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An initial boundary value problem for one-dimensional shallow water magnetohydrodynamics in the solar tachocline

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ABSTRACT

In this article we investigate the shallow water magnetohydrodynamic equations in space dimension one with Dirichlet boundary conditions only for the velocity. This model has been proposed to study the phenomena in the solar tachocline. In this article, the local well-posedness in time of the model is proven by constructing the approximate solutions and showing the strong convergence of the approximate solutions.

1. Introduction

In geophysical fluid dynamics, when the thickness of the shallow layer of fluid considered is much smaller than the horizontal scale, the Shallow Water (SW) equations are often commonly used to describe the evolution of such a system (see [1]). In general, SW equations are considered as a simplified version of the primitive equations and are mainly applied to model the large-scale dynamics of thinly stratified atmospheric or oceanic flows under the influence of Coriolis force and thermal forcing. From the mathematical point of view, SW equations have been extensively studied in the literature, in particular, on the questions such as the existence and uniqueness of solutions and related issues. We refer the interested reader to e.g. [2–9] and references therein.

In the context of astrophysical dynamics, the shallow water magnetohydrodynamic (SWMHD) equations for studying the phenomena in the solar tachocline were (to the best of our knowledge) first introduced by Gilman [10] and the word tachocline was introduced for the first time in [11]. In the literature, there are some theoretical and numerical works available for studying the shallow water magnetohydrodynamic equations (for theoretical works, see [12–14]; for numerical works, see [15–20]). However, as far as we know, there has been very few work on the subject of the well-posedness of the shallow water magnetohydrodynamic equations addressed in the literature. In the article, we are interested in studying the existence and uniqueness of solutions to the shallow water magnetohydrodynamic equations.

In this article, the one-dimensional shallow water magnetohydrodynamic equations are considered. For the Cauchy problem of the equations, the well-posed result follows from the general theory of quasilinear hyperbolic systems (see e.g. [21]). As for the periodic boundary condition or Dirichlet boundary condition, to the best of our knowledge, these problems have not been solved yet. More specifically, no general theory of quasilinear hyperbolic systems can be applied directly. In this work, we deal with the case of the Dirichlet boundary condition. The case of the periodic boundary condition will be addressed elsewhere.

In the presented work, we consider the one-dimensional shallow water magnetohydrodynamic equations with Dirichlet boundary conditions only imposed on the velocity. Under the assumption that the height remains strictly positive, as the ideas used in [9], we are able to show the existence and uniqueness of the strong solutions to the system on a certain time

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interval depending on the initial data. Indeed, we consider a new system derived from expressing the magnetic field in terms of the height due to the one-dimensional structure. Then, we construct approximate solutions by iteratively solving some linearized equations and show that these sequences are Cauchy in suitable Hilbert spaces. The difficulty of the estimates comes from the nonlinearity of the magnetic field, which can be solved under the suitable assumption on the height. At the end, we show that the sequences strongly converge to the limits which are the solutions of our problem. As for other higher dimensions, various boundary conditions or related mathematical issues to the shallow water magnetohydrodynamic equations, these problems will be studied elsewhere.

The rest of the article is organized as follows. In Section 2, we recall the equations and state our main results. In Section 3, we construct the approximate linear system, make various a priori estimates, derive the uniform boundedness in t and show the positivity of the approximate height. Finally, we conclude the proof of the existence and uniqueness of strong solutions in Section 4.

2. The shallow water magnetohydrodynamic equations: main results

In this section, we consider the one-dimensional shallow water magnetohydrodynamic equations on the interval $I = [0, 1]$ as [10,12]:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} - B\dfrac{\partial B}{\partial x} + g\dfrac{\partial h}{\partial x} = 0, \\ \dfrac{\partial B}{\partial t} + u\dfrac{\partial B}{\partial x} - B\dfrac{\partial u}{\partial x} = 0, \\ \dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(uh) = 0, \\ \dfrac{\partial}{\partial x}(hB) = 0. \end{cases} \tag{1}$$

Here the unknown functions u, B and h represent the fluid velocity, the magnetic field and the height of the conducting fluid, respectively. The constant g is the magnitude of the gravitational acceleration. Equations

$$u(x, 0) = u_0(x), \qquad B(x, 0) = B_0(x), \qquad h(x, 0) = h_0(x), \tag{2}$$

and with the following boundary conditions

$$u(0, t) = u(1, t) = 0. \tag{3}$$

We assume that the initial conditions satisfy the compatibility conditions. Namely, the functions $u_0(x)$, $B_0(x)$, and $h_0(x)$ satisfy

$$u_0(x = 0) = u_0(x = 1) = 0, \qquad \frac{\partial}{\partial x}(h_0 B_0) = 0. \tag{4}$$

We also assume that $h_0(x) \geq 2H_0 > 0$ for $x \in I$ and H_0 is a positive constant. In this article, no boundary conditions are imposed on the magnetic field and the height.

In what follows we show the following main result.

Theorem 1. *Given (u_0, B_0, h_0) in $H^2(0, 1)$ which satisfy (4) and $h_0 \geq 2H_0 > 0$. Then there exist a constant $T^* > 0$ depending on the initial data $|u_0|_{H^2(0,1)}$, $|B_0|_{H^2(0,1)}$ and $|h_0|_{H^2(0,1)}$ and a unique solution (u, B, h) of the problem (1) on the time interval $(0, T^*)$ with the initial conditions (2) and boundary conditions (3) such that*

$$(u, B, h) \in L^\infty(0, T^*; H^2(0, 1)^3). \tag{5}$$

Moreover, $h(x, t) \geq H_0$ for $t \in [0, T^*]$.

From (1), we obtain

$$\begin{aligned} \frac{\partial (hB)}{\partial t} &= \frac{\partial h}{\partial t}B + \frac{\partial B}{\partial t}h \\ &= -B\left(u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x}\right) + h\left(B\frac{\partial u}{\partial x} - u\frac{\partial B}{\partial x}\right) = -u\frac{\partial (Bh)}{\partial x} = 0. \end{aligned}$$

Combined with $(1)_3$, we can infer that

$$hB \equiv C = h_0 B_0$$

where C is assumed to be a nonzero constant due to the presence of the magnetic field.

Thus, we observe that Eq. (1) is equivalent to

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + g\dfrac{\partial h}{\partial x} + \dfrac{C^2}{h^3}\dfrac{\partial h}{\partial x} = 0, \\ \dfrac{\partial h}{\partial t} + u\dfrac{\partial h}{\partial x} + h\dfrac{\partial u}{\partial x} = 0, \\ hB = C. \end{cases} \tag{6}$$

Therefore, to prove Theorem 1, we only need to show the following theorem.

Theorem 2. *Given (u_0, h_0) in $H^2(0, 1)$ which satisfy (3) and $h_0 \geq 2H_0 > 0$. Then there exist a constant $T^* > 0$ depending on the initial data $|u_0|_{H^2(0,1)}$ and $|h_0|_{H^2(0,1)}$ and a unique solution (u, h) of the problem (6) on the time interval $(0, T^*)$ with the initial conditions (2) and boundary conditions (3) such that*

$$(u, h) \in L^\infty(0, T^*; H^2(0, 1)^2). \tag{7}$$

Moreover, $h(x, t) \geq H_0$ for $t \in [0, T^*]$.

Throughout this article, we denote c and c_i by various constants depending on the initial data and the size of the domain only and c can be different at different occurrences.

3. Approximate linear systems

In this section we prove the local existence of strong solutions of (6). The proofs are based on works related to shallow water equations in [9], which do not take the magnetic field into account. Similarly, we construct approximate linear systems of the problem (6), obtain various a priori estimates for them and then prove the uniform boundedness in t and the positivity of the approximate height.

Now, we define $u^0(x, t) = u_0(x)$, $h^0(x, t) = h_0(x)$ and then iteratively construct u^{k+1}, h^{k+1} as the solutions of the following linear problem:

$$\begin{cases} u_t^{k+1} + u^k u_x^{k+1} + \left(g + \dfrac{C^2}{(h^k)^3}\right) h_x^{k+1} = 0, \\ h_t^{k+1} + h^k u_x^{k+1} + u^k h_x^{k+1} = 0, \end{cases} \tag{8}$$

with the initial conditions

$$u^{k+1}(x, 0) = u_0(x), \qquad h^{k+1}(x, 0) = h_0(x). \tag{9}$$

and with the boundary conditions

$$u^{k+1}(0, t) = u^{k+1}(1, t) = 0. \tag{10}$$

Now we assume that $u^k, h^k \in L^\infty(0, T; H^2(0, 1))$, $u_t^k, h_t^k \in L^\infty(0, T; H^1(0, 1))$ and $h^k(x, t) \geq H_0$. We need to show that u^{k+1}, h^{k+1} enjoy the same properties.

