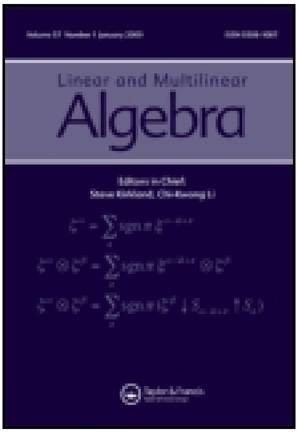

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Noncircular elliptic discs as numerical ranges of nilpotent operators

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Noncircular elliptic discs as numerical ranges of nilpotent operators†

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We show that (1) if A is a nonzero quasinilpotent operator with ran A^n closed for some $n \geq 1$, then its numerical range $W(A)$ contains 0 in its interior and has a differentiable boundary, and (2) a noncircular elliptic disc can be the numerical range of a nilpotent operator with nilpotency 3 on an infinite-dimensional separable space. (1) is a generalization of the known result for nonzero nilpotent operators, and (2) is in contrast to the finite-dimensional case, where the only elliptic discs which are the numerical ranges of nilpotent finite matrices are the circular ones centred at the origin.

Keywords: numerical range; nilpotent operator; quasinilpotent operator; essential numerical range

AMS Subject Classifications: 47A12; 15A60

For a bounded linear operator A on a complex Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$, its *numerical range* $W(A)$ is, by definition, the subset $\{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$ of the complex plane. The *numerical radius* $w(A)$ of A is $\sup\{|z| : z \in W(A)\}$. It is known that $W(A)$ is always bounded and convex, it is compact if H is finite dimensional, and $w(A)$ satisfies $\|A\|/2 \leq w(A) \leq \|A\|$. Other properties of the numerical range and numerical radius can be found in [11, Chapter 22] or [9].

The purpose of this article is to prove some results concerning the numerical ranges of nilpotent and quasinilpotent operators on infinite-dimensional spaces. Recall that an operator A is *nilpotent* with *nilpotency* n (≥ 1) if n is the smallest integer for which $A^n = 0$. It is *quasinilpotent* if its spectrum $\sigma(A)$ consists of 0 only. Obviously, nilpotent operators are quasinilpotent. The numerical ranges and numerical radii of nilpotent operators have been studied, e.g. in [8,10]. Among other things, it was shown in [8, Corollary 1.2] that if A is a nonzero nilpotent operator, then 0 belongs to the interior of $W(A)$ and $\partial W(A)$ is a differentiable curve. In Section 1, we first give its direct proof and then generalize it to certain

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quasinilpotent operators. We also prove an analogous result for the essential numerical range. The main result of this article is given in Section 2. We show that there exists a nilpotent operator A with nilpotency 3 such that $W(A)$ is an open noncircular elliptic disc. This is in striking contrast to the known finite-dimensional case: if the numerical range of a finite matrix A is an elliptic disc E, then the two foci of ∂E are the eigenvalues of A (cf. [15, Theorem 4.2] or [7, Theorem]), and thus if, in addition, A is nilpotent, then $W(A)$ must be a circular disc centred at the origin. Our result shows that the situation in the infinite-dimensional case is quite different. This is inspired by the recent work of Harris *et al.* [12] in which the authors show that there exists an operator A (on an infinite-dimensional separable space) such that $A^3 = I$ and $W(A)$ is an open circular disc centred at the origin. The general approach of our construction is similar to theirs, but the technical details are completely different.

1. Boundary of numerical range

We start by giving a direct matricial proof of [8, Corollary 1.2].

