of the arrangement graph $A_{n,k}$, $V(A_{n,k}) = \{\mathbf{p} \mid \mathbf{p} = p_1 p_2 \cdots p_k$ with $p_i \in \langle n \rangle$ for $1 \le i \le k$ and $p_i \ne p_j$ if $i \ne j\}$ and the edge set of $A_{n,k}$, $E(A_{n,k}) = \{(\mathbf{p}, \mathbf{q}) \mid \mathbf{p}, \mathbf{q} \in V(A_{n,k})$, $\mathbf{p}$ and $\mathbf{q}$ differ in exactly one position$\}$. Figure 1 illustrates $A_{4,2}$. By definition, $A_{n,k}$ is a regular graph of degree $k(n-k)$ with $\frac{n!}{(n-k)!}$ vertices. In Day and Tripathi (1992), it is proved that $A_{n,k}$ is vertex symmetric and edge symmetric. Moreover, $A_{n,1}$ is isomorphic to the complete graph K_n. Furthermore, $A_{n,n-1}$ is isomorphic to the famous n-dimensional star graph (Akers et al. 1986; Akers and Krishnamurthy 1989) if we allow $k = n - 1$.

Let i and j be two positive integers with $1 \le i, j \le n$, and k be an integer with $k > 1$. Let $V(A^i_{n,k}) = \{\mathbf{p} \mid \mathbf{p} = p_1 p_2 \cdots p_k$ and $p_k = i\}$. Let $A^i_{n,k}$ denote the subgraph of $A_{n,k}$ induced by $V(A^i_{n,k})$. It is easy to see that each $A^i_{n,k}$ is isomorphic to $A_{n-1,k-1}$. Thus, $A_{n,k}$ can be recursively constructed from n copies of $A_{n-1,k-1}$. Each $A^i_{n,k}$ represents a *subcomponent* of $A_{n,k}$. Let I be a subset of $\{1, 2, \ldots, n\}$. We use $A^I_{n,k}$ to denote the subgraph of $A_{n,k}$ induced by $\bigcup_{i \in I} V(A^i_{n,k})$. For $|I| < n$, $A^I_{n,k}$

Fig. 1 $A_{4,2}$



is called an *incomplete* arrangement graph. We use $E^{i,j}$ to denote the set of edges between $A^i_{n,k}$ and $A^j_{n,k}$. The following lemmas can be easily proved by definition.

Lemma 1 *Let i be any integer with $1 \le i \le k$. Let $P(i)$ be the function defined on $V(A_{n,k})$ into itself by assigning $\mathbf{x} = x_1x_2\cdots x_k$ to $\mathbf{y} = y_1y_2\cdots y_k$ where $y_i = x_k$, $y_k = x_i$, and $y_j = x_j$ if $j \notin \{i,k\}$. Then $P(i)$ is an isomorphism of $A_{n,k}$.*

Lemma 2 *Let n and k be two positive integers with $n-k \ge 2$, and let i and j be two distinct elements of $\langle n\rangle$. Then $|E^{i,j}| = \frac{(n-2)!}{(n-k-1)!}$. Suppose that $(\mathbf{u},\mathbf{v})$ and $(\mathbf{u}',\mathbf{v}')$ are distinct edges in $E^{i,j}$. Then $\{\mathbf{u},\mathbf{v}\} \cap \{\mathbf{u}',\mathbf{v}'\} = \emptyset$. Moreover, $(\mathbf{u},\mathbf{u}') \in E(A^i_{n,k})$ if and only if $(\mathbf{v},\mathbf{v}') \in E(A^j_{n,k})$.*

Lemma 3 $D(A_{n,k}) = \lfloor\frac{3k}{2}\rfloor$, *as shown in* Day and Tripathi (1992).

Lemma 4 *As shown in* Teng et al. (2008), *let $\mathbf{u} \in V(A^i_{n,k})$ and $\mathbf{v} \in V(A^j_{n,k})$ be two vertices in $A_{n,k}$ for some $i, j \in \langle n\rangle$ with $i \ne j$. Then a shortest path M connecting $\mathbf{u}$ to $\mathbf{v}$ can be written as $\langle \mathbf{u}, P_1, \mathbf{u}', \mathbf{v}', P_2, \mathbf{v}\rangle$ such that* (1) *$\langle \mathbf{u}, P_1, \mathbf{u}'\rangle$ is a path in $A^i_{n,k}$,* (2) *$\langle \mathbf{v}', P_2, \mathbf{v}\rangle$ is a path in $A^j_{n,k}$, and* (3) *$l(P_2) \le 1$.*

Let $\mathbf{u} \in V(A^i_{n,k})$ for some $i \in \langle n\rangle$. We use $N^i(\mathbf{u})$ and $N^*(\mathbf{u})$ to denote the neighbors of u in $V(A^i_{n,k})$ and $V(A_{n,k}) - V(A^i_{n,k})$, respectively. We call vertices in $N^*(\mathbf{u})$ the *outer neighbors* of u. It follows from the definitions, $|N^i(\mathbf{u})| = (k-1)(n-k)$ and $|N^*(\mathbf{u})| = n-k$. We define the *adjacent subcomponent* $AS(\mathbf{u})$ of $\mathbf{u}$ as $\{j \mid \mathbf{u}$ is adjacent to some vertices in $A^j_{n,k}$ and $\mathbf{u} \notin V(A^j_{n,k})\}$. By the definitions, $|AS(\mathbf{u})| = |N^*(\mathbf{u})| = n-k$. The following lemma can easily be proved by definition.

Lemma 5 *Suppose that $k \ge 2$, $n-k \ge 2$, and $i \in \langle n\rangle$. Let $\mathbf{u}$ and $\mathbf{v}$ be two distinct vertices in $A^i_{n,k}$ with $d(\mathbf{u},\mathbf{v}) = 1$. Then $|AS(\mathbf{u}) \cap AS(\mathbf{v})| = n-k-1$ and $AS(\mathbf{u}) \ne AS(\mathbf{v})$.*

Lemma 6 *Suppose that $k \ge 2$ and $n-k \ge 2$. Let $\mathbf{u}$ and $\mathbf{v}$ be two distinct vertices in $A_{n,k}$. Then $|Nbd_{A_{n,k}}(\mathbf{u}) \cap Nbd_{A_{n,k}}(\mathbf{v})| = n-k-1$ if $d_{A_{n,k}}(\mathbf{u},\mathbf{v}) = 1$, and $|Nbd_{A_{n,k}}(\mathbf{u}) \cap Nbd_{A_{n,k}}(\mathbf{v})| \le 2$ if $d_{A_{n,k}}(\mathbf{u},\mathbf{v}) = 2$.*

Proof Suppose that $d_{A_{n,k}}(\mathbf{u},\mathbf{v}) = 1$. Let $\mathbf{u} = x_1x_2\cdots x_k$. By Lemma 1, we may assume that $\mathbf{v} = x_{k+1}x_2\cdots x_k$ with $x_1 \ne x_{k+1}$. Obviously, $Nbd_{A_{n,k}}(\mathbf{u}) \cap Nbd_{A_{n,k}}(\mathbf{v}) = \{x_ix_2\cdots x_k \mid k+2 \le i \le n\}$. Thus $|Nbd_{A_{n,k}}(\mathbf{u}) \cap Nbd_{A_{n,k}}(\mathbf{v})| = n-k-1$.

Suppose that $d_{A_{n,k}}(\mathbf{u},\mathbf{v}) = 2$. Let $\mathbf{u} = x_1x_2\cdots x_k$. By Lemma 1, we may assume that either $\mathbf{v} = x_{k+1}x_{k+2}x_3\cdots x_k$ or $\mathbf{v} = x_2x_{k+1}x_3\cdots x_k$ with $x_{k+1} \ne x_1$, $x_{k+1} \ne x_2$, $x_{k+2} \ne x_1$, and $x_{k+2} \ne x_2$. Obviously,

$$\begin{aligned} &Nbd_{A_{n,k}}(\mathbf{u}) \cap Nbd_{A_{n,k}}(\mathbf{v}) \\ &\quad = \begin{cases} \{x_1x_{k+2}x_3\cdots x_k,\ x_{k+1}x_2x_3\cdots x_k\} & \text{if } \mathbf{v} = x_{k+1}x_{k+2}x_3\cdots x_k \\ \{x_1x_{k+1}x_3\cdots x_k\} & \text{if } \mathbf{v} = x_2x_{k+1}x_3\cdots x_k \end{cases} \end{aligned}$$

Thus $|Nbd_{A_{n,k}}(\mathbf{u}) \cap Nbd_{A_{n,k}}(\mathbf{v})| \le 2$. □

3 The 2RP-property and applications

3.1 The 2P-property and the 2RP-property

In Teng et al. (2007, 2008), we have the following result.

Lemma 7 *Let* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{x}$, *and* $\mathbf{y}$ *be any four distinct vertices of* $A_{n,k}$ *with* $n-k\geq 2$. *Suppose that* $I\subseteq\langle n\rangle$ *with* $|I|\geq 2$. *Then there exist two disjoint paths* P_1 *and* P_2 *such that* (1) P_1 *is a path joining* $\mathbf{u}$ *to* $\mathbf{v}$, (2) P_2 *is a path joining* $\mathbf{x}$ *to* $\mathbf{y}$, *and* (3) $P_1\cup P_2$ *spans* $A^I_{n,k}$.

As a corollary, we observed that the arrangement graphs $A_{n,k}$ satisfy the following property: *Let* $\mathbf{u}$*,* $\mathbf{v}$*,* $\mathbf{x}$*, and* $\mathbf{y}$ *be any four distinct vertices of* $A_{n,k}$ *with* $n-k\geq 2$*. Then there exist two disjoint paths* P_1 *and* P_2 *such that* (1) P_1 *is a path joining* $\mathbf{u}$ *to* $\mathbf{v}$*,* (2) P_2 *is a path joining* $\mathbf{x}$ *to* $\mathbf{y}$*, and* (3) $P_1\cup P_2$ *spans* $A_{n,k}$*.*

We refer the above property as the 2P-property of the arrangement graphs. Obviously, $l(P_1)\geq d_{A_{n,k}}(\mathbf{u},\mathbf{v})$, $l(P_2)\geq d_{A_{n,k}}(\mathbf{x},\mathbf{y})$, and $l(P_1)+l(P_2)=\frac{n!}{(n-k)!}-2$. We expect that both $l(P_1)$ and $l(P_2)$ can be any integers satisfying the above constraints. Such expectation is almost true for $(n,k)\notin\{(3,1),(4,2)\}$. In $A_{5,2}$, let $\mathbf{u}=15$, $\mathbf{v}=25$, $\mathbf{x}=35$, and $\mathbf{y}=45$. Obviously, $d_{A_{5,2}}(\mathbf{u},\mathbf{v})=1$ and $d_{A_{5,2}}(\mathbf{x},\mathbf{y})=1$. By brute force, we can find P_1 and P_2 with $(l(P_1),l(P_2))\in\{(1,17),(2,16),(3,15),\ldots,(17,1)\}-\{(2,16),(16,2)\}$. Note that $\{\mathbf{x},\mathbf{y}\}=Nbd_{A_{5,2}}(\mathbf{u})\cap Nbd_{A_{5,2}}(\mathbf{v})$ and we can not find P_1 with $l(P_1)=2$.

Now, we propose the 2RP-property of $A_{n,k}$: *Let* $\mathbf{u}$*,* $\mathbf{v}$*,* $\mathbf{x}$*, and* $\mathbf{y}$ *be any four distinct vertices of* $A_{n,k}$*. Let* l_1 *and* l_2 *be two integers with* $l_1\geq d_{A_{n,k}}(\mathbf{u},\mathbf{v})$*,* $l_2\geq d_{A_{n,k}}(\mathbf{x},\mathbf{y})$*, and* $l_1+l_2=\frac{n!}{(n-k)!}-2$*. Then there exist two disjoint paths* P_1 *and* P_2 *such that* (1) P_1 *is a path joining* $\mathbf{u}$ *to* $\mathbf{v}$ *with* $l(P_1)=l_1$*,* (2) P_2 *is a path joining* $\mathbf{x}$ *to* $\mathbf{y}$ *with* $l(P_2)=l_2$*, and* (3) $P_1\cup P_2$ *spans* $A_{n,k}$ *except for the following cases:* (a) $l_1=2$ *with* $d_{A_{n,k}}(\mathbf{u},\mathbf{v})\leq 2$ *and* $\{\mathbf{x},\mathbf{y}\}\supseteq Nbd_{A_{n,k}}(\mathbf{u})\cap Nbd_{A_{n,k}}(\mathbf{v})$*;* (b) $l_2=2$ *with* $d_{A_{n,k}}(\mathbf{x},\mathbf{y})\leq 2$ *and* $\{\mathbf{u},\mathbf{v}\}\supseteq Nbd_{A_{n,k}}(\mathbf{x})\cap Nbd_{A_{n,k}}(\mathbf{y})$.

Lemma 8 (Hsu et al. 2004) *Let* $F\subset V(A_{n,k})$ *with* $|F|\leq k(n-k)-3$. *Then there exists a Hamiltonian path of* $A_{n,k}-F$ *joining any two distinct vertices of* $A_{n,k}-F$.

Lemma 9 *Suppose that* $k\geq 2$, $n-k\geq 2$ *and* $I\subseteq\langle n\rangle$ *with* $|I|\geq 1$. *Then* $A^I_{n,k}$ *is Hamiltonian connected.*

Proof For $|I|=1$, $A^I_{n,k}$ is isomorphic to $A_{n-1,k-1}$. By Lemma 8, $A^I_{n,k}$ is Hamiltonian connected. Assume that $|I|\geq 2$. Let $\mathbf{x}$ and $\mathbf{y}$ be two arbitrary vertices in $A^I_{n,k}$. Let $\mathbf{u}$ and $\mathbf{v}$ be two adjacent vertices in $A^I_{n,k}-\{\mathbf{x},\mathbf{y}\}$. By Lemma 7, there exist two disjoint paths P_1 and P_2 such that (1) P_1 joins $\mathbf{x}$ to $\mathbf{u}$, (2) P_2 joins $\mathbf{v}$ to $\mathbf{y}$, and (3) $P_1\cup P_2$ spans $A^I_{n,k}$. Obviously, the path $\langle\mathbf{x},P_1,\mathbf{u},\mathbf{v},P_2,\mathbf{y}\rangle$ forms a Hamiltonian path of $A^I_{n,k}$. □

Theorem 1 *The arrangement graph* $A_{n,k}$ *satisfies the* 2*RP-property if and only if* $n-k\geq 2$ *and* $(n,k)\notin\{(3,1),(4,2)\}$.

We prove the theorem by induction. However, the proof of the theorem is rather long. We prove it in Sect. 4.

3.2 The applications of the 2RP-property

Now we apply the 2RP-property of $A_{n,k}$ to prove some properties of $A_{n,k}$. Suppose that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{x}$, and $\mathbf{y}$ are any four distinct vertices of a graph G. Then G is *globally two-equal-disjoint path coverable* if there exist two disjoint paths P and Q such that (1) P joins $\mathbf{u}$ and $\mathbf{v}$, and Q joins $\mathbf{x}$ and $\mathbf{y}$, (2) $l(P)=l(Q)$, and (3) $P\cup Q$ spans G. Lai and Hsu study the property on the crossed cube, the twisted cube, and the Möbius cube in Lai and Hsu (2008). Applying the 2RP-property of $A_{n,k}$, we have the following corollary.

