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4-ordered-Hamiltonian problems of the generalized Petersen graph $GP(n, 4)$[☆]

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A R T I C L E I N F O

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A B S T R A C T

A graph G is *k-ordered* if for every sequence of k distinct vertices of G, there exists a cycle in G containing these k vertices in the specified order. It is *k-ordered-Hamiltonian* if, in addition, the required cycle is a Hamiltonian cycle in G. The question of the existence of an infinite class of 3-regular 4-ordered-Hamiltonian graphs was posed in Ng and Schultz in 1997 [2]. At the time, the only known examples of such graphs were K_4 and $K_{3,3}$. Some progress was made by Mészáros in 2008 [21] when the Petersen graph was found to be 4-ordered and the Heawood graph was proved to be 4-ordered-Hamiltonian; moreover, an infinite class of 3-regular 4-ordered graphs was found. In 2010, a subclass of the generalized Petersen graphs was shown to be 4-ordered in Hsu et al. [9], with an infinite subset of this subclass being 4-ordered-Hamiltonian, thus answering the open question. However, these graphs are bipartite. In this paper we extend the result to another subclass of the generalized Petersen graphs. In particular, we find the first class of infinite non-bipartite graphs that are both 4-ordered-Hamiltonian and 4-ordered-Hamiltonian-connected, which can be seen as a solution to an extension of the question posted in Ng and Schultz in 1997 [2]. (A graph G is *k-ordered-Hamiltonian-connected* if for every sequence of k distinct vertices $a_1, a_2, \ldots, a_k$ of G, there exists a Hamiltonian path in G from a_1 to a_k where these k vertices appear in the specified order.)

1. Introduction and preliminaries

Throughout this paper we use standard graph theory terminology as in [1]. A graph G is *Hamiltonian* if it contains a Hamiltonian cycle, that is, a cycle that contains all vertices of G; a graph G is *Hamiltonian-connected* if, for every two distinct vertices u and v, there exists a path between u and v containing all vertices of G, that is, a *Hamiltonian path* between u and v. Clearly, if G is Hamiltonian connected, then G is Hamiltonian. If the graph is bipartite, then obviously it cannot be Hamiltonian-connected. The corresponding concept is the following: for every two distinct vertices u and v belonging to different partite sets, there is a Hamiltonian path between them; such a graph is called *Hamiltonian-laceable*.

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A graph G is called *k-ordered* if for any sequence of k distinct vertices of G, there exists a cycle in G containing these k vertices in the specified order. It is *k-ordered-Hamiltonian* if, in addition, the required cycle is Hamiltonian in G. This concept was introduced by Ng and Schultz [2], and the following open problem was posed: Find an infinite class of 3-regular 4-ordered-Hamiltonian graphs. (Incidentally, when the concept of k-ordered-Hamiltonian was introduced by Ng and Schultz [2], it was simply called k-ordered. Only later did researchers use the terms k-ordered and k-ordered-Hamiltonian as defined here.) Many papers [3–8] studied sufficient conditions for k-orderedness; in particular, Faudree [6] provides a comprehensive survey. Recently, Hsu et al. [9] answered affirmatively the open question of whether there exists an infinite class of 3-regular 4-ordered-Hamiltonian graphs posed by Ng and Schultz [2] by considering a subclass of the generalized Petersen graphs. The answer given in [9] is a class of bipartite graphs. In this paper we provide an example of a class of non-bipartite graphs. In addition, these graphs have important properties other than 4-ordered-Hamiltonian.

Similar to the concept of k-ordered-Hamiltonian, there are a number of related concepts. A graph G is *k-ordered-Hamiltonian-connected* if for any sequence of k distinct vertices $t_1, t_2, \ldots, t_k$ of G, there exists a Hamiltonian path between t_1 and t_k in G containing these k vertices in the specified order. If G is bipartite, then it is *k-ordered-Hamiltonian-laceable* if for any sequence of k distinct vertices $t_1, t_2, \ldots, t_k$ of G where t_1 and t_k are in different partite sets, there exists a Hamiltonian path between t_1 and t_k in G containing these k vertices in the specified order. Indeed, [2] posed the following question: study the existence of k-ordered-Hamiltonian-connected graphs with small degrees. Indeed, as an extension to the question whether there exists an infinite class of 3-regular 4-ordered-Hamiltonian graphs [2], one can ask whether there exists an infinite class of 3-regular (bipartite) graphs that are 4-ordered-Hamiltonian and 4-ordered-Hamiltonian-laceable. Similarly, one can ask whether there exists an infinite class of 3-regular (non-bipartite) graphs that are 4-ordered-Hamiltonian and 4-ordered-Hamiltonian-connected.