3.1. Estimates u^{k+1}, h^{k+1} in $L^\infty(0, T; L^2(0, 1))$

We multiply $(8)_1$ by $h^k u^{k+1}$ and $(8)_2$ by $(g + C^2/(h^k)^3)h^{k+1}$, add the resulting equations, integrate over the domain I and find that

$$\int_0^1 h^k u^{k+1} u_t^{k+1}\, dx + \int_0^1 h^k u^k u^{k+1} u_x^{k+1}\, dx + \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) h^k (u^{k+1}h^{k+1})_x\, dx$$
$$+ \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) h^{k+1} h_t^{k+1}\, dx + \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) u^k h^{k+1} h_x^{k+1}\, dx = 0. \tag{11}$$

The first term in (11) can be written as

$$\int_0^1 h^k u^{k+1} u_t^{k+1}\, dx = \frac{1}{2}\frac{d}{dt}\int_0^1 h^k (u^{k+1})^2\, dx - \frac{1}{2}\int_0^1 h_t^k (u^{k+1})^2\, dx. \tag{12}$$

To bound the last term in (12), due to the fact that $h^k \geq H_0$ and the Sobolev embedding theorem $H^1(0, 1) \subset L^\infty(0, 1)$, we have

$$\left|\int_0^1 h_t^k (u^{k+1})^2\, dx\right| \leq c|h_t^k|_{H^1(0,1)} \int_0^1 h^k (u^{k+1})^2\, dx. \tag{13}$$

To bound the second term in (11), we have

$$\left|\int_0^1 h^k u^k u^{k+1} u_x^{k+1}\,dx\right| = \left|\frac{-1}{2}\int_0^1 (h^k u^k)_x (u^{k+1})^2\,dx\right|$$
$$\le c(|u^k|_{L^\infty(0,1)}|h_x^k|_{L^\infty(0,1)} + |u_x^k|_{L^\infty(0,1)})\int_0^1 h^k (u^{k+1})^2\,dx$$
$$\le c(|u^k|_{H^1(0,1)}|h^k|_{H^2(0,1)} + |u^k|_{H^2(0,1)})\int_0^1 h^k (u^{k+1})^2\,dx. \tag{14}$$

To bound the third term in (11), we have

$$\left|\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) h^k (h^{k+1}u^{k+1})_x\,dx\right|$$
$$\le c\left|\int_0^1 \frac{h_x^k}{(h^k)^4} h^{k+1}u^{k+1}\,dx\right| + c\left|\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) h_x^k h^{k+1} u^{k+1}\,dx\right|$$
$$\le c|h^k|_{H^2(0,1)}\left(\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right)(h^{k+1})^2\,dx\right)^{1/2}\left(\int_0^1 h^k (u^{k+1})^2\,dx\right)^{1/2}. \tag{15}$$

To estimate the fourth term in (11), we obtain

$$\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) h^{k+1} h_t^{k+1}\,dx = \frac{1}{2}\frac{d}{dt}\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right)(h^{k+1})^2 dx - \frac{3}{2}\int_0^1 \frac{C^2}{(h^k)^4} h_t^k (h^{k+1})^2\,dx. \tag{16}$$

Similarly, the last term in (16) can be bounded by

$$\left|\int_0^1 \frac{C^2}{(h^k)^4} h_t^k (h^{k+1})^2\,dx\right| \le c|h_t^k|_{H^1(0,1)}\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right)(h^{k+1})^2\,dx. \tag{17}$$

Combining all the previous estimates, we find that

$$\frac{d}{dt} I_0^{k+1}(t) \le c_0 \xi_0^k(t) I_0^{k+1}(t), \tag{18}$$

where the function $I^{k+1}(t)$ is defined as

$$I_0^{k+1}(t) = \int_0^1 h^k (u^{k+1})^2\,dx + \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right)(h^{k+1})^2\,dx, \tag{19}$$

and

$$\xi_0^k(t) = |u^k|_{H^2(0,1)} + |u^k|_{H^1(0,1)}|h^k|_{H^2(0,1)} + |h^k|_{H^2(0,1)} + |h_t^k|_{H^1(0,1)}. \tag{20}$$

Then applying the Gronwall inequalities for (18), we obtain

$$I_0^{k+1}(t) \le I_0^{k+1}(0) e^{c_0 \int_0^t \xi_0^k(s)\,ds}. \tag{21}$$

3.2. Estimates u^{k+1}, h^{k+1} in $L^\infty(0,T;H^1(0,1))$

To make a priori estimates for u^{k+1}, h^{k+1} in $L^\infty(0,T;H^1(0,1))$, we differentiate the approximate equations (8) with respect to the variable x and find that

$$\begin{cases} u_{tx}^{k+1} + u_x^k u_x^{k+1} + u^k u_{xx}^{k+1} + \left(g + \dfrac{C^2}{(h^k)^3}\right) h_{xx}^{k+1} - \dfrac{3C^2}{(h^k)^4} h_x^k h_x^{k+1} = 0, \\ h_{tx}^{k+1} + h_x^k u_x^{k+1} + h^k u_{xx}^{k+1} + u_x^k h_x^{k+1} + u^k h_{xx}^{k+1} = 0. \end{cases} \tag{22}$$

We multiply $(22)_1$ by $h^k u_x^{k+1}$ and $(22)_2$ by $(g + C^2/(h^k)^3)h_x^{k+1}$, add the resulting equations, integrate over the domain I and find that

$$\int_0^1 h^k u_x^{k+1} u_{tx}^{k+1}\,dx + \int_0^1 h^k u_x^k (u_x^{k+1})^2\,dx + \int_0^1 u^k h^k u_x^{k+1} u_{xx}^{k+1}\,dx$$
$$+ \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) h^k (u_x^{k+1} h_x^{k+1})_x\,dx - 3\int_0^1 \frac{C^2}{(h^k)^3} h_x^k u_x^{k+1} h_x^{k+1}\,dx$$

$$+\int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right) h_x^{k+1}h_{tx}^{k+1}\,dx + \int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right) h_x^{k+1}h_x^k u_x^{k+1}\,dx$$
$$+\int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right) u_x^k (h_x^{k+1})^2\,dx + \int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right) h_x^{k+1}u^k h_{xx}^{k+1}\,dx = 0. \tag{23}$$

The first term in (23) can be written as

$$\int_0^1 h^k u_x^{k+1}u_{tx}^{k+1}\,dx = \frac{1}{2}\frac{d}{dt}\int_0^1 h^k (u_x^{k+1})^2\,dx - \frac{1}{2}\int_0^1 h_t^k (u_x^{k+1})^2\,dx. \tag{24}$$

The last term in (24) can be bounded by

$$\left|\int_0^1 h_t^k (u_x^{k+1})^2\,dx\right| \le c|h_t^k|_{H^1(0,1)} \int_0^1 h^k (u_x^{k+1})^2\,dx. \tag{25}$$

To bound the second term in (23), we have

$$\left|\int_0^1 h^k u_x^k (u_x^{k+1})^2\,dx\right| \le c|u^k|_{H^2(0,1)} \int_0^1 h^k (u_x^{k+1})^2\,dx. \tag{26}$$

To estimate the third term in (23), we obtain

$$\left|\int_0^1 u^k h^k u_x^{k+1}u_{xx}^{k+1}\,dx\right| = \frac{1}{2}\left|\int_0^1 (u^k h^k)_x (u_x^{k+1})^2\,dx\right|$$
$$\le c(|u^k|_{H^1(0,1)}|h^k|_{H^2(0,1)} + |u^k|_{H^2(0,1)}) \int_0^1 h^k (u_x^{k+1})^2\,dx. \tag{27}$$

To bound the fourth term in (23), due to integration by parts, we find that

$$\int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right) h^k (u_x^{k+1}h_x^{k+1})_x\,dx = \left(g+\frac{C^2}{(h^k)^3}\right) u_x^{k+1}h_x^{k+1}\,|_0^1 - \int_0^1 \left(g-\frac{2C^2}{(h^k)^3}\right) h_x^k u_x^{k+1}h_x^{k+1}\,dx. \tag{28}$$