Corollary 1 *The arrangement graph $A_{n,k}$ is globally two-equal-disjoint path coverable for $k\geq 2$, $n-k\geq 2$, and $(n,k)\neq(4,2)$.*

One of the major requirements in designing an interconnection network is the Hamiltonian property. The Hamiltonian property is fundamental to the deadlock-free routing algorithms of distributed systems. Recently, further attempts at Hamiltonian problems led some research into the study of super-Hamiltonian graphs, such as panpositionable Hamiltonian graphs and panconnected graphs. A Hamiltonian graph G is *panpositionable* if for any two different vertices u and v of G and for any integer l satisfying $d(u,v)\leq l\leq\frac{|V(G)|}{2}$, there exists a Hamiltonian cycle HC of G such that the relative distance between u and v on HC is l. A graph G is *panconnected* if there exists a path of length l joining any two different vertices u and v with $d(u,v)\leq l\leq|V(G)|-1$. We show some interesting properties about super-Hamiltonian for $A_{n,k}$ by applying the 2RP-property.

Lemma 10 *Suppose that $n-k\geq 2$. Assume that $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are three different vertices of the arrangement graph $A_{n,k}$ with $d_{A_{n,k}}(\mathbf{x},\mathbf{z})\geq 2$ and $(n,k)\neq(4,2)$. Let l be any integer with $d_{A_{n,k}}(\mathbf{x},\mathbf{y})\leq l\leq\frac{n!}{2(n-k)!}$. Then there exists a Hamiltonian path $R(\mathbf{x},\mathbf{y},\mathbf{z};l)$ from $\mathbf{x}$ to $\mathbf{z}$ such that $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x},\mathbf{y})=l$.*

Proof Since $d_{A_{n,k}}(\mathbf{x},\mathbf{z})\geq 2$, $k\neq 1$. Thus, $\deg(\mathbf{v})=k(n-k)\geq 4$ for any vertex $\mathbf{v}$ in $A_{n,k}$. We choose a vertex $\mathbf{w}$ as follows: Suppose that $d_{A_{n,k}}(\mathbf{x},\mathbf{y})\geq 3$. We choose any vertex $\mathbf{w}$ in $Nbd_{A_{n,k}}(\mathbf{y})-\{\mathbf{z}\}$. Suppose that $d_{A_{n,k}}(\mathbf{x},\mathbf{y})=2$. We choose any vertex $\mathbf{w}$ in $Nbd_{A_{n,k}}(\mathbf{y})-\{\mathbf{z}\}$ such that $d_{A_{n,k}-\{\mathbf{w}\}}(\mathbf{x},\mathbf{y})=2$. Suppose that $d_{A_{n,k}}(\mathbf{x},\mathbf{y})=1$. By Lemma 6, there exists a vertex $\mathbf{p}$ in $Nbd_{A_{n,k}}(\mathbf{x})\cap Nbd_{A_{n,k}}(\mathbf{y})$. We can choose a vertex $\mathbf{w}$ in $Nbd_{A_{n,k}}(\mathbf{y})-\{\mathbf{p},\mathbf{z}\}$. By Theorem 1, there exist two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(P_1)=l$, (2) P_2 is a path joining $\mathbf{w}$ to $\mathbf{z}$ with $l(P_2)=\frac{n!}{(n-k)!}-l-2$, and (3) $P_1\cup P_2$ spans $A_{n,k}$. Obviously, $\langle\mathbf{x},P_1,\mathbf{y},\mathbf{w},P_2,\mathbf{z}\rangle$ forms the required Hamiltonian path. □

Lemma 11 *Suppose that $n-k\geq 2$ and $(n,k)\notin\{(4,2),(3,1)\}$. Assume that $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are three different vertices of the arrangement graph $A_{n,k}$ with $d_{A_{n,k}}(\mathbf{x},\mathbf{y})\leq$*

$d_{A_{n,k}}(\mathbf{y},\mathbf{z})$, $d_{A_{n,k}}(\mathbf{x},\mathbf{z})=1$, *and* $\{\mathbf{z}\}\neq Nbd_{AQ_{n,k}}(\mathbf{x})\cap Nbd_{AQ_{n,k}}(\mathbf{y})$. *Then for any integer* l *with* $d_{A_{n,k}}(\mathbf{x},\mathbf{y})\le l\le \frac{n!}{2(n-k)!}$ *there exists a Hamiltonian path* $R(\mathbf{x},\mathbf{y},\mathbf{z};l)$ *from* $\mathbf{x}$ *to* $\mathbf{z}$ *such that* $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x},\mathbf{y})=l$.

Proof Since $\{\mathbf{z}\}\neq Nbd_{AQ_{n,k}}(\mathbf{x})\cap Nbd_{AQ_{n,k}}(\mathbf{y})$, we can choose a vertex $\mathbf{w}$ in $Nbd_{A_{n,k}}(\mathbf{y})-\{\mathbf{z}\}$. By Theorem 1, there exist two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(P_1)=l$, (2) P_2 is a path joining $\mathbf{w}$ to $\mathbf{z}$ with $l(P_2)=\frac{n!}{(n-k)!}-l-2$, and (3) $P_1\cup P_2$ spans $A_{n,k}$. Obviously, $\langle \mathbf{x},P_1,\mathbf{y},\mathbf{w},P_2,\mathbf{z}\rangle$ forms the required Hamiltonian path. □

Theorem 2 *Assume that* $n-k\ge 3$ *or* $(n,k)=(3,1)$. *Let* $\mathbf{x}$, $\mathbf{y}$, *and* $\mathbf{z}$ *be three different vertices of* $A_{n,k}$. *Let* l *be any integer with* $d_{A_{n,k}}(\mathbf{x},\mathbf{y})\le l\le \frac{n!}{(n-k)!}-1-d_{A_{n,k}}(\mathbf{y},\mathbf{z})$. *Then there exists a Hamiltonian path* $R(\mathbf{x},\mathbf{y},\mathbf{z};l)$ *from* $\mathbf{x}$ *to* $\mathbf{z}$ *such that* $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x},\mathbf{y})=l$.

Proof Obviously, the theorem holds for $(n,k)=(3,1)$. Assume that $n-k\ge 3$. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be three different vertices of the arrangement graph $A_{n,k}$. By the symmetric role between $\mathbf{x}$ and $\mathbf{z}$, we can assume that $l\le \frac{n!}{2(n-k)!}$.

Suppose that $d_{A_{n,k}}(\mathbf{x},\mathbf{z})\ge 2$. By Lemma 10, there exists a Hamiltonian path $R(\mathbf{x},\mathbf{y},\mathbf{z};l)$ from $\mathbf{x}$ to $\mathbf{z}$ such that $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x},\mathbf{y})=l$. Suppose that $d_{A_{n,k}}(\mathbf{x},\mathbf{z})=1$. By Lemma 6, $|Nbd_{A_{n,k}}(\mathbf{x})\cap Nbd_{A_{n,k}}(\mathbf{z})|\ge 2$. Thus, $\{\mathbf{z}\}\neq Nbd_{AQ_{n,k}}(\mathbf{x})\cap Nbd_{AQ_{n,k}}(\mathbf{y})$. By Lemma 11, there exists a Hamiltonian path $R(\mathbf{x},\mathbf{y},\mathbf{z};l)$ from $\mathbf{x}$ to $\mathbf{z}$ such that $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x},\mathbf{y})=l$.

The theorem is proved. □

Suppose that $\mathbf{x}$ and $\mathbf{y}$ are any two adjacent vertices in $A_{n,k}$ with $n-k=2$ and $n\ge 4$. By Lemma 6, there is only one vertex $\mathbf{z}$ in $Nbd_{A_{n,k}}(\mathbf{x})\cap Nbd_{A_{n,k}}(\mathbf{y})$. Obviously, there exists no Hamiltonian path $R(\mathbf{x},\mathbf{y},\mathbf{z};l)$ from $\mathbf{x}$ to $\mathbf{z}$ such that $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x},\mathbf{y})=2$. Thus, the above theorem does not hold if $n-k=2$ and $n\ge 4$. Yet, we still can easily get the following theorem by Lemmas 10 and 11.

Theorem 3 *Assume that* $n-k=2$ *with* $n\ge 5$. *Let* $\mathbf{x}$, $\mathbf{y}$, *and* $\mathbf{z}$ *be three different vertices of* $A_{n,k}$. *Let* l *be any integer with* $d_{A_{n,k}}(\mathbf{x},\mathbf{y})\le l\le \frac{n!}{(n-k)!}-1-d_{A_{n,k}}(\mathbf{y},\mathbf{z})$. *Then there exists a Hamiltonian path* $R(\mathbf{x},\mathbf{y},\mathbf{z};l)$ *from* $\mathbf{x}$ *to* $\mathbf{z}$ *such that* $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x},\mathbf{y})=l$ *except for the case that* $\mathbf{x}$, $\mathbf{y}$, *and* $\mathbf{z}$ *are adjacent to each other.*

In Fig. 1, we can check that $d_{A_{4,2}}(12,41)=2$, and there exists no path of length 2 joining 12 to 41 in $A_{4,2}-\{42\}$. Thus, there exists no Hamiltonian path P of $A_{4,2}$ joining 42 to 41 such that $d_P(42,13)=2$. Moreover, consider that $d_{A_{4,2}}(42,12)=1$ and $d_{A_{4,2}}(12,13)=1$. There exists no path of length 3 joining 42 to 12 in $A_{4,2}-\{12\}$ and there exists no path of length 3 joining 12 to 13 in $A_{4,2}-\{42\}$. Thus, there exists no Hamiltonian path P of $A_{4,2}$ joining 42 to 13 such that $d_P(42,13)\in\{3,8\}$. Thus, the above theorem does not hold for $(n,k)=(4,2)$.

With Theorems 2 and 3, we reprove the following theorems about the panpositionable Hamiltonian property and the panconnected property in Teng et al. (2008).

Theorem 4 *Suppose that $n-k\geq 2$. Assume that* $\mathbf{x}$ *and* $\mathbf{y}$ *are two different vertices of $A_{n,k}$. Let l be any integer with $d_{A_{n,k}}(\mathbf{x},\mathbf{y})\leq l\leq \frac{n!}{2(n-k)!}$. Then there exists a Hamiltonian cycle $C(\mathbf{x},\mathbf{y};l)$ such that $d_{C(\mathbf{x},\mathbf{y};l)}(\mathbf{x},\mathbf{y})=l$.*

Proof Obviously, the theorem holds for $(n,k)=(3,1)$ and $(4,1)$. By brute force, we can check that the theorem holds for $(n,k)=(4,2)$. Thus, we assume that $n\geq 5$. Suppose that $n-k\geq 3$ or $n-k=2$ with $(\mathbf{x},\mathbf{y})\notin E(A_{n,k})$. We choose $\mathbf{z}$ as any vertex of $Nbd_{A_{n,k}}(\mathbf{x})-\{\mathbf{y}\}$. Otherwise, we choose $\mathbf{z}$ as any vertex in $Nbd_{A_{n,k}}(\mathbf{x})-Nbd_{A_{n,k}}(\mathbf{y})$. By Theorems 2 and 3, there exists a Hamiltonian path $R(\mathbf{x},\mathbf{y},\mathbf{z};l)$ from $\mathbf{x}$ to $\mathbf{z}$ such that $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x},\mathbf{y})=l$. Obviously, $\langle \mathbf{x}, R(\mathbf{x},\mathbf{y},\mathbf{z};l),\mathbf{z},\mathbf{x}\rangle$ forms the required Hamiltonian cycle. □

Theorem 5 *Suppose that $n-k\geq 2$. Assume that* $\mathbf{x}$ *and* $\mathbf{y}$ *are two different vertices of $A_{n,k}$. Then there exists a path of length l joining* $\mathbf{x}$ *and* $\mathbf{y}$ *with $d_{A_{n,k}}(\mathbf{x},\mathbf{y})\leq l\leq \frac{n!}{(n-k)!}-1$.*

Proof Obviously, the theorem holds for $(n,k)=(3,1)$ and $(4,1)$. By brute force, we can check that the theorem holds for $(n,k)=(4,2)$. Thus, we assume that $n\geq 5$. Suppose that $n-k\geq 3$ or $n-k=2$ with $(\mathbf{x},\mathbf{y})\notin E(A_{n,k})$. We choose $\mathbf{z}$ as any vertex of $Nbd_{A_{n,k}}(\mathbf{y})-\{\mathbf{x}\}$. Otherwise, we choose $\mathbf{z}$ as any vertex in $Nbd_{A_{n,k}}(\mathbf{y})-Nbd_{A_{n,k}}(\mathbf{x})$. Suppose that $d_{A_{n,k}}(\mathbf{x},\mathbf{y})\leq l\leq \frac{n!}{(n-k)!}-2$. By Theorems 2 and 3, there exists a Hamiltonian path $R(\mathbf{x},\mathbf{y},\mathbf{z};l)=\langle \mathbf{x},P_1,\mathbf{y},P_2,\mathbf{z}\rangle$ from $\mathbf{x}$ to $\mathbf{z}$ such that $d_{R(\mathbf{x},\mathbf{y},\mathbf{z};l)}(\mathbf{x},\mathbf{y})=l$. Obviously, $\langle \mathbf{x},P_1,\mathbf{y}\rangle$ forms the required path. Suppose that $l=\frac{n!}{(n-k)!}-1$. By Lemma 8, there exists a Hamiltonian path HP from $\mathbf{x}$ to $\mathbf{y}$ such that $l(HP)=\frac{n!}{(n-k)!}-1$. Thus the theorem is proved. □

4 Proof of Theorem 1

Since $A_{3,1}$ contains exactly three vertices, it is meaningless to discuss the 2RP-property. Now we prove that $A_{4,2}$ does not satisfy the 2RP-property. Let $\mathbf{u}=12$, $\mathbf{v}=13$, $\mathbf{x}=24$, and $\mathbf{y}=34$ be four vertices in $A_{4,2}$. Let t be any integer with $3\leq t\leq 7$, and $\mathbf{w}=14$. Since $Nbd_{A_{4,2}}(\mathbf{w})=\{\mathbf{u},\mathbf{v},\mathbf{x},\mathbf{y}\}$, we can not find two disjoint paths P_1 and P_2 such that (1) P_1 is a path joining $\mathbf{u}$ to $\mathbf{v}$ with $l(P_1)=t$, (2) P_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(P_2)=|V(A_{4,2})|-t-2$, and (3) $P_1\cup P_2$ spans $A_{4,2}$.

For $n\geq 4$, $n-k\geq 2$, and $(n,k)\neq(4,2)$, we shall prove that $A_{n,k}$ satisfies the 2RP-property by induction. The induction basis are $A_{n,1}$ with $n\geq 4$, $A_{5,2}$, $A_{5,3}$, and $A_{6,2}$. Since $A_{n,1}$ is isomorphic to K_n, $A_{n,1}$ satisfies 2RP-property for $n\geq 4$. By brute force with a computer program, we can check that $A_{5,2}$, $A_{5,3}$, and $A_{6,2}$ satisfy the 2RP-property. Now, we assume that $A_{n-1,k-1}$ satisfies the 2RP-property with $n\geq 6$ and $n-k\geq 2$. We claim that $A_{n,k}$ also satisfies the 2RP-property.