The Petersen graph is an important graph in graph theory and there are several generalizations of it. One such generalization is the class of generalized Petersen graphs introduced by Watkins [10], which has attracted much research throughout the years. Some recent research include [11–15]. The *generalized Petersen graph* $GP(n, k)$, where $1 \le k \le (n-1)/2$, has $\{u_i, v_i : 0 \le i < n\}$ as its vertex set. There are three types of edges. The first is of the form (u_i, u_{i+1}) (with $i+1$ computed modulo n) for $0 \le i < n$. The second is of the form (v_i, v_{i+k}) (with $i+k$ computed modulo n) for $0 \le i < n$. The third is of the form (u_i, v_i) for $0 \le i < n$, which will be called *columns*. It is clear that $GP(n, k)$ is 3-regular. We note that the subgraph induced by the vertices u_i, $0 \le i < n$, form an n-cycle, and the subgraph induced by the vertices v_i, $0 \le i < n$, form essentially a circulant graph. So $GP(5, 2)$ is the Petersen graph. (We remark that in [12], the authors defined the $GP(n, k)$ for the range $1 \le k < n$. The two definitions are equivalent except for the case $k = n/2$ when n is even. If $n/2 < k < n$, then $GP(n, k)$ is isomorphic to $GP(n, n-k)$. If $k = n/2$, then the resulting graph is not trivalent.) One major task was to determine which of these graphs are Hamiltonian. There were incremental results in various papers [16,17]. The complete classification was finally solved by Alspach [18]: $GP(n, k)$ is Hamiltonian except for $GP(n, 2)$ for $n \equiv 5 \pmod 6$. We refer the reader to Alspach [18] for the history, motivation and development of this problem and its solution. For the related problem in classifying which of these graphs are Hamiltonian-connected/Hamiltonian-laceable, it is still unsolved. Alspach conjectured over twenty years ago that if $GP(n, k)$ is not isomorphic to $GP(6m+5, 2)$, and n and k are relatively prime, then $GP(n, k)$ is Hamiltonian-connected unless it is bipartite, in which case it is Hamiltonian-laceable. We note that $GP(n, k)$ is bipartite if and only if n is even and k is odd. Alspach [12] commented that this condition on n and k is not well understood, and further commented that this condition may be misleading after proving that the conjecture is true for $k = 1, 2, 3$ although the relatively prime condition is far from necessary. The more general 4-ordered-Hamiltonian-connected result of this paper will validate this conjecture for $k = 4$. At the same time, it also indicates that the relatively prime condition is somewhat misleading as Alspach has commented in [12]. We will propose a refinement of this conjecture in the conclusion.

It turns out that the generalized Petersen graphs form a rich class of examples for the k-ordered problem. In particular, Hsu et al. [9] proved that $GP(n, 2)$ is not 4-ordered-Hamiltonian for every n; for $n \ge 7$, $GP(n, 3)$ is 4-ordered-Hamiltonian if and only if n is even and either $n = 18$ or $n \ge 24$; if n is even, then $GP(n, 3)$ is 4-ordered-Hamiltonian-laceable if and only if $n \ge 10$; if n is odd, then $GP(n, 3)$ is 4-ordered-Hamiltonian-connected if and only if $n = 15$ or $n \ge 19$. We note that the 4-ordered-Hamiltonian graphs in this class are all bipartite and they are also 4-ordered-Hamiltonian-laceable. So this gives an infinite class of 3-regular (bipartite) graphs that are 4-ordered-Hamiltonian and 4-ordered-Hamiltonian-laceable. In this paper we give an infinite class of non-bipartite graphs that are 4-ordered-Hamiltonian and 4-ordered-Hamiltonian-connected. Given the result on $GP(n, 3)$, it is natural to consider $GP(n, 4)$. Indeed, $GP(n, 4)$ provides a good candidate. (Note that since n is even, $GP(n, 4)$ is not bipartite; thus 4-ordered-Hamiltonian-connectedness is the correct measure.) We will show that except for small n, $GP(n, 4)$ is 4-ordered-Hamiltonian and 4-ordered-Hamiltonian-connected.

2. The main results

In this section, we determine for which n the graph $GP(n, 4)$ is 4-ordered-Hamiltonian and which of them are 4-ordered-Hamiltonian-connected. We first note that these two properties are not comparable. Clearly 4-ordered-Hamiltonicity does not imply 4-ordered-Hamiltonian-connectedness as the latter concept only applies to non-bipartite graphs. On the other hand, $GP(20, 4)$ is 4-ordered-Hamiltonian-connected but it is not 4-ordered-Hamiltonian. Throughout this section we assume that $n \ge 9$. We first consider the 4-ordered-Hamiltonian-connectedness problem. We note that since $GP(n, 4)$ is not bipartite, this is the correct measure. Like other papers in this area such as Alspach [12], we use induction. We first