PROPOSITION 1.1 *If A is a nonzero nilpotent operator, then 0 is in the interior of $W(A)$ and $\partial W(A)$ is a differentiable curve.*

Proof We may assume that A is nilpotent with nilpotency n (≥ 2) on the separable space H. For each j, $1 \leq j \leq n$, let $\{x_k^{(j)}\}_k$ be an orthonormal basis of $\ker A^j \ominus \ker A^{j-1}$, and for each ℓ, $1 \leq \ell < \infty$, let H_ℓ be the closed subspace of H generated by $\{A^m x_k^{(j)} : 1 \leq j, k \leq \ell, 0 \leq m \leq j-1\}$. Then the H_ℓ's are all finite-dimensional invariant subspaces of A which are increasing with $\vee_\ell H_\ell = H$. Since $A|H_\ell$ is also nilpotent for each ℓ, we can find an orthonormal basis $\{e_j\}_j$ of H such that, for each ℓ, $\{e_1, \ldots, e_{\dim H_\ell}\}$ is a basis of H_ℓ with respect to which $A|H_\ell$ has an upper-triangular matrix representation. Thus A can be represented as $[a_{ij}]_{i,j=1}^{\infty}$ with $a_{ij} = 0$ for all $i \leq j$. Since A is nonzero, there exist some i_0 and j_0, $i_0 > j_0$, such that $a_{i_0 j_0} \neq 0$. Then the 2-by-2 matrix $\begin{bmatrix} 0 & a_{i_0 j_0} \\ 0 & 0 \end{bmatrix}$ can be dilated to A and thus $W\left(\begin{bmatrix} 0 & a_{i_0 j_0} \\ 0 & 0 \end{bmatrix}\right) = \{z \in \mathbb{C} : |z| \leq |a_{i_0 j_0}|/2\}$ is contained in $W(A)$. It follows that 0 is in the interior of $W(A)$.

If $\partial W(A)$ has a nondifferentiable point, say, λ, then λ must be in the spectrum of A (cf. [16, Theorem 2]). Since A is nilpotent, we have $\lambda = 0$, which contradicts what was proven above that 0 is in the interior of $W(A)$. Hence $\partial W(A)$ must be differentiable. ■

Operators with upper-triangular matrix representations were studied in more detail in [6, Section 2).

As was noted in [8, p. 718], the first assertion in the preceding proposition is not valid for a nonzero quasinilpotent operator. This is seen by the Volterra operator

$$(Af)(x) = \int_0^x f(t)\mathrm{d}t \quad \text{for } f \in L^2(0,1).$$

However, in this case, $\partial W(A)$ is still differentiable (cf. [11, p. 113]). With a slight modification, we can obtain a quasinilpotent counterexample to both assertions in Proposition 1.1. Indeed, let B be the (unique) operator on $L^2(0,1)$ with $B^2 = A$.

Then B is compact quasinilpotent with $W(B)$ contained in the sector in the first quadrant bounded by the lines $x = \pm y$ (cf. [14]). Hence 0 is in $\partial W(B)$ with the supporting lines $x = \pm y$ of $\overline{W(B)}$. In particular, $\partial W(B)$ is not differentiable at 0. In the following, we generalize Proposition 1.1 to cover a certain class of quasinilpotent operators.

Theorem 1.2 *If A is a nonzero quasinilpotent operator with* ran A^n *closed for some* $n \geq 1$, *then* 0 *is in the interior of* $W(A)$ *and* $\partial W(A)$ *is differentiable.*

Proof Assume that 0 is not in the interior of $W(A)$. Since 0 is in $\sigma(A)$ and hence in $\overline{W(A)}$, it must be in $\partial W(A)$. Then ker A = ker A^* (cf. [3, Lemma 1]). Hence $A = 0 \oplus B$ and $A^n = 0 \oplus B^n$ on ker $A \oplus \overline{\text{ran } A^*}$. Since B is one-to-one, the same is true for B^n. Together with the closedness of ran A^n = ran B^n, this implies that B^n is left invertible. Thus 0 is not in the left spectrum $\sigma_l(B^n)$ of B^n. However, since $\sigma_l(B^n)$ is contained in $\sigma(B^n) = \{0\}$ and is always nonempty, we must have $\sigma_l(B^n) = \{0\}$. This yields a contradiction. We conclude that 0 is in the interior of $W(A)$. The differentiability of $\partial W(A)$ follows as in Proposition 1.1. ∎

The preceding theorem generalizes the case $n = 1$ in [3, Corollary 2] and is indeed a generalization of Proposition 1.1 as there are nonnilpotent quasinilpotent operators A with ran A^n closed for all $n \geq 1$ (cf. [4, Example 5.4]).