Assume that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{x}$, and $\mathbf{y}$ are any four distinct vertices of $A_{n,k}$. Let l_1 and l_2 be any two positive integers with $l_1+l_2=\frac{n!}{(n-k)!}-2$ such that $l_1\geq d_{A_{n,k}}(\mathbf{u},\mathbf{v})$, $l_2\geq d_{A_{n,k}}(\mathbf{x},\mathbf{y})$, and $l_1+l_2=\frac{n!}{(n-k)!}-2$. Without loss of generality, we assume that $l_1\geq l_2$. Thus, $l_2\leq \frac{n!}{2(n-k)!}-1$.

We first consider the case $l_2 = 1$ with $d_{A_{n,k}}(\mathbf{x}, \mathbf{y}) = 1$. We set $P_2 = \langle \mathbf{x}, \mathbf{y} \rangle$. Obviously, $k(n-k) - 3 \geq 2$ for $n > 5$ and $n - k \geq 2$. By Lemma 8, there exists a Hamiltonian path P_1 of $A_{n,k} - V(P_2)$ joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, P_1 and P_2 form the required paths.

Then we consider the case $l_2 = 2$ with $d_{A_{n,k}}(\mathbf{x}, \mathbf{y}) \leq 2$ such that $\mathbf{u} \notin Nbd_{A_{n,k}}(\mathbf{x}) \cap Nbd_{A_{n,k}}(\mathbf{y})$ and $\mathbf{v} \notin Nbd_{A_{n,k}}(\mathbf{x}) \cap Nbd_{A_{n,k}}(\mathbf{y})$. Suppose that $\mathbf{w} \in Nbd_{A_{n,k}}(\mathbf{x}) \cap Nbd_{A_{n,k}}(\mathbf{y})$. We set $P_2 = \langle \mathbf{x}, \mathbf{w}, \mathbf{y} \rangle$. Obviously, $k(n-k) - 3 \geq 3$ for $n > 5$ and $n - k \geq 2$. By Lemma 8, there exists a Hamiltonian path P_1 of $A_{n,k} - V(P_2)$ joining $\mathbf{u}$ to $\mathbf{v}$. Obviously, P_1 and P_2 form the required paths.

Finally, we consider that $l_2 > \max\{2, d_{A_{n,k}}(\mathbf{x}, \mathbf{y})\}$. Let a, b, c, and d be the indices such that $\mathbf{u}$ is a vertex in $A^a_{n,k}$, $\mathbf{v}$ is a vertex in $A^b_{n,k}$, $\mathbf{x}$ is a vertex in $A^c_{n,k}$, and $\mathbf{y}$ is a vertex in $A^d_{n,k}$. By Lemma 1, we may assume that $a \neq b$. Depending on the distribution of a, b, c, and d, we have five cases. In each case, we have some subcases depending on the length of l_2. The reader can easily get the idea of the proof once s/he understand the first case.

In the following proof, we shall write l_2 as $q(l_2)\frac{(n-1)!}{(n-k)!} + r(l_2)$ with $0 \leq q(l_2) \leq \lceil \frac{n}{2} \rceil$ and $0 \leq r(l_2) < \frac{(n-1)!}{(n-k)!}$. That is, $q(l_2)$ and $r(l_2)$ is the quotient and remainder of l_2 divided by $|V(A_{n-1,k-1})| = \frac{(n-1)!}{(n-k)!}$, respectively.

Case 1 a, b, c, and d are four distinct elements in $\langle n \rangle$.

Subcase 1.1 $q(l_2) = 0$ and $r(l_2) \leq \frac{(n-1)!}{2(n-k)!} + D(A_{n,k})$. See Fig. 2(a) for an illustration. By Lemma 4, there exists a shortest path connecting $\mathbf{x}$ and $\mathbf{y}$ with the form $\langle \mathbf{x}, M_1, \mathbf{y}', \mathbf{y}'', M_2, \mathbf{y} \rangle$ such that (1) M_1 is a path in $A^c_{n,k}$, (2) M_2 is a path in $A^d_{n,k}$, and (3) $l(M_2) \leq 1$ with $\mathbf{y}' \in V(A^c_{n,k})$ and $\mathbf{y}'' \in V(A^d_{n,k})$. Let $I = \langle n \rangle - \{b, c, d\}$. Since $|E^{c,I}| = \frac{(n-3)(n-2)!}{(n-k-1)!} > \max\{n-k-1, 2\} + 3$, there exist $(\mathbf{w}, \mathbf{w}')$ and $(\mathbf{z}, \mathbf{z}')$ in $E^{c,I}$ such that $\{\mathbf{w}, \mathbf{z}\} \subset V(A^c_{n,k}) - \{\mathbf{x}, \mathbf{y}'\}$ and $\{\mathbf{w}', \mathbf{z}'\} \subset V(A^I_{n,k}) - \{\mathbf{u}\}$ with $\{\mathbf{w}, \mathbf{z}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{x}) \cap Nbd_{A_{n,k}}(\mathbf{y}')$. Obviously, $\frac{(n-1)!}{(n-k)!} - r(l_2) - 1 \geq \lfloor \frac{3k}{2} \rfloor$. By Lemma 3, $\frac{(n-1)!}{(n-k)!} - r(l_2) - 1 \geq D(A_{n,k})$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{x}$ to $\mathbf{y}'$ with $l(Q_1) = r(l_2) - 1$ if $l(M_2) = 0$, and $l(Q_1) = r(l_2) - 2$ if $l(M_2) = 1$, (2) Q_2 is a path joining $\mathbf{w}$ to $\mathbf{z}$ with $l(Q_2) = \frac{(n-1)!}{(n-k)!} - r(l_2) - 1$ if $l(M_2) = 0$, and $l(Q_2) = \frac{(n-1)!}{(n-k)!} - r(l_2)$ if $l(M_2) = 1$, and (3) $Q_1 \cup Q_2$ spans $A^c_{n,k}$. By Lemma 2, $|E^{b,d}| = \frac{(n-2)!}{(n-k-1)!} > 3$, hence there exists $(\mathbf{p}, \mathbf{p}')$ in $E^{b,d}$ such that $\mathbf{p} \in V(A^d_{n,k}) - \{\mathbf{y}, \mathbf{y}''\}$ and $\mathbf{p}' \in V(A^b_{n,k}) - \{\mathbf{v}\}$. Again, there exists $(\mathbf{q}, \mathbf{q}')$ in $E^{d,I}$ such that $\mathbf{q} \in V(A^d_{n,k}) - \{\mathbf{p}, \mathbf{y}, \mathbf{y}''\}$ and $\mathbf{q}' \in V(A^I_{n,k}) - \{\mathbf{u}, \mathbf{w}', \mathbf{z}'\}$. By Lemma 8, there exists a Hamiltonian path H_1 joining $\mathbf{p}$ to $\mathbf{q}$ in $A^d_{n,k} - \{\mathbf{y}, \mathbf{y}''\}$, and there exists a Hamiltonian path H_2 joining $\mathbf{p}'$ to $\mathbf{v}$ in $A^b_{n,k}$. By Lemma 7, there exist two disjoint paths S and T such that (1) S is a path joining $\mathbf{u}$ to $\mathbf{w}'$, (2) T is a path joining $\mathbf{z}'$ to $\mathbf{q}'$, and (3) $S \cup T$ spans $A^I_{n,k}$. We set $P_1 = \langle \mathbf{u}, S, \mathbf{w}', \mathbf{w}, Q_2, \mathbf{z}, \mathbf{z}', T, \mathbf{q}', \mathbf{q}, H_1, \mathbf{p}, \mathbf{p}', H_2, \mathbf{v} \rangle$ and set $P_2 = \langle \mathbf{x}, Q_1, \mathbf{y}', \mathbf{y}'', M_2, \mathbf{y} \rangle$. Obviously, P_1 and P_2 form the required paths.

Subcase 1.2 $q(l_2) = 0$ with $r(l_2) \geq \frac{(n-1)!}{2(n-k)!} + D(A_{n,k}) + 1$, or $q(l_2) = 1$ with $r(l_2) < D(A_{n,k})$. See Fig. 2(b) for an illustration. By Lemma 2, $|E^{c,d}| =$

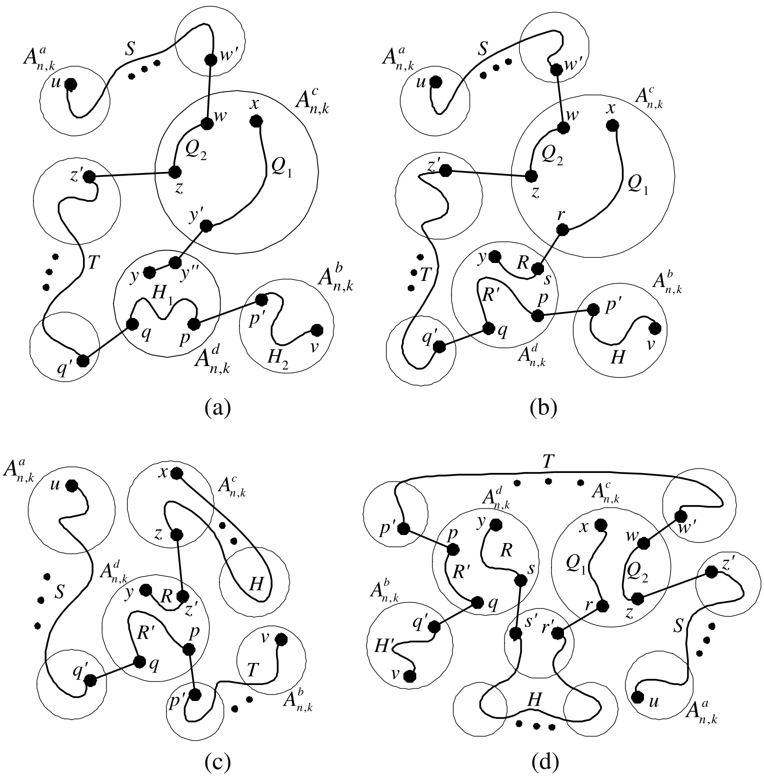


Fig. 2 Illustrations for Case 1 in Theorem 1

$\frac{(n-2)!}{(n-k-1)!} > 2$. Hence there exists $(\mathbf{r},\mathbf{s})$ in $E^{c,d}$ such that $\mathbf{r} \in V(A^c_{n,k}) - \{\mathbf{x}\}$ and $\mathbf{s} \in V(A^d_{n,k}) - \{\mathbf{y}\}$. By Lemma 2, $|E^{b,d}| = \frac{(n-2)!}{(n-k-1)!} > 3$. Again there exists $(\mathbf{p},\mathbf{p}')$ in $E^{b,d}$ such that $\mathbf{p} \in V(A^d_{n,k}) - \{\mathbf{s},\mathbf{y}\}$ and $\mathbf{p}' \in V(A^b_{n,k}) - \{\mathbf{v}\}$. Let $I = \langle n\rangle - \{b,c,d\}$. Since $|E^{c,I}| = \frac{(n-3)(n-2)!}{(n-k-1)!} > \max\{n-k-1, 2\} + 3$, there exist $(\mathbf{w},\mathbf{w}')$ and $(\mathbf{z},\mathbf{z}')$ in $E^{c,I}$ such that $\{\mathbf{w},\mathbf{z}\} \subset V(A^c_{n,k}) - \{\mathbf{r},\mathbf{x}\}$ and $\{\mathbf{w}',\mathbf{z}'\} \subset V(A^I_{n,k}) - \{\mathbf{u}\}$ with $\{\mathbf{w},\mathbf{z}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{x}) \cap Nbd_{A_{n,k}}(\mathbf{r})$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{x}$ to $\mathbf{r}$ with $l(Q_1) = \frac{(n-1)!}{2(n-k)!} - 1$, (2) Q_2 is a path joining $\mathbf{w}$ to $\mathbf{z}$ with $l(Q_2) = \frac{(n-1)!}{2(n-k)!} - 1$, and (3) $Q_1 \cup Q_2$ spans $A^c_{n,k}$. Again, there exists $(\mathbf{q},\mathbf{q}')$ in $E^{d,I}$ such that $\mathbf{q} \in V(A^d_{n,k}) - \{\mathbf{p},\mathbf{s},\mathbf{y}\}$ and $\mathbf{q}' \in V(A^I_{n,k}) - \{\mathbf{u},\mathbf{w}',\mathbf{z}'\}$ with $\{\mathbf{p},\mathbf{q}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{s}) \cap Nbd_{A_{n,k}}(\mathbf{y})$. By induction, there exist two paths R and R' such that (1) R is a path joining $\mathbf{s}$ to $\mathbf{y}$ with $l(R) = r(l_2) - \frac{(n-1)!}{2(n-k)!} - 1$ if $q(l_2) = 0$, and $l(R) = r(l_2) + \frac{(n-1)!}{2(n-k)!} - 1$ if $q(l_2) = 1$, (2) R' is a path joining $\mathbf{p}$ to $\mathbf{q}$ with $l(R') = \frac{3(n-1)!}{2(n-k)!} - r(l_2) - 1$ if $q(l_2) = 0$, and $l(R') = \frac{(n-1)!}{2(n-k)!} - r(l_2) - 1$ if $q(l_2) = 1$, and (3) $R \cup R'$ spans $A^d_{n,k}$. By Lemma 9, there exists a Hamiltonian path H joining $\mathbf{p}'$ to $\mathbf{v}$ in $A^b_{n,k}$. By Lemma 7, there exist two disjoint paths S and T such that (1) S is a path joining

$\mathbf{u}$ to $\mathbf{w}'$, (2) T is a path joining $\mathbf{z}'$ to $\mathbf{q}'$, and (3) $S \cup T$ spans $A^{I}_{n,k}$. We set $P_1 = \langle \mathbf{u}, S, \mathbf{w}', \mathbf{w}, Q_2, \mathbf{z}, \mathbf{z}', T, \mathbf{q}', \mathbf{q}, R', \mathbf{p}, \mathbf{p}', H, \mathbf{v}\rangle$ and set $P_2 = \langle \mathbf{x}, Q_1, \mathbf{r}, \mathbf{s}, R, \mathbf{y}\rangle$. Obviously, P_1 and P_2 form the required paths.

