A Worst Case of Circularity Test Algorithms for Attribute Grammars

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Although the circularity test problem for *attribute grammars* (AGs) has been proven to be intrinsically exponential, to date, a worst case for the existing circularity test algorithms has yet to be presented. This note presents a worst-case AG in which the number of incomparable dependency graphs induced at the root is exponential. The worst case can help to clarify the complexity of the problem.

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Additional Key Words and Phrases: Attribute grammars, circularity test, dependency graphs

1. INTRODUCTION

Jazayeri et al. [1975] first gave a sketch of a proof of the complexity of the circularity problem for *attribute grammars* (AGs). Since the original proof is very complex, Jazayeri [1981] later proposed another proof with a simpler AG. Unfortunately, this second proof does not include certain properties of the original proof. For this reason, the second proof was corrected and expanded by Dill [1989]. The complexity of the circularity problem is not obvious. A worst-case example is needed to help us understand the complexity of the problem.

Theoretically, such a worst case must exist. However, deriving a simple worst case from the proofs in Jazayeri [1981], Jazayeri et al. [1975], and Dill [1989] is not straightforward. The only exponential factor in existing circular-

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ity test algorithms [Deransart et al. 1984; Knuth 1968; Räihä and Saarinen 1982] is the number of (incomparable) *dependency graphs* for a nonterminal symbol [Räihä and Saarinen 1982]. Two dependency graphs are *incomparable* if none of them includes the other. Most practical AGs tend to have few dependency graphs [Deransart et al. 1984; Räihä and Saarinen 1982]. In this note, we present an example AG that contains an exponential number of incomparable dependency graphs. The example AG is a worst case for the existing circularity test algorithms.

2. A WORST-CASE AG FOR THE EXISTING ALGORITHMS

To show that an AG is a worst case, the size of the AG must be $O(n)$ and the number of incomparable dependency graphs be $O(2^n)$, for a given number n.

2.1 The Construction

In constructing our worst-case example, we partly follow the method in Jazayeri et al. [1975] and the simplification (the elimination of asterisk attributes) suggested in Dill [1989]. An AG $G(n)$ is defined as follows:

(1) There is a nonterminal symbol X. We ignore terminal symbols in the presentation.
(2) Symbol X has $C(j)$ and $P(j)$ pairs of attributes, $1 \le j \le 2n - 1$. Each $C(j)$ or $P(j)$ contains a pair $(\boldsymbol{a}, \boldsymbol{b})$, where attributes of $C(j)$ are inherited, and attributes of $P(j)$ are synthesized. The attributes $\boldsymbol{a}$ of $C(j)$ and $P(j)$ are denoted by $C(j, \boldsymbol{a})$ and $P(j, \boldsymbol{a})$, respectively. For a production rule containing more than one symbol X, the attribute occurrences are labeled $X_0.C(j, \boldsymbol{a})$, $X_1.C(j, \boldsymbol{a})$, etc.
(3) The attribution rules and production rules are defined as shown below. (The difference between production rules *p1* and *p2* is in their first attribution rule.)

p1: $X \rightarrow X$

$$X_1.C(n-1, \boldsymbol{a}) = X_0.C(n, \boldsymbol{a});$$
$$X_1.C(j-1, v) = X_0.C(j, v),\ v \in \{\boldsymbol{a}, \boldsymbol{b}\},\ 2 \le j \le 2n-1,\ j \ne n;$$
$$X_0.P(j, v) = X_1.P(j, v),\ v \in \{\boldsymbol{a}, \boldsymbol{b}\},\ 1 \le j \le 2n-1;$$

p2: $X \rightarrow X$

$$X_1.C(n-1, \boldsymbol{a}) = X_0.C(n, \boldsymbol{b});$$
$$X_1.C(j-1, v) = X_0.C(j, v),\ v \in \{\boldsymbol{a}, \boldsymbol{b}\},\ 2 \le j \le 2n-1,\ j \ne n;$$
$$X_0.P(j, v) = X_1.P(j, v),\ v \in \{\boldsymbol{a}, \boldsymbol{b}\},\ 1 \le j \le 2n-1;$$

p3: $X \rightarrow \varepsilon$

$$X.P(j, v) = X.C(j, v),\ v \in \{\boldsymbol{a}, \boldsymbol{b}\},\ 1 \le j \le 2n-1;$$

Note that the number of attributes of X, the size of the production rules, and the size of the attribution rules are all $O(n)$.

2.2 The Proof

We claim that the AG $G(n)$ is a worst case for three existing algorithms: Knuth [1968] and two improved algorithms of Räihä and Saarinen [1982] and Deransart et al. [1984]. The first part of the claim holds, since there is an exponential number of dependency graphs for X. Consider a derivation of length

$$n + 1: X \Rightarrow X \overset{p^{(1)}}{\Rightarrow} \dots \overset{p^{(n-1)}}{\Rightarrow} X \Rightarrow \varepsilon,$$

where $p^{(i)} \in \{p1, p2\}$, $1 \le i \le n - 1$. A dependency graph for the root X is shown as follows:

$$P(i, \boldsymbol{a}) \leftarrow C(n + i, \boldsymbol{a}) \text{ if } p^{(i)} = p1 \text{ and } P(i, \boldsymbol{a}) \leftarrow C(n + i, \boldsymbol{b}) \text{ if } p^{(i)} = p2,$$

where "$\leftarrow$" means "depends on." Because $P(i, \boldsymbol{a})$ depends on either $C(n + i, \boldsymbol{a})$ or $C(n + i, \boldsymbol{b})$, $1 \le i \le n - 1$, there are 2^{n-1} dependency graphs on the root X. Figure 1 shows a parse tree of $G(3)$ and its dependency graph. Figure 2 shows the dependency graph for the root X of the parse tree.