Recall that the *essential numerical range* $W_e(A)$ of an operator A on an infinite-dimensional separable space H is the intersection of the closures of the numerical ranges $W(A + K)$, where K is any compact operator on H. The next proposition gives the essential version of Proposition 1.1.

Proposition 1.3 *If A is a noncompact nilpotent operator on an infinite-dimensional separable space H, then* 0 *is in the interior of* $W_e(A)$ *and* $\partial W_e(A)$ *is a differentiable curve.*

Proof Let $\mathcal{B}(H)$ (resp., $\mathcal{K}(H)$) denote the C^*-algebra (resp., self-adjoint ideal) of all operators (resp., compact operators) on H. Let $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ be the Calkin algebra on H, and $\pi : \mathcal{B}(H) \to \mathcal{C}(H)$ be the quotient map $\pi(T) = \widehat{T}$ for T in $\mathcal{B}(H)$. We represent $\mathcal{C}(H)$ as the C^*-algebra of all operators on a (nonseparable) space H' via the $*$-isomorphism $\pi' : \mathcal{C}(H) \to \mathcal{B}(H')$ (cf. [5, Theorem VIII.5.17]). Then $A' \equiv (\pi' \circ \pi)(A)$ is a nonzero nilpotent operator on H' with $\overline{W(A')} = W_e(A)$. Hence Proposition 1.1 yields that 0 is in the interior of $W_e(A)$ $(= \overline{W(A')})$ and $\partial W_e(A)$ $(= \partial W(A'))$ is differentiable. ∎

2. Noncircular elliptic disc

The main open problem in our present discussion is to characterize all the numerical ranges of nilpotent (resp., quasinilpotent) operators. In this section, we move one step forward on this problem by showing that noncircular elliptic discs can be such numerical ranges.

Theorem 2.1 *For any a, $0 \leq a \leq 1/3$, the open elliptic disc* $E_a \equiv \{x + iy \in \mathbb{C} : x^2 + (1/(1 - a^2))y^2 < 1\}$ *is the numerical range of some nilpotent operator with nilpotency* 3 *on a separable space.*

The asserted operator is constructed by taking the direct sum (or direct integral) of 3-by-3 nilpotent matrices of the form

$$\begin{bmatrix} 0 & \lambda & \lambda \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{bmatrix}.$$

In the following, let B be the matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The properties of the numerical range of B were studied in [8, Lemma 1.3]. Among other things, it is known that (1) $W(B)$ is symmetric with respect to the x-axis, (2) $W(B)$ is contained in the closed rectangular region $[-1/2, 1] \times [-\sqrt{3}/2, \sqrt{3}/2]$, (3) $w(B) = 1$ and (4) $\partial W(B)$ contains a line segment on the line $x = -1/2$. The proof of Theorem 2.1 for $a = 1/3$ is via a series of lemmas, the first of which says that $W(B)$ is contained in the elliptic disc $\overline{E}_{1/3} = \{x + iy \in \mathbb{C} : x^2 + (9/8)y^2 \leq 1\}$.

Lemma 2.2 *For any* a, $0 \leq a \leq 1$, *let* $C_a = \begin{bmatrix} a & 2(1-a^2)^{1/2} \\ 0 & -a \end{bmatrix}$. *Then* $W(B) \subseteq W(C_a)$ *if and only if* $a \leq 1/3$.