Subcase 1.3 $q(l_2) > 0$ and $D(A_{n,k}) \le r(l_2) \le \frac{(n-1)!}{2(n-k)!} + D(A_{n,k})$. See Fig. 2(c) for an illustration. By Lemma 2, $|E^{c,d}| = \frac{(n-2)!}{(n-k-1)!} > 2$. Hence there exists $(\mathbf{z}, \mathbf{z}')$ in $E^{c,d}$ such that $\mathbf{z} \in V(A^{c}_{n,k}) - \{\mathbf{x}\}$ and $\mathbf{z}' \in V(A^{d}_{n,k}) - \{\mathbf{y}\}$. Let $I \subset \langle n\rangle - \{a, b, c, d\}$ with $|I| = q(l_2) - 1$, and let $J = \langle n\rangle - (I \cup \{c, d\})$. Obviously, there exist $(\mathbf{p}, \mathbf{p}')$ and $(\mathbf{q}, \mathbf{q}')$ in $E^{d,J}$ such that $\{\mathbf{p}, \mathbf{q}\} \subset V(A^{d}_{n,k}) - \{\mathbf{y}, \mathbf{z}'\}$ and $\{\mathbf{p}', \mathbf{q}'\} \subset V(A^{J}_{n,k}) - \{\mathbf{u}, \mathbf{v}\}$ with $\{\mathbf{p}, \mathbf{q}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{y}) \cap Nbd_{A_{n,k}}(\mathbf{z}')$. By induction, there exist two paths R and R' such that (1) R is a path joining $\mathbf{z}'$ to $\mathbf{y}$ with $l(R) = r(l_2)$, (2) R' is a path joining $\mathbf{p}$ to $\mathbf{q}$ with $l(R') = \frac{(n-1)!}{(n-k)!} - r(l_2) - 2$, and (3) $R \cup R'$ spans $A^{d}_{n,k}$. By Lemma 9, there exists a Hamiltonian path H joining $\mathbf{x}$ to $\mathbf{z}$ in $A^{I\cup\{a\}}_{n,k}$. By Lemma 7, there exist two disjoint paths S and T such that (1) S is a path joining $\mathbf{u}$ to $\mathbf{q}'$, (2) T is a path joining $\mathbf{p}'$ to $\mathbf{v}$, and (3) $S \cup T$ spans $A^{J}_{n,k}$. We set $P_1 = \langle \mathbf{u}, S, \mathbf{q}', \mathbf{q}, R', \mathbf{p}, \mathbf{p}', T, \mathbf{v}\rangle$ and set $P_2 = \langle \mathbf{x}, H, \mathbf{z}, \mathbf{z}', R, \mathbf{y}\rangle$. Obviously, P_1 and P_2 form the required paths.

Subcase 1.4 $q(l_2) > 0$ with $r(l_2) \ge \frac{(n-1)!}{2(n-k)!} + D(A_{n,k}) + 1$, or $q(l_2) > 1$ with $r(l_2) < D(A_{n,k})$. See Fig. 2(d) for an illustration. Let $e \in \langle n\rangle - \{a, b, c, d\}$. There exists $(\mathbf{r}, \mathbf{r}')$ in $E^{c,e}$ such that $\mathbf{r} \in V(A^{c}_{n,k}) - \{\mathbf{x}\}$ and $\mathbf{r}' \in V(A^{e}_{n,k})$. Again, there exists $(\mathbf{s}, \mathbf{s}')$ in $E^{d,e}$ such that $\mathbf{s} \in V(A^{d}_{n,k}) - \{\mathbf{v}\}$ and $\mathbf{s}' \in V(A^{e}_{n,k}) - \{\mathbf{r}'\}$. Let $I \subset \langle n\rangle - \{a, b, c, d, e\}$ with $|I| = q(l_2) - 1$ if $r(l_2) \ge \frac{(n-1)!}{2(n-k)!} + D(A_{n,k}) + 1$ and $|I| = q(l_2) - 2$ if $r(l_2) < D(A_{n,k})$. Let $J = \langle n\rangle - (I \cup \{b, c, d, e\})$. Obviously, there exist $(\mathbf{w}, \mathbf{w}')$ and $(\mathbf{z}, \mathbf{z}')$ in $E^{c,J}$ such that $\{\mathbf{w}, \mathbf{z}\} \subset V(A^{c}_{n,k}) - \{\mathbf{r}, \mathbf{x}\}$ and $\{\mathbf{w}', \mathbf{z}'\} \subset V(A^{J}_{n,k}) - \{\mathbf{u}\}$ with $\{\mathbf{w}, \mathbf{z}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{x}) \cap Nbd_{A_{n,k}}(\mathbf{r})$. Again, there exists $(\mathbf{p}, \mathbf{p}')$ in $E^{d,J}$ such that $\mathbf{p} \in V(A^{d}_{n,k}) - \{\mathbf{s}, \mathbf{y}\}$ and $\mathbf{p}' \in V(A^{J}_{n,k}) - \{\mathbf{w}', \mathbf{z}'\}$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{x}$ to $\mathbf{r}$ with $l(Q_1) = \frac{(n-1)!}{2(n-k)!} - 1$, (2) Q_2 is a path joining $\mathbf{w}$ to $\mathbf{z}$ with $l(Q_2) = \frac{(n-1)!}{2(n-k)!} - 1$, and (3) $Q_1 \cup Q_2$ spans $A^{c}_{n,k}$. Obviously, there exists $(\mathbf{q}, \mathbf{q}')$ in $E^{b,d}$ such that $\mathbf{q} \in V(A^{d}_{n,k}) - \{\mathbf{p}, \mathbf{s}, \mathbf{y}\}$ and $\mathbf{q}' \in V(A^{b}_{n,k}) - \{\mathbf{v}\}$ with $\{\mathbf{p}, \mathbf{q}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{s}) \cap Nbd_{A_{n,k}}(\mathbf{y})$. Again, by induction, there exist two paths R and R' such that (1) R is a path joining $\mathbf{s}$ to $\mathbf{y}$ with $l(R) = r(l_2) - \frac{(n-1)!}{2(n-k)!} - 1$ if $r(l_2) \ge \frac{(n-1)!}{2(n-k)!} + D(A_{n,k}) + 1$, and $l(R) = r(l_2) + \frac{(n-1)!}{2(n-k)!} - 1$ if $r(l_2) < D(A_{n,k})$, (2) R' is a path joining $\mathbf{p}$ to $\mathbf{q}$ with $l(R') = \frac{3(n-1)!}{2(n-k)!} - r(l_2) - 1$ if $r(l_2) \ge \frac{(n-1)!}{2(n-k)!} + D(A_{n,k}) + 1$, and $l(R') = \frac{(n-1)!}{2(n-k)!} - r(l_2) - 1$ if $r(l_2) < D(A_{n,k})$, and (3) $R \cup R'$ spans $A^{d}_{n,k}$. By Lemma 9, there exists a Hamiltonian path H joining $\mathbf{r}'$ to $\mathbf{s}'$ in $A^{I\cup\{e\}}_{n,k}$, and there exists a Hamiltonian path H' joining $\mathbf{q}'$ to $\mathbf{v}$ in $A^{b}_{n,k}$. By Lemma 7, there exist two disjoint paths S and T such that (1) S is a path joining $\mathbf{u}$ to $\mathbf{z}'$, (2) T is a path joining $\mathbf{w}'$ to $\mathbf{p}'$, and (3) $S \cup T$ spans $A^{J}_{n,k}$. We set $P_1 = \langle \mathbf{u}, S, \mathbf{z}', \mathbf{z}, Q_2, \mathbf{w}, \mathbf{w}', T, \mathbf{p}', \mathbf{p}, R', \mathbf{q}, \mathbf{q}', H', \mathbf{v}\rangle$ and set P_2 as $\langle \mathbf{x}, Q_1, \mathbf{r}, \mathbf{r}', H, \mathbf{s}', \mathbf{s}, R, \mathbf{y}\rangle$. Obviously, P_1 and P_2 form the required paths.

Case 2 $a = c$ and $b = d$.

Subcase 2.1 $q(l_2) = 0$ and $r(l_2) \leq \frac{(n-1)!}{2(n-k)!} + D(A_{n,k})$. By Lemma 4, there exists a shortest path connecting $\mathbf{x}$ and $\mathbf{y}$ with the form $\langle \mathbf{x}, M_1, \mathbf{y}', \mathbf{y}'', M_2, \mathbf{y}\rangle$ such that (1) M_1 is a path in $A^a_{n,k}$, (2) M_2 is a path in $A^b_{n,k}$, and (3) $l(M_2) \leq 1$ with $\mathbf{y}' \in V(A^a_{n,k})$ and $\mathbf{y}'' \in V(A^b_{n,k})$. Let $I = \langle n\rangle - \{a, b\}$. Obviously, there exists an edge $(\mathbf{z}, \mathbf{z}') \in E^{a,I}$ such that $\mathbf{z} \in V(A^a_{n,k}) - \{\mathbf{x}, \mathbf{y}', \mathbf{u}\}$ and $\mathbf{z}' \in V(A^I_{n,k})$ with $\{\mathbf{u}, \mathbf{z}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{x}) \cap Nbd_{A_{n,k}}(\mathbf{y}')$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{x}$ to $\mathbf{y}'$ with $l(Q_1) = r(l_2) - 1$ if $l(M_2) = 0$, and $l(Q_1) = r(l_2) - 2$ if $l(M_2) = 1$, (2) Q_2 is a path joining $\mathbf{u}$ to $\mathbf{z}$ with $l(Q_2) = \frac{(n-1)!}{(n-k)!} - r(l_2) - 1$ if $l(M_2) = 0$, and $l(Q_2) = \frac{(n-1)!}{(n-k)!} - r(l_2)$ if $l(M_2) = 1$, and (3) $Q_1 \cup Q_2$ spans $A^a_{n,k}$. Again, there exists $(\mathbf{w}, \mathbf{w}')$ in $E^{b,I}$ such that $\mathbf{w} \in V(A^b_{n,k}) - \{\mathbf{v}, \mathbf{y}, \mathbf{y}''\}$ and $\mathbf{w}' \in V(A^I_{n,k}) - \{\mathbf{z}'\}$. By Lemma 8, there exists a Hamiltonian path H joining $\mathbf{w}$ to $\mathbf{v}$ in $A^b_{n,k} - \{\mathbf{y}, \mathbf{y}''\}$. By Lemma 9, there exists a Hamiltonian path H' joining $\mathbf{z}'$ to $\mathbf{w}'$ in $A^I_{n,k}$. We set $P_1 = \langle \mathbf{u}, Q_2, \mathbf{z}, \mathbf{z}', H', \mathbf{w}', \mathbf{w}, H, \mathbf{v}\rangle$ and set $P_2 = \langle \mathbf{x}, Q_1, \mathbf{y}', \mathbf{y}'', M_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 form the required paths.

Subcase 2.2 $q(l_2) = 0$ with $r(l_2) \geq \frac{(n-1)!}{2(n-k)!} + D(A_{n,k}) + 1$, or $q(l_2) = 1$ with $r(l_2) < D(A_{n,k})$. Obviously, there exists $(\mathbf{w}, \mathbf{w}')$ in $E^{a,b}$ such that $\mathbf{w} \in V(A^a_{n,k}) - \{\mathbf{u}, \mathbf{x}\}$ and $\mathbf{w}' \in V(A^b_{n,k}) - \{\mathbf{v}, \mathbf{y}\}$. Let $I = \langle n\rangle - \{a, b\}$. There exists $(\mathbf{z}, \mathbf{z}')$ in $E^{a,I}$ such that $\mathbf{z} \in V(A^a_{n,k}) - \{\mathbf{u}, \mathbf{w}, \mathbf{x}\}$ and $\mathbf{z}' \in V(A^I_{n,k})$ with $\{\mathbf{u}, \mathbf{z}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{x}) \cap Nbd_{A_{n,k}}(\mathbf{w})$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{x}$ to $\mathbf{w}$ with $l(Q_1) = \frac{(n-1)!}{2(n-k)!} - 1$, (2) Q_2 is a path joining $\mathbf{u}$ to $\mathbf{z}$ with $l(Q_2) = \frac{(n-1)!}{2(n-k)!} - 1$, and (3) $Q_1 \cup Q_2$ spans $A^a_{n,k}$. Again, there exists $(\mathbf{p}, \mathbf{p}')$ in $E^{b,I}$ such that $\mathbf{p} \in V(A^b_{n,k}) - \{\mathbf{v}, \mathbf{w}', \mathbf{y}\}$ and $\mathbf{p}' \in V(A^I_{n,k}) - \{\mathbf{z}'\}$ with $\{\mathbf{p}, \mathbf{v}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{w}') \cap Nbdi_{A_{n,k}}(\mathbf{y})$. By induction, there exist two paths R and R' such that (1) R is a path joining $\mathbf{w}'$ to $\mathbf{y}$ with $l(R) = r(l_2) - \frac{(n-1)!}{2(n-k)!} - 1$ if $q(l_2) = 0$, and $l(R) = r(l_2) + \frac{(n-1)!}{2(n-k)!} - 1$ if $q(l_2) = 1$, (2) R' is a path joining $\mathbf{p}$ to $\mathbf{v}$ with $l(R') = \frac{3(n-1)!}{2(n-k)!} - r(l_2) - 1$ if $q(l_2) = 0$, and $l(R') = \frac{(n-1)!}{2(n-k)!} - r(l_2) - 1$ if $q(l_2) = 1$, and (3) $R \cup R'$ spans $A^b_{n,k}$. By Lemma 9, there exists a Hamiltonian path H joining $\mathbf{z}'$ to $\mathbf{p}'$ in $A^I_{n,k}$. We set $P_1 = \langle \mathbf{u}, Q_2, \mathbf{z}, \mathbf{z}', H, \mathbf{p}', \mathbf{p}, R', \mathbf{v}\rangle$ and set $P_2 = \langle \mathbf{x}, Q_1, \mathbf{w}, \mathbf{w}', R, \mathbf{y}\rangle$. Obviously, P_1 and P_2 form the required paths.

Subcase 2.3 $q(l_2) > 0$ and $D(A_{n,k}) \leq r(l_2) \leq \frac{(n-1)!}{2(n-k)!} + D(A_{n,k})$. Let $e \in \langle n\rangle - \{a, b\}$ such that $e \in AS(\mathbf{y})$. There exists $(\mathbf{y}, \mathbf{y}')$ in $E^{b,e}$ such that $\mathbf{y}' \in V(A^e_{n,k})$. Again, there exists $(\mathbf{w}, \mathbf{w}')$ in $E^{a,e}$ such that $\mathbf{w} \in V(A^a_{n,k}) - \{\mathbf{u}, \mathbf{x}\}$ and $\mathbf{w}' \in V(A^e_{n,k}) - \{\mathbf{y}'\}$. Let $I \subset \langle n\rangle - \{a, b, e\}$ with $|I| = q(l_2) - 1$, and let $J = \langle n\rangle - (I \cup \{a, b, e\})$. Obviously, there exists $(\mathbf{z}, \mathbf{z}')$ in $E^{a,J}$ such that $\mathbf{z} \in V(A^a_{n,k}) - \{\mathbf{u}, \mathbf{w}, \mathbf{x}\}$ and $\mathbf{z}' \in V(A^J_{n,k})$ with $\{\mathbf{u}, \mathbf{z}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{x}) \cap Nbd_{A_{n,k}}(\mathbf{w})$. Again, there exists $(\mathbf{p}, \mathbf{p}')$ in $E^{b,J}$ such that $\mathbf{p} \in V(A^b_{n,k}) - \{\mathbf{v}, \mathbf{y}\}$ and $\mathbf{p}' \in V(A^J_{n,k}) - \{\mathbf{z}'\}$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{x}$ to $\mathbf{w}$ with $l(Q_1) = r(l_2)$, (2) Q_2 is a path joining $\mathbf{u}$ to $\mathbf{z}$ with $l(Q_2) = \frac{(n-1)!}{(n-k)!} - r(l_2) - 2$, and (3) $Q_1 \cup Q_2$ spans $A^a_{n,k}$. By Lemma 8, there exists a Hamiltonian path R joining $\mathbf{p}$ to $\mathbf{v}$ in $A^b_{n,k} - \{\mathbf{y}\}$. By Lemma 9, there exists a Hamiltonian path H joining $\mathbf{w}'$

to $\mathbf{y}'$ in $A_{n,k}^{I\cup\{e\}}$, and there exists a Hamiltonian path H' joining $\mathbf{z}'$ to $\mathbf{p}'$ in $A_{n,k}^{J}$. We set $P_1 = \langle \mathbf{u}, Q_2, \mathbf{z}, \mathbf{z}', H', \mathbf{p}', \mathbf{p}, R, \mathbf{v}\rangle$ and set $P_2 = \langle \mathbf{x}, Q_1, \mathbf{w}, \mathbf{w}', H, \mathbf{y}', \mathbf{y}\rangle$. Obviously, P_1 and P_2 form the required paths.