Thus, we need to calculate the boundary terms from (28). Indeed, we know that $u^{k+1}(0,t) = u^{k+1}(1,t) = 0$ and this implies that

$$u_t^{k+1}(0,t) = u_t^{k+1}(1,t) = 0. \tag{29}$$

Then, it can be inferred from (8), and (29) that

$$h_x^{k+1}(0,t) = h_x^{k+1}(1,t) = 0. \tag{30}$$

Applying (30) in (28), we have

$$\left|\int_0^1 u^k h^k u_x^{k+1}u_{xx}^{k+1}\,dx\right| \le c|h^k|_{H^2(0,1)}|h^k|_{H^1(0,1)}^{3/2} \left(\int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right)(h_x^{k+1})^2\,dx\right)^{1/2} \left(\int_0^1 h^k (u_x^{k+1})^2\,dx\right)^{1/2}. \tag{31}$$

Similarly, the fifth term in (23) can be bounded by

$$\left|\int_0^1 \left|\frac{C^2}{(h^k)^3}h_x^k u_x^{k+1}h_x^{k+1}\,dx\right|\right. \le c|h^k|_{H^2(0,1)}|h^k|_{H^1(0,1)}^{3/2} \left(\int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right)(h^{k+1})^2\,dx\right)^{1/2} \left(\int_0^1 h^k (u^{k+1})^2\,dx\right)^{1/2}. \tag{32}$$

The sixth term in (23) can be written as

$$\int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right) h_x^{k+1}h_{tx}^{k+1}\,dx = \frac{1}{2}\frac{d}{dt}\int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right)(h_x^{k+1})^2\,dx + \frac{3}{2}\int_0^1 \frac{C^2}{(h^k)^4}h_t^k (h_x^{k+1})^2\,dx. \tag{33}$$

The last term in (33) can be bounded by

$$\left|\int_0^1 \frac{C^2}{(h^k)^4}h_t^k (h_x^{k+1})^2\,dx\right| \le c|h_t^k|_{H^1(0,1)} \int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right)(h_x^{k+1})^2\,dx. \tag{34}$$

Similarly, the seventh and eighth terms can be bounded by

$$\left|\int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right) h_x^{k+1}h_x^k u_x^{k+1}\,dx\right|$$
$$\le c|h^k|_{H^2(0,1)}|h^k|_{H^1(0,1)}^{3/2} \left(\int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right)(h^{k+1})^2\,dx\right)^{1/2} \left(\int_0^1 h^k (u^{k+1})^2\,dx\right)^{1/2}, \tag{35}$$

and

$$\left|\int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right) u_x^k (h_x^{k+1})^2\,dx\right| \le c|u^k|_{H^2(0,1)} \int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right)(h_x^{k+1})^2\,dx. \tag{36}$$

To estimate the ninth term in (23), we obtain

$$\begin{aligned}\left|\int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right) h_x^{k+1} u^k h_{xx}^{k+1}\,dx\right| &= \left|\frac{1}{2}\int_0^1 \left[\left(g+\frac{C^2}{(h^k)^3}\right)u_x^k - 3\frac{C^2}{(h^k)^4}u^k h_x^k\right](h_x^{k+1})^2\,dx\right| \\ &\le c(|u^k|_{H^2(0,1)} + |u^k|_{H^1(0,1)}|h^k|_{H^2(0,1)}|h^k|_{H^1(0,1)}^{3/2}) \int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right)(h_x^{k+1})^2\,dx. \end{aligned}\tag{37}$$

Combining all the previous estimates, we find that

$$\frac{d}{dt}I_1^{k+1}(t) \le c_1\xi_1^k(t)I_1^{k+1}(t), \tag{38}$$

where the function $I_1^{k+1}(t)$ is defined as

$$I_1^{k+1}(t) = \int_0^1 h^k(u_x^{k+1})^2\,dx + \int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right)(h^{k+1})^2\,dx, \tag{39}$$

and

$$\xi_1^k(t) = \xi_0^k(t) + (1+|u^k|_{H^1(0,1)})|h^k|_{H^2(0,1)}|h^k|_{H^1(0,1)}^{3/2}. \tag{40}$$

Using the Gronwall inequalities to (38), we obtain

$$I_1^{k+1}(t) \le I_1^{k+1}(0)e^{\left(c_1\int_0^t \xi^k(s)\,ds\right)}. \tag{41}$$

3.3. Estimates u^{k+1}, h^{k+1} in $L^\infty(0,T;H^2(0,1))$

Again, to make a priori estimates for u^{k+1}, h^{k+1} in $L^\infty(0,T;H^2(0,1))$, we differentiate (22) with respect to the variable x and obtain

$$\begin{cases} u_{txx}^{k+1} + u_{xx}^k u_x^{k+1} + 2u_x^k u_{xx}^{k+1} + u^k u_{xxx}^{k+1} - \dfrac{6C^2}{(h^k)^4}h_x^k h_{xx}^{k+1} + \left(g+\dfrac{C^2}{(h^k)^3}\right)h_{xxx}^{k+1} \\ \qquad + \dfrac{12C^2}{(h^k)^5}(h_x^k)^2 h_x^{k+1} - \dfrac{3C^2}{(h^k)^4}h_{xx}^k h_x^{k+1} = 0, \\ h_{txx}^{k+1} + 2h_{xx}^k u_x^{k+1} + h_x^k u_{xx}^{k+1} + h^k u_{xxx}^{k+1} + u_{xx}^k h_x^{k+1} + 2u_x^k h_{xx}^{k+1} + u^k h_{xxx}^{k+1} = 0. \end{cases}\tag{42}$$

We multiply $(42)_1$ by $h^k u_{xx}^{k+1}$ and $(42)_2$ by $(g+C^2/(h^k)^3)h_{xx}^{k+1}$, add the resulting equations, integrate over the domain I, and find that

$$\begin{aligned} &\int_0^1 h^k u_{xx}^{k+1}u_{txx}^{k+1}\,dx + \int_0^1 h^k u_{xx}^{k+1}u_{xx}^k u_x^{k+1}\,dx + 2\int_0^1 h^k u_x^k (u_{xx}^{k+1})^2\,dx \\ &\quad + \int_0^1 h^k u^k u_{xx}^{k+1}u_{xxx}^{k+1}\,dx - \int_0^1 \frac{6C^2}{(h^k)^3}h_x^k u_{xx}^{k+1}h_{xx}^{k+1}\,dx + \int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right)h^k(u_{xx}^{k+1}h_{xx}^{k+1})_x\,dx \\ &\quad + \int_0^1 \frac{12C^2}{(h^k)^4}(h_x^k)^2 u_{xx}^{k+1}h_x^{k+1}\,dx - \int_0^1 \frac{3C^2}{(h^k)^3}h_{xx}^k h_x^{k+1}u_{xx}^{k+1}\,dx + \int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right)h_{xx}^{k+1}h_{txx}^{k+1}\,dx \\ &\quad +2\int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right)h_{xx}^k u_x^{k+1}h_{xx}^{k+1}\,dx + \int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right)h_x^k h_{xx}^{k+1}u_{xx}^{k+1}\,dx \\ &\quad + \int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right)u_{xx}^k h_x^{k+1}h_{xx}^{k+1}\,dx + 2\int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right)u_x^k (h_{xx}^{k+1})^2\,dx \\ &\quad + \int_0^1 \left(g+\frac{C^2}{(h^k)^3}\right)u^k h_{xx}^{k+1}h_{xxx}^{k+1}\,dx = 0. \end{aligned}\tag{43}$$

The first term in (43) can be written as

$$\int_0^1 h^k u_{xx}^{k+1}u_{txx}^{k+1}\,dx = \frac{1}{2}\frac{d}{dt}\int_0^1 h^k(u_{xx}^{k+1})^2\,dx - \frac{1}{2}\int_0^1 h_t^k(u_{xx}^{k+1})\,dx, \tag{44}$$

and the last term can be bounded by

$$\left|\int_0^1 h_t^k (u_{xx}^{k+1})\, dx\right| \le c|h_t^k|_{H^1(0,1)} \int_0^1 h^k (u_{xx}^{k+1})^2\, dx. \tag{45}$$