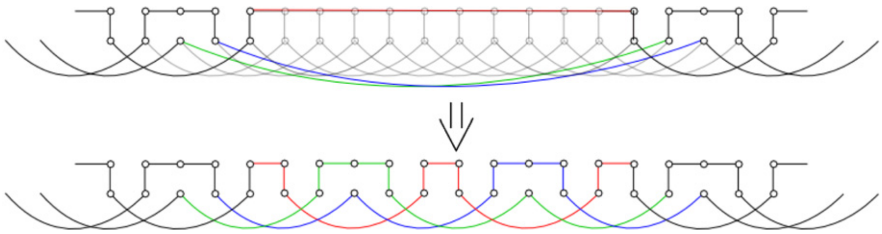

Fig. 1. Example of a substitution. The faded vertices represent the vertices in F'. Notice how the colored edges are replaced by paths spanning the vertices of F'.

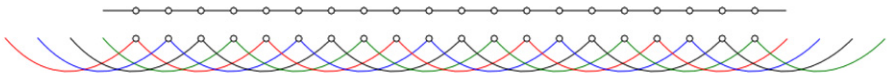

Fig. 2. Case 11111.

have to establish the base cases. This is done by a computer verification in the next lemma. We defer the discussion of the computer program to Section 3.

Lemma 2.1. *Let $n \in [9, 96]$. Then $GP(n, 4)$ is not 4-ordered-Hamiltonian-connected if $n \in [9, 17]$. Moreover, $GP(n, 4)$ is 4-ordered-Hamiltonian-connected if $n \in [18, 96]$.*

Theorem 2.2. *Let $n \geq 9$. Then $GP(n, 4)$ is 4-ordered-Hamiltonian-connected if and only if $n \geq 18$.*

Proof. We apply induction on n, with base cases for n up to 96 established in Lemma 2.1. Let $m > 96$. Assume that the statement holds for $GP(p, 4)$ for all $p < m$. Let x_1, x_2, x_3, x_4 be the four prescribed vertices in this specific order. We now select a set of 48 vertices $F = \{u_i, u_{i+1}, \ldots, u_{i+23}, v_i, v_{i+1}, \ldots, v_{i+23}\}$ forming 24 consecutive columns from $GP(m, 4)$ such that none of x_1, x_2, x_3, x_4 is in F. We can find such a set because of the assumption that $m > 96$. Without loss of generality (for notational convenience), assume that $i = 5$. We also define the set $F' = \{u_7, u_8, \ldots, u_{26}, v_7, v_8, \ldots, v_{26}\} \subseteq F$ of twenty columns for our induction step. We use only 20 columns to form F', but for later cases the reason for selecting 24 columns will become apparent.

Take $GP(m, 4)$ and delete the vertices of F', u_7 through u_{26} and v_7 through v_{26}. We then add edges (u_6, u_{27}), (v_3, v_{27}), (v_4, v_{28}), (v_5, v_{29}), (v_6, v_{30}), resulting in a graph, G', isomorphic to $GP(m - 20, 4)$. By our induction hypothesis, $GP(m, 4) - F'$ is 4-ordered-Hamiltonian-connected, that is, there is a Hamiltonian path P_0 between x_1 and x_4 containing x_1, x_2, x_3, x_4 in the given order. To complete the induction step, we obtain a Hamiltonian path P between x_1 and x_4 of $GP(m, 4)$ using P_0 by inserting the vertices $u_7, u_8, \ldots, u_{26}, v_7, v_8, \ldots, v_{26}$ in P_0. Therefore, we consider which of the following five edges (u_6, u_{27}), (v_3, v_{27}), (v_4, v_{28}), (v_5, v_{29}), (v_6, v_{30}) are in P_0. For each of these edges that is in P_0, we find a path consisting of vertices of F' whose endpoints are the same as the given edge; moreover, we need to ensure that all vertices of F' are spanned by the union of all these paths and that these paths are vertex-disjoint. This will give a Hamiltonian path between x_1 and x_4 of $GP(m, 4)$ containing x_1, x_2, x_3, x_4 in the given order.

For example, if (u_6, u_{27}) is in C_0, we find a path p_1 consisting only of vertices in F', with endpoints u_7 and u_{26}. We can extend our path of G' by replacing (u_6, u_{27}) with p_1 combined with the edges (u_6, u_7) and (u_{26}, u_{27}). Similarly, if (v_6, v_{30}) is in C_0, then we include a path p_6 from v_{10} to v_{26} and replace (v_6, v_{30}) with path p_2 and the edges (v_6, v_{10}) and (v_{26}, v_{30}). Similarly the paths p_3, p_4, p_5 replace (v_3, v_{27}), (v_4, v_{28}), (v_5, v_{29}), respectively, if necessary. Moreover, p_i's span the vertices of F' and they are vertex-disjoint. So this gives a Hamiltonian path between x_1 and x_4 of $GP(m, 4)$ containing x_1, x_2, x_3, x_4 in the given order. Fig. 1 illustrates a substitution of an edge in P_0 with a path. (Note: throughout the paper, each figure of this type consists of consecutive columns in the graph where the upper vertices are the u_i's and the lower vertices are the v_i's.)