Subcase 2.4 $q(l_2) > 0$ with $r(l_2) \geq \frac{(n-1)!}{2(n-k)!} + D(A_{n,k}) + 1$, or $q(l_2) > 1$ with $r(l_2) < D(A_{n,k})$. Let $e \in \langle n\rangle - \{a, b\}$. There exists $(\mathbf{r}, \mathbf{r}')$ in $E^{a,e}$ such that $\mathbf{r} \in V(A_{n,k}^a) - \{\mathbf{u}, \mathbf{x}\}$ and $\mathbf{r}' \in V(A_{n,k}^e)$. Again, there exists $(\mathbf{s}, \mathbf{s}')$ in $E^{b,e}$ such that $\mathbf{s} \in V(A_{n,k}^b) - \{\mathbf{v}, \mathbf{y}\}$ and $\mathbf{s}' \in V(A_{n,k}^e) - \{\mathbf{r}'\}$. Let $I \subset \langle n\rangle - \{a, b, e\}$ with $|I| = q(l_2) - 1$ if $r(l_2) \geq \frac{(n-1)!}{2(n-k)!} + D(A_{n,k}) + 1$ and $|I| = q(l_2) - 2$ if $r(l_2) < D(A_{n,k})$. Let $J = \langle n\rangle - (I \cup \{a, d, e\})$. Obviously, there exists $(\mathbf{z}, \mathbf{z}')$ in $E^{a,J}$ such that $\mathbf{z} \in V(A_{n,k}^a) - \{\mathbf{r}, \mathbf{u}, \mathbf{x}\}$ and $\mathbf{z}' \in V(A_{n,k}^J)$ with $\{\mathbf{u}, \mathbf{z}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{x}) \cap Nbd_{A_{n,k}}(\mathbf{r})$. Again, there exists $(\mathbf{p}, \mathbf{p}')$ in $E^{b,J}$ such that $\mathbf{p} \in V(A_{n,k}^b) - \{\mathbf{s}, \mathbf{v}, \mathbf{y}\}$ and $\mathbf{p}' \in V(A_{n,k}^J) - \{\mathbf{z}'\}$ with $\{\mathbf{p}, \mathbf{v}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{s}) \cap Nbd_{A_{n,k}}(\mathbf{y})$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{x}$ to $\mathbf{r}$ with $l(Q_1) = \frac{(n-1)!}{2(n-k)!} - 1$, (2) Q_2 is a path joining $\mathbf{u}$ to $\mathbf{z}$ with $l(Q_2) = \frac{(n-1)!}{2(n-k)!} - 1$, and (3) $Q_1 \cup Q_2$ spans $A_{n,k}^a$. Again, by induction, there exist two paths R and R' such that (1) R is a path joining $\mathbf{s}$ to $\mathbf{y}$ with $l(R) = r(l_2) - \frac{(n-1)!}{2(n-k)!} - 1$ if $r(l_2) \geq \frac{(n-1)!}{2(n-k)!} + D(A_{n,k}) + 1$, and $l(R) = r(l_2) + \frac{(n-1)!}{2(n-k)!} - 1$ if $r(l_2) < D(A_{n,k})$, (2) R' is a path joining $\mathbf{p}$ to $\mathbf{v}$ with $l(R') = \frac{3(n-1)!}{2(n-k)!} - r(l_2) - 1$ if $r(l_2) \geq \frac{(n-1)!}{2(n-k)!} + D(A_{n,k}) + 1$, and $l(R') = \frac{(n-1)!}{2(n-k)!} - r(l_2) - 1$ if $r(l_2) < D(A_{n,k})$, and (3) $R \cup R'$ spans $A_{n,k}^b$. By Lemma 9, there exists a Hamiltonian path H joining $\mathbf{r}'$ to $\mathbf{s}'$ in $A_{n,k}^{I\cup\{e\}}$, and there exists a Hamiltonian path H' joining $\mathbf{z}'$ to $\mathbf{p}'$ in $A_{n,k}^J$. We set $P_1 = \langle \mathbf{u}, Q_2, \mathbf{z}, \mathbf{z}', H', \mathbf{p}', \mathbf{p}, R', \mathbf{v}\rangle$ and set P_2 as $\langle \mathbf{x}, Q_1, \mathbf{r}, \mathbf{r}', H, \mathbf{s}', \mathbf{s}, R, \mathbf{y}\rangle$. Obviously, P_1 and P_2 form the required paths.

Case 3 $a = c$, $a \neq d$, and $b \neq d$.

Subcase 3.1 $q(l_2) = 0$ and $r(l_2) \leq \frac{(n-1)!}{2(n-k)!} + D(A_{n,k})$. By Lemma 4, there exists a shortest path connecting $\mathbf{x}$ and $\mathbf{y}$ with the form $\langle \mathbf{x}, M_1, \mathbf{y}', \mathbf{y}'', M_2, \mathbf{y}\rangle$ such that (1) M_1 is a path in $A_{n,k}^a$, (2) M_2 is a path in $A_{n,k}^d$, and (3) $l(M_2) \leq 1$ with $\mathbf{y}' \in V(A_{n,k}^a)$ and $\mathbf{y}'' \in V(A_{n,k}^d)$. Let $I = \langle n\rangle - \{a, d\}$. Obviously, there exists an edge $(\mathbf{z}, \mathbf{z}') \in E^{a,I}$ such that $\mathbf{z} \in V(A_{n,k}^a) - \{\mathbf{x}, \mathbf{y}', \mathbf{u}\}$ and $\mathbf{z}' \in V(A_{n,k}^I) - \{\mathbf{v}\}$ with $\{\mathbf{u}, \mathbf{z}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{x}) \cap Nbd_{A_{n,k}}(\mathbf{y}')$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{x}$ to $\mathbf{y}'$ with $l(Q_1) = r(l_2) - 1$ if $l(M_2) = 0$, and $l(Q_1) = r(l_2) - 2$ if $l(M_2) = 1$, (2) Q_2 is a path joining $\mathbf{u}$ to $\mathbf{z}$ with $l(Q_2) = \frac{(n-1)!}{(n-k)!} - r(l_2) - 1$ if $l(M_2) = 0$, and $l(Q_2) = \frac{(n-1)!}{(n-k)!} - r(l_2)$ if $l(M_2) = 1$, and (3) $Q_1 \cup Q_2$ spans $A_{n,k}^a$. Again, there exist $(\mathbf{p}, \mathbf{p}')$ and $(\mathbf{q}, \mathbf{q}')$ in $E^{d,I}$ such that $\{\mathbf{p}, \mathbf{q}\} \subset V(A_{n,k}^d) - \{\mathbf{y}, \mathbf{y}''\}$ and $\{\mathbf{p}', \mathbf{q}'\} \subset V(A_{n,k}^I) - \{\mathbf{v}, \mathbf{z}'\}$. By Lemma 8, there exists a Hamiltonian path H joining $\mathbf{p}$ to $\mathbf{q}$ in $A_{n,k}^d - \{\mathbf{y}, \mathbf{y}''\}$. By Lemma 7, there exist two disjoint paths S and T such that (1) S is a path joining $\mathbf{z}'$ to $\mathbf{p}'$, (2) T is a path joining $\mathbf{q}'$ to $\mathbf{v}$, and (3) $S \cup T$ spans $A_{n,k}^I$. We set $P_1 = \langle \mathbf{u}, Q_2, \mathbf{z}, \mathbf{z}', S, \mathbf{p}', \mathbf{p}, H, \mathbf{q}, \mathbf{q}', T, \mathbf{v}\rangle$ and set $P_2 = \langle \mathbf{x}, Q_1, \mathbf{y}', \mathbf{y}'', M_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 form the required paths.

Subcase 3.2 $q(l_2) = 0$ with $r(l_2) \geq \frac{(n-1)!}{2(n-k)!} + D(A_{n,k}) + 1$, or $q(l_2) = 1$ with $r(l_2) < D(A_{n,k})$. Obviously, there exists $(\mathbf{w}, \mathbf{w}')$ in $E^{a,d}$ such that $\mathbf{w} \in V(A_{n,k}^a) -$

$\{\mathbf{u}, \mathbf{x}\}$ and $\mathbf{w}' \in V(A^d_{n,k}) - \{\mathbf{y}\}$. Let $I = \langle n \rangle - \{a, d\}$. There exists $(\mathbf{z}, \mathbf{z}')$ in $E^{a,I}$ such that $\mathbf{z} \in V(A^a_{n,k}) - \{\mathbf{u}, \mathbf{w}, \mathbf{x}\}$ and $\mathbf{z}' \in V(A^I_{n,k}) - \{\mathbf{v}\}$ with $\{\mathbf{u}, \mathbf{z}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{x}) \cap Nbd_{A_{n,k}}(\mathbf{w})$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{x}$ to $\mathbf{w}$ with $l(Q_1) = \frac{(n-1)!}{2(n-k)!} - 1$, (2) Q_2 is a path joining $\mathbf{u}$ to $\mathbf{z}$ with $l(Q_2) = \frac{(n-1)!}{2(n-k)!} - 1$, and (3) $Q_1 \cup Q_2$ spans $A^a_{n,k}$. Again, there exist $(\mathbf{p}, \mathbf{p}')$ and $(\mathbf{q}, \mathbf{q}')$ in $E^{d,I}$ such that $\{\mathbf{p}, \mathbf{q}\} \subset V(A^d_{n,k}) - \{\mathbf{w}', \mathbf{y}\}$ and $\{\mathbf{p}', \mathbf{q}'\} \subset V(A^I_{n,k}) - \{\mathbf{v}, \mathbf{z}'\}$ with $\{\mathbf{p}, \mathbf{q}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{w}') \cap Nbd_{A_{n,k}}(\mathbf{y})$. By induction, there exist two paths R and R' such that (1) R is a path joining $\mathbf{w}'$ to $\mathbf{y}$ with $l(R) = r(l_2) - \frac{(n-1)!}{2(n-k)!} - 1$ if $q(l_2) = 0$, and $l(R) = r(l_2) + \frac{(n-1)!}{2(n-k)!} - 1$ if $q(l_2) = 1$, (2) R' is a path joining $\mathbf{p}$ to $\mathbf{q}$ with $l(R') = \frac{3(n-1)!}{2(n-k)!} - r(l_2) - 1$ if $q(l_2) = 0$, and $l(R') = \frac{(n-1)!}{2(n-k)!} - r(l_2) - 1$ if $q(l_2) = 1$, and (3) $R \cup R'$ spans $A^d_{n,k}$. By Lemma 7, there exist two disjoint paths S and T such that (1) S is a path joining $\mathbf{z}'$ to $\mathbf{p}'$, (2) T is a path joining $\mathbf{q}'$ to $\mathbf{v}$, and (3) $S \cup T$ spans $A^I_{n,k}$. We set $P_1 = \langle \mathbf{u}, Q_2, \mathbf{z}, \mathbf{z}', S, \mathbf{p}', \mathbf{p}, R', \mathbf{q}, \mathbf{q}', T, \mathbf{v} \rangle$ and set $P_2 = \langle \mathbf{x}, Q_1, \mathbf{w}, \mathbf{w}', R, \mathbf{y} \rangle$. Obviously, P_1 and P_2 form the required paths.

Subcase 3.3 $q(l_2) > 0$ and $D(A_{n,k}) \le r(l_2) \le \frac{(n-1)!}{2(n-k)!} + D(A_{n,k})$. Obviously, there exists $(\mathbf{w}, \mathbf{w}')$ in $E^{a,d}$ such that $\mathbf{w} \in V(A^a_{n,k}) - \{\mathbf{u}, \mathbf{x}\}$ and $\mathbf{w}' \in V(A^d_{n,k}) - \{\mathbf{y}\}$. Let $I \subset \langle n \rangle - \{a, b, d\}$ with $|I| = q(l_2) - 1$, and let $J = \langle n \rangle - (I \cup \{a, d\})$. There exists $(\mathbf{z}, \mathbf{z}')$ in $E^{a,J}$ such that $\mathbf{z} \in V(A^a_{n,k}) - \{\mathbf{u}, \mathbf{w}, \mathbf{x}\}$ and $\mathbf{z}' \in V(A^J_{n,k}) - \{\mathbf{v}\}$ with $\{\mathbf{u}, \mathbf{z}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{x}) \cap Nbd_{A_{n,k}}(\mathbf{w})$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{x}$ to $\mathbf{w}$ with $l(Q_1) = r(l_2)$, (2) Q_2 is a path joining $\mathbf{u}$ to $\mathbf{z}$ with $l(Q_2) = \frac{(n-1)!}{(n-k)!} - r(l_2) - 2$, and (3) $Q_1 \cup Q_2$ spans $A^a_{n,k}$. By Lemma 9, there exists a Hamiltonian path H joining $\mathbf{w}'$ to $\mathbf{y}$ in $A^{I \cup \{d\}}_{n,k}$, and there exists a Hamiltonian path H' joining $\mathbf{z}'$ to $\mathbf{v}$ in $A^J_{n,k}$. We set $P_1 = \langle \mathbf{u}, Q_2, \mathbf{z}, \mathbf{z}', H', \mathbf{v} \rangle$ and set $P_2 = \langle \mathbf{x}, Q_1, \mathbf{w}, \mathbf{w}', H, \mathbf{y} \rangle$. Obviously, P_1 and P_2 form the required paths.