To bound the second, seventh, eighth, tenth and twelfth terms in (42), we obtain

$$\begin{aligned}\left|\int_0^1 h^k u_{xx}^{k+1} u_{xx}^k u_x^{k+1}\, dx\right| &\le |h^k|_{L^\infty(0,1)} |u_x^{k+1}|_{L^\infty(0,1)} |u_{xx}^k|_{L^2(0,1)} |u_{xx}^{k+1}|_{L^2(0,1)}\\ &\le c|h^k|_{H^1(0,1)} |u^k|_{H^2(0,1)} (|u_x^{k+1}|_{L^2(0,1)} + |u_{xx}^{k+1}|_{L^2(0,1)}) |u_{xx}^{k+1}|_{L^2(0,1)}\\ &\le c|h^k|_{H^1(0,1)} |u^k|_{H^2(0,1)} \left(\int_0^1 h^k (u_x^{k+1})^2\, dx + \int_0^1 h^k (u_{xx}^{k+1})^2\, dx\right).\end{aligned} \tag{46}$$

$$\begin{aligned}\left|\int_0^1 \frac{12C^2}{(h^k)^4} (h_x^k)^2 u_{xx}^{k+1} h_x^{k+1}\, dx\right| &\le c|h^k|^2_{H^2(0,1)} |h^k|^{3/2}_{H^1(0,1)} \left(\int_0^1 h^k (u_{xx}^{k+1})^2\, dx\right)^{1/2}\\ &\quad\times \left(\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) (h_{xx}^{k+1})^2\, dx\right)^{1/2}.\end{aligned} \tag{47}$$

$$\begin{aligned}\left|\int_0^1 \frac{3C^2}{(h^k)^3} h_{xx}^k u_{xx}^{k+1} h_x^{k+1}\, dx\right| &\le c|h^k|_{H^2(0,1)} |h^k|^{3/2}_{H^1(0,1)} \left(\int_0^1 h^k (u_{xx}^{k+1})\right)^2 dx\\ &\quad + \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) (h_x^{k+1})^2\, dx + \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) (h_{xx}^{k+1})^2\, dx.\end{aligned} \tag{48}$$

$$\begin{aligned}\left|\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) h_{xx}^k u_x^{k+1} h_{xx}^{k+1}\, dx\right| &\le c|h^k|_{H^2(0,1)} |h^k|^{3/2}_{H^1(0,1)} \left(\int_0^1 h^k (u_{xx}^{k+1})^2\, dx\right.\\ &\quad \left.+ \int_0^1 h^k (u_x^{k+1})^2\, dx + \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) (h_{xx}^{k+1})^2\, dx\right).\end{aligned} \tag{49}$$

and

$$\begin{aligned}\left|\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) u_{xx}^k h_x^{k+1} h_{xx}^{k+1}\, dx\right| &\le c|u^k|_{H^2(0,1)} |h^k|^{3/2}_{H^1(0,1)} \left(\int_0^1 h^k (u_{xx}^{k+1})^2\, dx\right.\\ &\quad \left.+ \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) (h_x^{k+1})^2\, dx + \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) (h_{xx}^{k+1})^2\, dx\right).\end{aligned} \tag{50}$$

To estimate the third and thirteenth terms in (42), we have

$$\left|\int_0^1 h^k u_x^k (u_{xx}^{k+1})^2\, dx\right| \le c|u^k|_{H^2(0,1)} \int_0^1 h^k (u_{xx}^{k+1})^2\, dx, \tag{51}$$

and

$$\left|\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) u_x^k (h_{xx}^{k+1})^2\, dx\right| \le c|u^k|_{H^2(0,1)} \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) (h_{xx}^{k+1})^2\, dx. \tag{52}$$

To bound the fourth and fourteenth terms in (42), due to integration by parts, we find that

$$\left|\int_0^1 h^k u^k u_{xx}^{k+1} u_{xxx}^{k+1}\, dx\right| \le c(|u^k|_{H^2(0,1)} + |h^k|_{H^2(0,1)} |u^k|_{H^1(0,1)}) \int_0^1 h^k (u_{xx}^{k+1})^2\, dx, \tag{53}$$

and

$$\left|\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) u^k h_{xx}^{k+1} h_{xxx}^{k+1}\, dx\right| \le c(|u^k|_{H^2(0,1)} + |h^k|_{H^2(0,1)} |u^k|_{H^1(0,1)}) \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) (h_{xx}^{k+1})^2\, dx. \tag{54}$$

To estimate the fifth and eleventh terms in (42), we obtain

$$\left|\int_0^1 \frac{C^2}{(h^k)^3} h_x^k u_{xx}^{k+1} h_{xx}^{k+1}\, dx\right| \le c|h^k|_{H^2(0,1)} \left(\int_1^0 h^k (u_{xx}^{k+1})^2\, dx\right)^{1/2} \left(\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) (h_{xx}^{k+1})^2\, dx\right)^{1/2}, \tag{55}$$

and

$$\left|\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) h^k_x h^{k+1}_{xx} u^{k+1}_{xx}\, dx\right| \le c|h^k|_{H^2(0,1)} \left(\int_1^0 h^k (u^{k+1}_{xx})^2\, dx\right)^{1/2} \left(\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right)(h^{k+1}_{xx})^2\, dx\right)^{1/2}. \tag{56}$$

The sixth term in (42) can be written as

$$\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) h^k (u^{k+1}_{xx} h^{k+1}_{xx})_x\, dx = \left(g + \frac{C^2}{(h^k)^3}\right) h^k u^{k+1}_{xx} h^{k+1}_{xx} \,|_0^1 - \int_0^1 \left(g - \frac{2C^2}{(h^k)^3}\right) h^k_x u^{k+1}_{xx} h^{k+1}_{xx}\, dx. \tag{57}$$

Thus, we need to calculate the boundary terms from (57). Indeed, we can infer from (30) that

$$h^{k+1}_{xt}(0,t) = h^{k+1}_{xt}(1,t) = 0. \tag{58}$$

Due to (22), (58) implies that

$$u^{k+1}_{xx}(0,t) = u^{k+1}_{xx}(1,t) = 0. \tag{59}$$

We can now estimate (57) as

$$\left|\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) h^k (u^{k+1}_{xx} h^{k+1}_{xx})_x\, dx\right| \le c|h^k|_{H^2(0,1)} |h^k|^{3/2}_{H^1(0,1)} \left(\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right)(h^{k+1})^2\, dx\right)^{1/2} \times \left(\int_0^1 h^k (u^{k+1})^2\, dx\right)^{1/2}. \tag{60}$$

The ninth term in (42) can be written as

$$\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) h^{k+1}_{xx} h^{k+1}_{txx}\, dx = \frac{1}{2}\frac{d}{dt}\int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right)(h^{k+1}_{xx})^2\, dx + \frac{1}{2}\int_0^1 \frac{3C^2}{(h^k)^4} h^k_t (h^{k+1}_{xx})^2\, dx. \tag{61}$$

The last term in (61) can be bounded by

$$\left|\int_0^1 \frac{3C^2}{(h^k)^4} h^k_t (h^{k+1}_{xx})^2\, dx\right| \le c|h^k_t|_{H^1(0,1)} \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right)(h^{k+1}_{xx})^2\, dx. \tag{62}$$

Combining all the previous estimates, we find that

$$\frac{d}{dt} I^{k+1}_2(t) \le c_2 \xi^k_2(t) I^{k+1}_2(t) + c_2 \xi^k_2(t) I^{k+1}_1(t), \tag{63}$$

where the function $I^{k+1}_2(t)$ is defined as

$$I^{k+1}_2(t) = \int_0^1 h^k (u^{k+1}_{xx})^2\, dx + \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right)(h^{k+1}_{xx})^2\, dx, \tag{64}$$

and

$$\xi^k_2(t) = \xi^k_1(t) + |h^k|^{3/2}_{H^1(0,1)} |u^k|_{H^2(0,1)}. \tag{65}$$

Applying the Gronwall lemma to (63), we have

$$I^{k+1}_2(t) \le I^{k+1}_2(0) e^{\left(c_2 \int_0^t \xi^k_2(s)\, ds\right)} + \int_0^t c_2 \xi^k_2(s) I^{k+1}_1(s) e^{-c_2 \int_t^s \xi^k_2(z)\, dz}\, ds.$$