Proof Note that $W(B) \subseteq W(C_a)$ if and only if $\max \sigma(\mathrm{Re}(e^{-i\theta}B)) \leq \max \sigma(\mathrm{Re}(e^{-i\theta}C_a))$ for all $\theta, -\pi \leq \theta \leq \pi$. A simple computation yields that the characteristic polynomial of $\mathrm{Re}(e^{-i\theta}B)$ (resp., $\mathrm{Re}(e^{-i\theta}C_a)$) is $z^3 - (3/4)z - (1/4)\cos\theta$ (resp., $z^2 - (1 - a^2 \sin^2\theta)$). Thus we obtain $\max \sigma(\mathrm{Re}(e^{-i\theta}B)) = \cos(\theta/3)$ (resp., $\max \sigma (\mathrm{Re}(e^{-i\theta}C_a)) = (1 - a^2 \sin^2\theta)^{1/2}$) for all θ, $-\pi \leq \theta \leq \pi$. Since $\cos(\theta/3) \leq (1 - a^2 \sin^2\theta)^{1/2}$ if and only if

$$a^2 \leq \frac{\sin^2(\theta/3)}{\sin^2\theta} = \frac{1}{(3 - 4\sin^2(\theta/3))^2},$$

and since $1/(3 - 4\sin^2(\theta/3)) \geq 1/3$ for all θ, we infer that $W(B) \subseteq W(C_a)$ if and only if $a \leq 1/3$ as asserted. ■

We now rotate scalar multiples $(re^{it})B$ of B around the origin so that their numerical ranges are all contained in $\overline{E}_{1/3}$ and the boundaries are all tangent to $\partial E_{1/3}$. The next two lemmas find, for each fixed t, the corresponding r and the corresponding tangent point, respectively.

Lemma 2.3 *For* $r \geq 0$ *and* $0 \leq t \leq \pi/2$, *the numerical range* $W(re^{it}B)$ *is contained in* $\overline{E}_{1/3}$ *if and only if* $r^2 \leq \min\{(8 + \cos^2\theta)/(9\cos^2((\theta - t)/3)) : -\pi \leq \theta \leq \pi\}$.

Proof Note that $\overline{E}_{1/3} = W(C_{1/3})$, where $C_{1/3} = \begin{bmatrix} 1/3 & 4\sqrt{2}/3 \\ 0 & -1/3 \end{bmatrix}$. As in the proof of Lemma 2.2, we have $\max \sigma(\mathrm{Re}(e^{-i\theta}re^{it}B)) = r\cos((\theta - t)/3)$ and $\max \sigma(\mathrm{Re}(e^{-i\theta}C_{1/3})) = (1 - (1/9)\sin^2\theta)^{1/2}$. Thus $W(re^{it}B) \subseteq W(C_{1/3})$ if and only if $r^2\cos^2((\theta - t)/3) \leq 1 - (1/9)\sin^2\theta$ for all θ. Our assertion follows immediately. ■

Lemma 2.4 *For each fixed* t, $0 \leq t \leq \pi/2$, *assume that* $f_t(\theta) = (8 + \cos^2\theta)/(9\cos^2((\theta - t)/3))$ *attains its minimum value over* $[-\pi, \pi]$ *at* θ_t. *Then* $\theta_0 = 0$, $\theta_{\pi/2} = \pi/2$ *and* $t \leq \theta_t \leq \pi/2$.

Proof We first prove that $\theta_0 = 0$. This is equivalent to showing that $8 + \cos^2\theta \geq 9\cos^2(\theta/3)$ for all θ. Since $\cos\theta = 4\cos^3(\theta/3) - 3\cos(\theta/3)$, this is the same as $2\cos^6(\theta/3) - 3\cos^4(\theta/3) + 1 \geq 0$ for all θ. Letting $x = \theta/3$ and $g(x) = 2\cos^6 x - 3\cos^4 x + 1$, we derive from $g'(x) = (3/2)\sin^3(2x) = 0$ that $x = 0$ gives the minimum value 0 of $g(x)$ on $[-\pi/3, \pi/3]$. Hence $\theta_0 = 0$.