Subcase 3.4 $q(l_2) > 0$ with $r(l_2) \ge \frac{(n-1)!}{2(n-k)!} + D(A_{n,k}) + 1$, or $q(l_2) > 1$ with $r(l_2) < D(A_{n,k})$. Let $I \subset \langle n \rangle - \{a, b, d\}$ with $|I| = q(l_2)$ if $r(l_2) \ge \frac{(n-1)!}{2(n-k)!} + D(A_{n,k}) + 1$ and $|I| = q(l_2) - 1$ if $r(l_2) < D(A_{n,k})$. Let $J = \langle n \rangle - (I \cup \{a, d\})$. Obviously, there exists $(\mathbf{r}, \mathbf{r}')$ in $E^{a,I}$ such that $\mathbf{r} \in V(A^a_{n,k}) - \{\mathbf{u}, \mathbf{x}\}$ and $\mathbf{r}' \in V(A^I_{n,k})$. Again, there exists $(\mathbf{z}, \mathbf{z}')$ in $E^{a,J}$ such that $\mathbf{z} \in V(A^a_{n,k}) - \{\mathbf{u}, \mathbf{r}, \mathbf{x}\}$ and $\mathbf{z}' \in V(A^J_{n,k}) - \{\mathbf{v}\}$ with $\{\mathbf{u}, \mathbf{z}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{x}) \cap Nbd_{A_{n,k}}(\mathbf{r})$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{x}$ to $\mathbf{r}$ with $l(Q_1) = \frac{(n-1)!}{2(n-k)!} - 1$, (2) Q_2 is a path joining $\mathbf{u}$ to $\mathbf{z}$ with $l(Q_2) = \frac{(n-1)!}{2(n-k)!} - 1$, and (3) $Q_1 \cup Q_2$ spans $A^a_{n,k}$. Obviously, there exists $(\mathbf{s}, \mathbf{s}')$ in $E^{d,I}$ such that $\mathbf{s} \in V(A^d_{n,k}) - \{\mathbf{y}\}$ and $\mathbf{s}' \in V(A^I_{n,k}) - \{\mathbf{r}'\}$. Again, there exist $(\mathbf{p}, \mathbf{p}')$ and $(\mathbf{q}, \mathbf{q}')$ in $E^{d,J}$ such that $\{\mathbf{p}, \mathbf{q}\} \subset V(A^d_{n,k}) - \{\mathbf{s}, \mathbf{y}\}$ and $\{\mathbf{p}', \mathbf{q}'\} \subset V(A^J_{n,k}) - \{\mathbf{v}, \mathbf{z}'\}$ with $\{\mathbf{p}, \mathbf{q}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{s}) \cap Nbd_{A_{n,k}}(\mathbf{y})$. By induction, there exist two paths R and R' such that (1) R is a path joining $\mathbf{s}$ to $\mathbf{y}$ with $l(R) = r(l_2) - \frac{(n-1)!}{2(n-k)!} - 1$ if $r(l_2) \ge \frac{(n-1)!}{2(n-k)!} + D(A_{n,k}) + 1$, and $l(R) = r(l_2) + \frac{(n-1)!}{2(n-k)!} - 1$ if $r(l_2) < D(A_{n,k})$, (2) R' is a path joining $\mathbf{p}$ to $\mathbf{q}$ with $l(R') = \frac{3(n-1)!}{2(n-k)!} - r(l_2) - 1$ if $r(l_2) \ge$

$\frac{(n-1)!}{2(n-k)!} + D(A_{n,k}) + 1$, and $l(R') = \frac{(n-1)!}{2(n-k)!} - r(l_2) - 1$ if $r(l_2) < D(A_{n,k})$, and (3) $R \cup R'$ spans $A^d_{n,k}$. By Lemma 9, there exists a Hamiltonian path H joining $\mathbf{r}'$ to $\mathbf{s}'$ in $A^I_{n,k}$. Again, by Lemma 7, there exist two disjoint paths S and T such that (1) S is a path joining $\mathbf{z}'$ to $\mathbf{p}'$, (2) T is a path joining $\mathbf{q}'$ to $\mathbf{v}$, and (3) $S \cup T$ spans $A^J_{n,k}$. We set $P_1 = \langle \mathbf{u}, Q_2, \mathbf{z}, \mathbf{z}', S, \mathbf{p}', \mathbf{p}, R', \mathbf{q}, \mathbf{q}', T, \mathbf{v}\rangle$ and set P_2 as $\langle \mathbf{x}, Q_1, \mathbf{r}, \mathbf{r}', H, \mathbf{s}', \mathbf{s}, Q_1, \mathbf{y}\rangle$. Obviously, P_1 and P_2 form the required paths.

Case 4 $a \neq c$, $b \neq c$, and $c = d$.

Subcase 4.1 $q(l_2) = 0$ and $r(l_2) \le \frac{(n-1)!}{2(n-k)!}$. Obviously, there exist two vertices $\{\mathbf{w}, \mathbf{z}\} \subset V(A^c_{n,k}) - \{\mathbf{x}, \mathbf{y}\}$ such that $\{\mathbf{w}, \mathbf{z}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{x}) \cap Nbd_{A_{n,k}}(\mathbf{y})$ and $\{\mathbf{w}, \mathbf{z}\} \not\subseteq N^c(\mathbf{u}) \cup N^c(\mathbf{v})$. Let $I = \langle n\rangle - \{c\}$. There exist two edges $(\mathbf{w}, \mathbf{w}')$ and $(\mathbf{z}, \mathbf{z}')$ in $E^{c,I}$ such that $\{\mathbf{w}', \mathbf{z}'\} \subset V(A^I_{n,k})$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{w}$ to $\mathbf{z}$ with $l(Q_1) = \frac{(n-1)!}{(n-k)!} - r(l_2) - 2$, (2) Q_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(Q_2) = r(l_2)$, and (3) $Q_1 \cup Q_2$ spans $A^c_{n,k}$. By Lemma 7, there exist two disjoint paths S and T such that (1) S is a path joining $\mathbf{u}$ to $\mathbf{w}'$, (2) T is a path joining $\mathbf{z}'$ to $\mathbf{v}$, and (3) $S \cup T$ spans $A^I_{n,k}$. We set $P_1 = \langle \mathbf{u}, S, \mathbf{w}', \mathbf{w}, Q_1, \mathbf{z}, \mathbf{z}', T, \mathbf{v}\rangle$ and set P_2 as Q_2. Obviously, P_1 and P_2 form the required paths.

Subcase 4.2 $q(l_2) = 0$ with $r(l_2) > \frac{(n-1)!}{2(n-k)!}$, or $q(l_2) = 1$ with $r(l_2) < \max\{3, d_{A_{n,k}}(\mathbf{x}, \mathbf{y})\}$. There exist two vertices $\mathbf{w}$ and $\mathbf{z}$ in $A^c_{n,k} - \{\mathbf{x}, \mathbf{y}\}$ such that $\{a, b\} \not\subseteq AS(\mathbf{w})$, $\{a, b\} \subseteq AS(\mathbf{z})$, and $\mathbf{z} \notin N^c(\mathbf{u}) \cup N^c(\mathbf{v})$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{w}$ to $\mathbf{z}$ with $l(Q_1) = \frac{(n-1)!}{2(n-k)!} - 1$, (2) Q_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(Q_2) = \frac{(n-1)!}{2(n-k)!} - 1$, and (3) $Q_1 \cup Q_2$ spans $A^c_{n,k}$. Since $l(Q_2) = \frac{(n-1)!}{2(n-k)!} - 1$, we can write Q_2 as $\langle \mathbf{x}, Q^1_2, \mathbf{r}, \mathbf{s}, Q^2_2, \mathbf{y}\rangle$ for some vertices $\mathbf{r}$ and $\mathbf{s}$ such that $\{\mathbf{u}, \mathbf{v}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{r}) \cup Nbd_{A_{n,k}}(\mathbf{s})$. By Lemma 5, $|AS(\mathbf{r}) \cap AS(\mathbf{s})| = n - k - 1$. Let $e \in AS(\mathbf{r}) \cap AS(\mathbf{s})$ for some $e \in \langle n\rangle - \{c\}$. Let $\mathbf{r}'$ be the vertex adjacent to $\mathbf{r}$ in $A^e_{n,k}$ and $\mathbf{s}'$ be the vertex adjacent to $\mathbf{s}$ in $A^e_{n,k}$. By Lemma 2, $d_{A_{n,k}}(\mathbf{r}', \mathbf{s}') = 1$. Then consider the following subcases.

Subcase 4.2.1 $e \in \{a, b\}$. Without loss of generality, we assume that $e = a$. Obviously, there exists a vertex $\mathbf{t} \in V(A^a_{n,k}) - \{\mathbf{u}, \mathbf{r}', \mathbf{s}'\}$ such that $\{\mathbf{r}', \mathbf{s}'\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{u}) \cap Nbd_{A_{n,k}}(\mathbf{t})$, $\mathbf{v} \notin N^b(\mathbf{t})$, $d_{A_{n,k}}(\mathbf{w}, \mathbf{t}) > 2$, and $d_{A_{n,k}}(\mathbf{z}, \mathbf{t}) > 2$. By induction, there exist two paths R and R' such that (1) R is a path joining $\mathbf{u}$ to $\mathbf{t}$ with $l(R) = \frac{3(n-1)!}{2(n-k)!} - r(l_2) - 2$ if $q(l_2) = 0$, and $l(R) = \frac{(n-1)!}{2(n-k)!} - r(l_2) - 2$ if $q(l_2) = 1$, (2) R' is a path joining $\mathbf{r}'$ to $\mathbf{s}'$ with $l(R') = r(l_2) - \frac{(n-1)!}{2(n-k)!}$ if $q(l_2) = 0$, and $l(R') = \frac{(n-1)!}{2(n-k)!} + r(l_2)$ if $q(l_2) = 1$, and (3) $R \cup R'$ spans $A^a_{n,k}$. Let $I = \langle n\rangle - \{a, c\}$. We have $\{\mathbf{w}, \mathbf{z}\} \not\subseteq N^c(\mathbf{v})$, $\mathbf{v} \notin N^b(\mathbf{t})$, $d_{A_{n,k}}(\mathbf{w}, \mathbf{t}) > 2$ and $d_{A_{n,k}}(\mathbf{z}, \mathbf{t}) > 2$. Hence there exist $(\mathbf{w}, \mathbf{w}')$, $(\mathbf{z}, \mathbf{z}')$ in $E^{c,I}$, and $(\mathbf{t}, \mathbf{t}')$ in $E^{a,I}$ such that $\{\mathbf{w}', \mathbf{z}', \mathbf{t}'\} \subset V(A^I_{n,k}) - \{\mathbf{v}\}$. By Lemma 7, there exist two disjoint paths S and T such that (1) S is a path joining $\mathbf{t}'$ to $\mathbf{w}'$, (2) T is a path joining $\mathbf{z}'$ to $\mathbf{v}$, and (3) $S \cup T$ spans $A^I_{n,k}$. We set $P_1 = \langle \mathbf{u}, R, \mathbf{t}, \mathbf{t}', S, \mathbf{w}', \mathbf{w}, Q_1, \mathbf{z}, \mathbf{z}', T, \mathbf{v}\rangle$ and set $P_2 = \langle \mathbf{x}, Q^1_2, \mathbf{r}, \mathbf{r}', R', \mathbf{s}', \mathbf{s}, Q^2_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 form the required paths.

Subcase 4.2.2 $e \notin \{a, b\}$. Since $\{a, b\} \not\subseteq AS(\mathbf{w})$, there exists a vertex $\mathbf{w}' \in V(A^f_{n,k})$ adjacent to $\mathbf{w}$ for some $f \in \langle n\rangle - \{a, b, c, e\}$. Since $b \in AS(\mathbf{z})$ and $\mathbf{z} \notin N^c(\mathbf{v})$, there exists $\mathbf{z}'$ in $V(A^b_{n,k}) - \{\mathbf{v}\}$ adjacent to $\mathbf{z}$. By Lemma 9, there exists a Hamiltonian path H joining $\mathbf{z}'$ to $\mathbf{v}$ in $A^b_{n,k}$. Obviously, there exist $\mathbf{p}$ and $\mathbf{q}$ in $V(A^e_{n,k}) - \{\mathbf{r}', \mathbf{s}'\}$ such that $b \notin AS(\mathbf{p}) \cup AS(\mathbf{q})$, $\{\mathbf{u}, \mathbf{w}'\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{p}) \cup Nbd_{A_{n,k}}(\mathbf{q})$, and $\{\mathbf{p}, \mathbf{q}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{r}') \cap Nbd_{A_{n,k}}(\mathbf{s}')$. Let $I = \langle n\rangle - \{b, c, e\}$. Thus there exist $(\mathbf{p}, \mathbf{p}')$ and $(\mathbf{q}, \mathbf{q}')$ in $E^{e,I}$ such that $\{\mathbf{p}', \mathbf{q}'\} \subset V(A^I_{n,k}) - \{\mathbf{u}, \mathbf{w}'\}$. By induction, there exist two paths R and R' such that (1) R is a path joining $\mathbf{p}$ to $\mathbf{q}$ with $l(R) = \frac{3(n-1)!}{2(n-k)!} - r(l_2) - 2$ if $q(l_2) = 0$, and $l(R) = \frac{(n-1)!}{2(n-k)!} - r(l_2) - 2$ if $q(l_2) = 1$, (2) R' is a path joining $\mathbf{r}'$ to $\mathbf{s}'$ with $l(R') = r(l_2) - \frac{(n-1)!}{2(n-k)!}$ if $q(l_2) = 0$, and $l(R') = \frac{(n-1)!}{2(n-k)!} + r(l_2)$ if $q(l_2) = 1$, and (3) $R \cup R'$ spans $A^e_{n,k}$. By Lemma 7, there exist two disjoint paths S and T such that (1) S is a path joining $\mathbf{u}$ to $\mathbf{p}'$, (2) T is a path joining $\mathbf{q}'$ to $\mathbf{w}'$, and (3) $S \cup T$ spans $A^I_{n,k}$. We set $P_1 = \langle \mathbf{u}, S, \mathbf{p}', \mathbf{p}, R, \mathbf{q}, \mathbf{q}', T, \mathbf{w}', \mathbf{w}, Q_1, \mathbf{z}, \mathbf{z}', H, \mathbf{v}\rangle$ and set P_2 as $\langle \mathbf{x}, Q^1_2, \mathbf{r}, \mathbf{r}', R', \mathbf{s}', \mathbf{s}, Q^2_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 form the required paths.