Due to (41), the above inequality implies that

$$I^{k+1}_2(t) \le \left(I^{k+1}_2(0) + c_3 I^{k+1}_1(0) \int_0^t \xi^k_2(s)\, ds\right) e^{\left(c_4 \int_0^t \xi^k_2(s)\, ds\right)}. \tag{66}$$

3.4. Estimates u_t^{k+1}, h_t^{k+1} in $L^\infty(0,T;L^2(0,1))$

From (8), we obtain

$$\begin{aligned}|u_t^{k+1}|_{L^2(0,1)} &\le |u^k|_{L^\infty(0,1)}|u_x^{k+1}|_{L^2(0,1)} + c|h_x^{k+1}|_{L^2(0,1)}\\ &\le c|u^k|_{H^1(0,1)}\left(\int_0^1 h^k(u_x^{k+1})^2\,dx\right)^{1/2} + c|h^k|_{H^1(0,1)}^{3/2}\left(\int_0^1\left(g+\frac{C^2}{(h^k)^3}\right)(h_x^{k+1})^2\,dx\right)^{1/2},\end{aligned} \tag{67}$$

and

$$|h_t^{k+1}|_{L^2(0,1)} \le c|h^k|_{H^1(0,1)}\left(\int_0^1 h^k(u_x^{k+1})^2\,dx\right)^{1/2} + c|h^k|_{H^1(0,1)}^{3/2}|u^k|_{H^1(0,1)}\left(\int_0^1\left(g+\frac{C^2}{(h^k)^3}\right)(h_x^{k+1})^2\,dx\right)^{1/2}. \tag{68}$$

The inequalities (67) and (68) imply that

$$|u_t^{k+1}|_{L^2(0,1)}^2 \le c_5(|u^k|_{H^1(0,1)}^2 + |h^k|_{H^1(0,1)}^3)I_1^{k+1}(0)e^{\left(c_1\int_0^t \xi_1^k(s)\,ds\right)}, \tag{69}$$

$$|h_t^{k+1}|_{L^2(0,1)}^2 \le c_5(|h^k|_{H^1(0,1)}^2 + |u^k|_{H^1(0,1)}^2|h^k|_{H^1(0,1)}^3)I_1^{k+1}(0)e^{\left(c_1\int_0^t \xi_1^k(s)\,ds\right)}. \tag{70}$$

3.5. Estimates u_t^{k+1}, h_t^{k+1} in $L^\infty(0,T;H^1(0,1))$

To derive a priori estimates for u_t^{k+1} and h_t^{k+1} in $L^\infty(0,T;H^1(0,1))$, we need to estimate u_{tx}^{k+1} and h_{tx}^{k+1} in $L^\infty(0,T;L^2(0,1))$. For that purpose, it can be inferred from (22) that

$$|u_{tx}^{k+1}|_{L^2(0,1)} \le c(|u^k|_{H^2(0,1)} + |h^k|_{H^2(0,1)})(I_1^{k+1})^{1/2}(t) + c(|u^k|_{H^1(0,1)}+1)(I_2^{k+1})^{1/2}(t), \tag{71}$$

and

$$\begin{aligned}|h_{tx}^{k+1}|_{L^2(0,1)} &\le c(|h^k|_{H^2(0,1)} + |u^k|_{H^2(0,1)}|h^k|_{H^1(0,1)}^{3/2})(I_1^{k+1})^{1/2}(t)\\ &\quad + c(|h^k|_{H^1(0,1)} + |u^k|_{H^1(0,1)}|h^k|_{H^1(0,1)}^{3/2})(I_2^{k+1})^{1/2}(t).\end{aligned} \tag{72}$$

Using (41) and (66), we have

$$\begin{aligned}|u_{tx}^{k+1}|_{L^2(0,1)}^2 &\le c_6(|u^k|_{H^2(0,1)}^2 + |h^k|_{H^2(0,1)}^2)I_1^{k+1}(0)e^{c_1\int_0^t \xi_1^k(s)\,ds}\\ &\quad + c_6(|u^k|_{H^1(0,1)}^2+1)(I_2^{k+1}(0) + c_3I_1^{k+1}(0)\int_0^t \xi_2^k(s)\,ds)e^{\left(c_4\int_0^t \xi_2^k(s)\,ds\right)},\end{aligned} \tag{73}$$

and

$$\begin{aligned}|h_{tx}^{k+1}|_{L^2(0,1)}^2 &\le c_6(|h^k|_{H^2(0,1)}^2 + |u^k|_{H^2(0,1)}^2|h^k|_{H^1(0,1)}^3)I_1^{k+1}(0)e^{c_1\int_0^t \xi_1^k(s)\,ds}\\ &\quad + c_6(|h^k|_{H^1(0,1)}^2 + |u^k|_{H^1(0,1)}^2|h^k|_{H^1(0,1)}^3)\left(I_2^{k+1}(0) + c_3I_1^{k+1}(0)\int_0^t \xi_2^k(s)\,ds\right)e^{\left(c_4\int_0^t \xi_2^k(s)\,ds\right)}.\end{aligned} \tag{74}$$

3.6. Uniform estimates on u^{k+1} and h^{k+1}

Now, we would like to obtain uniform estimates on u^{k+1} and h^{k+1} by induction. This also gives us the existence of the solutions for the approximate linear system (8)–(10). For that purpose, let two constants α and β satisfy

$$H_0|u_0|_{H^2(0,1)}^2 = \frac{\alpha}{12}, \qquad \left(g+\frac{C^2}{H_0}\right)|h_0|_{H^2(0,1)}^2 = \frac{\beta}{12}. \tag{75}$$

We assume that

$$H_0|u^k|_{L^\infty(0,T;H^2(0,1))}^2 + g|h^k|_{L^\infty(0,T;H^2(0,1))}^2 \le \alpha+\beta, \tag{76}$$

$$|u_t^k|_{L^\infty(0,T;H^1(0,1))}^2 \le (c_5+c_6)\left(1+\frac{\alpha+\beta}{H_0}\right)\frac{\alpha+\beta}{2}, \tag{77}$$

$$|h_t^k|_{L^\infty(0,T;H^1(0,1))}^2 \le (c_5+c_6)\left(\frac{1}{g}+\frac{1}{H_0}\right)\frac{(\alpha+\beta)^2}{2}. \tag{78}$$

We will show that the inequalities (76)–(78) hold for all k on a small time $T > 0$ that does not depend on k by induction. It is easy to see that inequalities (76)–(78) hold for $k = 0$. Now we suppose that the inequalities (76)–(78) are true for $k = i$. Then, we can infer that there exists a constant γ independent of k such that

$$|\xi_2^k(t)|_{L^\infty(0,1)} \leq \gamma(\alpha, \beta).$$

We will now show that for a small time $T > 0$ that depends only on α and β, we still have

$$H_0|u^{i+1}|^2_{L^\infty(0,T;H^2(0,1))} + g|h^{i+1}|^2_{L^\infty(0,T;H^2(0,1))} \leq \alpha + \beta, \tag{79}$$

$$|u_t^{i+1}|^2_{L^\infty(0,T;H^1(0,1))} \leq (c_5 + c_6)\left(1 + \frac{\alpha + \beta}{H_0}\right)\frac{\alpha + \beta}{2}, \tag{80}$$

$$|h_t^{i+1}|^2_{L^\infty(0,T;H^1(0,1))} \leq (c_5 + c_6)\left(\frac{1}{g} + \frac{1}{H_0}\right)\frac{(\alpha + \beta)^2}{2}. \tag{81}$$

By using the inequalities for I_0^{i+1}, I_1^{i+1} and I_2^{i+1}, and the fact that $h^i \geq H_0$, we obtain

$$H_0|u^{i+1}|^2_{L^2(0,1)} + g|h^{i+1}|^2_{L^2(0,1)} \leq I_0^{i+1}(t) \leq \frac{\alpha + \beta}{12}e^{(c_0\gamma T)}, \tag{82}$$

$$H_0|u_x^{i+1}|^2_{L^2(0,1)} + g|h_x^{i+1}|^2_{L^2(0,1)} \leq I_1^{i+1}(t) \leq \frac{\alpha + \beta}{12}e^{(c_1\gamma T)}, \tag{83}$$

$$H_0|u_{xx}^{i+1}|^2_{L^2(0,1)} + g|h_{xx}^{i+1}|^2_{L^2(0,1)} \leq I_2^{i+1}(t) \leq \frac{\alpha + \beta}{12}e^{(c_4\gamma T)}(1 + c_3\gamma T). \tag{84}$$