From now on, we consider only $0 < t \leq \pi/2$. Let $r_t = \min\{\sqrt{f_t(\theta)} : -\pi \leq \theta \leq \pi\}$. If $-t < \theta < t$, then $\cos^2\theta > \cos^2 t$ and hence

$$f_t(\theta) > \frac{1}{9}(8 + \cos^2 t) = f_t(t) \geq r_t^2,$$

which shows that θ_t is not in $(-t, t)$. Next assume that $\pi/2 < \theta \leq \pi$. Then $0 < \cos((\theta - t)/3) < \cos(((\pi/2) - t)/3)$ and hence

$$f_t(\theta) > \frac{8}{9\cos^2(((\pi/2) - t)/3)} = f_t(\pi/2) \geq r_t^2.$$

Thus θ_t is not in $(\pi/2, \pi]$. Finally, if $-\pi \leq \theta \leq -t$, then

$$-\frac{\pi}{2} \leq -\frac{\pi + t}{3} \leq \frac{\theta - t}{3} < \frac{\theta + t}{3} \leq 0.$$

Hence

$$f_t(\theta) > \frac{8 + \cos^2\theta}{9\cos^2((\theta + t)/3)} = f_t(-\theta) \geq r_t^2,$$

and thus θ_t is not in $[-\pi, -t]$. We conclude that $t \leq \theta_t \leq \pi/2$ as asserted. ■

It follows from the above that, for each t, $0 \leq t \leq \pi/2$, the quantity $r_t \equiv \sqrt{f_t(\theta_t)}$ is the minimum of $\sqrt{f_t(\theta)}$ over $[-\pi, \pi]$, and $W(r_t e^{it} B)$ is contained in $\overline{E}_{1/3}$ with their boundaries tangent to each other at the common tangent point α_t of the line $x\cos\theta_t + y\sin\theta_t = (1 - (1/9)\sin^2\theta_t)^{1/2}$ with $\partial W(r_t e^{it} B)$ and $\partial E_{1/3}$ (see Figure 1).

We next show that the θ_t in the preceding lemma is unique. This is done via the following lemma.

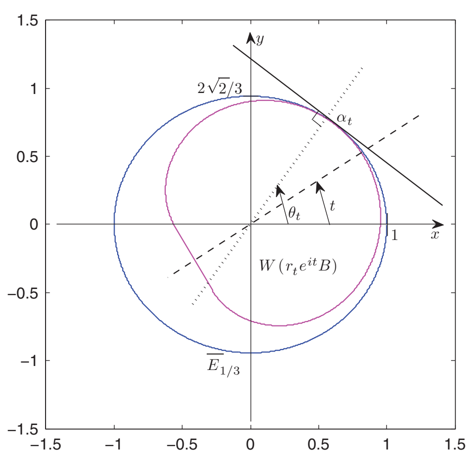

Figure 1. $W(r_t e^{it} B), \overline{E}_{1/3}$ and the common tangent point α_t.

LEMMA 2.5 *For each* $t, 0 \le t \le \pi/2$, *let* $g_t(\theta) = (\sin(2\theta))/\sin(2(\theta - t)/3)$ *for* $t < \theta < \pi/2$. *Then* g_t *is strictly decreasing on* $(t, \pi/2)$.

Proof We check that $g_t'(\theta) < 0$ on $(t, \pi/2)$. Since

$$g_t'(\theta) = \frac{\sin(2(\theta - t)/3) \cdot 2\cos(2\theta) - \sin(2\theta) \cdot (2/3)\cos(2(\theta - t)/3)}{\sin^2(2(\theta - t)/3)},$$

we need only check the negativity of its numerator. Note that

$$\begin{aligned}
&\sin\left(\frac{2}{3}(\theta - t)\right) \cdot \cos(2\theta) - \frac{1}{3}\cos\left(\frac{2}{3}(\theta - t)\right) \cdot \sin(2\theta) \\
&= \sin\left(\frac{2}{3}(\theta - t) - 2\theta\right) + \frac{2}{3}\cos\left(\frac{2}{3}(\theta - t)\right) \cdot \sin(2\theta) \\
&= -\sin\left(\frac{2}{3}(2\theta + t)\right) + \frac{1}{3}\left(\sin\left(\frac{2}{3}(4\theta - t)\right) + \sin\left(\frac{2}{3}(2\theta + t)\right)\right) \\
&= -\frac{2}{3}\sin\left(\frac{2}{3}(2\theta + t)\right) + \frac{1}{3}\sin\left(\frac{2}{3}(4\theta - t)\right).
\end{aligned}$$