Therefore, we must consider which of the five edges are in the Hamiltonian path P_0 of G'. Since each of the five edges is either in P_0 or not in P_0, there are a total of 32 cases. To help keep track of these cases, we will assign a 5-digit binary code to each case, depending on which of the edges

$$(u_6, u_{27}), (v_3, v_{27}), (v_4, v_{28}), (v_5, v_{29}), (v_6, v_{30})$$

are in C_0. The first digit corresponds to (u_6, u_{27}) with a "0" indicating the absence of this edge and a "1" indicating its presence in C_0. The second, third, fourth, and fifth digits correspond to (v_3, v_{27}), (v_4, v_{28}), (v_5, v_{29}), (v_6, v_{30}), respectively.

We now consider these 32 cases separately. Many of these cases occur in pairs that are symmetric. To be precise, $(a_0, a_1, a_2, a_3, a_4)$ and $(a_0, a_4, a_3, a_2, a_1)$ are symmetric under reflection. We now present the solutions to the majority of these cases in Figs. 2–16; each solution is presented in the form of a collection of replacement paths. In each figure, the top row of vertices represents, from left to right, u_7 through u_{26}, and the bottom row of vertices represents, also from left to right, v_7 through v_{26}. Each different path is color coded for easy reference. We note that these paths do not intersect and they span the vertices of F'. For cases that are symmetric, only one will be presented.

Fig. 3. Case 11110, becomes 10111 when reflected.

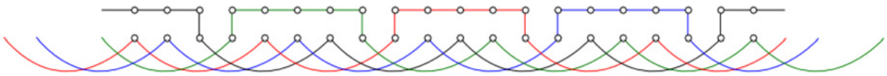

Fig. 4. Case 11101, becomes 11011 when reflected.

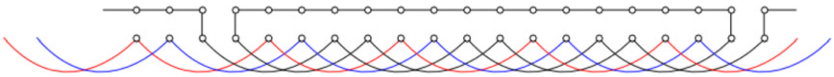

Fig. 5. Case 11100, becomes 10011 when reflected.

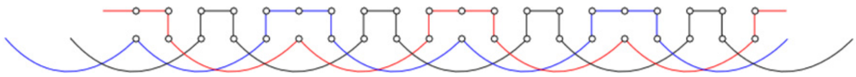

Fig. 6. Case 11010, becomes 10101 when reflected.

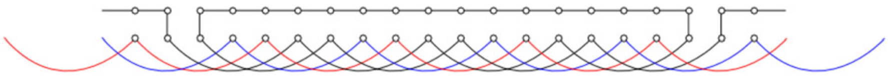

Fig. 7. Case 11001.

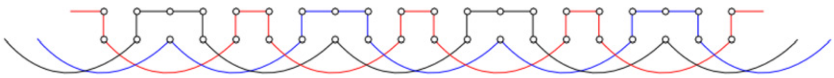

Fig. 8. Case 10110.

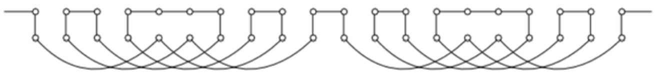

Fig. 9. Case 10000.

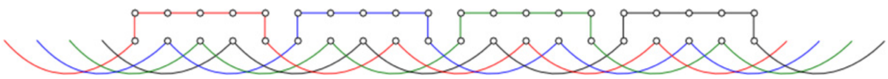

Fig. 10. Case 01111.

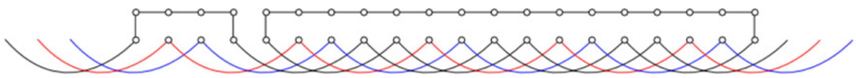

Fig. 11. Case 01110, becomes 00111 when reflected.

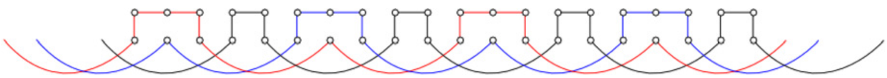

Fig. 12. Case 01101, becomes 01011 when reflected.

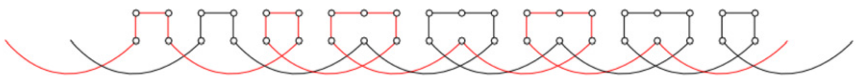

Fig. 13. Case 01010, becomes 00101 when reflected.

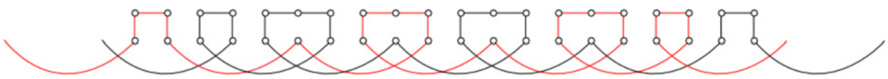

Fig. 14. Case 01001.