Adding all the inequalities (82)–(84) and choosing suitable T so that $c_3\gamma T < 1$, we find that

$$H_0|u^{i+1}|^2_{H^2(0,1)} + g|h^{i+1}|^2_{H^2(0,1)} \leq \frac{\alpha + \beta}{2}e^{(\max(c_0,c_1,c_4)\gamma T)}.$$

By choosing T_1 small enough such that $e^{(\max(c_0,c_1,c_4)\gamma T)} < 2$, we obtain that

$$H_0|u^{i+1}|^2_{H^2(0,1)} + g|h^{i+1}|^2_{H^2(0,1)} \leq \alpha + \beta, \quad \forall t \leq T_1.$$

To find T_2 small enough such that the inequalities (80) and (81) are true, we use the inequalities (69), (70), (73) and (74) which yield

$$|u_t^{i+1}|^2_{H^1(0,1)} \leq (c_5 + c_6)\left(\frac{\alpha + \beta}{H_0} + 1\right)\frac{\alpha + \beta}{12}e^{(\max(c_1,c_2)\gamma T)},$$

and

$$|h_t^{i+1}|^2_{H^1(0,1)} \leq (c_5 + c_6)\left(\frac{1}{H_0} + \frac{1}{g}\right)\frac{(\alpha + \beta)^2}{12}e^{(\max(c_1,c_2)\gamma T)}.$$

By taking T_2 small enough such that $e^{(\max c_1,c_2)\gamma T_2)} \leq 2$, the inequalities (80) and (81) are true for $t \leq t_2$.

To complete the induction, we take $T^* = \min(T_1, T_2)$ and find that all the inequalities (82)–(84) are satisfied for $t \leq T^*$ where α, β, γ and T^* do not depend on k.

Now, we need to show that the relation that $h^{k+1} \geq H_0$ is still true since we assume the same relation satisfied by h^k. This can be proven by showing that there exists a small time (independent of k and possibly smaller than T^*) such that

$$|h^{k+1}(x, t) - h_0(x)| < H_0,$$

which implies that $h^{k+1} \geq H_0$. Note that we still denote this small time as T^*. Similar proofs can be found in [9].

4. Existence and uniqueness of strong solutions

In this section, we will show the existence and uniqueness of the strong solutions for Eq. (6) for a small time T^*.

4.1. Existence

To prove the existence of the solutions, the main key is to verify that the sequences $\{u^k\}_{k=1}^\infty$ and $\{h^k\}_{k=1}^\infty$ constructed by solving the approximate linear systems (8)–(10) are Cauchy in $C([0, T^*]; H^1(0, 1))$. For that purpose, we write

$$v^{k+1} = u^{k+1} - u^k,$$
$$\Phi^{k+1} = h^{k+1} - h^k,$$

and find that v^{k+1} and Φ^{k+1} satisfy

$$\begin{cases} v_t^{k+1} + u^k v_x^{k+1} + v^k u_x^k + \left(g + \dfrac{C^2}{(h^k)^3}\right)\Phi_x^{k+1} + C^2\left(\dfrac{1}{(h^k)^3} - \dfrac{1}{(h^k - \Phi^k)^3}\right)h_x^k = 0, \\ \Phi_t^{k+1} + u^k \Phi_x^{k+1} + v^k h_x^k + h^k v_x^{k+1} + u_x^k \Phi^k = 0, \end{cases} \tag{85}$$

with the initial conditions

$$v^{k+1}(x, 0) = 0, \qquad \Phi^{k+1} = 0, \tag{86}$$

and the boundary conditions

$$v^{k+1}(0, t) = v^{k+1}(1, t) = 0. \tag{87}$$

A priori estimates for v^{k+1} and Φ^{k+1} in $L^\infty(0, T^*; H^1(0, 1))$. Multiplying (85)$_1$ by $h^k v^{k+1}$, (85)$_2$ by $\left(g + \frac{C^2}{(h^k)^3}\right)\Phi^{k+1}$, adding the resulting equations, integrating the equation over the domain I and applying the integration by parts and boundary conditions, we find that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left(h^k (v^{k+1})^2 + \left(g + \frac{C^2}{(h^k)^3}\right)(\Phi^{k+1})^2\right) dx - \frac{1}{2}\int_0 h_t^k (v^{k+1})^2\, dx \\ &\quad - \frac{1}{2}\int_0^1 (u^k h^k)_x (v^{k+1})^2\, dx + \int_0^1 h^k v^k u_x^k v^{k+1}\, dx \\ &\quad - \int_0^1 \left(\left(g + \frac{C^2}{(h^k)^3}\right)h^k\right)_x v^{k+1}\Phi^{k+1}\, dx + \int_0^1 C^2 h^k v^{k+1} h_x^k \left(\frac{1}{(h^k)^3} - \frac{1}{(h^k - \Phi^k)^3}\right) dx \\ &\quad + \frac{3}{2}\int_0^1 \frac{C^2}{(h^k)^4} h_t^k (\Phi^{k+1})^2\, dx - \frac{1}{2}\int_0^1 \left(\left(g + \frac{C^2}{(h^k)^3}\right)u^k\right)_x (\Phi^{k+1})^2\, dx \\ &\quad + \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) v^k h_x^k \Phi^{k+1}\, dx + \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) h^k \Phi^{k+1} v_x^{k+1}\, dx \\ &\quad + \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) u_x^k \Phi^k \Phi^{k+1}\, dx = 0. \end{aligned} \tag{88}$$

To estimate Eq. (88), we point out how to deal with the sixth term in (88) since other terms can be estimated as before. Due to the condition that $h^k - \Phi^k = h^{k-1}$ is bounded below, we obtain

$$\begin{aligned} &\left|\int_0^1 C^2 h^k v^{k+1} h_x^k \left(\frac{1}{(h^k)^3} - \frac{1}{(h^k - \Phi^k)^3}\right) dx\right| \\ &\quad \le c|h^k|_{H^1(0,1)}^{1/2} |h^k|_{H^2(0,1)} |h^{k-1}|_{H^1(0,1)}^{3/2} \left(\int_0^1 h^k (v^{k+1})^2\, dx\right)^{1/2} \left(\int_0^1 \left(g + \frac{C^2}{(h^{k-1})^3}\right)(\Phi^k)^2\, dx\right)^{1/2}. \end{aligned}$$

We have the following inequality:

$$\frac{d}{dt} J_0^{k+1}(t) \le c_7 \xi_2^k J_0^{k+1}(t) + c_8(\xi_2^k + |h^{k-1}|_{H^1(0,1)}^3) J_0^k(t), \tag{89}$$

where the term J_0^{k+1} is defined as

$$J_0^{k+1}(t) = \int_0^1 h^k (v^{k+1})^2\, dx + \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right)(\Phi^{k+1})^2\, dx.$$

Since we have that the term $|h^{k-1}|_{H^1(0,1)}^7$ is bounded and independent of all k, we set

$$|h^{k-1}|_{H^1(0,1)}^3 \le \gamma_1 = \gamma_1(\alpha, \beta),$$

where γ_1 is a constant that does not depend on k.

Then, using the Gronwall inequality and the fact that $J_0^{k+1}(0) = 0$, we obtain

$$\begin{aligned} J_0^{k+1}(t) &\le c_8 \int_0^t (\xi_2^k(s) + \gamma_1) J_0^k(s)\, ds\, e^{(c_7 \int_0^t \xi_2^k(s)\, dx)} \\ &\le c_8(\gamma e^{c_7 \gamma T^*} + \gamma_1) T^* \sup_{0 \le s \le T^*} J_0^k(s). \end{aligned}$$

By taking T^* small enough so that $c_8(\gamma+\gamma_1)e^{c_7\gamma T^*}T^* \le \frac{1}{2}$, we find that

$$H_0|v^{k+1}|^2_{L^2(0,1)} + g|\Phi^{k+1}|^2_{L^2(0,1)} \le J_0^{k+1}(t) \le \frac{c_9}{2^{k+1}}, \quad \forall 0 \le t \le T^* \text{ and } k. \tag{90}$$

Thus, we show that the sequences $\{u^k\}$ and $\{h^k\}$ are Cauchy in $C([0,T^*];L^2(0,1))$, which implies that the strong convergence of u^k and h^k in $C([0,T^*];L^2(0,1))$ is verified.