Hence we need to show that $h_t(\theta) \equiv 2\sin((2/3)(2\theta + t)) - \sin((2/3)(4\theta - t)) > 0$ on $(t, \pi/2)$. We have $h_t'(\theta) = (8/3)[\cos((2/3)(2\theta + t)) - \cos((2/3)(4\theta - t))]$, $0 \le 2t < (2/3)(2\theta + t) < (2/3)(\pi + t) \le \pi$ and $0 \le 2t < (2/3)(4\theta - t) < (2/3)(2\pi - t) \le (4/3)\pi$. If $0 \le (2/3)(4\theta - t) \le \pi$, then, since $(2/3)(2\theta + t) < (2/3)(4\theta - t)$, we have $h_t'(\theta) > 0$. Otherwise, if $\pi < (2/3)(4\theta - t) \le (4/3)\pi$, then, since $(2/3)(4\theta - t) - \pi < \pi - (2/3)(2\theta + t)$, we also have $h_t'(\theta) > 0$. Thus $h_t(\theta)$ is strictly increasing and hence $h_t(\theta) > \lim_{\theta \to t^+} h_t(\theta) = \sin(2t) \ge 0$ for θ in $(t, \pi/2)$. This yields that $g_t'(\theta) < 0$ on $(t, \pi/2)$ and our assertion follows. ∎

LEMMA 2.6 *For each fixed* $t, 0 \le t \le \pi/2$, *let* $f_t(\theta) = (8 + \cos^2\theta)/(9\cos^2((\theta - t)/3))$ *and* $r_t = \min\{\sqrt{f_t(\theta)} : -\pi \le \theta \le \pi\}$. *Then there exists a unique* θ_t *in* $[t, \pi/2]$ *such that* $f_t(\theta_t) = r_t^2$.

Proof To check the uniqueness of θ_t, let θ_1 and θ_2 in $[t, \pi/2]$ be such that $f_t(\theta_1) = f_t(\theta_2) = r_t^2$ (by Lemma 2.4). If $u_t(\theta) = 8 + \cos^2\theta - 9r_t^2\cos^2((\theta - t)/3)$ for θ in $[t, \pi/2]$, then $u_t(\theta_1) = u_t(\theta_2) = 0 = \min\{u_t(\theta) : t \le \theta \le \pi/2\}$. We must have $u_t'(\theta_1) = u_t'(\theta_2) = 0$. This is the same as

$$-2\cos\theta_j \sin\theta_j + 6r_t^2 \cos\frac{\theta_j - t}{3} \cdot \sin\frac{\theta_j - t}{3} = 0$$

or $\sin(2\theta_j) = 3r_t^2 \sin((2/3)(\theta_j - t))$, $j = 1, 2$. Lemma 2.5 implies that $\theta_1 = \theta_2$, completing the proof. ∎

Finally, we prove that the function $t \mapsto \theta_t$ maps $[0, \pi/2]$ onto itself.

LEMMA 2.7 *If* θ_t *is defined as in Lemma* 2.6, *then* θ_t *is continuous for* t *in* $[0, \pi/2]$ *and* $\{\theta_t : 0 \le t \le \pi/2\} = [0, \pi/2]$.

Proof To prove the continuity of θ_t, let $\{t_n\}_n$ be a sequence in $[0,\pi/2]$ which converges to t_0 as n approaches ∞. We may assume that $\{\theta_{t_n}\}_n$ also converges, say, to θ_0. We have