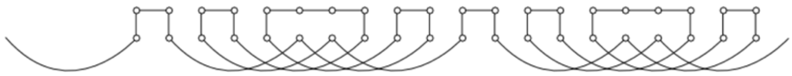

Fig. 15. Case 01000, becomes 00001 when reflected.

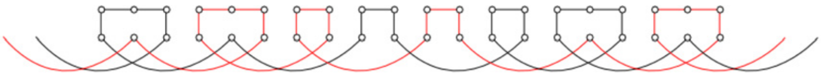

Fig. 16. Case 00110.

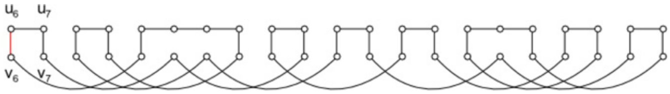

Fig. 17. Case 00000—replace (u_6, v_6) with the black path.

The remaining cases to be covered are 00000, 10100, 10010, 01100, 00011, 00100, 00010, 11000, and 10001. We now consider these cases separately. For the case concerning 00000, none of the five edges are to be replaced by paths. So P_0 must contain the edges (u_5, u_6), (u_6, v_6), (v_2, v_6). Note that P_0 contains the path $u_5 - u_6 - v_6 - v_2$. We now replace (u_6, v_6) (in red) by a path consisting of precisely the vertices of F'. This is shown in Fig. 17.

For the other remaining cases, we proceed as before. We have deleted the set F', which contains vertices of 20 consecutive columns. Now F' is a subset of F in $GP(m, 4)$. We note that G' contains the consecutive columns $\{u_5, v_5\}$, $\{u_6, v_6\}$, $\{u_{27}, v_{27}\}$, and $\{u_{28}, v_{28}\}$. As long as we insert 20 columns in a space between two adjacent columns listed above and perform the replacement of edges by paths with the constraint given above, the relative ordering of x_1, x_2, x_3, x_4 remains the same, hence our induction remains valid. In the remaining cases we insert the 20 columns between $\{u_4, v_4\}$ and $\{u_5, v_5\}$ or between $\{u_5, v_5\}$ and $\{u_6, v_6\}$ instead of the original $\{u_6, v_6\}$ and $\{u_{27}, v_{27}\}$ and show that the resulting graph is similar to earlier cases. This is the reason why we needed at least 22 consecutive columns containing none of x_1, x_2, x_3, x_4.

We begin with the case 10100. In P_0, vertex v_6 must have two incident edges, so P_0 contains (v_2, v_6) and (u_6, v_6). Now consider vertex u_6. Since (u_6, v_6) and (u_6, u_{27}) are in P_0, (u_5, u_6) is not in P_0. By assumption, this case implies that (v_3, v_{27}), (v_5, v_{29}) are not in P_0 and (v_4, v_{28}) is in P_0. Thus the only edges in P_0 of the form (x_i, y_j) where x and y can be either u or v, $i \in \{2, 3, 4, 5\}$ and $j \in \{6, 27, 28, 29\}$ are (v_2, v_6) and (v_4, v_{28}). This is equivalent to inserting our columns of F' between columns $\{u_5, v_5\}$ and $\{u_6, v_6\}$ for the case 01010, so we are done. We now note that 10010 is a reflection of 10100, so it is done by symmetry. (This is the reason to ensure that we have 24 consecutive columns containing none of x_1, x_2, x_3, x_4.)

For case 01100, we note that (u_5, u_6) and (u_6, v_6) must be in P_0 by considering u_6. So (v_2, v_6) is in P_0. We can now conclude that the only edges in P_0 of the form (x_i, y_j) where x and y can be either u or v, $i \in \{2, 3, 4, 5\}$ and $j \in \{6, 27, 28, 29\}$ are (u_5, u_6), (v_2, v_6), (v_3, v_{27}), (v_4, v_{28}). Therefore we may insert F' between columns $\{u_5, v_5\}$ and $\{u_6, v_6\}$ for the case 11110. We now note that 00011 is a reflection of 01100, so it is done by symmetry.

For case 00100, we see that (u_5, u_6) must be in P_0 by considering u_6. Now from v_6, we see that (v_2, v_6) must be in P_0. Thus we may insert F' between columns $\{u_5, v_5\}$ and $\{u_6, v_6\}$ as in case 11010. We now note that 00010 is a reflection of 00100, so it is done by symmetry.

Finally for case 11000, we see that (u_6, v_6) and (v_2, v_6) must be in P_0 by considering v_6. Then since (u_6, v_6) and (u_6, u_7) are in P_0, (u_5, u_6) cannot be in P_0, so (u_4, u_5) and (u_5, v_5) must be in P_0. Now consider v_5 to conclude that (v_1, v_5) must also be in P_0. Thus, we insert F' between columns $\{u_4, v_4\}$ and $\{u_5, v_5\}$, leaving us with case 11110. We now note that 10001 is a reflection of 11000, so it is done by symmetry.