Subcase 4.3 $q(l_2) > 0$ and $\max\{3, d_{A_{n,k}}(\mathbf{x}, \mathbf{y})\} \le r(l_2) \le \frac{(n-1)!}{2(n-k)!}$. Obviously, there exist two adjacent vertices $\mathbf{r}$ and $\mathbf{s}$ in $V(A^c_{n,k}) - \{\mathbf{x}, \mathbf{y}\}$ such that $e \in AS(\mathbf{r}) \cap AS(\mathbf{s})$ for some $e \in \langle n\rangle - \{a, b, c\}$. Let $\mathbf{r}'$ be the vertex adjacent to $\mathbf{r}$ in $A^e_{n,k}$ and $\mathbf{s}'$ be the vertex adjacent to $\mathbf{s}$ in $A^e_{n,k}$. By Lemma 2, $d_{A_{n,k}}(\mathbf{r}', \mathbf{s}') = 1$. Again, there exist two vertices $\{\mathbf{p}, \mathbf{q}\} \subset V(A^e_{n,k}) - \{\mathbf{r}', \mathbf{s}'\}$ such that $\{\mathbf{p}, \mathbf{q}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{r}') \cap Nbd_{A_{n,k}}(\mathbf{s}')$ and $\{\mathbf{p}, \mathbf{q}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{u}) \cup Nbd_{A_{n,k}}(\mathbf{v})$. Let $I \subseteq \langle n\rangle - \{a, b, c, e\}$ with $|I| = q(l_2) - 1$ and $J = \langle n\rangle - (I \cup \{c, e\})$. Thus there exist two edges $(\mathbf{p}, \mathbf{p}')$ and $(\mathbf{q}, \mathbf{q}')$ in $E^{e,J}$ such that $\{\mathbf{p}', \mathbf{q}'\} \subset V(A^J_{n,k}) - \{\mathbf{u}, \mathbf{v}\}$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{p}$ to $\mathbf{q}$ with $l(Q_1) = \frac{(n-1)!}{(n-k)!} - r(l_2) - 2$, (2) Q_2 is a path joining $\mathbf{r}'$ to $\mathbf{s}'$ with $l(Q_2) = r(l_2)$, and (3) $Q_1 \cup Q_2$ spans $A^e_{n,k}$. By Lemma 7, there exist two disjoint paths H_1 and H_2 such that (1) H_1 is a path joining $\mathbf{x}$ to $\mathbf{r}$, (2) H_2 is a path joining $\mathbf{s}$ to $\mathbf{y}$, and (3) $H_1 \cup H_2$ spans $A^{I\cup\{c\}}_{n,k}$. Again, by Lemma 7, there exist two disjoint paths S and T such that (1) S is a path joining $\mathbf{u}$ to $\mathbf{p}'$, (2) T is a path joining $\mathbf{q}'$ to $\mathbf{v}$, and (3) $S \cup T$ spans $A^J_{n,k}$. We set $P_1 = \langle \mathbf{u}, S, \mathbf{p}', \mathbf{p}, Q_1, \mathbf{q}, \mathbf{q}', T, \mathbf{v}\rangle$ and set $P_2 = \langle \mathbf{x}, H_1, \mathbf{r}, \mathbf{r}', Q_2, \mathbf{s}', \mathbf{s}, H_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 form the required paths.

Subcase 4.4 $q(l_2) > 0$ with $r(l_2) > \frac{(n-1)!}{2(n-k)!}$, or $q(l_2) > 1$ with $r(l_2) < \max\{3, d_{A_{n,k}}(\mathbf{x}, \mathbf{y})\}$. Let e be an element in $\langle n\rangle - \{a, c\}$. Obviously, there exists an edge $(\mathbf{r}, \mathbf{w}) \in E^{c,e}$ such that $\mathbf{r} \in V(A^e_{n,k})$ and $\mathbf{w} \in V(A^c_{n,k}) - \{\mathbf{x}, \mathbf{y}\}$. Obviously, there exists a vertex $\mathbf{s} \in V(A^e_{n,k})$ adjacent to $\mathbf{r}$ such that $f \in AS(\mathbf{s})$ for some $f \in \langle n\rangle - \{a, b, c, e\}$. Thus there exists an edge $(\mathbf{s}, \mathbf{s}') \in E^{e,f}$ such that $\mathbf{s} \in V(A^e_{n,k}) - \{\mathbf{r}\}$ and $\mathbf{s}' \in V(A^f_{n,k})$. Obviously, there exists an edge $(\mathbf{z}, \mathbf{z}') \in E^{c,f}$ such that $\mathbf{z} \in V(A^f_{n,k}) - \{\mathbf{s}'\}$ and $\mathbf{z}' \in V(A^c_{n,k}) - \{\mathbf{w}, \mathbf{x}, \mathbf{y}\}$. Let $I \subseteq \langle n\rangle - \{a, b, c, e, f\}$ with $|I| = q(l_2) - 1$ and $J = \langle n\rangle - (I \cup \{a, c, e, f\})$. Again, there exist two vertices $\{\mathbf{p}, \mathbf{q}\} \subset V(A^e_{n,k}) - \{\mathbf{r}, \mathbf{s}\}$ such that $\{\mathbf{p}, \mathbf{q}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{r}) \cap Nbd_{A_{n,k}}(\mathbf{s})$, $\mathbf{v} \notin Nbd_{A_{n,k}}(\mathbf{p}) \cup Nbd_{A_{n,k}}(\mathbf{q})$, $J \cap AS(\mathbf{p}) \neq \emptyset$, and $J \cap AS(\mathbf{q}) \neq \emptyset$. Obviously, there exist two edges $\{(\mathbf{p}, \mathbf{p}'), (\mathbf{q}, \mathbf{q}')\} \subseteq E^{e,J}$ such that $\{\mathbf{p}', \mathbf{q}'\} \subset V(A^J_{n,k}) - \{\mathbf{v}\}$. By induction, there exist two paths R and R' such that (1) R is a path joining $\mathbf{p}$ to $\mathbf{q}$ with $l(R) = \frac{3(n-1)!}{2(n-k)!} - r(l_2) - 2$ if $r(l_2) > \frac{(n-1)!}{2(n-k)!}$, and $l(R) = \frac{(n-1)!}{2(n-k)!} - r(l_2) - 2$ if $r(l_2) < \max\{3, d_{A_{n,k}}(\mathbf{x},\mathbf{y})\}$, (2) R' is a path joining $\mathbf{r}$ to $\mathbf{s}$ with $l(R') = r(l_2) - \frac{(n-1)!}{2(n-k)!}$ if $r(l_2) > \frac{(n-1)!}{2(n-k)!}$, and $l(R') = \frac{(n-1)!}{2(n-k)!} + r(l_2)$ if $r(l_2) < \max\{3, d_{A_{n,k}}(\mathbf{x},\mathbf{y})\}$, and (3) $R \cup R'$ spans $A^e_{n,k}$. Again, there exists an edge $(\mathbf{u}', \mathbf{v}') \in E^{a,f}$ such that $\mathbf{u}' \in V(A^a_{n,k}) - \{\mathbf{u}\}$ and $\mathbf{v}' \in V(A^f_{n,k}) - \{\mathbf{s}', \mathbf{z}\}$. Obviously, there exists a vertex $\mathbf{t} \in V(A^f_{n,k}) - \{\mathbf{s}', \mathbf{v}', \mathbf{z}\}$ such that $\{\mathbf{t}, \mathbf{v}'\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{s}') \cap Nbd_{A_{n,k}}(\mathbf{z})$, $\mathbf{v} \notin Nbd_{A_{n,k}}(\mathbf{t})$, and $J \cap AS(\mathbf{t}) \neq \emptyset$. Thus there exists an edge $(\mathbf{t}, \mathbf{t}')$ in $E^{f,J}$ such that $\mathbf{t}' \in V(A^J_{n,k}) - \{\mathbf{p}', \mathbf{q}', \mathbf{v}\}$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{s}'$ to $\mathbf{z}$ with $l(Q_1) = \frac{(n-1)!}{2(n-k)!} - 1$, (2) Q_2 is a path joining $\mathbf{v}'$ to $\mathbf{t}$ with $l(Q_2) = \frac{(n-1)!}{2(n-k)!} - 1$, and (3) $Q_1 \cup Q_2$ spans $A^f_{n,k}$. By Lemma 7, there exist two disjoint paths H_1 and H_2 such that (1) H_1 is a path joining $\mathbf{x}$ to $\mathbf{w}$, (2) H_2 is a path joining $\mathbf{z}'$ to $\mathbf{y}$, and (3) $H_1 \cup H_2$ spans $A^I_{n,k}$. By Lemma 9, there exists a Hamiltonian path H_3 joining $\mathbf{u}$ to $\mathbf{u}'$ in $A^a_{n,k}$. Again, by Lemma 7, there exist two disjoint paths S and T such that (1) S is a path joining $\mathbf{t}'$ to $\mathbf{q}'$, (2) T is a path joining $\mathbf{p}'$ to $\mathbf{v}$, and (3) $S \cup T$ spans $A^J_{n,k}$. We set $P_1 = \langle \mathbf{u}, H_3, \mathbf{u}', \mathbf{v}', Q_2, \mathbf{t}, \mathbf{t}', S, \mathbf{q}', \mathbf{q}, R, \mathbf{p}, \mathbf{p}', T, \mathbf{v}\rangle$ and set P_2 as $\langle \mathbf{x}, H_1, \mathbf{w}, \mathbf{r}, R', \mathbf{s}, \mathbf{s}', Q_1, \mathbf{z}, \mathbf{z}', H_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 form the required paths.

Case 5 $a = c = d$.

Subcase 5.1 $q(l_2) = 0$ and $r(l_2) \le \frac{(n-1)!}{2(n-k)!}$. Let e be an element in $\langle n\rangle - \{a, b\}$. Obviously, there exists an edge $(\mathbf{w}, \mathbf{z})$ in $E^{a,e}$ such that $\mathbf{w} \in V(A^a_{n,k}) - \{\mathbf{x}, \mathbf{y}, \mathbf{u}\}$ and $\mathbf{z} \in V(A^e_{n,k})$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{u}$ to $\mathbf{w}$ with $l(Q_1) = \frac{(n-1)!}{(n-k)!} - r(l_2) - 2$, (2) Q_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(Q_2) = r(l_2)$, and (3) $Q_1 \cup Q_2$ spans $A^a_{n,k}$. Let $I = \langle n\rangle - \{a\}$. By Lemma 9, there exists a Hamiltonian path R of $A^I_{n,k}$ joining $\mathbf{z}$ to $\mathbf{v}$. We set $P_1 = \langle \mathbf{u}, Q_1, \mathbf{w}, \mathbf{z}, R, \mathbf{v}\rangle$ and set P_2 as Q_2. Obviously, P_1 and P_2 form the required paths.

Subcase 5.2 $q(l_2) = 0$ with $r(l_2) > \frac{(n-1)!}{2(n-k)!}$, or $q(l_2) = 1$ with $r(l_2) < \max\{3, d_{A_{n,k}}(\mathbf{x},\mathbf{y})\}$. Obviously, there exists a vertex $\mathbf{z}$ in $V(A^a_{n,k}) - \{\mathbf{u}, \mathbf{x}, \mathbf{y}\}$ such that $\mathbf{v} \notin Nbd_{A_{n,k}}(\mathbf{z})$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{u}$ to $\mathbf{z}$ with $l(Q_1) = \frac{(n-1)!}{2(n-k)!} - 1$, (2) Q_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(Q_2) = \frac{(n-1)!}{2(n-k)!} - 1$, and (3) $Q_1 \cup Q_2$ spans $A^a_{n,k}$. Since $l(Q_2) > 2$, we can write Q_2 as $\langle \mathbf{x}, Q_2^1, \mathbf{r}, \mathbf{s}, Q_2^2, \mathbf{y}\rangle$ for some vertices $\mathbf{r}$ and $\mathbf{s}$ such that $\mathbf{v} \notin Nbd_{A_{n,k}}(\mathbf{r}) \cup Nbd_{A_{n,k}}(\mathbf{s})$. By Lemma 5, $|AS(\mathbf{r}) \cap AS(\mathbf{s})| = n - k - 1 \ge 1$. There exists an element $e \in AS(\mathbf{r}) \cap AS(\mathbf{s})$. Obviously, $e \neq a$. Let $\mathbf{r}'$ be the vertex adjacent to $\mathbf{r}$ in $A^e_{n,k}$, and $\mathbf{s}'$ be the vertex adjacent to $\mathbf{s}$ in $A^e_{n,k}$. By Lemma 2, $d_{A_{n,k}}(\mathbf{r}', \mathbf{s}') = 1$. Then consider the following subcases.

Subcase 5.2.1 $e = b$. Since $|N^*(\mathbf{z})| = n - k \ge 2$, there exists a vertex $\mathbf{t} \in N^*(\mathbf{z})$ in $A^f_{n,k}$ for some $f \in \langle n\rangle - \{a, b\}$. Obviously, there exists a vertex $\mathbf{z}' \in V(A^b_{n,k}) - \{\mathbf{s}', \mathbf{r}', \mathbf{v}\}$ such that $\{\mathbf{z}', \mathbf{v}\} \not\supseteq Nbd_{A^b_{n,k}}(\mathbf{r}') \cap Nbd_{A^b_{n,k}}(\mathbf{s}')$ and $f \in$

$AS(\mathbf{z}')$ with $\mathbf{t} \notin N^f(\mathbf{z}')$. Let $\mathbf{t}'$ be the vertex in $A^f_{n,k}$ adjacent to $\mathbf{z}'$. By induction, there exist two paths R and R' such that (1) R is a path joining $\mathbf{z}'$ to $\mathbf{v}$ with $l(R) = \frac{3(n-1)!}{2(n-k)!} - r(l_2) - 2$ if $q(l_2) = 0$, and $l(R) = \frac{(n-1)!}{2(n-k)!} - r(l_2) - 2$ if $q(l_2) = 1$, (2) R' is a path joining $\mathbf{r}'$ to $\mathbf{s}'$ with $l(R') = r(l_2) - \frac{(n-1)!}{2(n-k)!}$ if $q(l_2) = 0$, and $l(R') = \frac{(n-1)!}{2(n-k)!} + r(l_2)$ if $q(l_2) = 1$, and (3) $R \cup R'$ spans $A^b_{n,k}$. By Lemma 9, there exists a Hamiltonian path S of $A^{\langle n\rangle - \{a,b\}}_{n,k}$ joining $\mathbf{t}$ to $\mathbf{t}'$. We set $P_1 = \langle \mathbf{u}, Q_1, \mathbf{z}, \mathbf{t}, S, \mathbf{t}', \mathbf{z}', R, \mathbf{v}\rangle$ and set $P_2 = \langle \mathbf{x}, Q^1_2, \mathbf{r}, \mathbf{r}', R', \mathbf{s}', \mathbf{s}, Q^2_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 form the required paths.

Subcase 5.2.2 $e \neq b$. Since $|N^*(\mathbf{z})| = n - k \geq 2$, there exists a vertex $\mathbf{t} \in N^*(\mathbf{z})$ in $A^f_{n,k}$ for some $f \in \langle n\rangle - \{a, e\}$. By Lemma 6, $|Nbd_{A^e_{n,k}}(\mathbf{r}') \cap Nbd_{A^e_{n,k}}(\mathbf{s}')| = n - k - 1$. Obviously, $|E^{e,g}| - (n - k - 1) = \frac{(n-2)!}{(n-k-1)!} - (n - k - 1) > 3$ for some $g \in \langle n\rangle - \{a, b, e\}$. Thus there exist two vertices $\mathbf{p}$ and $\mathbf{w}$ in $A^e_{n,k} - \{\mathbf{r}', \mathbf{s}'\}$ such that $\{\mathbf{p}, \mathbf{w}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{r}') \cap Nbd_{A_{n,k}}(\mathbf{s}')$ and $g \in AS(\mathbf{p}) \cap AS(\mathbf{w})$. There exist two edges $(\mathbf{p}, \mathbf{p}')$ and $(\mathbf{w}, \mathbf{w}')$ in $E^{e,g}$. By induction, there exist two paths R and R' such that (1) R is a path joining $\mathbf{p}$ to $\mathbf{w}$ with $l(R) = \frac{3(n-1)!}{2(n-k)!} - r(l_2) - 2$ if $q(l_2) = 0$, and $l(R) = \frac{(n-1)!}{2(n-k)!} - r(l_2) - 2$ if $q(l_2) = 1$, (2) R' is a path joining $\mathbf{r}'$ to $\mathbf{s}'$ with $l(R') = r(l_2) - \frac{(n-1)!}{2(n-k)!}$ if $q(l_2) = 0$, and $l(R') = \frac{(n-1)!}{2(n-k)!} + r(l_2)$ if $q(l_2) = 1$, and (3) $R \cup R'$ spans $A^e_{n,k}$. By Lemma 7, there exist two disjoint paths S and T such that (1) S is a path joining $\mathbf{t}$ to $\mathbf{p}'$, (2) T is a path joining $\mathbf{w}'$ to $\mathbf{v}$, and (3) $S \cup T$ spans $A^{\langle n\rangle - \{a,e\}}_{n,k}$. We set $P_1 = \langle \mathbf{u}, Q_1, \mathbf{z}, \mathbf{t}, S, \mathbf{p}', \mathbf{p}, R, \mathbf{w}, \mathbf{w}', T, \mathbf{v}\rangle$ and set P_2 as $\langle \mathbf{x}, Q^1_2, \mathbf{r}, \mathbf{r}', R', \mathbf{s}', \mathbf{s}, Q^2_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 form the required paths.