$$f_{t_0}(\theta_{t_0}) = r_{t_0}^2 \leq f_{t_0}(\theta_0). \tag{$*$}$$

To prove that the above relation is actually an equality, note that for any $\varepsilon > 0$ there is some θ in $[-\pi,\pi]$ such that $f_{t_0}(\theta) < f_{t_0}(\theta_{t_0}) + \varepsilon$. Since $\lim_n t_n = t_0$, we have $f_{t_n}(\theta) < f_{t_0}(\theta) + \varepsilon$ for all large n. These, together with $f_{t_n}(\theta_{t_n}) = r_{t_n}^2 \leq f_{t_n}(\theta)$, yield $f_{t_n}(\theta_{t_n}) < f_{t_0}(\theta_{t_0}) + 2\varepsilon$ for all large n. Letting n approach ∞, we obtain $f_{t_0}(\theta_0) \leq f_{t_0}(\theta_{t_0}) + 2\varepsilon$ for all $\varepsilon > 0$. It follows that $f_{t_0}(\theta_0) \leq f_{t_0}(\theta_{t_0})$. Together with $(*)$, this shows the equality of $f_{t_0}(\theta_0)$ and $f_{t_0}(\theta_{t_0})$. Lemma 2.6 then yields that $\theta_0 = \theta_{t_0}$. Hence $\lim_n t_n = t_0$ implies that $\lim_n \theta_{t_n} = \theta_{t_0}$. Thus θ_t is continuous in t as asserted.

Since $\theta_0 = 0$ and $\theta_{\pi/2} = \pi/2$ by Lemma 2.3, the continuity of θ_t yields that the function $t \mapsto \theta_t$ maps $[0,\pi/2]$ onto itself. ■

Now we are ready for the proof of Theorem 2.1.

Proof of Theorem 2.1 We only prove for the case $a = 1/3$; other values of a can be done similarly. Let $\{\theta_n\}_{n=1}^{\infty}$ be a countable dense subset of $[0,\pi/2]$, and let $\{t_n\}_n$ be in $[0,\pi/2]$ such that $\theta_{t_n} = \theta_n$ for all n (by Lemma 2.7). If $r_n = \sqrt{f_{t_n}(\theta_n)} = \min\{\sqrt{f_{t_n}(\theta)} : -\pi \leq \theta \leq \pi\}$, then the 3-by-3 nilpotent matrix $B_n \equiv r_n e^{it_n} B$ is such that $W(B_n) \subseteq \overline{E}_{1/3}$ and $\partial W(B_n) \cap \partial E_{1/3}$ consists of the intersection point α_n of $W(B_n)$ (or $\overline{E}_{1/3}$) with its supporting line $x\cos\theta_n + y\sin\theta_n = (1-(1/9)\sin^2\theta_n)^{1/2}$ (cf. Figure 2.5). If $C = \sum_n \oplus B_n$, then $C^3 = 0$ and $W(C) = (\cup_n W(B_n))^\wedge$, the convex hull of $\cup_n W(B_n)$ (cf. [13, Corollary 3.5]). Note that the denseness of $\{\theta_n\}_n$ in $[0,\pi/2]$ implies the same for $\{\alpha_n\}_n$. Hence $W(C)$ is contained in $\overline{E}_{1/3}$ and contains $\{\alpha_n : n \geq 1\} \cup (E_{1/3} \cap \{x+iy \in \mathbb{C}: x, y \geq 0\})$. Let $D = C \oplus (-C) \oplus C^* \oplus (-C^*)$. Then $W(D) = \{\pm\alpha_n, \pm\overline{\alpha_n} : n \geq 1\} \cup E_{1/3}$. Finally, if $A = \sum_{n=1}^{\infty} \oplus (1-(1/n))D$, then A is nilpotent with nilpotency 3 and $W(A) = E_{1/3}$. ■

We remark that in the preceding proof, we may take the direct integral, instead of the direct sum, of the $r_t e^{it}B$'s. Indeed, if C' is the direct integral $\int_{[0,\pi/2]}^{\oplus} r_t e^{it} B \, dt$, then $W(C') = \cap\{(\cup_{t \in [0,\pi/2] \setminus \triangle} W(r_t e^{it} B))^\wedge : \triangle$ Borel subset of $[0,\pi/2]$ with Lebesgue measure zero$\}$ (cf. [13, Theorem 3.3]). Hence if $A' = C' \oplus (-C') \oplus C'^* \oplus (-C'^*)$, then $A'^3 = 0$ and $W(A') = E_{1/3}$.