This completes the proof. □

Obviously, Theorem 2.2 implies that $GP(n, 4)$ is Hamiltonian-connected for $n \geq 18$. This together with a computer search for $n \in [9, 17]$ gives the following result.

Corollary 2.3. [1] *Let $n \geq 9$. Then $GP(n, 4)$ is Hamiltonian-connected if and only if $n \neq 12$.*

We note that Corollary 2.3 verifies that Alspach's conjecture for $k = 4$ is correct as 12 and 4 are not relatively prime. However, it also shows that the relatively prime condition is far from necessary.

We now turn our attention to the 4-ordered-Hamiltonicity for $GP(n, 4)$. As before, we start with the following computer verified result, and we will defer the discussion to Section 3.

Lemma 2.4. *Let $n \in [9, 96]$. Then $GP(n, 4)$ is not 4-ordered-Hamiltonian if $n \in [9, 17]$ and $n = 20$. Moreover, $GP(n, 4)$ is 4-ordered-Hamiltonian if $n \in \{18, 19\}$ or $n \in [21, 96]$.*

[1] This corollary was presented as an individual result at the Forty-Second Southeastern International Conference on Combinatorics, Graph Theory and Computing [19].

Theorem 2.5. *Let $n \geq 9$. Then $GP(n, 4)$ is 4-ordered-Hamiltonian if and only if $n \in \{18, 19\}$ or $n \geq 21$.*

Proof. The proof is the same as the proof of Theorem 2.2. The initial step is given by Lemma 2.4. The induction step is exactly as in the proof of Theorem 2.2 by replacing the Hamiltonian path P_0 with a Hamiltonian cycle C_0. We note that despite the gap at $n = 20$, the induction step still starts at $m = 97$ as the induction step is applied to $GP(m-20, 4)$ and $m-20 > 20$. □

3. Computer verification of initial steps

Since Lemmas 2.1 and 2.4 rely on computer programs as justification, we will now describe this approach. As in any computer search, an existence result is easy to justify as it is unnecessary to check the details of the program as long as the existence of the desired object, in this case, a Hamiltonian cycle/path with restriction, is given, and the condition can be easily verified.

We first consider Lemma 2.1. For $n \in [18, 96]$, for every four vertices x_1, x_2, x_3, x_4 in $GP(n, 4)$, we need to find a Hamiltonian path between x_1 and x_4 containing x_1, x_2, x_3, x_4 in the prescribed order. The first observation is that we do not have to consider all such 4-tuples as $GP(n, 4)$ is highly symmetric. Let $U = \{u_i : 0 \leq i < n\}$ and $V = \{v_i : 0 \leq i < n\}$. Then it is clear that the vertices in U form a vertex-transitive class and the vertices in V form a vertex-transitive class. It is possible that for some n, the vertices in U and V form one vertex-transitive class, that is, $GP(n, 4)$ is vertex-transitive. (Indeed, it is known, in general, the precise condition for which $GP(n, k)$ is vertex-transitive.) Thus, we only have to consider the following possibilities:

(1) $x_1 = u_0$, and
(2) $x_1 = v_0$.

We note that we can further tighten the cases into (1) $x_1 = u_0$ and (2) $x_1 = v_0$ and $x_4 \in V$.

We now consider Lemma 2.4. For $n \in \{18, 19\} \cup [21, 96]$, for every four vertices x_1, x_2, x_3, x_4 in $GP(n, 4)$, we need to find a Hamiltonian cycle containing x_1, x_2, x_3, x_4 in the prescribed order. Again, since the vertices in U form a vertex-transitive class and the vertices in V form a vertex-transitive class, we only have to consider the following possibilities:

(1) $x_1 = u_0$, and
(2) $x_1 = v_0$.

We note that we can further tighten the cases into (1) $x_1 = v_0$ and (2) $x_1 = v_0$ and $x_2, x_3, x_4 \in V$.