To estimate v^{k+1} and Φ^{k+1} in $C([0,T^*];H^1(0,1))$, we differentiate (85) with respect to the variable x, multiply the first equation by $h^k v_x^{k+1}$, the second equation by $\left(g+\frac{C^2}{(h^k)^3}\right)\Phi_x^{k+1}$, add the resulting equation and integrating the equation over the domain I, we find that, after using the integration by parts and boundary conditions,

$$\begin{aligned}
&\frac{1}{2}\left\{\int_0^1 h^k(v_x^{k+1})^2\,dx + \int_0^1\left(g+\frac{C^2}{(h^k)^3}\right)(\Phi_x^{k+1})^2\,dx\right\} - \frac{1}{2}\int_0^1 h_t^k(v_x^{k+1})^2\,dx\\
&\quad + \int_0^1 h^k u_x^k(v_x^{k+1})^2\,dx - \frac{1}{2}\int_0^1 (h^k u^k)_x (v_x^{k+1})^2\,dx + \int_0^1 h^k u_x^k v_x^k v_x^{k+1}\,dx\\
&\quad + \int_0^1 h^k u_{xx}^k v_x^{k+1} v^k\,dx - \int_0^1\left(\left(g+\frac{C^2}{(h^k)^3}\right)h^k\right)_x \Phi_x^{k+1} v_x^{k+1}\,dx - 3C^2\int_0^1 v_x^{k+1}\frac{h_x^k}{(h^k)^3}\Phi_x^{k+1}\,dx\\
&\quad + C^2\int_0^1 h^k v_x^{k+1}\left(\frac{-3h_x^k}{(h^k)^4} + \frac{3(h_x^k-\Phi_x^k)}{(h^k-\Phi^k)^4}\right)h_x^k\,dx + C^2\int_0^1\left(\frac{1}{(h^k)^3} - \frac{1}{(h^k-\Phi^k)^3}\right)h_{xx}^k h^k v_x^{k+1}\,dx\\
&\quad + \frac{3C^2}{2}\int_0^1 \frac{h_t^k}{(h^k)^4}(\Phi_x k+1)^2\,dx + \int_0^1\left(g+\frac{C^2}{(h^k)^3}\right)u_x^k(\Phi_x^{k+1})^2\,dx\\
&\quad - \frac{1}{2}\int_0^1\left(\left(g+\frac{C^2}{(h^k)^3}\right)u^k\right)_x(\Phi_x^{k+1})^2\,dx + \int_0^1\left(g+\frac{C^2}{(h^k)^3}\right)\Phi_x^{k+1}v_x^k h_x^k\,dx\\
&\quad + \int_0^1\left(g+\frac{C^2}{(h^k)^3}\right)\Phi_x^{k+1}v^k h_{xx}^k\,dx + \int_0^1\left(g+\frac{C^2}{(h^k)^3}\right)\Phi_x^{k+1}h_x^k v_x^{k+1}\,dx\\
&\quad + \int_0^1\left(g+\frac{C^2}{(h^k)^3}\right)\Phi_x^{k+1}u_{xx}^k\Phi^k\,dx + \int_0^1\left(g+\frac{C^2}{(h^k)^3}\right)\Phi_x^{k+1}u_x^k\Phi_x^k\,dx = 0.
\end{aligned} \tag{91}$$

To estimate the terms in (91), we only present how to estimate the ninth and tenth terms in (91) since other terms can be estimated by applying previous techniques. To bound the ninth term in (91), we have

$$\begin{aligned}
&\left|3C^2\int_0^1 h^k v_x^{k+1}\left(\frac{-h_x^k}{(h^k)^4} + \frac{(h_x^k-\Phi_x^k)}{(h^k-\Phi^k)^4}\right)h_x^k\,dx\right|\\
&\le \left|3C^2\int_0^1 h^k v_x^{k+1}\left(\frac{-1}{(h^k)^4} + \frac{1}{(h^k-\Phi^k)^4}\right)(h_x^k)^2\,dx\right| + \left|3C^2\int_0^1 \frac{h^k v_x^{k+1}h_x^k\Phi_x^k}{(h^k-\Phi^k)^4}\,dx\right|\\
&\le c|h^k|^2_{H^2(0,1)}|h^{k-1}|^{3/2}_{H^1(0,1)}(J_0^k)^{1/2}\left(\int_0^1 h^k(v_x^{k+1})^2\,dx\right)^{1/2}\\
&\quad + c|h^k|_{H^1(0,1)}|h^k|_{H^2(0,1)}|h^{k-1}|^{3/2}_{H^1(0,1)}\left(\int_0^1 h^k(v_x^{k+1})^2\,dx\right)^{1/2}\left(\int_0^1\left(g+\frac{C^2}{(h^{k-1})^3}\right)(\Phi_x^k)^2\,dx\right)^{1/2}.
\end{aligned} \tag{92}$$

As for the tenth term in (91), we find that

$$\begin{aligned}
&\left|C^2\int_0^1\left(\frac{1}{(h^k)^3} - \frac{1}{(h^k-\Phi^k)^3}\right)h_{xx}^k h^k v_x^{k+1}\,dx\right|\\
&\le c\int_0^1 |\Phi^k||h_{xx}^k||v_x^{k+1}|\,dx\\
&\le c(|\Phi^k|_{L^2(0,1)} + |\Phi_x^k|_{L^2(0,1)})|h^k|_{H^2(0,1)}\left(\int_0^1 h^k(v_x^{k+1})^2\,dx\right)^{1/2}\\
&\le c|h^k|_{H^2(0,1)}|h^{k-1}|^{3/2}_{H^2(0,1)}\left(\left(\int_0^1\left(g+\frac{C^2}{(h^{k-1})^3}\right)(\Phi_x^k)^2\,dx\right)^{1/2} + (J_0^k)^{1/2}\right)\left(\int_0^1 h^k(v_x^{k+1})^2\,dx\right)^{1/2}.
\end{aligned} \tag{93}$$

Therefore, combining (92) and (93), we obtain that

$$\frac{d}{dt}J_1^{k+1}(t) \le c\xi_3(t)(J_1^{k+1}(t) + J_1^k(t) + J_0^k(t)), \tag{94}$$

where the function $J_1^{k+1}(t)$ is defined as

$$J_1^{k+1}(t) = \int_0^1 h^k (v_x^{k+1})^2\,dx + \int_0^1 \left(g + \frac{C^2}{(h^k)^3}\right) (\Phi_x^{k+1})^2\,dx,$$

and

$$\xi_3(t) = \xi_2^k(t) + (1 + |h^k|_{H^1(0,1)})|h^k|_{H^2(0,1)}|h^{k-1}|^{3/2}_{H^2(0,1)}.$$

Adding (89) and (94), we find that

$$\frac{d}{dt}(J_0^{k+1}(t) + J_0^{k+1}(t)) \le c_{10}\xi_3(t)(J_0^{k+1}(t) + J_1^{k+1}(t)) + c_{11}(\xi_3(t) + |h^{k-1}|^7_{H^1(0,1)})(J_0^k(t) + J_1^k(t)). \tag{95}$$

Since $\xi_3 + |h^{k-1}|^7_{H^1(0,1)}$ is uniformly bounded in $[0, T^*]$, after using the Gronwall inequality for (95) and the fact that $J_0^{k+1}(0) = J_1^{k+1}(0) = 0$, we obtain that

$$\begin{aligned} J_0^{k+1}(t) + J_1^{k+1}(t) &\le c_{11}\int_0^t (\xi_3(s) + |h^{k-1}(s)|^7_{H^1(0,1)})(J_0^k(s) + J_1^k(s))\,ds\, e^{\int_0^t c_{10}\xi_3(s)\,s} \\ &\le c_{11}(\gamma_2 + \gamma_1)e^{c_{10}\gamma_2 T^*} T^* \sup_{0<s<T^*}(J_0^k(s) + J_1^k(s)), \end{aligned} \tag{96}$$

where γ_2 is a constant independent of k and satisfying

$$\xi_3(t) \le \gamma_2.$$

By taking T^* small enough such that $c_{11}T^*(\gamma_1 + \gamma_2)e^{c_{10}\gamma_2 T^*} < 1/2$, we find that for all k

$$H_0|v^{k+1}|^2_{H^1(0,1)} + g|\Phi^{k+1}|^2_{H^1(0,1)} \le J_0^{k+1}(t) + J_1^{k+1}(t) \le \frac{c}{2^{k+1}}, \quad \text{for } t \in [0, T^*]. \tag{97}$$

Thanks to (97), we can deduce that the sequences $\{u^k\}$ and $\{h^k\}$ are Cauchy in $C([0, T^*]; H^1(0, 1))$. Thus, there exist functions $(u, h) \in C([0, T^*]; H^1(0, 1))$ such that u^k and h^k strongly converge to u and h in $C([0, T^*]; H^1(0, 1))$, respectively. To show that u and h are the solutions to the system (6), we can indeed pass to the limit in the approximate linear system (8) and find that u and h satisfy the system (6). Furthermore, since $\{u^k\}$ and $\{h^k\}$ are bounded in $L^\infty(0, T^*; H^2(0, 1))$, we can deduce that the solutions u and h also belong to $L^\infty(0, T^*; H^2(0, 1))$. Then, the existence of the solution is established.