The second part holds as well, since the algorithm in Räihä and Saarinen [1982] applies several improved techniques without changing the basic circularity test scheme.

The third part calls for more detailed treatment. One major improvement in the algorithm in Deransart et al. [1984] is based on the concept of a *covering*. The covering of a set of dependency graphs is the set of incomparable elements. To show $G(n)$ is a worst case, we need to show there are an exponential number of incomparable dependency graphs. Because the algorithm works from the bottom up, we need to show that the dependency graphs induced on X for derivations of length $n + 1$ are incomparable with those of length less than $n + 1$. The dependency graph of a derivation of length 1, i.e., $X \Rightarrow \varepsilon$, is a trivial case and is incomparable with those of length $n + 1$.

Consider a dependency graph of a derivation of length

$$k + 1, 1 \le k < n: X \Rightarrow X \overset{p^{(1)}}{\Rightarrow} \dots \overset{p^{(k-1)}}{\Rightarrow} X \Rightarrow \varepsilon.$$

These dependency graphs are included in

$$\{P(i, \boldsymbol{a}) \leftarrow C(k + i, \boldsymbol{a}) \mid 1 \le i \le 2n - 1 - k\} \cup \{P(i, \boldsymbol{a}) \leftarrow C(k + i, \boldsymbol{b}) \mid 1 \le i \le 2n - 1 - k\}.$$

Thus, they are incomparable with the dependency graphs of derivations of length $n + 1$. The size of the covering of $G(n)$ is then exponential. The third part of the claim holds.

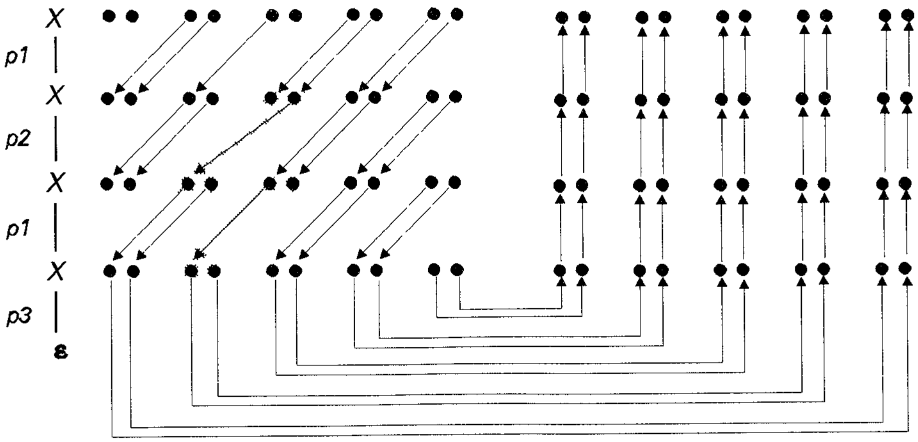

Fig. 1. A parse tree of $G(3)$ and its dependency graph.

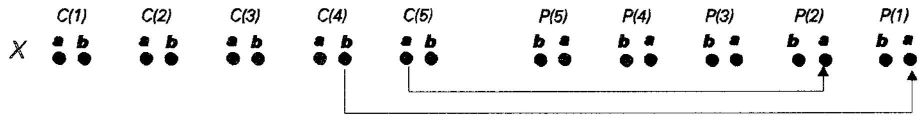

Fig. 2. The dependency graph for the root X of the parse tree in Figure 1.

3. CONCLUSION

We have presented a worst-case AG for the existing circularity test algorithms. The worst-case example contains an exponential number of incomparable dependency graphs. The example is very simple and can help us to understand the complexity of the circularity problem.

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REFERENCES

DERANSART, P., JOURDAN, M., AND LORHO, B. 1984. Speeding up circularity tests for attribute grammars. *Acta Informatica 21*, 375–391.

DILL, J. M. 1989. A counterexample for "A Simpler Construction for Showing the Intrinsically Exponential Complexity of the Circularity Problem for Attribute Grammars." *J. ACM 36*, 1 (Jan.), 92–96.

JAZAYERI, M. 1981. A simple construction for showing the intrinsically exponential complexity of the circularity problem for attribute grammars. *J. ACM 28*, 4 (Oct.), 715–720.

JAZAYERI, M., OGDEN, W. F., AND ROUNDS, W. C. 1975. The intrinsically exponential complexity of the circularity problem for attribute grammars. *Commun. ACM 18*, 12 (Dec.), 697–706.

KNUTH, D. E. 1968. Semantics of context-free languages. *Math. Syst. Theory 2*, 2, 127–145. Correction: *Mathematical Systems Theory*, Vol. 5, No. 1, 1971, pp. 95–96.

RÄIHÄ, K.-J. AND SAARINEN, M. 1982. Testing attribute grammars for circularity. *Acta Informatica 17*, 185–192.

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