Obviously, in the verification of Lemmas 2.1 and 2.4, one can generate all such Hamiltonian cycles/paths and use them to verify the claim. However, the number of Hamiltonian cycle/paths increases rapidly when n increases. We considered both depth-first search (DFS) and breadth-first search (BFS). The idea is to find a small set of Hamiltonian cycles/paths and hope that they are enough to certify that for every x_1, x_2, x_3, x_4, there is a Hamiltonian cycle containing them in the prescribed Hamiltonian path between x_1 and x_4 containing x_1, x_2, x_3, x_4 in the prescribed order. At first glance, DFS may be ideal for this task as one can find a Hamiltonian cycle/path quickly. Then once we have found what we estimate to be a large enough number of Hamiltonian cycles/paths, we will test whether they are sufficient for all certifications. Unfortunately, although one may be able to find a group of Hamiltonian cycles/paths quickly, there is not enough variation among them, thus rendering them insufficient. Instead, we use BFS. Here we need a queue to store every path that we find during the execution of the algorithm. We need to copy every path and determine the next vertex to be in the path. The number of paths is factorially large. In fact, the space we have allocated for the queue is not large enough to store every path of BFS. In our algorithm, the path is discarded when the queue is full. Thus, the algorithm is unable to find every Hamiltonian cycle/path of $GP(n, 4)$. Nevertheless, some Hamiltonian cycles/paths can be found rapidly, and fortunately, the program found enough Hamiltonian cycles/paths to certify that for every x_1, x_2, x_3, x_4, there is a Hamiltonian cycle containing them in the prescribed order or a Hamiltonian path between x_1 and x_4 containing x_1, x_2, x_3, x_4 in the prescribed order. For example, for the graph $GP(88, 4)$, the program found 311 Hamiltonian cycles and they are enough to certify that $GP(88, 4)$ is 4-ordered-Hamiltonian.

We now turn our attention to the negative statements asserted by the lemmas. For example, for $n \in [9, 17] \cup \{20\}$, we want to show that $GP(n, 4)$ is not 4-ordered-Hamiltonian. Here the computer program needs to verify a negative result for which there is no simple certificate as it is not known (and is probably unlikely) that 4-ordered-Hamiltonicity is in co-NP. Fortunately, the graphs are small and an exhaustive search is feasible.

We will now mention the computation time. These results were verified on two computers running Windows. Each machine has four processors with 3 GB of memory. The computation for Lemma 2.1 was done in 15 days and the computation for Lemma 2.4 was done in about 6 days. The sets of Hamiltonian cycles/paths that are sufficient to certify these results are given in http://www.csie.dyu.edu.tw/~spring/Fourordered/index.html.

4. Conclusion

We remark that although the proof of Theorems 2.2 and 2.5 relies on a computer search, these cases (except $n \leq 12$) are positive in the sense that the program verified the existence of a Hamiltonian cycle/path with the required properties by providing one. If one is not comfortable with using computers to verify negative results, one can simply replace these theorems by slightly weaker statements, that is, $GP(n, 4)$ is 4-ordered-Hamiltonian if $n \geq 21$ and $GP(n, 4)$ is 4-ordered-Hamiltonian-connected if $n \geq 18$. On the other hand, $GP(n, 4)$ is small enough if $n \in [9, 17]$ or $n = 20$ so that the claim is checkable by hand using an ad hoc argument although we have not done so. (Note that even here, a computer program is helpful as it gives us the proper x_1, x_2, x_3, x_4 to show that $GP(n, 4)$ is not 4-ordered-Hamiltonian (connected).)

Based on the results for $k = 2, 3, 4$ and additional computational data, we propose the following refinement of the conjecture of Alspach. We omit the case $k = 2$ as the result does not fit the pattern, whose classification is proved in [12].

Conjecture 4.1. *Let $k \geq 3$ be a fixed integer. Then there exists a finite set of integers $S \subseteq [2k+1, \infty)$ such that $GP(n, k)$ is Hamiltonian-connected/laceable if and only if $n \notin S$. Moreover, if $n \in S$, then n and k are not relatively prime.*

The conjecture does not specify S. An open problem is to classify S. In our studies, we found a subclass of $GP(n, k)$ with graphs that are not Hamiltonian-connected.

Proposition 4.2. *Let $k \geq 3$ and k be even. Then $GP(3k, k)$ is not Hamiltonian-connected.*

Proof. We claim that there is no Hamiltonian path between u_0 and u_2. Suppose there is a Hamiltonian path P between u_0 and u_2. The subgraph induced by the v_i's form k disjoint triangles. Note that the three vertices on such a triangle must appear consecutively on P. Construct G' by contracting each triangle into a vertex. Then P will become P', a Hamiltonian path between u_0 and u_2. It is easy to see that G' is bipartite with u_0 and u_2 on the same bipartition set. So there is no Hamiltonian path between u_0 and u_2, which is a contradiction. □

With this result, computer data and known results, a possible classification is the following. Let $k \geq 2$ and $n \geq 2k+1$. Then $GP(n, k)$ is Hamiltonian-connected/laceable except for the following: (1) $k = 2$ and $n \equiv 0, 4, 5 \pmod 6$, (2) $n = 3k$ and k is even, and (3) $GP(n, k)$ that is isomorphic to a graph in (1).