Corollary 2.9 *For any a, $0 \leq a \leq 1/3$, and any countable subset $\{\alpha_n\}_{n=1}^{\infty}$ of ∂E_a, there is a nilpotent operator A with nilpotency 3 (on a separable space) such that $W(A) = E_a \cup \{\alpha_n : n \geq 1\}$.*

Proof We assume that $a = 1/3$. For each α_n in the first quadrant of $\partial E_{1/3}$, let θ_n in $[0,\pi/2]$ be the angle from the positive x-axis to the ray from the origin which is perpendicular to the tangent line of $\partial E_{1/3}$ at α_n (see Figure 1). If t_n in $[0,\pi/2]$ is such that $\theta_{t_n} = \theta_n$ (by Lemma 2.7) and $r_n = (8+\cos^2\theta_n)^{1/2}/(3\cos((\theta_n - t_n)/3))$, then $B_1 \equiv \sum_n \oplus (r_n e^{it_n} B)$ is such that $B_1^3 = 0$, $W(B_1) \subseteq \overline{E}_{1/3}$ and $\partial W(B_1) \cap \partial E_{1/3} = \{\alpha_n : n \geq 1\} \cap \{x+iy \in \mathbb{C}: x, y \geq 0\}$. By symmetry, we obtain $B_j, j = 2, 3, 4$, with similar properties for the α_n's in the jth quadrant. If A_1 is the nilpotent operator with nilpotency 3 such that $W(A_1) = E_{1/3}$ (by Theorem 2.1), then $A = A_1 \oplus (\sum_{j=1}^{4} \oplus B_j)$ is the asserted operator. ■

Following a similar procedure as above, we can also construct, for any a, $0 \leq a \leq 1/2$, and any countably many α_n's on the boundary of E_a, a nilpotent operator A with nilpotency 4 on a separable space with $W(A) = E_a \cup \{\alpha_n : n \geq 1\}$. In comparison, for a nilpotent operator with nilpotency 2, that is, a square-zero operator, its numerical range can only be an (open or closed) circular disc centred at the origin (cf. [17, Theorem 2.1 (1)]). It is thus natural to ask whether there is a nilpotent operator A with nilpotency at least 3 for which $W(A)$ is a closed noncircular elliptic disc. If we allow such an A to act on a nonseparable space, then the answer is affirmative. This is seen by using the Berberian [1] representation. Namely, if A on the (infinite-dimensional separable) space H is such that $A^3 = 0$ and $W(A) = E_{1/3}$, and K is a (necessarily nonseparable) space containing H with a unital $*$-isomorphism $\alpha : \mathcal{B}(H) \to \mathcal{B}(K)$ such that $W(\alpha(T)) = \overline{W(T)}$ for all T in $\mathcal{B}(H)$ (among many other properties), then $\alpha(A)$ on K is such that $\alpha(A)^3 = 0$ and $W(\alpha(A)) = \overline{E}_{1/3}$ (cf. [2, Proposition]). The problem remains as to whether such an operator can exist on a separable space.

In view of the square-zero case, we may also ask whether there is a nilpotent operator A with nilpotency n (≥ 3) on a separable space with $W(A) \subsetneq \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $W(A) \cap \partial\mathbb{D}$ an arc of $\partial\mathbb{D}$. Note that if we only require that $\partial W(A) \cap \partial\mathbb{D}$ be an arc of $\partial\mathbb{D}$, then such an A indeed exists. For example, if A is the direct integral $\int_{[0,\pi/4]}^{\oplus} e^{it} B \, \mathrm{d}t$, then $W(A) \subsetneq \overline{\mathbb{D}}$ and $\partial W(A) \cap \partial\mathbb{D} = \{e^{it}: 0 \leq t \leq \pi/4\}$ by [13, Theorem 3.3]. In the most general case, a characterization of subsets of the plane which are the numerical ranges of some nilpotent operators (with nilpotency ≥ 3) or some quasinilpotent operators is desirable.

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