Subcase 5.3 $q(l_2) = 1$ and $\max\{3, d_{A_{n,k}}(\mathbf{x}, \mathbf{y})\} \leq r(l_2) \leq \frac{(n-1)!}{2(n-k)!}$. Obviously, there exists a vertex $\mathbf{u}' \in Nbd_{A^a_{n,k}}(\mathbf{u})$ in $A^a_{n,k} - \{\mathbf{x}, \mathbf{y}\}$ such that $\mathbf{u}' \notin N^*(\mathbf{v})$. By Lemma 8, there exists a Hamiltonian path H of $A^a_{n,k} - \{\mathbf{u}, \mathbf{u}'\}$ joining $\mathbf{x}$ to $\mathbf{y}$. We can write H as $\langle \mathbf{x}, H^1, \mathbf{r}, \mathbf{s}, H^2, \mathbf{y}\rangle$ for some vertices $\mathbf{r}$ and $\mathbf{s}$ such that $\mathbf{v} \notin Nbd_{A_{n,k}}(\mathbf{r}) \cup Nbd_{A_{n,k}}(\mathbf{s})$. By Lemma 5, $|AS(\mathbf{r}) \cap AS(\mathbf{s})| = n - k - 1 \geq 1$. There exists an element $e \in AS(\mathbf{r}) \cap AS(\mathbf{s})$. Obviously, $e \neq a$. Let $\mathbf{r}'$ be the vertex adjacent to $\mathbf{r}$ in $A^e_{n,k}$ and $\mathbf{s}'$ be the vertex adjacent to $\mathbf{s}$ in $A^e_{n,k}$. By Lemma 2, $d_{A_{n,k}}(\mathbf{r}', \mathbf{s}') = 1$. Then consider the following subcases.

Subcase 5.3.1 $e = b$. Since $|N^*(\mathbf{u}')| = n - k \geq 2$, there exists a vertex $\mathbf{z} \in N^*(\mathbf{u}')$ in $A^f_{n,k}$ for some $f \in \langle n\rangle - \{a, b\}$. Let $\mathbf{v}'$ be a vertex in $A^b_{n,k} - \{\mathbf{r}', \mathbf{s}', \mathbf{v}\}$ such that $\{\mathbf{v}, \mathbf{v}'\} \not\supseteq Nbd_{A^b_{n,k}}(\mathbf{r}') \cap Nbd_{A^b_{n,k}}(\mathbf{s}')$ with $\mathbf{z} \notin N^f(\mathbf{v}')$. Let $\mathbf{t}$ be the vertex in $A^g_{n,k}$ adjacent to $\mathbf{v}'$ for some $g \in \langle n\rangle - \{a, b\}$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{r}'$ to $\mathbf{s}'$ with $l(Q_1) = r(l_2) + 2$, (2) Q_2 is a path joining $\mathbf{v}'$ to $\mathbf{v}$ with $l(Q_2) = \frac{(n-1)!}{(n-k)!} - r(l_2) - 4$, and (3) $Q_1 \cup Q_2$ spans $A^b_{n,k}$. By Lemma 9, there exists a Hamiltonian path R of $A^{\langle n\rangle - \{a,b\}}_{n,k}$ joining $\mathbf{z}$ to $\mathbf{t}$. We set $P_1 = \langle \mathbf{u}, \mathbf{u}', \mathbf{z}, R, \mathbf{t}, \mathbf{v}', Q_2, \mathbf{v}\rangle$ and set $P_2 = \langle \mathbf{x}, H^1, \mathbf{r}, \mathbf{r}', Q_1, \mathbf{s}', \mathbf{s}, H^2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 form the required paths.

Subcase 5.3.2 $e \neq b$. Since $|N^*(\mathbf{u}')| = n - k \geq 2$, there exists a vertex $\mathbf{z} \in N^*(\mathbf{u}')$ in $A^f_{n,k}$ for some $f \in \langle n\rangle - \{a, e\}$. Since $\mathbf{u}' \notin N^*(\mathbf{v})$, $\mathbf{z} \neq \mathbf{v}$. By Lemma 6,

$|Nbd_{A^e_{n,k}}(\mathbf{r}') \cap Nbd_{A^e_{n,k}}(\mathbf{s}')| = n-k-1$. Obviously, $|E^{e,g}| - (n-k-1) = \frac{(n-2)!}{(n-k-1)!} - (n-k-1) > 3$ for some $g \in \langle n\rangle - \{a,b,e,f\}$. Thus there exist two vertices $\mathbf{t}$ and $\mathbf{w}$ in $A^e_{n,k} - \{\mathbf{r}', \mathbf{s}'\}$ such that $\{\mathbf{t}, \mathbf{w}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{r}') \cap Nbd_{A_{n,k}}(\mathbf{s}')$ and $g \in AS(\mathbf{t}) \cap AS(\mathbf{w})$. Let $\mathbf{t}'$ and $\mathbf{w}'$ be the vertices in $A^g_{n,k}$ such that $\{(\mathbf{t}, \mathbf{t}'), (\mathbf{w}, \mathbf{w}')\} \subseteq E^{e,g}$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{r}'$ to $\mathbf{s}'$ with $l(Q_1) = r(l_2) + 2$, (2) Q_2 is a path joining $\mathbf{w}$ to $\mathbf{t}$ with $l(Q_2) = \frac{(n-1)!}{(n-k)!} - r(l_2) - 4$, and (3) $Q_1 \cup Q_2$ spans $A^e_{n,k}$. By Lemma 7, there exist two disjoint paths S and T such that (1) S is a path joining $\mathbf{z}$ to $\mathbf{w}'$, (2) T is a path joining $\mathbf{t}'$ to $\mathbf{v}$, and (3) $S \cup T$ spans $A^{\langle n\rangle - \{a,e\}}_{n,k}$. We set $P_1 = \langle \mathbf{u}, \mathbf{u}', \mathbf{z}, S, \mathbf{w}', \mathbf{w}, Q_2, \mathbf{t}, \mathbf{t}', T, \mathbf{v}\rangle$ and set P_2 as $\langle \mathbf{x}, H^1, \mathbf{r}, \mathbf{r}', Q_1, \mathbf{s}', \mathbf{s}, H^2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 form the required paths.

Subcase 5.4 $q(l_2) > 1$ and $\max\{3, d_{A_{n,k}}(\mathbf{x}, \mathbf{y})\} \le r(l_2) \le \frac{(n-1)!}{2(n-k)!}$. Let $I \subseteq \langle n\rangle - \{a,b\}$ with $|I| = q(l_2)$ and $J = \langle n\rangle - (I \cup \{a\})$. Obviously, there exists an edge $(\mathbf{t}, \mathbf{z}) \in E^{a,J}$ such that $\mathbf{z} \in V(A^a_{n,k}) - \{\mathbf{x}, \mathbf{y}, \mathbf{u}\}$ and $\mathbf{t} \in V(A^J_{n,k}) - \{\mathbf{v}\}$ with $\mathbf{z} \notin Nbd_{A_{n,k}}(\mathbf{x}) \cap Nbd_{A_{n,k}}(\mathbf{y})$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{u}$ to $\mathbf{z}$ with $l(Q_1) = \frac{(n-1)!}{(n-k)!} - r(l_2) - 2$, (2) Q_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(Q_2) = r(l_2)$, and (3) $Q_1 \cup Q_2$ spans $A^a_{n,k}$. We write Q_2 as $\langle \mathbf{x}, Q^1_2, \mathbf{r}, \mathbf{s}, Q^2_2, \mathbf{y}\rangle$ for some vertices $\mathbf{r}$ and $\mathbf{s}$. Since $|AS(\mathbf{r})| = |AS(\mathbf{s})| = n-k \ge 2$, there exists a vertex $\mathbf{r}'$ adjacent to $\mathbf{r}$ in $A^e_{n,k}$ and a vertex $\mathbf{s}'$ adjacent to $\mathbf{s}$ in $A^f_{n,k}$ for some $\{e,f\} \subseteq \langle n\rangle - \{a,b\}$. By Lemma 9, there exists a Hamiltonian path S of $A^I_{n,k}$ joining $\mathbf{r}'$ to $\mathbf{s}'$, and there exists a Hamiltonian path T of $A^J_{n,k}$ joining $\mathbf{t}$ to $\mathbf{v}$. We set $P_1 = \langle \mathbf{u}, Q_1, \mathbf{z}, \mathbf{t}, T, \mathbf{v}\rangle$ and set $P_2 = \langle \mathbf{x}, Q^1_2, \mathbf{r}, \mathbf{r}', S, \mathbf{s}', \mathbf{s}, Q^2_2, \mathbf{y}\rangle$. Obviously, P_1 and P_2 form the required paths.

Subcase 5.5 $q(l_2) > 0$ with $r(l_2) > \frac{(n-1)!}{2(n-k)!}$, or $q(l_2) > 1$ with $r(l_2) < \max\{3, d_{A_{n,k}}(\mathbf{x}, \mathbf{y})\}$. Obviously, there exists an edge $(\mathbf{t}, \mathbf{z}) \in E^{a,b}$ such that $\mathbf{z} \in V(A^a_{n,k}) - \{\mathbf{x}, \mathbf{y}, \mathbf{u}\}$ and $\mathbf{t} \in V(A^b_{n,k}) - \{\mathbf{v}\}$. By induction, there exist two paths Q_1 and Q_2 such that (1) Q_1 is a path joining $\mathbf{u}$ to $\mathbf{z}$ with $l(Q_1) = \frac{(n-1)!}{2(n-k)!} - 1$, (2) Q_2 is a path joining $\mathbf{x}$ to $\mathbf{y}$ with $l(Q_2) = \frac{(n-1)!}{2(n-k)!} - 1$, and (3) $Q_1 \cup Q_2$ spans $A^a_{n,k}$. We write Q_2 as $\langle \mathbf{x}, Q^1_2, \mathbf{r}, \mathbf{s}, Q^2_2, \mathbf{y}\rangle$ for some vertices $\mathbf{r}$ and $\mathbf{s}$. Since $|AS(\mathbf{r})| = |AS(\mathbf{s})| = n-k \ge 2$, there exist a vertex $\mathbf{r}'$ adjacent to $\mathbf{r}$ in $A^e_{n,k}$ and a vertex $\mathbf{s}'$ adjacent to $\mathbf{s}$ in $A^f_{n,k}$ for some $\{e,f\} \subseteq \langle n\rangle - \{a,b\}$. Obviously, there exists a vertex $\mathbf{s} \in V(A^e_{n,k})$ adjacent to $\mathbf{r}$ such that $f \in AS(\mathbf{s})$ for some $f \in \langle n\rangle - \{a,b,c,e\}$. Again, there exists a vertex $\mathbf{w} \in V(A^e_{n,k})$ adjacent to $\mathbf{r}'$ such that $f \in AS(\mathbf{w})$ with $\mathbf{s}' \notin Nbd_{A_{n,k}}(\mathbf{w})$. There exist two vertices $\mathbf{p}$ and $\mathbf{q}$ in $A^e_{n,k}$ such that $g \in AS(\mathbf{p}) \cap AS(\mathbf{q})$ for some $g \in \langle n\rangle - \{a,b,e,f\}$, and $\{\mathbf{p}, \mathbf{q}\} \not\supseteq Nbd_{A_{n,k}}(\mathbf{w}) \cap Nbd_{A_{n,k}}(\mathbf{r}')$. Let $\mathbf{w}'$ be the vertex in $A^f_{n,k} - \{\mathbf{s}'\}$ such that $(\mathbf{w}, \mathbf{w}') \in E^{e,f}$. There exist two edges $(\mathbf{p}, \mathbf{p}')$ and $(\mathbf{q}, \mathbf{q}')$ in $E^{e,g}$ such that $\{\mathbf{p}', \mathbf{q}'\} \subseteq V(A^g_{n,k})$. Let $I \subseteq \langle n\rangle - \{a,b,e,g\}$ with $|I| = q(l_2)$ if $r(l_2) > \frac{(n-1)!}{2(n-k)!}$, and $|I| = q(l_2) - 1$ if $r(l_2) < \max\{3, d_{A_{n,k}}(\mathbf{x}, \mathbf{y})\}$. By Lemma 9, there exists a Hamiltonian path H joining $\mathbf{w}'$ to $\mathbf{s}'$ in $A^I_{n,k}$. By induction, there exist two paths R and R' such that (1) R is a path joining $\mathbf{r}'$ to $\mathbf{w}$ with $l(R) = r(l_2) - \frac{(n-1)!}{2(n-k)!}$ if $r(l_2) > \frac{(n-1)!}{2(n-k)!}$, and $l(R) = \frac{(n-1)!}{2(n-k)!} + r(l_2)$ if $r(l_2) < \max\{3, d_{A_{n,k}}(\mathbf{x}, \mathbf{y})\}$,

(2) R' is a path joining $\mathbf{q}$ to $\mathbf{p}$ with $l(R') = \frac{3(n-1)!}{2(n-k)!} - r(l_2) - 2$ if $r(l_2) > \frac{(n-1)!}{2(n-k)!}$, and $l(R') = \frac{(n-1)!}{2(n-k)!} - r(l_2) - 2$ if $r(l_2) < \max\{3, d_{A_{n,k}}(\mathbf{x}, \mathbf{y})\}$, and (3) $R \cup R'$ spans $A^e_{n,k}$. Let $J = \langle n \rangle - (I \cup \{a, e\})$. By Lemma 7, there exist two disjoint paths S and T such that (1) S is a path joining $\mathbf{p}'$ to $\mathbf{v}$, (2) T is a path joining $\mathbf{t}$ to $\mathbf{q}'$, and (3) $S \cup T$ spans $A^J_{n,k}$. We set $P_1 = \langle \mathbf{u}, Q_1, \mathbf{z}, \mathbf{t}, T, \mathbf{q}', \mathbf{q}, R', \mathbf{p}, \mathbf{p}', S, \mathbf{v} \rangle$ and set P_2 as $\langle \mathbf{x}, Q^1_2, \mathbf{r}, \mathbf{r}', R, \mathbf{w}, \mathbf{w}', H, \mathbf{s}', \mathbf{s}, Q^2_2, \mathbf{y} \rangle$. Obviously, P_1 and P_2 form the required paths.

Thus Theorem 1 holds.

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