Conjecture 4.3. *Let $k \geq 3$ be a fixed integer. Then there exists a finite set of integers $S \subseteq [2k+1, \infty)$ such that $GP(n, k)$ is 4-ordered-Hamiltonian-connected/laceable if and only if $n \notin S$.*

These types of problems are difficult. It took considerable effort to finally completely solve the Hamiltonian problem for $P(n, k)$ as given in [18]. The result can actually be extended to the case when an edge or a vertex is deleted from the graph as given in [20]. The corresponding Hamiltonian-connected/laceable problems seem even more difficult as the original conjecture of Alspach is still not settled after over 20 years. This would indicate that the corresponding 4-ordered-Hamiltonian and 4-ordered-Hamiltonian-connected/laceable problem is even more difficult. We hope that the recent paper of [9,12] and the result in this paper would lead to an eventual complete classification of $P(n, k)$ for Hamiltonian-connectedness/laceability, 4-ordered-Hamiltonicity and 4-ordered-Hamiltonian connectedness/laceability.

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References

[1] L.-H. Hsu, C.-K. Lin, Graph Theory and Interconnection Networks, CRC Press, 2009.
[2] L. Ng, M. Schultz, *k*-ordered Hamiltonian graphs, Journal of Graph Theory 24 (1997) 45–57.
[3] G. Chen, R.J. Gould, F. Pfender, New conditions for *k*-ordered-Hamiltonian graphs, Ars Combinatoria 70 (2003) 245–255.
[4] J.R. Faudree, R.J. Faudree, Forbidden subgraphs that imply *k*-ordered and *k*-ordered Hamiltonian, Discrete Mathematics 243 (2002) 91–108.
[5] J.R. Faudree, R.J. Gould, F. Pfender, A. Wolf, On *k*-ordered bipartite graphs, Electronic Journal of Combinatorics 10 (2003) 11.
[6] R.J. Faudree, Survey of results on *k*-ordered graphs, Discrete Mathematics 229 (2001) 73–87.
[7] J.R. Faudree, R.J. Gould, A.V. Kostochka, L. Lesniak, I. Schiermeyer, A. Saito, Degree conditions for *k*-ordered Hamiltonian graphs, Journal of Graph Theory 42 (2003) 199–210.
[8] R. Li, S. Li, Y. Guo, Degree conditions on distance 2 vertices that imply *k*-ordered Hamiltonian, Discrete Applied Mathematics 158 (2010) 331–339.
[9] L.-H. Hsu, J.J.M. Tan, E. Cheng, L. Lipták, C.-K. Lin, M. Tsai, Solution to an open problem on 4-ordered Hamiltonian graphs, Discrete Mathematics 312 (2012) 2356–2370.
[10] M.E. Watkins, A theorem on Tait colorings with an application to the generalized Petersen graphs, Journal of Combinatorial Theory 6 (1969) 152–164.
[11] D.R. Silaban, A. Parestu, B.N. Herawati, K.A. Sugeng, Slamin, Vertex-magic total labelings of union of generalized Petersen graphs and union of special circulant graphs, Journal of Combinatorial Mathematics and Combinatorial Computing 71 (2009) 201–207.
[12] B. Alspach, J. Liu, On the Hamilton connectivity of generalized Petersen graphs, Discrete Mathematics 309 (2009) 5461–5473.
[13] G. Xu, 2-rainbow domination in generalized Petersen graphs $P(n, 3)$, Discrete Applied Mathematics 157 (2009) 2570–2573.
[14] C. Tong, X. Lin, Y. Yang, M. Luo, 2-rainbow domination of generalized Petersen graphs $P(n, 2)$, Discrete Applied Mathematics 157 (2009) 1932–1937.
[15] M. Behzad, P. Hatami, E.S. Mahmoodian, Minimum vertex covers in the generalized Petersen graphs $P(n, 2)$, Bulletin of the Institute of Combinatorics and its Applications 56 (2009) 98–102.
[16] G.N. Robertson, Graphs minimal under girth, valency and connectivity constraints, Ph.D. Thesis, University of Waterloo, 1968.
[17] J.A. Bondy, Variations on the Hamiltonian theme, Canadian Mathematical Bulletin 15 (1972) 57–62.
[18] B. Alspach, The classification of Hamiltonian generalized Petersen graphs, Journal of Combinatorial Theory, Series B 34 (1983) 293–312.
[19] D. Lu, R. Jia, E. Cheng, L. Lipták, M. Tsai, C.-K. Lin, J.M. Tan, L.-H. Hsu, Hamiltonian connectedness of the generalized Petersen graph $GP(n, 4)$, Congressus Numerantium 209 (2011) 5–15.
[20] T.-C. Mai, J.-J. Wang, L.-H. Hsu, Hyper Hamiltonian generalized Petersen graphs, Computers and Mathematics with Applications 55 (2008) 2076–2085.
[21] K. Mészáros, On 3-regular 4-ordered graphs, Discrete Mathematics 308 (2008) 2149–2155.