4.2. Uniqueness

To show the uniqueness of the solution to (6), we assume that two solutions (u_1, h_1) and $(u_2, h_2) \in L^\infty(0, T^*; H^2(0, 1))$ satisfy (6). Then we write

$$\begin{cases} v = u_1 - u_2, \\ \Phi = h_1 - h_2, \end{cases}$$

and find that v and Φ satisfy

$$\begin{cases} v_t + u_{1,x}v + u_2 v_x + \left(g + \dfrac{C^2}{(h_1)^3}\right)\Phi_x + C^2\left(\dfrac{1}{(h_1)^3} - \dfrac{1}{(h_1 - \Phi)^3}\right)h_{2,x} = 0, \\ \Phi_t + h_{1,x}v + u_2\Phi_x + h_1 v_x + u_{2,x}\Phi = 0, \end{cases} \tag{98}$$

with the initial conditions

$$v(x, 0) = 0, \qquad \Phi(x, 0) = 0, \tag{99}$$

and with the boundary conditions

$$v(0, t) = v(1, t) = 0, \quad \forall t \in [0, T^*]. \tag{100}$$

Multiplying $(98)_1$ by $h_1 v$, $(98)_2$ by $\left(g + \frac{C^2}{(h_1)^3}\right)\Phi$, integrating over the domain I, adding the resulting equations and using the integration by parts and the boundary conditions, we obtain

$$\begin{aligned} &\frac{1}{2}\int_0^1 \left\{h_1 v^2 + \left(g + \frac{C^2}{(h_1)^3}\right)\Phi^2\right\}dx - \frac{1}{2}\int_0^1 h_{1,t}v^2\,dx + \int_0^1 h_1 u_{1,x}v^2\,dx \\ &- \frac{1}{2}\int_0^1 (h_1u_2)_x v^2\,dx + \int_0^1 \left(g + \frac{C^2}{(h_1)^3}\right)h_1(v\Phi)_x\,dx + \int_0^1 C^2\left(\frac{1}{(h_1)^3} - \frac{1}{(h_1 - \Phi)^3}\right)h_{1,x}v\,dx \\ &+ \frac{3}{2}\int_0^1 \frac{C^2}{(h_1)^4}h_{1,t}\Phi^2\,dx + \int_0^1 \left(g + \frac{C^2}{(h_1)^3}\right)h_{1,x}v\Phi\,dx - \frac{1}{2}\int_0^1 \left(\left(g + \frac{C^2}{(h_1)^3}\right)u_2\right)_x \Phi^2\,dx \\ &+ \int_0^1 \left(g + \frac{C^2}{(h_1)^3}\right)u_{2,x}\Phi^2\,dx = 0. \end{aligned} \tag{101}$$

As before, using similar techniques, we derive the following inequality:

$$\frac{d}{dt}\left\{\int_0^1 \left(h_1 v^2 + \left(g + \frac{C^2}{(h_1)^3}\right)\Phi^2\right) dx\right\} \leq c(|h_{1,t}|_{H^1(0,1)} + |u_1|_{H^2(0,1)} + |h_1|_{H^2(0,1)}|u_2|_{H^1(0,1)} + |u_2|_{H^2(0,1)} + |h_1|_{H^2(0,1)}|h_1|^{3/2}_{H^1(0,1)})\left\{\int_0^1 \left(h_1 v^2 + \left(g + \frac{C^2}{(h_1)^3}\right)\Phi^2\right) dx\right\}. \quad (102)$$

Applying the Gronwall inequality for (102) and using the initial conditions, we obtain that

$$\int_0^1 (H_0 v^2 + g\Phi^2)\, dx \leq \int_0^1 \left(h_1 v^2 + \left(g + \frac{C^2}{(h_1)^3}\right)\Phi^2\right) dx = 0, \quad \forall\, t \in [0, T^*],$$

which implies that the solution is unique.

Finally, the proof of Theorem 2 is complete.

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References

[1] J. Pedlosky, Geophysical Fluid Dynamics, Springer-Verlag, New York, 1987.
[2] P. Orenga, Un the'ore'me d'existence de solutions d'un proble'me de shallow water, Arch. Ration. Mech. Anal. 130 (1995) 183–204.
[3] A. Majda, Compressible Fluid Flows and Systems of Conservation Laws in Several Space Variables, Springer-Verlag, New York, 1984.
[4] A. Majda, Vorticity and the mathematical theory of incompressible fluid flow, Comm. Pure Appl. Math. 39 (1986) S187–S220.
[5] C.D. Levermore, M. Sammartino, A shallow water model with eddy viscosity for basins with varying bottom topography, Nonlinearity 14 (2001) 1493–1515.
[6] A. Majda, A.L. Bertozzi, Vorticity and Incompressible Flow, in: Cambridge Texts in Applied Mathematics, Cambridge University Press, New York, 2002.
[7] M.J.P. Cullen, Analysis of the semi-geostrophic shallow water equations, Physica D 237 (2008) 1461–1465.
[8] J.M. Rakotoson, R. Temam, J. Tribbia, Remarks on the nonviscous shallow water equations, Indiana Univ. Math. J. 57 (2008) 2969–2998.
[9] M. Petcu, R. Temam, The one dimensional shallow water equations with Dirichlet boundary conditions on the velocity, Discrete Contin. Dyn. Syst. Ser. S 4 (2011) 209–222.
[10] P.A. Gilman, Magnetohydrodynamic "shallow water" equations for the solar tachocline, The Astrophy. J. Lett. 544 (2000) L79–L82.
[11] E.A. Spiegler, J.-P. Zahn, The solar tachocline, Astron. Astrophys. 265 (1992) 106–114.
[12] H. De Sterck, Hyperbolic theory of the "shallow water" magnetohydrodynamics equations, Phys. Plasmas 8 (2001) 3293–3304.
[13] P.J. Dellar, Hamiltonian and symmetric hyperbolic structures of shallow water magnetohydrodynamics, Phys. Plasmas 9 (2002) 1130–1136.
[14] P.J. Dellar, Dispersive shallow water magnetohydrodynamics, Phys. Plasmas 10 (2003) 581–590.
[15] Y.-J. Lee, C.-D. Munz, Advanced hyperbolic divergence cleaning scheme for shallow water magnetohydrodynamics, J. Hyperbolic Differ. Equ. 1 (2004) 171–195.
[16] T. Kroger, M. Lukacova-Medvidova, An evolution Galerkin scheme for the shallow water magnetohydrodynamic equations in two space dimensions, J. Comput. Phys. 206 (2005) 122–149.
[17] S. Qamar, G. Warnecke, Application of space–time CE/SE method to shallow water magnetohydrodynamic equations, J. Comput. Appl. Math. 196 (2006) 132–149.
[18] S. Qamar, S. Mudasser, A kinetic flux-vector splitting method for the shallow water magnetohydrodynamics, Comput. Phys. Comm. 181 (2010) 1109–1122.
[19] R. Touma, Unstaggered central schemes with constrained transport treatment for ideal and shallow water magnetohydrodynamics, Appl. Numer. Math. 60 (2010) 752–766.
[20] R. Touma, P. Arminjon, Central finite volume methods for ideal and shallow water magnetohydrodynamics, Internat. J. Numer. Methods Fluids 67 (2011) 155–174.
[21] S. Benzoni-Gavage, D. Serre, Multi-dimensional Hyperbolic Partial Differential Equations: First-order Systems and Applications, Oxford University Press, USA, 2007.